ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
*Classe di Scienze*

GIOVANNI LEONI

**Asymptotic stability for perturbed hamiltonian systems, II**

*Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série*, tome 23, nº 3 (1996), p. 531-549

<http://www.numdam.org/item?id=ASNSP_1996_4_23_3_531_0>

# Asymptotic Stability for Perturbed Hamiltonian Systems, II

GIOVANNI LEONI

## 1. – Introduction

The asymptotic stability of the rest state of the perturbed lagrangian system

$$(1.1)\qquad (\mathcal{L}_p(t,u,u'))' - \mathcal{L}_u(t,u,u') = Q(t,u,u'), \qquad J = [T,\infty),$$

has been studied extensively in the literature. See in particular the work of Artstein and Infante [1], Ballieu and Peiffer [2], Burton [3], Duffin [5], Hatvani [6], Levin and Nohel [12], Salvatori [21], Thurston and Wong [24], and Yoshizawa [26] for $\mathcal{L}(t,u,p) = \frac{1}{2}|p|^2 - F(t,u)$ and of Pucci and Serrin [18-20] for general lagrangians of the natural form

$$(1.2)\qquad \mathcal{L}(t,u,p) = G(u,p) - F(t,u).$$

It has been shown in [17] that, solely under the hypotheses that $\mathcal{L}$ is of class $C^1$ and that $\mathcal{L}(t,u,\cdot)$ is strictly convex in $\mathbb{R}^N$, the change of variables

$$t = t, \qquad u = u, \qquad v = \mathcal{L}_p(t,u,p)$$

transforms the system (1.1) into the *hamiltonian form*

$$(1.3)\qquad \begin{aligned} u' &= \mathcal{H}_v(t,u,v) \\ v' &= -\mathcal{H}_u(t,u,v) + P(t,u,v), \end{aligned}$$

where the functions $\mathcal{H}$ and $P$ are defined on the open set

$$D = \left\{(t,u,v) \in J \times \mathbb{R}^N \times \mathbb{R}^N : v = \mathcal{L}_p(t,u,p) \quad \text{for some} \quad p \in \mathbb{R}^N\right\}.$$

Note that if $|\mathcal{L}_p(t,u,p)| \to \infty$ as $|p| \to \infty$ for all $(t,u) \in J \times \mathbb{R}^N$, then $D = J \times \mathbb{R}^N \times \mathbb{R}^N$ (see the proof of Theorem 1.6.1 in Chapter I of [10]). A canonical example for this kind of behavior given when the function $G$ in (1.2)

Pervenuto alla Redazione il 14 giugno 1995.

is the $m$-laplacian, namely $G(p) = \frac{1}{m}|p|^m$, $m > 1$. Indeed the corresponding hamiltonian function is

$$\mathcal{H}(t, u, v) = \frac{m-1}{m}|v|^{m/(m-1)} + F(t, u) \quad \text{and} \quad D = J \times \mathbb{R}^N \times \mathbb{R}^N .$$

On the other hand, when $G$ in (1.2) is the *mean curvature operator* $G(p) = \sqrt{1 + |p|^2} - 1$, then

$$\mathcal{H}(t, u, v) = 1 - \sqrt{1 - |v|^2} + F(t, u) ,$$

and thus $D = \{(t, u, v) \in J \times \mathbb{R}^N \times \mathbb{R}^N : |v| < 1\}$.

With the exception of these simple examples, where the set $D$ can be found explicitly, in general the shape of $D$ can be rather complicated. Therefore for simplicity in what follows we will confine our interest to lagrangians $\mathcal{L}(t, u, p)$ for which

$$D \supset J \times \Omega , \tag{1.4}$$

where $\Omega$ is a bounded open subset of $\mathbb{R}^{2N}$ containing the origin.

This technical assumption places only a mild restriction on the form of $\mathcal{L}(t, u, p)$; thus it is satisfied in the majority of important cases and, in particular, by lagrangians of the general form (1.2).

[An example of a lagrangian function which does not satisfy (1.4) is given by

$$\mathcal{L}(t, u, p) = \left(\sqrt{1 + e^{-2t}|p|^2} - 1\right) - F(t, u), \qquad J = [1, \infty) .$$

Indeed, an easy calculation shows that

$$D = \{(t, u, v) \in [1, \infty) \times \mathbb{R}^N \times \mathbb{R}^N : |v| < e^{-t}\} .]$$

The structure of (1.3) leads one to introduce the symmetrized set of equations

$$\begin{aligned} u' &= \mathcal{H}_v(t, u, v) + S(t, u, v) \\ v' &= -\mathcal{H}(t, u, v) + P(t, u, v) , \end{aligned} \tag{1.5}$$

whose theory turns out to be particularly interesting. If we write $x = \begin{pmatrix} u \\ v \end{pmatrix}$, then the perturbed hamiltonian system (1.5) can be rewritten in the compact form

$$x' = \mathcal{J}\mathcal{H}_x(t, x) + R(t, x) , \tag{1.6}$$

where

$$\mathcal{H} : J \times \Omega \to \mathbb{R}, \qquad R : J \times \Omega \to \mathbb{R}^{2N} ,$$

with $\Omega \subset \mathbb{R}^{2N}$ a bounded open set containing the origin, and where

$$\mathcal{J} = \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix}$$

is the simplectic matrix.

A (classical) solution of (1.6) is a function $u : J \to \Omega$ of class $C^1$ which satisfies the system (1.6) for all $t \in J$.

Under the assumptions $\mathcal{H}_x(t,0) = R(t,0) = 0$ the rest state $x \equiv 0$ is a solution of (1.6). This state is said to be *stable* if for every $\epsilon > 0$ there exists $\varepsilon_0 = \varepsilon_0(\epsilon) > 0$ such that any solution $x(t)$ of (1.6) with $|x(T)| \le \varepsilon_0$ is defined on all of $J$ and satisfies

$$|x(t)| \le \epsilon \qquad \text{for all} \quad t \ge T\,.$$

The rest state is *asymptotically stable* if it is stable and if there exists $\varepsilon_1 > 0$ such that for any solution $x(t)$ of (1.6) with $|x(T)| \le \varepsilon_1$ we have $x(t) \to 0$ as $t \to \infty$.

It is worth noting that when the hamiltonian $\mathcal{H}(t,x)$ is defined on all of $J \times \mathbb{R}^{2N}$ our techniques can be used to show that, under appropriate assumptions on $\mathcal{H}$ and $R$, all bounded solutions of (1.6) tend to zero as $t \to \infty$, or, in other words, that the rest state $x \equiv 0$ is a global attractor, see also [9].

For $R(t,x)$ to represent a damping term we require the inequality

$$(1.7) \qquad (\mathcal{H}_x(t,x), R(t,x)) \le 0 \qquad \text{for all} \quad (t,x) \in J \times \Omega\,. \quad ^{(1)}$$

Note that for hamiltonian systems (1.3) arising from perturbed lagrangian systems (1.1), condition (1.7) becomes

$$(1.7') \qquad (\mathcal{H}_v(t,x), P(t,x)) \le 0 \quad \text{or equivalently} \quad (p, Q(t,u,p)) \le 0\,.$$

We allow the damping to oscillate in magnitude as a function of time between zero and infinity. However, it is well know that when the damping is unbounded in magnitude we may lose asymptotic stability due to the phenomenon of overdamping. Therefore, in order to obtain asymptotic stability it is necessary to place appropriate growth restrictions on the damping as the time increases to infinity. This is accomplished by the main condition $(A_2)$ in Section 2 below. For parabolic systems a similar hypothesis, although somewhat stronger, was introduced by Pucci and Serrin in a remark of [16]. Note that $(A_2)$ combines a growth condition on the damping $R(t,x)$ as a function of time together with a geometrical condition on the vectors $R(t,x)/|R(t,x)|$ and $\mathcal{H}_x(t,x)/|\mathcal{H}_x(t,x)|$. See the remarks at the end of Section 2 below.

Furthermore the special form of $(A_2)$ allow us to use a new integral condition, of the type introduced for hyperbolic systems by Pucci and Serrin in [16],

(1) By $(\cdot,\cdot)$ we mean the inner product either in $\mathbb{R}^N$ or $\mathbb{R}^{2N}$, depending on the context. Thus in particular (1.7) uses the inner product in $\mathbb{R}^{2N}$, while in (1.7′) the inner product is in $\mathbb{R}^N$.

which improves upon earlier work. See particular [18, Theorem 4.1], [19, Theorem 2] and [10, Theorem 2.8.1].

While our concerns are theoretical, the stabilization of hamiltonian systems is also of physical interest. In addition to applications to standard electrical and mechanical networks, other examples arise from feedback methods for dynamic control of robot manipulators. This theory turns out be particularly interesting, see for example the work of Van Der Schaft [25] and of Takegaki and Arimoto [23], though we shall not dwell on this here. We only point the fact that for robot manipulators the generalized coordinates $u_i$, $i = 1, \ldots, N$ represent either joint displacements or joint angles, depending on whether the joint $i$ is translational or rotational. Therefore even for the simplest models of robot manipulators we must have $N > 1$. This fact alone fully justified the choice to study differential systems ($N > 1$) rather than scalar differential equations ($N = 1$).

When $\mathcal{H}$ has the form

$$\mathcal{H}(t, x) = \frac{1}{2}(H(t)x, x),$$

where

$$H(t) = (h_{ij}(t))_{i,j=1}^{2N}$$

is a symmetric $2N \times 2N$ matrix, the system (1.6) reduces to the classical form

$$x' = \mathcal{J}H(t)x + R(t, x). \tag{1.8}$$

To described our results, consider the special case in (1.8) where

$$H(t) = b\, I_{2N}, \qquad b > 0.$$

The system can then be written as

$$\begin{aligned} u' &= \phantom{-}b\, v + S(t, u, v) \\ v' &= -b\, u + P(t, u, v). \end{aligned} \tag{1.9}$$

When $N = 1$ and

$$\lim_{x \to 0} \frac{S(t, x)}{|x|} = \lim_{x \to 0} \frac{P(t, x)}{|x|} = 0,$$

the linearized system for (1.9) is given by

$$\begin{aligned} u' &= \phantom{-}b\, v \\ v' &= -b\, u. \end{aligned} \tag{1.10}$$

Since the eigenvalues for this system are purely immaginary, the origin $x = 0$ is called a *center* for (1.10).

The perturbation of a center has been treated extensively in the literature (see e.g. Chapter 2 in [13] and Chapter 15 in [4]), but only for the autonomous case, that is for the special case when $S = S(x)$ and $P = P(x)$. Here we study the (new) situation in which $S$ and $P$ strongly depend on the time variable $t$.

To illustrate our results we take for simplicity

$$S(t,x) = -\sigma_1(t)|u|^{\alpha_1}|v|^{\beta_1}u, \quad P(t,x) = -\sigma_2(t)|u|^{\alpha_2}|v|^{\beta_2}v\,,$$

where $\alpha_1$, $\beta_2 > -1$, $\alpha_2$, $\beta_1 \geq 0$ and $\sigma_1$, $\sigma_2$ are nonnegative functions. The system (1.9) then reduces to

$$\begin{aligned} u' &= \phantom{-}b\,v - \sigma_1(t)|u|^{\alpha_1}|v|^{\beta_1}u \\ v' &= -b\,u - \sigma_2(t)|u|^{\alpha_2}|v|^{\beta_2}v\,. \end{aligned} \tag{1.11}$$

The case $\beta_1 = \alpha_2 = 0$ has been studied in [10, Chapter III]. Let

$$\delta(t) = \sigma_1(t) + \sigma_2(t)\,.$$

If $\delta$ is too small, say $\delta \in L^1(J)$, then asymptotic stability fails. To see this, let $b = 1$, $\alpha_1 = \beta_1 = \alpha_2 = \beta_2 = 0$ and $\sigma_1 = \sigma_2 = \delta$. The system (1.11) has the form

$$x' = \mathcal{J}x - \delta(t)x\,.$$

If $x(t)$ is a solution of this system with $|x(T)| = \varepsilon_0 \neq 0$, then $x(t)$ does not tend to 0 as $t \to \infty$, since

$$|x(t)|^2 = \varepsilon_0^2 \exp\left(-2\int_T^t \delta(s)ds\right) \to \varepsilon_0^2 \exp(-2\|\delta\|_{L^1(J)}) \quad \text{as} \quad t \to \infty\,,$$

although its $L^\infty(T,\infty)$-norm can be made arbitrarily small, by letting $\varepsilon_0 \to 0$.

On the other hand, if $\delta$ is too large, then again we lose asymptotic stability. Consider the following system

$$\begin{aligned} u' &= \phantom{-}v \\ v' &= -u - \sigma_2(t)|u|^{1/2}|v|^{-1/2}v\,, \end{aligned}$$

where $b = 1$, $\alpha_2 = \frac{1}{2}$, $\beta_2 = -\frac{1}{2}$ and

$$\sigma_1(t) \equiv 0, \quad \sigma_2(t) = \delta(t) = \frac{t^2(t^\epsilon + 1) + \epsilon(\epsilon + 1)}{\sqrt{\epsilon}\,t^{3/2}(t^\epsilon + 1)^{1/2}} \quad \text{for} \quad t \geq T > 0\,.$$

A solution of this system is

$$u(t) = u_0(1 + t^{-\varepsilon}), \quad v(t) = -u_0\,\varepsilon\,t^{-\varepsilon - 1}\,,$$

whose $L^\infty(T,\infty)$-norm can be made arbitrarily small, by letting $u_0 \to 0$, but which does not tend to zero as $t \to \infty$ if $u_0 \neq 0$.

The following conclusions hold for the system (1.11), as immediate consequences of our main results in Section 3 and 6.

THEOREM A. (i) *Suppose $\delta$ is bounded on $J$ and $\delta \in L^1(J)$. If $\delta$ is absolutely continuous and*

$$\int_T^t |\delta'| ds \leq \text{Const.} \int_T^t \delta\, ds \quad \textit{for all} \quad t \in J \quad \textit{sufficiently large},$$

*then the rest state of* (1.11) *is asymptotically stable.*

(ii) *Suppose that $1/\delta$ is bounded on $J$ and let*

$$m = \min\{\alpha_1, \beta_2\} + 2, \quad \lambda = 1/\max\{1, \alpha_1 + 2, \alpha_2, \beta_1, \beta_2 + 2\}. \tag{1.12}$$

*If $1/\delta \notin L^{1/(m-1)}(J)$ and $|\delta'| \leq \text{Const.}\, \delta^{1+\lambda m/(m-1)}$, then the rest state of* (1.11) *is asymptotically stable.*

When $\delta$ is at the same time neither bounded from above nor away from zero, the situation is more delicate.

THEOREM B. *Let $m$ and $\lambda$ as in* (1.12). *Then the rest state of* (1.11) *is asymptotically stable if one of the following conditions* (a), (b), (c) *is satisfied.*

(a) *There exists a number $a \in [0, 1]$ such that for all $t \in J$ sufficiently large*

$$\int_T^t (\delta + \delta^{1-m}) \frac{ds}{s^{am}} \leq \text{Const.} \begin{cases} t^{(1-a)m} & \textit{for} \quad a < 1 \\ \log^m t & \textit{for} \quad a = 1. \end{cases}$$

(b) *There exists a nonnegative, bounded absolutely continuous function $k = k(t)$ such that*

$$k \notin L^1(J), \quad \int_T^t |k'| ds \leq \text{Const.} \int_T^t k\, ds \quad \textit{for all } t \in J \textit{ sufficiently large}, \tag{1.13}$$

*and*

$$\text{(i)} \quad (\delta + \delta^{1-m}) k^{m-1} \in L^\infty(J) \quad \textit{or} \quad \text{(ii)} \quad (\delta + \delta^{1-m}) k^m \in L^1(J).$$

(c) *There exists a nonnegative, bounded absolutely continuous unction $k = k(t)$ on $J$ satisfying $(1.13)_1$ and* (i) *or* (ii), *such that*

$$|k'| \leq \text{Const.}\ \delta^\lambda k^{1-\lambda} \quad \textit{a.e. on} \quad J.$$

Theorem A follows from Theorem B with $k = \delta$ in the first part and $k = \delta^{-1/(m-1)}$ in the second. Theorem B(a) comes from Corollary 3 in Section 3, Theorem B(b) follows from Theorem 3.1, while Theorem B(c) is a consequence of Theorem 6.1.

This paper is organized as follows. The following section contains our basic assumptions. In Section 3 we present the main conclusions of the paper and in Section 4-5 their proofs. Finally in Section 6 some additional results are given.

I wish to tank Prof. J. Serrin for his valuable help and constant encouragement and Prof. P. Pucci for carefully reading the manuscript and for many helpful suggestions. I also want to thank them for giving me a copy of their preprint papers. *Asymptotic stability for non-autonomous dissipative wave systems* and *Asymptotic stability for nonlinear parabolic systems.*

## 2. – Preliminaries

We consider the system (1.6) under the following specific hypotheses which are assumed to hold throughout the paper.
$(H_1)$ $\mathcal{H} \in C^1(J \times \Omega; \mathbb{R})$; $\mathcal{H}(t, 0) = 0$ for all $t \in J$. For all $x_0 > 0$ there exists a nonnegative function $\psi \in L^1(J)$ and a constant $\kappa > 0$ such that

$$(2.1) \qquad \mathcal{H}_t(t, x) \leq \psi(t) \quad \text{when} \quad t \in J \text{ (a.e.)} \quad \text{and} \quad x \in \Omega\,,$$

$$(2.2) \qquad (\mathcal{H}_x(t, x), x) \geq \kappa \quad \text{when} \quad (t, x) \in J \times \Omega \quad \text{and} \quad |x| \geq x_0\,.$$

$(H_2)$ $R \in C(J \times \Omega; \mathbb{R}^{2N})$; $R(t, 0) = 0$ for all $t \in J$.

$$(2.3) \qquad (\mathcal{H}_x(t, x), R(t, x)) \leq 0 \quad \text{for all} \quad (t, x) \in J \times \Omega\,.$$

Condition (2.2) implies in particular that

$$(\mathcal{H}_x(t, x), x) \geq 0 \quad \text{for all} \quad (t, x) \in J \times \Omega\,.$$

If we now fix $t \in J$ and set $x = -s\mathcal{H}_x(t, 0)$ in the previous inequality, where $s > 0$ is so small that $x \in \Omega$, we obtain, by letting $s \to 0^+$, that $\mathcal{H}_x(t, 0) = 0$. Since $R(t, 0) = 0$, it is clear that (1.6) admits the rest state $x \equiv 0$ as a solution.

Hypotheses $(H_1)$-$(H_2)$, together with the uniqueness of the initial value problem $x(T) = 0$, are all that is needed for the stability of the rest state of (1.6). Indeed we have the following result (for a proof see [10])

THEOREM 2.1. *Assume that the function $\mathcal{H}$ satisfies conditions $(H_1)$-$(H_2)$. Suppose moreover that the only solution of the initial value problem $x(T) = 0$ is $x \equiv 0$. Then the rest state of (1.6) is stable.*

REMARK. When condition (2.1) is strengthened to $\mathcal{H}_t(t, x) \leq 0$ a.e. in $J$ and for all $x \in \Omega$ then it is well known that there is no need to assume uniqueness of the initial value problem $x(T) = 0$. In this case we can also apply a standard Liapunov Theorem to obtain stability of (1.6), where $\mathcal{H}$ is the natural candidate for the Liapunov function.

To prove asymptotic stability we require also the following conditions $(A_1)$-$(A_3)$. Set

$$u = (x_1, \dots, x_N) \quad \text{and} \quad v = (x_{N+1}, \dots, x_{2N})\,.$$

$(A_1)$ There exists a number $D > 0$ such that

$$|\mathcal{H}_x(t, x)| \leq D \quad \text{for all} \quad (t, x) \in J \times \Omega\,.$$

$(A_2)$ For all $x_0 > 0$ there exist a measurable control set $I \subset J$, a measurable function $\delta : I \to [0, \infty)$ and a number $m > 1$ such that

$$|R(t, x)| \leq \delta^{1/m}(t) \left|\left(\mathcal{H}_x(t, x), R(t, x)\right)\right|^{1/m'} + \text{Const.} \left|\left(\mathcal{H}_x(t, x), R(t, x)\right)\right|$$

for all $(t, x) \in I \times \Omega$ with $|x| \geq x_0$, where $m'$ denotes the Hölder conjugate exponent of $m$.

$(A_3)$ For all $x_0 > 0$ there exist a function $\sigma \in L^\infty_{\mathrm{loc}}(J; [0, \infty))$ and a continuous function $\phi : \Omega \to [0, \infty)$ such that

$$\left|(\mathcal{H}_x(t, x), R(t, x))\right| \geq \sigma(t)\phi(x) \tag{2.4}$$

for all $(t, x) \in J \times \Omega$ with $|x| \geq x_0$, and

$$\phi(x) > 0 \quad \text{when} \quad u \neq 0 \quad \text{and} \quad v \neq 0\,. \tag{2.5}$$

REMARKS. A condition of the type $(A_2)$ first appears in a remark of the work [16] of Pucci and Serrin, where they treated the asymptotic stability of parabolic systems. Note that it implies in particular that $R(t, x) = 0$ whenever the vectors $R(t, x)$ and $\mathcal{H}_x(t, x)$ are perpendicular. For hamiltonian systems (1.3) arising from perturbed lagrangian systems (1.1), condition $(A_2)$ becomes

$$|Q(t, u, p)| \leq \delta^{1/m}(t)|(p, Q(t, u, p))|^{1/m'} + \text{Const.}\,|(p, Q(t, u, p))|\,, \tag{2.6}$$

which in turn yields

$$|Q(t, u, p)| \leq \delta(t)|p|^{m-1} \quad \text{for all} \quad |p| \quad \text{sufficiently small}\,. \tag{2.7}$$

This last hypothesis was used in [10, Chapter II], [15] and [18-19], together with the *tameness* assumption.

$$|p| \cdot |Q(t, u, p)| \leq \text{Const.}\,|(p, Q(t, u, p))|\,. \tag{2.8}$$

If we denote by $\Theta$ the angle between the vectors $-p$ and $Q$, then (2.8) is equivalent to

$$|\cos\Theta(t, u, p)| \geq \text{Const.} > 0\,.$$

Thus the geometrical meaning of (2.8) is simply that the vectors $-p$ and $Q$ are bounded away from orthogonality. Note that this always true when $N = 1$ and indeed whenever the vectors $-p$ and $Q$ have the same direction.

It is easy to see that (2.7) and (2.8) imply (2.6), while (2.6) does not imply the tameness condition (2.8). Indeed, let $N = 2$, $J = [1, \infty)$ and $p = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$.

The damping term

$$Q(t, u, p) = -t\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} + t^{1+\epsilon}\begin{pmatrix} p_2 \\ -p_1 \end{pmatrix}$$

does not very (2.8), since

$$|Q(t, u, p)| = |p| t\sqrt{1+t^{2\epsilon}} \quad \text{and} \quad (p, Q(t, u, p)) = -t|p|^2,$$

so that

$$|\cos\Theta(t, u, p)| = 1/\sqrt{1+t^{2\epsilon}} \to 0 \quad \text{as} \quad t \to \infty.$$

On the other hand condition (2.6) clearly holds with $m = 2$ and $\delta(t) = 2t^{1+2\epsilon}$.

Condition $(A_1)$ is a natural consequence of hypotheses (2.1) and (2.2) for systems arising from lagrangian functions of the general form

$$\mathcal{L}(t, u, p) = G(u, p) - \ell(t)F(u), \tag{2.9}$$

as well as for the canonical system (1.8) in the introduction. Indeed in the first case the hamiltonian corresponding to (2.9) is

$$\mathcal{H}(t, u, v) = G^*(u, v) + \ell(t)F(u),$$

defined on a set $J \times A$, where $A$ is an open set of $\mathbb{R}^{2N}$ containing the origin. Consequently

$$\mathcal{H}_u(t, x) = G^*_u(u, v) + \ell(t)F_u(u) \quad \text{and} \quad \mathcal{H}_v(t, x) = G^*_v(u, v).$$

If we now take the open bounded set $\Omega$ such that

$$\bar{\Omega} \subset A,$$

condition $(A_1)$ holds, if we show that $\ell$ is bounded. To see this note that hypothesis (2.2) implies in particular that $\ell(t) \geq$ Const. $> 0$, while (2.1) is equivalent to $(\ell')^+ \in L^1(J)$. Therefore

$$0 < \ell(t) \leq \ell(T) + \int_T^\infty (\ell')^+(s)ds < \infty,$$

as claimed.

Similarly for (1.8), property (2.1) holds if and only if

$$(h'_{ii})^+ \in L^1(J) \quad \text{and} \quad h'_{ij} \in L^1(J) \quad \text{for} \quad j \neq i. \tag{2.10}$$

By $(2.10)_1$ the $h_{ii}$'s are bounded above, while by $(2.10)_2$ all the $h_{ij}$'s are bounded when $i \neq j$. Let $h(t)$ be the least eigenvalue of the matrix $H(t)$ and $x = x(t) \in R^{2N}$ a corresponding unitary eigenvector. It now follows from (2.2) that

$$(H(t)x, x) = h(t) \geq \kappa = \kappa(1) > 0 \quad \text{for all} \quad t \in J.$$

Therefore the matrix $H(t)$ is *uniformly* positive definite for $t \in J$. In turn this implies that $h_{ii}(t) \geq \kappa$ in $J$ so that all the entries of $H(t)$ are uniformly bounded on $J$ and $(A_1)$ follows.

## 3. – Main results

In this section we present our principal asymptotic stability results. Here and for the proofs in Section 4-5, *we assume that conditions* $(H_1)$-$(H_3)$ *and* $(A_1)$-$(A_3)$ *hold and that the only solution of the initial value problem* $x(T) = 0$ *is* $x \equiv 0$.

In what follows $\sigma$ and $\delta$ are the functions given in $(A_2)$ and $(A_3)$ corresponding to any fixed $x_0$. The function $k$ in Theorem 3.1 below depends on the functions $\sigma$ and $\delta$ and, in turn, on $x_0$. For simplicity in the notation we do not indicate explicitly this dependence.

In stating the following theorems we agree that the function $\delta k$ is extended to all of $J$ by the definition $\delta(t)k(t) = 0$ for $t \in J \backslash I$ and that $\sigma^{1-m}(t)k^m(t) = 0$ whenever $k(t) = 0$.

THEOREM 3.1. *Assume that there exists a nonnegative, bounded absolutely continuous function k on J such that*

$$k \notin L^1(J), \qquad k = 0 \text{ on } J\backslash I. \tag{3.1}$$

*Suppose also that*

$$\liminf_{t\to\infty} \frac{\left(\int_T^t (\delta + \sigma^{1-m})k^m \, ds\right)^{1/m} + \int_T^t |k'| ds}{\int_T^t k \, ds} = M < \infty. \tag{3.2}$$

*Then the rest state of* (1.6) *is asymptotically stable.*

COROLLARY 1. *Assume that* $\sigma$ *and* $\delta\,\sigma^{m-1}$ *are bounded in* $J$. *If* $\sigma$ *is absolutely continuous,* $\sigma \notin L^1(J)$ *and*

$$\int_T^t |\sigma'| ds \leq \text{Const.} \int_T^t \sigma \, ds \quad \text{for all} \quad t \in J \quad \text{sufficiently large}, \tag{3.3}$$

*then the rest state of* (1.6) *is asymptotically stable.*

PROOF. Take $k = \sigma$ in Theorem 3.1. □

REMARKS. Hypothesis (3.3) is implied by the easier condition

$$|\sigma'(t)| \leq \text{Const.}\,\sigma(t) \quad \text{a.e. in } J. \tag{3.4}$$

On the other hand there are functions $\sigma$ for which (3.3) holds without (3.4) being satisfied. An easy example is given by the function $\sigma = (\cos t)^+$.

COROLLARY 2. *Assume that either* $k \in BV(J)$ *or* $I = J$ *and* $\log k \in \text{Lip}(J)$. *Then condition* (3.2) *in Theorem 3.1 can be weakened to*

$$\liminf_{t\to\infty} \frac{\left(\int_T^t (\delta + \sigma^{1-m})k^m \, ds\right)^{1/m}}{\int_T^t k \, ds} < \infty. \tag{3.5}$$

PROOF. When $k \in BV(J)$ then $k' \in L^1(J)$, while when $\log k \in \mathrm{Lip}(J)$ then $|k'| \le \mathrm{Const.}\, k$. Therefore in both cases (using also $(3.1)_1$) we have

$$\int_T^t |k'| ds \le \mathrm{Const.} \int_T^t k\, ds \quad \text{for all} \quad t \in J \quad \text{sufficiently large},$$

which, together with (3.5) implies (3.2). □

REMARKS. Under the additional hypothesis

$$k(t) \le \mathrm{Const.}\, \sigma(t) \quad \text{for all} \quad t \in J\,, \tag{3.6}$$

condition (3.5) in Corollary 2 can be simplified to

$$\liminf_{t \to \infty} \frac{\left( \int_T^t \delta\, k^m\, ds \right)^{1/m}}{\int_T^t k\, ds} < \infty\,. \tag{3.7}$$

Condition (3.5) was introduced by Pucci and Serrin in [15], where they treated the asymptotic stability for dissipative wave systems. It significantly improves Theorem 2.8.1 in [10], where (3.6) and (3.7) were assumed in place of (3.5) and the exponent $1/m$ in (3.7) was missing. A condition of the type (3.5) also appears in Theorem 2 in [20]. Condition (3.2) is new.

COROLLARY 3. *There exists a number* $a \in [0, 1]$ *such that for all* $t \in J$ *sufficiently large*

$$\int_T^t (\delta + \sigma^{1-m}) \frac{ds}{s^{am}} \le \mathrm{Const.} \begin{cases} t^{(1-a)m} & \text{for} \quad a < 1 \\ \log^m t & \text{for} \quad a = 1\,. \end{cases}$$

PROOF. Apply Corollary 2 with $k(t) = \min\{1, 1/t^a\}$. □

REMARK. Results corresponding to Corollaries 2 and 3 have been given in [15] for hyperbolic systems.

## 4. – Preliminary lemmas

We present here some elementary consequences of the assumptions in Theorem 3.1.

LEMMA 4.1. *Let* $\mathcal{G}(t, x) = -(\mathcal{H}_u(t, x), u) + (\mathcal{H}_v(t, x), v)$. *Then for every* $x_0 > 0$ *there exist two positive numbers* $\kappa$, $w_0$ *such that*

$$\mathcal{G}(t, x) \ge \kappa \qquad \text{for all} \quad (t, x) \in J \times \Omega \ \text{ with } \ |x| \ge x_0 \ \text{ and } \ |u| \le w_0\,;$$

*and*

$$\mathcal{G}(t, x) \le -\kappa \quad \text{for all} \quad (t, x) \in J \times \Omega \ \text{ with } \ |x| \ge x_0 \ \text{ and } \ |v| \le w_0\,.$$

Proof. By (2.2) there is a constant $\kappa_1 = \kappa_1(x_0)$ such that

$$(\mathcal{H}_x(t,x), x) \geq \kappa_1 \quad \text{when} \quad (t,x) \in J \times \Omega \quad \text{and} \quad |x| \geq x_0 . \tag{4.1}$$

Let $w_0 = \kappa_1/4D$ and $\kappa = \kappa_1/2$, then by $(A_1)$

$$|(\mathcal{H}_u(t,x), u))| \leq \frac{\kappa_1}{4} \text{ for all } (t,x) \in J \times \Omega \text{ with } |x| \geq x_0 \text{ and } |u| \leq w_0 . \tag{4.2}$$

Therefore

$$\mathcal{G}(t,x) = -2(\mathcal{H}_u(t,x), u) + (\mathcal{H}_x(t,x), x) \geq -2\frac{\kappa_1}{4} + \kappa_1 = \kappa > 0$$

by (4.1) and (4.2). This proves the first part of the lemma. The second part follows in a similar fashion by writing $\mathcal{G}(t,x) = -(\mathcal{H}_x(t,x), x) + 2(\mathcal{H}_v(t,x), v)$ and using (4.1) and $(A_1)$. □

Lemma 4.2. *Let $F = F_{C,u_0,v_0} = \{x \in C : |u| \geq u_0, |v| \geq v_0\}$, for some fixed compact set $C \subset \Omega$ and some constant $u_0, v_0 > 0$. Then*

$$1 \leq \text{Const.}\, \sigma^{(1-m)/m}(t)|(\mathcal{H}_x(t,x), R(t,x))|^{1/m'} \quad \textit{for } (t,x) \in J \times F ,$$

*where the constant depends on $C$, $u_0$ and $v_0$.*

Proof. By (2.5)

$$\phi_0 = \min_{x \in F} \phi(x) > 0 .$$

Now, from (2.4) we obtain

$$\begin{aligned} 1 &\leq \sigma^{(1-m)/m} \sigma^{(m-1)/m} \left( \frac{\phi(x)}{\phi_0} \right)^{(m-1)/m} \\ &\leq \phi_0^{(1-m)/m} \sigma^{(1-m)/m} |(\mathcal{H}_x(t,x), R(t,x))|^{1/m'} \end{aligned}$$

for $(t,x) \in J \times F$, which is the desired conclusion. □

## 5. – Proof of Theorem 3.1

If $r \in (0,1)$ is a sufficiently small constant, then

$$\overline{B(0,r)} \subset \Omega .$$

Moreover, since by Theorem 2.1 the rest state of (1.6) is stable, there is an $\varepsilon_1 > 0$ such that any solution $x = x(t)$ of (1.6) with $|x(T)| \leq \varepsilon_1$ is defined on all of $J$ and satisfies

$$|x(t)| \leq r \qquad \text{for all} \quad t \in J . \tag{5.1}$$

Thus to prove asymptotic stability it is sufficient to show that any solution $x = x(t)$ of (1.6) satisfying (5.1) has the property

$$x(t) \to 0 \quad \text{as} \quad t \to \infty .$$

The following three lemmas hold solely under assumptions $(H_1)$ and $(H_2)$.

LEMMA 5.1. *Let* $x = x(t)$ *be any solution of* (1.6) *satisfying* (5.1).
(i) *There exists* $\ell \geq 0$ *such that*

$$\mathcal{H}(t, x) \to \ell \quad as \quad t \to \infty,$$

(ii) $(\mathcal{H}_x(t, x), R(t, x)) \in L^1(J)$.

LEMMA 5.2. $x(t) \to 0$ *as* $t \to \infty$ *if and only if* $\ell = 0$ *in Lemma 5.1*(i).

LEMMA 5.3. *If* $\ell > 0$ *in Lemma 5.1* (i), *then there exist* $T_1 \geq T$ *and a positive number* $q$ *(depending on* $\ell$*) such that*

$$|u(t)| + |v(t)| \geq 4q \quad for\ all \quad t \in J_1 = [T_1, \infty).$$

We omit the proofs of Lemmas 5.1-5.3 since they can be found in [10, Chapter III] and are easily derived from $(H_1)$-$(H_2)$. In particular Lemma 5.1(i) is based on the primary identity

$$\{\mathcal{H}(t, x)\}' = (\mathcal{H}_x(t, x), R(t, x)) + \mathcal{H}_t(t, x) \quad \text{in} \quad J.$$

LEMMA 5.4. *Let* $x = x(t)$ *be any solution of* (1.6) *satisfying* (5.1) *and for which* $\ell > 0$ *in Lemma 5.1. Then, under the hypotheses of Theorem 3.1, there exists a positive constant* $\kappa$ *(depending on* $\ell$*) such that*

$$\begin{aligned} k\,h(u, v) \leq \text{Const.} &\int_{T_2}^{t} (\delta + \sigma^{1-m})^{1/m} k |(\mathcal{H}_x, R)|^{1/m'} ds \\ &+ \frac{\kappa}{M+1} \int_{T_2}^{t} |k'| ds - \kappa \int_{T}^{t} k\, ds + \text{Const.}(T_2), \qquad t \in [T_2, \infty), \end{aligned} \tag{5.2}$$

*where* $T_2 \geq T_1$ *and* $h(t) = -g(u(t)) + g(v(t))$, *with* $g : \mathbb{R}^N \to [0, 1]$ *a function of class* $C^1$ *such that*

$$g(w) = \begin{cases} 1 & if \quad |w| \leq \varepsilon \\ 0 & if \quad |w| \geq 2\varepsilon, \end{cases} \tag{5.3}$$

*for a sufficiently small constant* $\varepsilon$ *(depending on* $\ell$*).*

PROOF. Let $x = x(t)$ be a solution of (1.6) satisfying (5.1) and for which $\ell > 0$ in Lemma 5.1. By Lemma 5.3 we can take the functions $\delta$, $\sigma$, $\phi$ in $(A_2)$-$(A_3)$ corresponding to the choice $x_0 = 2\sqrt{2}q$. These choices determine the function $k$ in Theorem 3.1. In view of (1.6) the following identity holds in $J$ along $x$

$$\{k\,h(u, v)\}' = k\,h\,\mathcal{G}(t, x) + k\,h(R(t, x), x^*) + k'\,h(u, v) + k\,h'(u, v),$$

where

$$\mathcal{G}(t, x) = -(\mathcal{H}_u(t, x), u) + (\mathcal{H}_v(t, x), v), \qquad x^* = \begin{pmatrix} v \\ u \end{pmatrix}$$

and the constant $\varepsilon$ in (5.3) remains to be chosen. We now estimate the various terms in the right-hand side of this identity.

STEP 1. Estimation of $k\,h\,\mathcal{G}(t,x)$. We claim that

$$k\,h\,\mathcal{G}(t,x) \le -\kappa\,k + \text{Const.}\,\sigma^{(1-m)/m}k|(\mathcal{H}_x, R)|^{1/m'} \tag{5.4}$$

for all $t \in J_1$, provided $\varepsilon$ is sufficiently small. To see this we first apply Lemma 4.1 with $x_0 = 2\sqrt{2}q$ to obtain two positive numbers $\kappa$, $w_0$ depending on $q$ such that

(5.5) $\mathcal{G}(t,x) \ge \kappa$ for all $(t,x) \in J \times \Omega$ with $|x| \ge 2\sqrt{2}q$ and $|u| \le w_0$;

(5.6) $\mathcal{G}(t,x) \le -\kappa$ for all $(t,x) \in J \times \Omega$ with $|x| \ge 2\sqrt{2}q$ and $|v| \le w_0$.

Next we take $I_1 = \{t \in J_1 : |u(t)| \le 2\varepsilon\}$ and let $\varepsilon \le q$. By Lemma 5.3

$$|v(t)| \ge 4q - |u(t)| \ge 2q \ge 2\varepsilon \quad \text{for} \quad t \in I_1,$$

so that $g(v(t)) = 0$ by $(5.3)_2$ and in turn $h(t) = -g(u(t))$. Therefore by (5.5)

$$k\,h\,\mathcal{G}(t,x) \le -\kappa\,k\,g(u(t)) \le 0, \tag{5.7}$$

provided $\varepsilon \le w_0/2$.

There are now two cases. For those $t \in I_1$ such that $|u(t)| \le \varepsilon$ we get $g(u(t)) = 1$ by $(5.3)_1$ and thus by the first inequality in (5.7)

$$k\,h\,\mathcal{G}(t,x) \le -\kappa\,k.$$

For those $t \in I_1$ such that $\varepsilon < |u(t)| \le 2\varepsilon$ we obtain

$$k\,h\,\mathcal{G}(t,x) \le 0 = -\kappa\,k + \kappa\,k \le -\kappa\,k + \text{Const.}\,\sigma^{(1-m)/m}k|(\mathcal{H}_x, R)|^{1/m'}$$

by (5.7) and Lemma 4.2 with $C = \overline{B(0,r)}$, $u_0 = \varepsilon$ and $v_0 = 2q$.

In the remaining set $J_1 \backslash I_1 = \{t \in J_1 : |u(t)| > 2\varepsilon\}$ by $(5.3)_2$ we have $g(u(t)) = 0$ and thus

$$k\,h\,\mathcal{G}(t,x) = k\,g(v(t))\mathcal{G}(t,x).$$

Hence by (5.6) for $|v(t)| \le 2\varepsilon$ we get

$$k\,h\,\mathcal{G}(t,x) \le -\kappa\,k\,g(v(t)) \le 0. \tag{5.8}$$

By $(5.3)_2$ this estimate continues to hold for those $t \in J_1 \backslash I_1$ such that $|v(t)| \ge 2\varepsilon$. Again there are two cases. If $|v(t)| \le \varepsilon$ then $g(v(t)) = 1$ by $(5.3)_1$ and thus by the first inequality in (5.8)

$$k\,h\,\mathcal{G}(t,x) \le -\kappa\,k.$$

If $\varepsilon < |v(t)|$ then again by (5.8)

$$k\,h\,\mathcal{G}(t,x) \le 0 = -\kappa\,k + \kappa\,k \le -\kappa\,k + \text{Const.}\,\sigma^{(1-m)/m}k|(\mathcal{H}_x, R)|^{1/m'}$$

by Lemma 4.2 with $C = \overline{B(0,r)}$, $u_0 = 2\varepsilon$ and $v_0 = \varepsilon$.

In summary, with

$$\varepsilon \le \frac{1}{2}\min\{q, w_0\}, \tag{5.9}$$

the above estimates for the sets $I_1$ and $J_1\backslash I_1$ are simultaneously valid. Combining these gives (5.4).

STEP 2. Estimation of $k\,h(R(t,x),x^*)$. By (5.1), the fact that $|h(t)| \le 1$ and $(A_2)$

$$k\,h(R(t,x),x^*) \le r\,\delta^{1/m}k|(\mathcal{H}_x, R)|^{1/m'} + r(\sup_J k)\,\text{Const.}\,|(\mathcal{H}_x, R)|\,.$$

STEP 3. Estimation of $k'\,h(u,v)$. Let the constant $\varepsilon$ in (5.3) be chosen to satisfy $\varepsilon \le \kappa/2r(M+1)$ as well as (5.9), where $M$ is given in (3.2). If either $|u(t)| \le 2\varepsilon$ or $|v(t)| \le 2\varepsilon$ then by (5.1) and the fact that $|h(t)| \le 1$

$$|k'\,h(u,v)| \le \frac{\kappa}{M+1}|k'|\,.$$

Otherwise, if $|u(t)| > 2\varepsilon$ and $|v(t)| > 2\varepsilon$ then $h(t) = 0$ again by $(5.3)_2$.

STEP 4. Estimation of $k\,h'(v,u)$. Observe that $g_w(w) \equiv 0$ whenever $0 \le |w| \le \varepsilon$ or $|w| \ge 2\varepsilon$ by (5.3). Consequently

$$\|g_w\|_\infty = \sup\{|g_w(w)| : w \in \mathbb{R}^N\} = \max\{|g_w(w)| : \varepsilon < |w| < 2\varepsilon\} < \infty\,.$$

Moreover

$$h'(t) = 0 \quad \text{in} \quad J_1\backslash\{t \in J_1 : \varepsilon < |u(t)| < 2\varepsilon \text{ or } \varepsilon < |v(t)| < 2\varepsilon\}\,.$$

If $\varepsilon < |u(t)| < 2\varepsilon$, then $|v(t)| > 2q \ge 2\varepsilon$ by Lemma 5.3 and (5.9) and in turn $g_w(v(t)) = 0$ by $(5.3)_2$. Therefore, by (5.3), $(1.5)_1$, $(A_1)$, (5.1) we have

$$\begin{aligned} k\,h'(u,v) &= -k(g_w(u), \mathcal{H}_v(t,x) + S(t,x))\cdot(u,v) \\ &\le \|g_w\|_\infty D\,r^2\,k + \|g_w\|_\infty r^2\,k|S(t,x)| \\ &\le \text{Const.}\left(\sigma^{(1-m)/m} + \delta^{1/m}\right)k|(\mathcal{H}_x, R)|^{1/m'} + \text{Const.}\,|(\mathcal{H}_x, R)| \end{aligned}$$

respectively by Lemma 4.2 with $C = \overline{B(0,r)}$, $u_0 = \varepsilon$ and $v_0 = 2q$, and by $(A_2)$, where

$$R = \begin{pmatrix} S \\ P \end{pmatrix}, \qquad \text{with } S, P : J \times \Omega \to \mathbb{R}^N\,.$$

A similar estimate can be obtained when $\varepsilon < |v(t)| < 2\varepsilon$.

By the previous STEPS 1-4 we obtain

$$\{k\,h(u,v)\}' \leq \text{Const.}(\delta + \sigma^{1-m})^{1/m} k |(\mathcal{H}_x, R)|^{1/m'} + \frac{\kappa}{M+1}|k'| - \kappa\, k + \text{Const.}\,|(\mathcal{H}_x, R)|$$

for almost all $t \in J_1$. To complete the proof it is now sufficient to integrate the previous inequality from $T_2$ to $t$, where $T_2 \geq T_1$, and use Lemma 5.1(ii) for the last term. □

We now turn the proof of Theorem 3.1. By (5.2) and Hölder's inequality

$$k\,h(u,v) \leq \text{Const.}\left(\int_{T_2}^{t} |(\mathcal{H}_x, R)| ds\right)^{1/m'} \left(\int_{T_2}^{t} (\delta + \sigma^{1-m}) k^m\, ds\right)^{1/m} + \frac{\kappa}{M+1}\int_{T_2}^{t} |k'| ds - \kappa \int_{T}^{t} k\, ds + \text{Const.}(T_2)$$

for all $t \geq T_2 \geq T_1$. Moreover by Lemma 5.1(ii) we can choose $T_2$ so large that

$$\text{Const.}\left(\int_{T_2}^{t} |(\mathcal{H}_x, R)| ds\right)^{1/m'} \leq \frac{\kappa}{M+1},$$

and also by $(3.1)_1$ take $t \geq T_2$ so large that $\int_T^t k(s) ds > 0$. Then

$$k\,h(u,v) \leq \frac{\kappa}{M+1}\int_{T}^{t} k(s) ds \left(\frac{\left(\int_T^t (\delta + \sigma^{1-m}) k^m ds\right)^{1/m} + \int_T^t |k'| ds}{\int_T^t k(s) ds} - M - 1\right) + \text{Const.}$$

By $(3.1)_1$ and (3.2) it now follows that

$$\liminf_{t\to\infty} k\,h(u,v) = -\infty,$$

which is a contradiction by (5.1) and the boundedness of the functions $k$ and $h$. Hence the case $\ell > 0$ cannot occur, and the proof is complete. □

## 6. – An alternative hypothesis

As in the previous sections *we suppose that conditions* $(H_1)$-$(H_3)$ *and* $(A_1)$-$(A_3)$ *hold and that the only solution of the initial value problem* $x(T) = 0$ *is* $x \equiv 0$. We also assume that for all $x_0 > 0$ there are positive exponents $\alpha$, $\beta$ such that

$$\phi(x) \geq |u|^{\alpha} \cdot |v|^{\beta} \quad \text{for all} \quad x \in \Omega \quad \text{with} \quad |x| \geq x_0, \tag{6.1}$$

where $\phi$ is the function introduced in $(A_3)$.

THEOREM 6.1. *Let the hypotheses of Theorem 3.1 hold, except that* (3.2) *is weakened to* (3.5) *and we suppose that*

$$|k'| \le B\,\sigma^{\lambda} k^{1-\lambda} \quad a.e. \text{ in } \quad J, \tag{6.2}$$

*where B is a positive constant and*

$$\lambda = \min\{1, 1/\alpha, 1/\beta\}.$$

*Then the conclusion of Theorem 3.1 remains valid.*

PROOF. As before, in view of Lemma 5.3 we can take $x_0 = 2\sqrt{2}q$ in $(A_2)$, $(A_3)$ and (6.1). The proof of Theorem 3.1 then applies essentially word-for-word, except for STEP 3 in Lemma 5.4. Here, in $J_1$ we see by (6.2) that

$$k' h(u, v) \le B\,\sigma^{\lambda} k^{1-\lambda}|h|\,|u| \cdot |v| \tag{6.3}$$

$$\le \begin{cases} Br\,\sigma(k/\sigma)^{1-\lambda}|u| & \text{if } |u| < 2\varepsilon \\ Br\,\sigma(k/\sigma)^{1-\lambda}|v| & \text{if } |v| < 2\varepsilon \\ 0 & \text{otherwise} \end{cases} \tag{6.4}$$

by (5.1), the fact that $|h(t)| \le 1$ and $(5.3)_2$.

Recall that $I_1 = \{t \in J_1 : |u(t)| < 2\varepsilon\}$ and that $|v(t)| \ge 2q$ in $I_1$ by Lemma 5.3 and (5.9). If $\lambda = 1$ then $\alpha \le 1$ and thus $|u| \le |u|^{\alpha}$ in $I_1$, since $r < 1$ in (5.1). Hence by (6.3), (6.1) and (2.4)

$$k' h(u, v) \le \text{Const.}\,\sigma|u| \le \text{Const.}\,\sigma|u|^{\alpha} \cdot \frac{|v|^{\beta}}{(2q)^{\beta}} \le \text{Const.}\,|(\mathcal{H}_x(t, x), R(t, x))| \text{ in } I_1.$$

If $\lambda < 1$ set

$$I_1' = \left\{t \in I_1 : (k/\sigma)^{1-\lambda}\tau \le |u|^{\lambda'}\right\}, \quad I_1'' = \left\{t \in I_1 : |u|^{\lambda'} < (k/\sigma)^{1-\lambda}\tau\right\},$$

where $\tau$ is a positive constant and $\lambda' = (1-\lambda)/\lambda$. For $t \in I_1'$ by (6.4), (6.1) and (2.4)

$$\begin{aligned} k' h(u, v) &\le \text{Const.}\,\sigma(k/\sigma)^{1-\lambda}|u| \\ &\le \text{Const.}\,\sigma|u|^{1+\lambda'} \cdot \frac{|v|^{\beta}}{(2q)^{\beta}} \le \text{Const.}\,|\mathcal{H}_x(t, x), R(t, x)|, \end{aligned}$$

where we have used the fact that $|u|^{1+\lambda'} \le |u|^{\alpha}$ in $I_1$, since $r < 1$ and $1 + \lambda' = 1/\lambda \ge \alpha$.

By (6.3), (5.1) and the fact that $|h| \le 1$ for $t \in I_1''$

$$k' h(u, v) \le \text{Const.}\,\sigma^{\lambda} k^{1-\lambda}[(k/\sigma)^{1-\lambda}\tau]^{1/\lambda'} = \text{Const.}\,\tau^{1/\lambda'} k.$$

If we now take $\tau^{1/\lambda'} \le \kappa/2\,\text{Const.}$, also by the previous estimate in $I_1'$, we conclude that

$$k' h(u, v) \le \frac{1}{2}\kappa\, k + \text{Const.}\,|(\mathcal{H}_x(t, x), R(t, x))| \quad \text{in} \quad I_1.$$

The same estimate can be obtain in the set where $|v| < 2\varepsilon$. We omit the proof since it is quite similar to the previous one.

The rest of the proof is now the same as before with the constant $\kappa$ in (5.2) now replaced by $\kappa/2$.

COROLLARY 4. *Assume that* $1/\delta$ *and* $1/(\sigma^{m-1}\delta)$ *are bounded on* $J$. *If* $1/\delta$ *is absolutely continuous,* $1/\delta \notin L^{1/(m-1)}(J)$ *and*

$$|\delta'| \leq \text{Const.}\, \delta(\sigma^{m-1}\delta)^{\lambda/(m-1)},$$

*then the rest state of* (1.6) *is asymptotically stable.*

PROOF. Take $k = 1/\delta^{1/(m-1)}$ in Theorem 6.1

## REFERENCES


[1] A. ARTESTEIN - E.F. INFANTE, *On the asymptotic stability of oscillators with unbounded damping*, Quart. Appl. Math. **34** (1976), 195-199.

[2] R.J. BALLIEU - K. PEIFFER, *Attractivity of the origin for the equation* $\ddot{x} + f(t, x, \dot{x})|\dot{x}|^{\alpha}\dot{x} + g(x) = 0$, J. Math. Anal. Appl. **65** (1978), 321-332.

[3] T.A. BURTON, *On the equation* $x'' + f(x)h(x')x' + g(x) = e(t)$, Ann. Math. Pura Appl. **85** (1970), 227-285.

[4] E.A. CODDINGTON - N. LEVINSON, Theory of Ordinary Differential Equations, McGraw-Hill, New York, 1955.

[5] R.J. DUFFIN, *Stability of systems with nonlinear damping*, J. Math. Anal. Appl. **23** (1968), 428-439.

[6] L. HATVANI, *Nonlinear oscillation with large damping*, Dynamical Systems and Applications **1** (1992), 257-270.

[7] V. LAKSHMIKANTHAM - V.M. MATROSOV - S. SIVASUNDARAM, Vector Liapunov Functions and Stability Analysis of Nonlinear Systems, Kluwer Academic Pub., 1991.

[8] G. LEONI, *A note on a theorem of Pucci& Serrin*, J. Differential Equations **113** (1994), 535-542.

[9] G. LEONI, *Asymptotic stability for perturbed Hamiltonian systems*, Arch. Rational Mech. Anal. **128** (1994), 105-125.

[10] G. LEONI, *Existence and asymptotic stability for perturbed Lagrangian and Hamiltonian systems*, (1995), Ph.D. thesis, University of Minesota.

[11] G. LEONI - M. MANFREDINI - P. PUCCI, *Stability properties for solutions of general Euler-Lagrange systems*, Differential Integral Equations **5** (1992), 537-552.

[12] J.J. LEVIN - J.A. NOHEL, *Global asymptotic stability for nonlinear systems of differential equations and applications to reactor dynamics*, Arch. Rational Mech. Anal. **5** (1960), 194-211.

[13] L. PERKO, Differential Equations and Dynamical Systems, Springer-Verlag, New York, 1992.

[14] P. PUCCI - J. SERRIN, *A general variational identity*, Indiana Univ. Math. J. **35** (1986), 681-703.

[15] P. PUCCI - J. SERRIN, *Asymptotic stability for non-autonomous damped wave systems*, Comm. Pure Appl. Math. **49** (1996), 177-216.

[16] P. PUCCI - J. SERRIN, *Asymptotic stability for nonlinear parabolic systems*, Proc. of Conference on "Energy Methods in Continuum Mechanics", Kluwer Pl., 1995.

[17] P. PUCCI - J. SERRIN, *On the derivation of Hamiltonian's equations*, Arch. Rational Mech. Anal. **125** (1994), 297-310.

[18] P. PUCCI - J. SERRIN, *Precise damping conditions for global asymptotic stability for nonlinear second order systems*, Acta Math. **170** (1993), 275-307.

[19] P. PUCCI - J. SERRIN, *Precise damping conditions for global asymptotic stability for nonlinear second order systems, II*, J. Differential Equations **113** (1994), 505-534.

[20] P. PUCCI - J. SERRIN, *Remarks on Lyapunov stability*, Differential Integral Equations **8** (1995), 1265-1277.

[21] L. SALVATORI, *Famiglie ad un parametro di funzioni di Liapunov nello studio della stabilità*, Symposia Math. Pubbl. INdAM **6** (1971), 309-330.

[22] R.A. SMITH, *Asymptotic stability of* $x'' + a(t)x' + x = 0$, Quart. J. Math. Oxford **12** (1961), 123-126.

[23] M. TAKEGAKI - S. ARIMOTO, *A new feedback method for dynamic control of manipulators*, J. Dyn. Systems, Meas. Control **103** (1981), 119-125.

[24] L.H. THURSTON - J.W. WONG, *On global asymptotic stability of certain second order differential equations with integrable forcing terms*, SIAM J. Appl. Math. **24** (1973), 50-61.

[25] A.J. DER SCHAFT, *Stabilization of Hamiltonian systems*, Nonlinear Anal. **10** (1986), 1021-1035.

[26] T. YOSHIZAWA, *Asymptotic behavior of solutions of a system of differential equations*, Contributions to Differential Equations **1** (1963), 371-387.

Dipartimento di Matematica
Università di Perugia
Via Vanvitelli, 1
06123 Perugia, Italia
E-mail: leoni@ gauss.dipmat.unipg.it