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C^∞ regularity of solutions of a quasilinear equation related to the Levi operator

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C^∞ Regularity of Solutions of a Quasilinear Equation Related to the Levi Operator

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1. – Introduction

We will study the regularity of the solutions of the equation

$$(1) \qquad \mathcal{L}u = q\frac{1+|\nabla u|^2}{1+u_t^2} \quad \text{in} \quad \Omega \subset \mathbb{R}^3$$

where

$$(2) \qquad \mathcal{L}u = u_{xx} + u_{yy} + 2\frac{u_y - u_x u_t}{1+u_t^2}u_{xt} - 2\frac{u_x + u_y u_t}{1+u_t^2}u_{yt} + \frac{u_x^2 + u_y^2}{1+u_t^2}u_{tt}\,,$$

and $q \in C^\infty(\Omega)$. Here we have denoted (x, y, t) a point of $\mathbb{R}^3$, u_x the first derivative with respect to x, and ∇ the euclidean gradient of u. The operator $\mathcal{L}$ is the Levi one, and it naturally arises in the study of the curvature of a hypersurface in $\mathbb{R}^4$. Indeed if $\Omega \subset \mathbb{R}^3$, $u : \Omega \to \mathbb{R}$ is a smooth function, and q is the Levi curvature of the surface $\{(x, y, t, u(x, y, t)) : (x, y, t) \in \Omega\}$, then u satisfies the equation

$$(3) \qquad \mathcal{L}u = q\frac{(1+|\nabla u|^2)^{3/2}}{1+u_t^2}$$

(see for example [T] for some more details on the geometrical meaning of the equation). Hence (1) is just a simplification of this one. It is a quasilinear degenerate elliptic equation, whose characteristic form is positively semidefinite and has the minimum eigenvalue identically 0. Hence, $\forall \epsilon > 0$ the operator

$$\mathcal{L}u + \epsilon \Delta u$$

is elliptic, and the equation has been initially studied with elliptic techniques, and letting $\epsilon \to 0$. In this way some geometric properties of (3) were established, as for example, the weak, and the strong maximum principle (see [DG] and [T]).

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Existence results are known under particular conditions on q: indeed Bedford and Gaveau proved that, if $q = 0$, Ω is psuedoconvex and $\phi \in C^{m+5}(\partial\Omega)$, then the problem

$$\begin{cases} \mathcal{L}(u) = 0 & \text{in } \Omega \\ u = \phi & \text{on } \partial\Omega \end{cases} \tag{4}$$

has a solution in $C^{m+\alpha}(\Omega) \cap \mathrm{Lip}(\bar{\Omega})$ (see [BG]).

More recently Slodkowsky and Tomassini proved that, if Ω is pseudoconvex, and q satisfies a geometric hypothesis related to the Levi curvature of $\partial\Omega \times \mathbb{R}$, the Dirichlet problem associated to equation (3), has at least a viscous solution $u \in \mathrm{Lip}(\bar{\Omega})$ (see [ST]).

The theorem of Debiard and Gaveau provides a regularity result when $q = 0$, but otherwise the problem was still open. However, when $q(\xi) \neq 0$ for all $\xi \in \Omega$, the equation can be studied with a completely different approach, introduced by [C]. Indeed it is possible to define two vector fields $X = X(u)$ and $Y = Y(u)$, naturally associated to L, and such that

X, Y and $[X, Y]$ are linearly independent at every point.

This is the Hörmander condition for hypoellipticity for linear operators with C^∞ coefficients in the form

$$Lu = X^2u + Y^2u + c[X, Y]u\,, \tag{5}$$

and it seems crucial for the regularity of solutions also in that nonlinear case. Indeed, with the maximum propagation principle of Bony, (see [B]) a strong comparison principle for the solutions of (1) and (3) were extablished.

In this paper we use the same idea, with a different choice of vector fields X and Y, which allows us to represent $\mathcal{L}$ as a sum of squares of vector fields plus a commutator. Indeed if u is a smooth solution we set

$$X = \begin{pmatrix} 1 \\ 0 \\ \dfrac{u_y - u_x u_t}{1 + u_t^2} \end{pmatrix} \quad Y = \begin{pmatrix} 0 \\ 1 \\ -\dfrac{u_x + u_y u_t}{1 + u_t^2} \end{pmatrix}. \tag{6}$$

Since X and Y are the first two columns of the characteristic form of $\mathcal{L}$, then it is easy to see that

$$\mathcal{L}u = X^2u + Y^2u - c(u)\partial_t u\,, \tag{7}$$

where

$$c(u) = X\left(\frac{u_y - u_x u_t}{1 + u_t^2}\right) - Y\left(\frac{u_x + u_y u_t}{1 + u_t^2}\right).$$

Besides we will show that

$$[X, Y] = -q\frac{1 + (Xu)^2 + (Yu)^2}{1 + (\partial_t u)^2}\partial_t$$

(see Section 7). Since $q(\xi) \neq 0 \forall \xi \in \Omega$ this means that $X, Y, [X, Y]$ are linearly independent at every point, and we will prove that:

THEOREM 1.1. *If* $\alpha > \frac{1}{2}$, $q(\xi) \neq 0$ *for every* $\xi \in \Omega$ *and* u *is a solution of* (1) *of class* $C^{2,\alpha}(\Omega)$, *then* u *is of class* $C^\infty(\Omega)$.

Sum of squares of vector fields have been intensively studied only in the linear case with C^∞ coefficients, and in this context many regularity results are known (see [F], [FS], [RS]). However, for the application to the quasilinear case, we need to consider operators with less regular coefficients. Hence we will start by studying a particular class of linear operators, defined in terms of suitable vector fields.

If a and b are continuous functions on an open set Ω, we will denote:

$$X = \begin{pmatrix} 1 \\ 0 \\ a \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 0 \\ 1 \\ b \end{pmatrix}, \tag{8}$$

and we will consider the operator formally defined as

$$Lu = X^2u + Y^2u - (Xa + Yb)\partial_t u\,. \tag{9}$$

For this kind of operators the first order differential operators X and Y are the natural analogue of the derivatives in the elliptic case. Hence we will denote $C^1_L(\Omega)$ the set of functions such that Xu and Yu are continuous, and we will call Xu and Yu derivatives of u in the direction X and Y.

Assume that a and b are of class $C^1_L(\Omega)$. Then for every function $f \in C^\infty_0(\Omega)$

$$[X, Y]f = (XY - YX)f = (Xb - Ya)\partial_t f\,,$$

so that, if we assume that

$$(Xb - Ya) \neq 0 \;\forall\, \xi \in \Omega\,, \tag{10}$$

then X, Y, and $[X, Y]$ are linearly independent at every point, and L is written in the form (5). In Section 2 we will show that there exists a distance d, naturally associated to L, so that we can define the Lipschitz classes in terms of it: a function u is of class $C^\alpha_L(\Omega)$ with $0 < \alpha \leq 1$ if

$$|u(\xi) - u(\xi_0)| \leq C d^\alpha(\xi, \xi_0) \quad \forall\, \xi, \xi_0 \in \Omega\,. \tag{11}$$

In this setting, if a and b are only continuous, a function of class $u \in C^\infty(\Omega)$ is of class $C^1_L(\Omega)$, but it is not further differentiable, since

$$Xf = \partial_x f + a\partial_t f \quad \forall\, f \in C^\infty\,,$$

which in general isn't differentiable, if a is not. Hence it does not seem natural to introduce the class $C^2_L(\Omega)$ unless a and b are of class $C^1_L(\Omega)$. If this last condition holds, however, we say that a function f is of class $C^2_L(\Omega)$ if Xf and Yf are of class $C^1_L(\Omega)$. More generally we will introduce the classes $C^{k+2}_L(\Omega)$ only if a, $b \in C^{k+1}_L(\Omega)$. Finally we will denote $C^{k,\alpha}_L(\Omega)$ the class of functions with all derivatives of order k in the directions X and Y of class $C^{k,\alpha}_L(\Omega)$.

With these notations we will prove:

THEOREM 1.2. *Assume that a and b are of class* $C_L^{k+1,\alpha}(\Omega)$, (10) *is satisfied, and* $c \in C_L^{k,\alpha}(\Omega)$. *If u is a solution of class* $C_L^{2,\alpha}(\Omega)$ *of the equation* $Lu = f \in C_L^{k,\alpha}(\Omega)$, *then* $u \in C_{L,\mathrm{loc}}^{k+2,\alpha}(\Omega)$.

The proof of this theorem is achieved with a suitable adaptation of the freezing method: if L is elliptic, its frozen operator is obtained just evaluating its coefficients at a given point ξ_0. Here we call frozen operator of L the operator whose coefficients are the first order Taylor expansion of the coefficients of L, in the directions X and Y. The operator L_{ξ_0} obtained in this way is - up to a change of variable - the Kohn Laplacian on the Heisenberg group: an hypoelliptic second order operator, which has already been intensively studied. In particular its fundamental solution is explicitly known, and we will write in terms of it a representation formula for functions of class $C_L^{2,\alpha}$. Differentiating this formula we will deduce Theorem 1.2.

After that we will go back to study equation (1). If u is a fixed solution of class $C_{\mathcal{L}}^{2,\alpha}(\Omega)$, then $\mathcal{L}$ can be considered as a linear operator belonging to the previous class, and using the regularity results just proved, we get Theorem 1.1.

The paper is organized as follows: in Section 2-6 we will study only the linear operator L: in Section 2 we will describe in detail the properties of the distance associated to it, and we define the frozen operator L_{ξ_0}. In Section 3 we will prove the representation formula. In Section 4 we will compute the derivatives of u in the directions X_{ξ_0}, Y_{ξ_0}, naturally associated to L_{ξ_0} and we will make their Hölder estimate in Section 5. Theorem 1.2 will be proved in Section 6, and Theorem 1.1 in Section 7.

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2. – The linear operator

Let Ω be an open set in $\mathbb{R}^3$, a and b be continuous functions in Ω, and X and Y the vector fields defined in (8). Recall that the derivative in the direction X is defined as follows: assume that a is Lipschitz in euclidean sense, and let γ be the integral curve of X such that $\gamma(0) = \xi \in \Omega$. If the derivative

$$Xu(\xi) = \frac{d}{dh}(u \circ \gamma)|_{h=0} \tag{12}$$

exists, it is called partial derivative of u in the direction X. (The derivative in the direction Y is defined in the same way).

We assume that

$$\text{(H1)} \qquad a \text{ and } b \text{ are of class } C_L^1(\Omega) \cap \mathrm{Lip}(\Omega)\,,$$

$$\text{(H2)} \qquad Xb - Ya \neq 0 \quad \forall\, x \in \Omega\,,$$

and will call L the operator defined in (9):

$$Lu = X^2u + Y^2u - (Xa + Yb)\partial_t u\,.$$

Then we will prove that we can introduce a control distance associated to L, and we will show the relation between the Lipschitz classes associated to L, and the classic ones. In the second part of the section we will define the frozen operator in this particular case.

REMARK 2.1. For every $\xi = (x, y, t)$ and $\xi_0 = (x_0, y_0, t_0) \in \Omega$ there exists a function γ, connecting ξ and ξ_0, piecewise integral curve of X or Y.

Let $e^{tX}(\xi_0)$ denote the integral curve of X such that $\gamma(0) = \xi_0$, and set

$$\xi_1 = e^{(x-x_0)X}(\xi_0), \quad \text{and} \quad \xi_2 = e^{(y-y_0)Y}(\xi_1)\,.$$

A direct computation shows that ξ_2 has the first two components respectively equal to ξ. If

$$\xi_3 = e^{dX}(\xi_2),\ \xi_4 = e^{dY}(\xi_3),\ \xi_5 = e^{-dX}(\xi_4),\ \xi_6 = e^{-dY}(\xi_5)\,, \tag{13}$$

it is not difficult to show that ξ_6 has the same property, and that

$$\xi_6 = \xi_3 + d^2\begin{pmatrix} 0 \\ 0 \\ Xb - Ya \end{pmatrix} + o(d^2)\,. \tag{14}$$

Since $\inf_\Omega |Xb - Ya| > 0$, then ξ_6 and ξ_3 differ in the third component, and applying again the choice of points in (13) a finite number k of times, we can assume that $\xi = \xi_{2+4k}$ and we have found the required path.

The previous remark ensures that we can define a distance as follows: for all $\xi, \xi_0 \in \Omega$ we call $G(\xi, \xi_0)$ the set of paths γ, piecewise integral curves of X or Y such that $\gamma(T) = \xi$, and $\gamma(0) = \xi_0$. Then the control distance naturally associated to L is

$$d(\xi, \xi_0) = \inf\{T > 0 | \exists\, \gamma : [0, T] \to \mathbb{R}^3,\ \gamma \in G(\xi, \xi_0)\} \tag{15}$$

(see [NSW] for the definition in the regular case). Let us estimate it:

REMARK 2.2. If

$$\tilde{d}(\xi, \xi_0) = (((x - x_0)^2 + (y - y_0)^2)^2 + (t - t_0 - a(\xi_0)(x - x_0) - b(\xi_0)(y - y_0))^2)^{1/4}\,,$$

there exist two constants C_1 and C_2 such that

$$\forall\, \xi, \xi_0 \in \Omega \quad C_1 d(\xi, \xi_0) \le \tilde{d}(\xi, \xi_0) \le C_2 d(\xi, \xi_0)\,. \tag{16}$$

PROOF. Using (14), and the properties of the distance d proved in [NSW], we deduce that

$$d(\xi, \xi_0) = ((A^2 + B^2)^2 + S^2)^{1/4},$$

where A, B, C satisfy:

$$\begin{cases} \dot{\gamma}_L = AX + BY + S \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} (\gamma_L) \\ \gamma_L(0) = \xi_0 \quad \gamma_L(1) = \xi . \end{cases}$$

If follows that

$$A = x - x_0, \quad B = y - y_0,$$

while

$$(\gamma_L)_3(1) = A \int_0^1 a(\gamma_L(\tau))d\tau + B \int_0^1 b(\gamma_L(\tau))d\tau + S + t_0 =$$
$$= (x - x_0) \int_0^1 a(\gamma_L(\tau))d\tau + (y - y_0) \int_0^1 b(\gamma_L(\tau))d\tau + S + t_0 .$$

Hence if we set

$$T = t - t_0 - a(\xi_0)(x - x_0) - b(\xi_0)(y - y_0),$$

we get

$$|T - S| = (x - x_0) \int_0^1 (a(\gamma_L(\tau)) - a(\xi_0))d\tau$$
$$+ (y - y_0) \int_0^1 (b(\gamma_L(\tau)) - b(\xi_0))d\tau \le C((x - x_0)^2 + (y - y_0)^2),$$

and (16) immediately follows. □

Since d is well defined, then we can consider the Lipschitz classes $C_L^{1,\alpha}$ and $C_L^{2,\alpha}$ associated to L and defined as in (11); let us prove some properties of their elements.

REMARK 2.3. If a and b satisfy (H1), (H2) and are of class $C_L^{1,\alpha}(\Omega)$, then for any $f \in C_L^{1,\alpha}(\Omega)$

$$f(\xi) = f(\xi_0) + Xf(\xi_0)(x - x_0) + Yf(\xi_0)(y - y_0) + O(d^{1+\alpha}(\xi, \xi_0)). \tag{17}$$

Hence we will call the function

$$P^1_{\xi_0} f(\xi) = f(\xi_0) + Xf(\xi_0)(x - x_0) + Yf(\xi_0)(y - y_0)$$

Taylor polynomial of order 1 of f with initial point ξ_0.

PROOF. If ξ and ξ_0 are fixed, we choose $\xi_1, \dots, \xi_6$ as in Remark 2.1, and assume that $\xi_6 = \xi$. In particular $\gamma(s) = e^{s(x-x_0)X}\xi_0$ is an integral curve of $(x-x_0)X$ connecting ξ_0 and ξ_1. Then $f \circ \gamma$ is of class $C^{1,\alpha}$ in ordinary sense, and we have

$$f(\xi_1) = f(\xi_0) + Xf(\xi_0)(x-x_0) + O(d^{1+\alpha}(\xi, \xi_0)).$$

Analogously

$$\begin{aligned} f(\xi_2) &= f(\xi_1) + Yf(\xi_1)(y-y_0) + O(d^{1+\alpha}(\xi, \xi_0)) \\ &= f(\xi_0) + Xf(\xi_0)(x-x_0) + Yf(\xi_1)(y-y_0) + O(d^{1+\alpha}(\xi, \xi_0)) \\ &= f(\xi_0) + Xf(\xi_0)(x-x_0) + Yf(\xi_0)(y-y_0) + O(d^{1+\alpha}(\xi, \xi_0)). \end{aligned} \tag{18}$$

By Remark 2.2, $d(\xi_0, \xi_2) \leq d(\xi, \xi_0)$, and since $\xi = \xi_6$, then $d = d(\xi_2, \xi_6) \leq d(\xi_2, \xi_0) + d(\xi_0, \xi_6) \leq 2d(\xi, \xi_0)$. Hence we have:

$$\begin{aligned} f(\xi) = f(\xi_6) &= f(\xi_5) - dYf(\xi_5) + O(d^{1+\alpha}) \\ &= f(\xi_4) - dYf(\xi_5) - dXf(\xi_4) + O(d^{1+\alpha}) \\ &= f(\xi_3) - dYf(\xi_5) - dXf(\xi_4) + dYf(\xi_3) + O(d^{1+\alpha}) \\ &= f(\xi_2) - dYf(\xi_5) - dXf(\xi_4) + dYf(\xi_3) + dXf(\xi_2) + O(d^{1+\alpha}) \\ &= f(\xi_2) + d(Xf(\xi_2) - Xf(\xi_4)) + d(Yf(\xi_3) - Yf(\xi_5)) + O(d^{1+\alpha}) \\ &= f(\xi_2) + O(d^{1+\alpha}) = f(\xi_2) + O(d^{1+\alpha}(\xi, \xi_0)). \end{aligned}$$

The thesis follows immediately, using (18). □

REMARK 2.4. If a and b satisfy (H1), (H2) and are of class $C_L^{1,\alpha}(\Omega)$, for every $f \in C_L^{2,\alpha}(\Omega)$ and for every $\xi \in \Omega$ there exists $\partial_t f(\xi)$ and

$$\partial_t f = \frac{[X, Y]f}{Xb - Ya} = \frac{(XY - YX)f}{Xb - Ya}.$$

In particular $\partial_t f \in C_L^\alpha(\Omega)$.

PROOF. Let $\xi_0 \in \Omega$ and $h > 0$ be fixed; by (H2) we can assume that $(Xb - Ya)(\xi_0) > 0$ and set $d = \sqrt{\frac{h}{(Xb-Ya)(\xi_0)}}$. Then put

$$\xi_1 = e^{dX}(\xi_0), \quad \xi_2 = e^{dY}(\xi_1), \quad \xi_3 = e^{-dX}(\xi_2), \quad \xi_4 = e^{-dY}(\xi_3).$$

We have already noted that ξ_4 and ξ_0 have the same first two components, and

$$\xi_4 = \xi_0 + (Xb - Ya)d^2 e_3 + o(d^2) = \xi_0 + he_3 + o(h),$$

where e_3 is the third vector in the canonical basis; hence

$$f(\xi_0 + he_3) = f(\xi_4) + o(h) =$$

(by the regularity of f)

$$\begin{aligned}
&= f(\xi_3) - dYf(\xi_3) + \frac{d^2}{2}Y^2 f(\xi_3) + o(d^2) \\
&= f(\xi_2) - dXf(\xi_2) + \frac{d^2}{2}X^2 f(\xi_0) - dYf(\xi_2) \\
&\qquad + d^2 XYf(\xi_2) + \frac{d^2}{2}Y^2 f(\xi_3) + o(d^2) =
\end{aligned}$$

(using the Taylor expansions, with initial points ξ_2 and ξ_1)

$$= f(\xi_0) + d^2(XY - YX)f(\xi_0) + o(h^2).$$

Hence the thesis immediately follows. □

Arguing in the same way as in the preceding remark, it can be proved that.

REMARK 2.5. If a and b satisfy (H1), (but not necessarily (H2)), then for all $f \in C^2_L(\Omega) \cap C^1(\Omega)$ we have

$$[X, Y]f = (Xb - Ya)\partial_t f.$$

REMARK 2.6. If a and b satisfy (H1), (H2) and are of class $C^{1,\alpha}_L(\Omega)$ any function $f \in C^{2,\alpha}_L(\Omega)$ has the following Taylor expansion of order 2, and initial point ξ_0:

$$f(\xi) = P^2_{\xi_0} f(\xi) + O(d(\xi, \xi_0)^{2+\alpha}),$$

where

$$\begin{aligned}
P^2_{\xi_0} f(\xi) &= f(\xi_0) + Xf(\xi_0)(x - x_0) + Yf(\xi_0)(y - y_0) \\
&\quad + \frac{1}{2}X^2 f(\xi_0)(x - x_0)^2 + \frac{1}{2}Y^2 f(\xi_0)(y - y_0)^2 \\
&\quad + \frac{1}{2}XYf(\xi_0)(x - x_0)(y - y_0) + \frac{1}{2}YXf(\xi_0)(x - x_0)(y - y_0) \\
&\quad + \partial_t f(\xi_0)\Big(t - t_0 - a(\xi_0)(x - x_0) - \frac{Xa(\xi_0)}{2}(x - x_0)^2 \\
&\quad - b(\xi_0)(y - y_0) - \frac{1}{2}Yb(\xi_0)(y - y_0)^2 \\
&\quad - \frac{Xb(\xi_0)}{2}(y - y_0)(x - x_0) - \frac{Ya(\xi_0)}{2}(y - y_0)(x - x_0)\Big).
\end{aligned}$$

From the previous remarks it easily follows that, if $f \in C_L^{2,\alpha}(\Omega)$, then $f \in C^{1,\alpha/2}(\Omega)$, and viceversa, if $f \in C^{2,\alpha}(\Omega)$, then $f \in C_L^{2,\alpha}(\Omega)$.

Because of Remark 2.3 it seems natural to call frozen operator of L

$$L_{\xi_0} = X_{\xi_0}^2 + Y_{\xi_0}^2 - (Xa + Yb)(\xi_0)\partial_t \,, \tag{19}$$

where

$$X_{\xi_0} = \begin{pmatrix} 1 \\ 0 \\ P_{\xi_0}^1 a \end{pmatrix} \quad \text{and} \quad Y_{\xi_0} = \begin{pmatrix} 0 \\ 1 \\ P_{\xi_0}^1 b \end{pmatrix} . \tag{20}$$

We will show that - up to a change of variable - L_{ξ_0} is the Kohn operator on the Heisenberg group. Let us however begin by recalling some properties of this last operator. H^3 is the group, defined by $\mathbb{R}^3$, with the following composition law:

$$(x, y, t) \cdot (x_0, y_0, t_0) = (x + x_0, y + y_0, t + t_0 + 2(xy_0 - yx_0)) .$$

If

$$X_H = \frac{\partial}{\partial x} + 2y\frac{\partial}{\partial t} \quad \text{and} \quad Y_H = \frac{\partial}{\partial y} - 2x\frac{\partial}{\partial t}$$

the Kohn Laplacian (or subelliptic Laplacian) is defined by

$$\Delta_H = X_H^2 + Y_H^2 + c\partial_t \,,$$

where c is constant. This operator is invariant with respect to the left translations of the group, and, if $\delta_\lambda : H^3 \to H^3$ is the group of dilations

$$\delta_\lambda(x, y, t) = (\lambda x, \lambda y, \lambda^2 t), \quad \lambda > 0 \,,$$

Δ_H is homogeneous of degree 2 with respect to δ_λ. The control distance has the following expression:

$$d_H(\xi, \xi_0) = n(\xi_0^{-1} \circ \xi) \,,$$

where

$$n(\xi) = ((x^2 + y^2)^2 + t^2)^{1/4} \,,$$

and the measure of the balls in this metric is

$$|B_H(\xi_0, r)| = C_0 r^N$$

where $N = 4$. This number N is called the homogeneous dimension of H^3, and it is the natural analogue of the Euclidean dimension for the elliptic case. Indeed $\forall\, f : \mathbb{R} \to \mathbb{R}$,

$$\int_{B_H(\xi_0, r)} f(d_H(\xi_0, \zeta)) d\zeta = \int_0^r f(\rho)\rho^{N-1} d\rho \,.$$

The fundamental solution Δ_H can be explicitly written in terms of d and N:

$$\Gamma_H(\xi,\xi_0) = d_H(\xi,\xi_0)^{-N+2} \exp\left[c \arccos \frac{t - t_0 - 2(xy_0 - yx_0)}{d_H(\xi,\xi_0)^2}\right]$$

and its derivatives satisfy the following estimates:

$$\Gamma_H(\xi,\xi_0) \simeq d_H(\xi,\xi_0)^{-N+2} \quad Y_H\Gamma_H(\xi,\xi_0) \simeq d_H(\xi,\xi_0)^{-N+1}$$
$$\partial_t\Gamma_H(\xi,\xi_0) \simeq d_H(\xi,\xi_0)^{-N}.$$

Thus X_H has the homogeneity of a first derivative, while $\partial_t = [X, Y]$ is a second derivative, in the direction of these vector fields.

Now the change of variable

$$\phi_{\xi_0}(x,y,t) = \Bigg(x, y, \frac{2(2t - Xa(\xi_0)x^2 - Yb(\xi_0)y^2 - (Xb(\xi_0) + Ya(\xi_0))xy)}{Ya(\xi_0) - Xb(\xi_0)}$$
$$- \frac{4(a(\xi_0) - Ya(\xi_0)y_0 - Xa(\xi_0)x_0)x}{Ya(\xi_0) - Xb(\xi_0)} - \frac{4(b(\xi_0) - Xb(\xi_0)x_0 - Xb(\xi_0)y_0)y}{Ya(\xi_0) - Xb(\xi_0)}\Bigg)$$

changes L_{ξ_0} (defined in (19)) into the subelliptic Laplacian. Indeed $\forall u : \mathbb{R}^3 \to \mathbb{R}$ if

$$u_H = u \circ \phi_{\xi_0}^{-1}$$

then

$$X_{\xi_0}u(\xi) = X_H u_H(\phi_{\xi_0}(\xi)) \quad \text{and} \quad L_{\xi_0}u(\xi) = \Delta_H u_H(\xi)$$

with $c = \frac{4}{(Ya - Xb)(\xi_0)}$. Consequently for every ξ_0 it is explicitly known the fundamental solution of L_{ξ_0}, and it is precisely $\Gamma_{\xi_0} = \Gamma_H(\phi_{\xi_0}, \phi_{\xi_0})$, while the control distance is:

$$d_{\xi_0}(\xi,\zeta) = d_H(\phi_{\xi_0}(\xi), \phi_{\xi_0}(\zeta)).$$

Hence it can be written as:

$$d_{\xi_0}(\xi,\zeta) = \left(\left((x_\xi - x_\zeta)^2 + (y_\xi - y_\zeta)^2\right)^2 + \frac{4}{(Ya - Xb)(\xi_0)^2} T_{\xi_0}(\xi,\zeta)^2\right)^{1/4},$$

where

$$\begin{aligned} T_{\xi_0}(\xi,\zeta) = 2(t_\xi - t_\zeta) &- Xa(\xi_0)((x_\xi - x_0) + (x_\zeta - x_0))(x_\xi - x_\zeta) \\ &- Xb(\xi_0)((y_\xi - y_0) + (y_\zeta - y_0))(y_\xi - y_\zeta) \\ &- Yb(\xi_0)((x_\xi - x_0) + (x_\zeta - x_0))(y_\xi - y_\zeta) \\ &- Ya(\xi_0)((y_\xi - y_0) + (y_\zeta - y_0))(x_\xi - x_\zeta) \\ &- 2a(\xi_0)(x_\xi - x_\zeta) - 2b(\xi_0)(y_\xi - y_\zeta). \end{aligned}$$

In particular, if $\xi = \xi_0$, $d_\xi(\xi, \zeta)$ has the following expression:

$$\begin{aligned} d_\xi(\xi,\zeta) = \big(&((x-x_\zeta)^2 + (y-y_\zeta)^2)^2 \\ &+ \frac{4}{(Ya - Xb)(\xi)^2}\big(2(t-t_\zeta) - 2a(\xi)(x-x_\zeta) - 2b(\xi)(y-y_\zeta) \\ &- Xa(\xi)(x-x_\zeta)^2 - Xb(\xi)(y-y_\zeta)^2 \\ &- (Xb+Ya)(\xi)(x-x_\zeta)(y-y_\zeta)\big)^2\big)^{1/4}. \end{aligned}$$

An easy computation shows that there exist constants C_0 and C_1 such that

$$C_0 d_{\xi_0}(\xi, \xi_0) \le \tilde{d}(\xi, \xi_0) \le C_1 d_{\xi_0}(\xi, \xi_0), \tag{21}$$

where $\tilde{d}(\xi, \xi_0)$ has been introduced in Remark 2.2. Hence, because of (16) also $d_{\xi_0}(\xi, \xi_0)$ provides an estimate of $d(\xi, \xi_0)$.

Let us study some relations between the distances we have introduced.

REMARK 2.7. If $\xi_0 \in \Omega$ is fixed, there exist C_0, $C_1 > 0$ such that for every ξ, ζ

$$C_0 d_\xi(\xi,\zeta) \le d_{\xi_0}^{1/2}(\xi,\zeta) \le C_1 d_\xi^{1/4}(\xi,\zeta).$$

REMARK 2.8. From the expression of $d_{\xi_0}(\xi,\zeta)$ it follows that there exist C_0 and C_1 such that

$$d_{\xi_0}(\xi,\zeta) \le d_\xi(\xi,\zeta) + (|x-x_\zeta| + |y-y_\zeta|)^{1/4}(|x-x_0| + |y-y_0|)^{1/4}$$

and

$$d_\xi(\xi,\zeta) \le d_{\xi_0}(\xi,\zeta) + (|x-x_\zeta| + |y-y_\zeta|)^{1/4}(|x-x_0| + |y-y_0|)^{1/4}.$$

In particular

$$\begin{aligned} d_\xi(\xi,\zeta) &\le d_{\xi_0}(\xi,\zeta) + d_{\xi_0}^{1/2}(\xi,\zeta)(d_{\xi_0}^{1/2}(\xi_0,\zeta) + d_{\xi_0}^{1/2}(\xi,\xi_0)) \\ &\quad + d_{\xi_0}^{1/2}(\xi,\xi_0)d_{\xi_0}^{1/2}(\xi,\zeta) \le C_1 d_{\xi_0}(\xi,\zeta) + d_{\xi_0}^{1/2}(\xi,\xi_0)d_{\xi_0}^{1/2}(\xi,\zeta). \end{aligned} \tag{22}$$

REMARK 2.9. We can assume that $Ya - Xb > 0$ in Ω; then

$$\begin{aligned} &\frac{4}{Ya(\xi) - Xb(\xi)} T_\xi(\xi,\zeta) - \frac{4}{Ya(\xi_0) - Xb(\xi_0)} T_{\xi_0}(\xi_0,\zeta) \\ &= \left(\frac{4}{Ya(\xi) - Xb(\xi)} - \frac{4}{Ya(\xi_0) - Xb(\xi_0)}\right) T_{\xi_0}(\xi_0,\zeta) \\ &\quad + 2(t_\xi - t_{\xi_0} - 2a(\xi_0)(x_\xi - x_\zeta) - 2b(\xi_0)(y_\xi - y_{\xi_0}) \\ &\quad - 2(a(\xi) - a(\xi_0))(x_\xi - x_\zeta) - 2(b(\xi) - b(\xi_0))(y_\xi - y_\zeta) \\ &\quad - (Xa(\xi) - Xa(\xi_0))(x_\xi - x_\zeta)^2 - Xa(\xi_0)((x_\xi - x_\zeta)^2 - (x_\xi - x_\zeta)^2) \\ &\quad - (Ya(\xi) - Ya(\xi_0))(y_\xi - y_\zeta)^2 - Ya(\xi_0)((y_\xi - y_\zeta)^2 - (y_\xi - y_\zeta)^2) \\ &\quad - ((Xb + Ya)(\xi) - (Xb + Ya)(\xi_0))(x_\xi - x_\zeta)(y_\xi - y_\zeta) \\ &\quad - (Xb + Ya)(\xi_0)((x_\xi - x_\zeta)(y_\xi - y_\zeta) - (x_{\xi_0} - x_\zeta)(y_{\xi_0} - y_\zeta)) \\ &\le d_{\xi_0}^\alpha(\xi,\xi_0)d_{\xi_0}^2(\zeta,\xi_0) + d_{\xi_0}(\xi,\xi_0)(d_\xi(\xi,\zeta) + d_{\xi_0}(\xi_0,\zeta)). \end{aligned}$$

Hence

$$
\begin{aligned}
&d_{\xi}^2(\xi,\zeta) - d_{\xi_0}^2(\xi_0,\zeta) \\
&\le d_{\xi_0}^{\alpha}(\xi,\xi_0)d_{\xi_0}^2(\zeta,\xi_0) + d_{\xi_0}(\xi,\xi_0)(d_{\xi}(\xi,\zeta) + d_{\xi_0}(\xi_0,\zeta)).
\end{aligned} \tag{23}
$$

3. – Representation formulas

In this section we will prove a representation formula for functions of class $C_L^{2,\alpha}(\Omega)$, which will be the main tool in the proof of our regularity result.

Let us first note that:

REMARK 3.1. If $a \in C_L^{1,\alpha}(\Omega)$, then

$$
\begin{aligned}
&|(a(\zeta) - P_{\xi_0}^1 a(\zeta)) - (a(\xi) - P_{\xi_0}^1 a(\xi))| \\
&\le |(a(\zeta) - P_{\xi}^1 a(\zeta))| + |(Xa(\xi) - Xa(\xi_0))(x_\zeta - x_\xi) \\
&\quad + (Ya(\xi) - Ya(\xi_0))(y_\zeta - y_\xi)| \\
&\le d_{\xi}^{1+\alpha}(\xi,\zeta) + d_{\xi_0}(\xi,\zeta)d_{\xi_0}^{\alpha}(\xi,\xi_0) \le
\end{aligned}
$$

(by (22))

$$
\le d_{\xi_0}^{1+\alpha}(\xi,\zeta) + d_{\xi_0}^{\frac{1+\alpha}{2}}(\xi,\zeta)d_{\xi_0}^{\frac{1+\alpha}{2}}(\xi,\xi_0) + d_{\xi_0}(\xi,\zeta)d_{\xi_0}^{\alpha}(\xi,\xi_0).
$$

Moreover the following assertion holds:

REMARK 3.2. If a and b satisfy (H1), (H2) and are of class $C_L^{1,\alpha}(\Omega)$ and $f \in C_L^{1,\alpha}(\Omega)$, then for every compact set $K \subset \Omega$ there exists a sequence f_h in $C^\infty(\Omega)$ such that

$$
f_h \to f \quad \text{and} \quad D_i f_h \to D_i f,
$$

uniformly in K (recall that $D = (X, Y)$). Analogously if f is a fixed function in $C_L^{2,\alpha}(\Omega)$, then for every compact set $K \subset \Omega$ there exists a sequence f_h in $C_L^\infty(\Omega)$ such that

$$
f_h \to f, \quad D_i f_h \to D_i f, \quad D_i^2 f_h \to D_i^2 f
$$

as $h \to +\infty$ uniformly in K.

PROOF. Arguing as in [GL], it is possible to see that there exists $M > 0$ such that for every $h \in N$ there exist N_h points $\xi_1^h, \dots, \xi_N^h$, such that the union of the balls with center in these points, and radius $\frac{1}{h}$ covers K, and every point

ξ belongs at most to M of these spheres. If $\phi \in C_0^\infty([0,2],\mathbb{R})$, is such that $\phi = 1$ on [0,1], and

$$\phi_j^h(\xi) = \phi(kd_{\xi_j^h}(\xi, \xi_j^h)),$$

then the sequence

$$f_h(\xi) = \sum_{j=1}^{N_h} P^1_{\xi_j,h} f(\xi)\phi_n(\xi)$$

satisfies the assertion. □

The following relation holds between the derivatives of Γ_{ξ_0}

REMARK 3.3. If we denote

$$\tilde{X}_{\xi_0} = X_{\xi_0} + (Xb - Ya)(\xi_0)(y_\xi - y_\zeta)\partial_t$$

and

$$\tilde{Y}_{\xi_0} = Y_{\xi_0} + (Ya - Xb)(\xi_0)(x_\xi - x_\zeta)\partial_t\,.$$

then the following condition is satisfied: $X^\zeta_{\xi_0}\Gamma_{\xi_0}(\xi,\zeta) = -\tilde{X}^\xi_{\xi_0}\Gamma_{\xi_0}(\xi,\zeta)$ and $Y^\zeta_{\xi_0}\Gamma_{\xi_0}(\xi,\zeta) = -\tilde{Y}^\xi_{\xi_0}\Gamma_{\xi_0}(\xi,\zeta)$, where $Y^\xi_{\xi_0}$ denote the derivate with respect to the variable ξ.

With the same notations of the previous section we have:

THEOREM 3.1. *If $v \in C_L^{2,\beta}$ and $\phi \in C_0^\infty(\Omega)$, then $v\phi$ can be represented in the following way:*

$$\phi v(\xi) = A(\xi,\xi_0) + B(\xi,\xi_0) + \partial_t v(\xi_0)C(\xi,\xi_0) + E(\xi,\xi_0)\,,$$

where

$$A(\xi,\xi_0) = \int \Gamma_{\xi_0}(\xi,\zeta)Lv(\zeta)\phi(\zeta)d\zeta\,,$$

$$\begin{aligned}
B(\xi,\xi_0) = {} & 2\int \Gamma_{\xi_0}(\xi,\zeta)(Xa(\zeta) - Xa(\xi_0))\phi(\zeta)(\partial_t v(\zeta) - \partial_t v(\xi_0))d\zeta \\
& - 2\int \tilde{X}_{\xi_0}\Gamma_{\xi_0}(\xi,\zeta)(a(\zeta) - P^1_{\xi_0}a(\zeta))\phi(\zeta)(\partial_t v(\zeta) - \partial_t v(\xi_0))d\zeta \\
& - \int \partial_t\Gamma_{\xi_0}(\xi,\zeta)(a(\zeta) - P^1_{\xi_0}a(\zeta))^2\phi(\zeta)(\partial_t v(\zeta) - \partial_t v(\xi_0))d\zeta \\
& + 2\int \Gamma_{\xi_0}(\xi,\zeta)(Yb(\zeta) - Yb(\xi_0))\phi(\zeta)(\partial_t v(\zeta) - \partial_t v(\xi_0))d\zeta \\
& - 2\int \tilde{Y}_{\xi_0}\Gamma_{\xi_0}(\xi,\zeta)(b(\zeta) - P^1_{\xi_0}b(\zeta))\phi(\zeta)(\partial_t v(\zeta) - \partial_t v(\xi_0))d\zeta \\
& - \int \partial_t\Gamma_{\xi_0}(\xi,\zeta)(b(\zeta) - P^1_{\xi_0}b(\zeta))^2\phi(\zeta)(\partial_t v(\zeta) - \partial_t v(\xi_0))d\zeta\,,
\end{aligned}$$

$$
\begin{aligned}
C(\xi,\xi_0) = &-2\int \Gamma_{\xi_0}(\xi,\zeta)(a(\zeta) - P^1_{\xi_0}a(\zeta))X_{\xi_0}\phi(\zeta)d\zeta \\
&-2\int \Gamma_{\xi_0}(\xi,\zeta)(b(\zeta) - P^1_{\xi_0}b(\zeta))Y_{\xi_0}\phi(\zeta)d\zeta \\
&-\int \Gamma_{\xi_0}(\xi,\zeta)(a(\zeta) - P^1_{\xi_0}a(\zeta))^2\partial_t\phi(\zeta)d\zeta \\
&-\int \Gamma_{\xi_0}(\xi,\zeta)(b(\zeta) - P^1_{\xi_0}b(\zeta))^2\partial_t\phi(\zeta)d\zeta\,,
\end{aligned}
$$

$$
\begin{aligned}
E(\xi,\xi_0) = &\,2\int \Gamma_{\xi_0}(\xi,\zeta)Xv(\zeta)X_{\xi_0}\phi(\zeta)d\zeta \\
&+2\int \Gamma_{\xi_0}(\xi,\zeta)Yv(\zeta)Y_{\xi_0}\phi(\zeta)d\zeta + \int \Gamma_{\xi_0}(\xi,\zeta)v(\zeta)L_{\xi_0}\phi(\zeta)d\zeta \\
&+\int \Gamma_{\xi_0}(\xi,\zeta)(a(\zeta) - P^1_{\xi_0}a(\zeta))^2\partial_t v(\zeta)\partial_t\phi(\zeta)d\zeta \\
&+\int \Gamma_{\xi_0}(\xi,\zeta)(b(\zeta) - P^1_{\xi_0}b(\zeta))^2\partial_t v(\zeta)\partial_t\phi(\zeta)d\zeta\,.
\end{aligned}
$$

REMARK 3.4. We note explicitly that

$$
\int \partial_t\Gamma_{\xi_0}(\xi,\zeta)(a(\zeta) - P^1_{\xi_0}a(\zeta))^2(\partial_t v(\zeta) - \partial_t v(\xi_0))\phi(\zeta)d\zeta
$$

is a principal value integral, defined as

$$
\begin{aligned}
&\lim_{\epsilon\to 0}\int_{d_{\xi_0}(\xi,\zeta)>\epsilon} \partial_t\Gamma_{\xi_0}(\xi,\zeta)\big((a(\zeta) - P^1_{\xi_0}a(\zeta))^2(\partial_t v(\zeta) - \partial_t v(\xi_0))\phi(\zeta) \\
&-(a(\xi) - P^1_{\xi_0}a(\xi))^2(\partial_t v(\xi) - \partial_t v(\xi_0))\phi(\xi)\big)d\zeta\,,
\end{aligned}
$$

and the limit exists because v is of class $C^{2,\beta}_L(\Omega)$, and

$$
|\partial_t v(\zeta) - \partial_t v(\xi)| \le d^\beta_\xi(\xi,\zeta) \le d^\beta_{\xi_0}(\xi,\zeta) + d^{\beta/2}_{\xi_0}(\xi,\zeta)d^{\beta/2}_{\xi_0}(\xi,\xi_0) \le d^{\beta/2}_{\xi_0}(\xi,\zeta)\,,
$$

(by (22)). Moreover

$$
\big|(a(\zeta) - P^1_{\xi_0}a(\zeta)) - (a(\xi) - P^1_{\xi_0}a(\xi))\big| \le C d^{\frac{1+\alpha}{2}}_{\xi_0}(\xi,\zeta)\,,
$$

by Remark 3.1.

PROOF OF THEOREM 3.1. Due to Remark 3.2, we only have to prove the theorem for smooth v, a and b. By definition of fundamental solution

$$\begin{aligned} v\phi(\xi) &= \int \Gamma_{\xi_0}(\xi,\zeta)L_{\xi_0}(v\phi)(\zeta)d\zeta \\ &= \int \Gamma_{\xi_0}(\xi,\zeta)L_{\xi_0}v(\zeta)\phi(\zeta)d\zeta + 2\int \Gamma_{\xi_0}(\xi,\zeta)X_{\xi_0}v(\zeta)X_{\xi_0}\phi(\zeta)d\zeta \\ &\quad + 2\int \Gamma_{\xi_0}(\xi,\zeta)Y_{\xi_0}v(\zeta)Y_{\xi_0}\phi(\zeta)d\zeta + \int \Gamma_{\xi_0}(\xi,\zeta)v(\zeta)L_{\xi_0}\phi(\zeta)d\zeta \\ &= \int \Gamma_{\xi_0}(\xi,\zeta)Lv(\zeta)\phi(\zeta)d\zeta + 2\int \Gamma_{\xi_0}(\xi,\zeta)Xv(\zeta)X_{\xi_0}\phi(\zeta)d\zeta \\ &\quad + 2\int \Gamma_{\xi_0}(\xi,\zeta)Yv(\zeta)Y_{\xi_0}\phi(\zeta)d\zeta + \int \Gamma_{\xi_0}(\xi,\zeta)v(\zeta)L_{\xi_0}\phi(\zeta)d\zeta \\ &\quad - 2\int \Gamma_{\xi_0}(\xi,\zeta)(X - X_{\xi_0})v(\zeta)X_{\xi_0}\phi(\zeta)d\zeta \\ &\quad - 2\int \Gamma_{\xi_0}(\xi,\zeta)(Y - Y_{\xi_0})v(\zeta)Y_{\xi_0}\phi(\zeta)d\zeta \\ &\quad - \int \Gamma_{\xi_0}(\xi,\zeta)(L - L_{\xi_0})v(\zeta)\phi(\zeta)d\zeta\,. \end{aligned} \tag{24}$$

Let us compute $L - L_{\xi_0}$ in terms of X_{ξ_0}, Y_{ξ_0} and ∂_t. Since

$$Xv = X_{\xi_0}v + (a - P^1_{\xi_0}a)\partial_t v\,, \tag{25}$$

then

$$\begin{aligned} (L - L_{\xi_0})v &= X^2v + Y^2v - (Xa + Yb)\partial_t v \\ &\quad - X^2_{\xi_0}v - Y^2_{\xi_0}v + (Xa + Yb)(\xi_0)\partial_t v \\ &= X(X_{\xi_0} + (a - P^1_{\xi_0}a)\partial_t)v + Y(Y_{\xi_0} + (b - P^1_{\xi_0}b)\partial_t)v \\ &\quad - (Xa - Xa(\xi_0))\partial_t v - (Yb - Yb(\xi_0))\partial_t v - X^2_{\xi_0}v - Y^2_{\xi_0}v \\ &= XX_{\xi_0}v + (a - P^1_{\xi_0}a)X\partial_t v + YY_{\xi_0}v + (b - P^1_{\xi_0}b)Y\partial_t - X^2_{\xi_0}v - Y^2_{\xi_0}v = \end{aligned}$$

(applying again (25))

$$\begin{aligned} &= 2(a - P^1_{\xi_0}a)X_{\xi_0}\partial_t v + 2(b - P^1_{\xi_0}b)Y_{\xi_0}\partial_t v + (a - P^1_{\xi_0}a)^2\partial^2_{tt}v + (b - P^1_{\xi_0}b)^2\partial^2_{tt}v \\ &= 2(a - P^1_{\xi_0}a)X_{\xi_0}(\partial_t v - \partial_t v(\xi_0)) + (a - P^1_{\xi_0}a)^2\partial_t(\partial_t v - \partial_t v(\xi_0)) \\ &\quad + 2(b - P^1_{\xi_0}b)Y_{\xi_0}(\partial_t v - \partial_t v(\xi_0)) + (b - P^1_{\xi_0}b)^2\partial_t(\partial_t v - \partial_t v(\xi_0))\,. \end{aligned}$$

In order to study the last integral in (24) we start with

$$\begin{aligned} &- 2\int \Gamma_{\xi_0}(\xi,\zeta)(a(\zeta) - P^1_{\xi_0}a(\zeta))X_{\xi_0}(\partial_t v(\zeta) - \partial_t v(\xi_0))\phi(\zeta)d\zeta \\ &- \int \Gamma_{\xi_0}(\xi,\zeta)(a(\zeta) - P^1_{\xi_0}a(\zeta))^2\partial_t(\partial_t v(\zeta) - \partial_t v(\xi_0))(\zeta)\phi(\zeta)d\zeta = \end{aligned} \tag{26}$$

(integrating by parts, using the fact that $X_{\xi_0}(a(\zeta)-P^1_{\xi_0}a(\zeta)) = X_{\xi_0}a(\zeta)-Xa(\xi_0)$, and denoting $X^{\zeta}_{\xi_0}\Gamma_{\xi_0}(\xi,\zeta)$ the derivative with respect to ζ)

$$
\begin{aligned}
&= 2\int X^{\zeta}_{\xi_0}\Gamma_{\xi_0}(\xi,\zeta)(a(\zeta)-P^1_{\xi_0}a(\zeta))(\partial_t v(\zeta)-\partial_t v(\xi_0))\phi(\zeta)d\zeta\\
&\quad +2\int \Gamma_{\xi_0}(\xi,\zeta)(X_{\xi_0}a(\zeta)-Xa(\xi_0))(\partial_t v(\zeta)-\partial_t v(\xi_0))\phi(\zeta)d\zeta\\
&\quad +2\int \Gamma_{\xi_0}(\xi,\zeta)(a(\zeta)-P^1_{\xi_0}a(\zeta))(\partial_t v(\zeta)-\partial_t v(\xi_0))X_{\xi_0}\phi(\zeta)d\zeta\\
&\quad +\int \partial^{\zeta}_t\Gamma_{\xi_0}(\xi,\zeta)(a(\zeta)-P^1_{\xi_0}a(\zeta))^2(\partial_t v(\zeta)-\partial_t v(\xi_0))\phi(\zeta)d\zeta\\
&\quad +2\int \Gamma_{\xi_0}(\xi,\zeta)(a(\zeta)-P^1_{\xi_0}a(\zeta))\partial_t a(\zeta)(\partial_t v(\zeta)-\partial_t v(\xi_0))\phi(\zeta)d\zeta\\
&\quad +\int \Gamma_{\xi_0}(\xi,\zeta)(a(\zeta)-P^1_{\xi_0}a(\zeta))^2(\partial_t v(\zeta)-\partial_t v(\xi_0))\partial_t\phi(\zeta)d\zeta =
\end{aligned}
$$

(since $(a-P^1_{\xi_0}a)\partial_t a = Xa - X_{\xi_0}a$ and $(a-P^1_{\xi_0}a)\partial_t v = Xv - X_{\xi_0}v$ and $\partial^{\zeta}_t\Gamma_{\xi_0}(\xi,\zeta) = -\partial^{\xi}_t\Gamma_{\xi_0}(\xi,\zeta)$ and $X^{\zeta}_{\xi_0}\Gamma_{\xi_0}(\xi,\zeta) = -\tilde{X}^{\xi}_{\xi_0}\Gamma_{\xi_0}(\xi,\zeta)$)

$$
\begin{aligned}
&= -2\int \tilde{X}^{\xi}_{\xi_0}\Gamma_{\xi_0}(\xi,\zeta)(a(\zeta)-P^1_{\xi_0}a(\zeta))(\partial_t v(\zeta)-\partial_t v(\xi_0))\phi(\zeta)d\zeta\\
&\quad +2\int \Gamma_{\xi_0}(\xi,\zeta)(X_{\xi_0}a(\zeta)-Xa(\xi_0))(\partial_t v(\zeta)-\partial_t v(\xi_0))\phi(\zeta)d\zeta\\
&\quad +2\int \Gamma_{\xi_0}(\xi,\zeta)(X-X_{\xi_0})v(\zeta)X_{\xi_0}\phi(\zeta)d\zeta\\
&\quad -2\partial_t v(\xi_0)\int \Gamma_{\xi_0}(\xi,\zeta)(a(\zeta)-P^1_{\xi_0}a(\zeta))X_{\xi_0}\phi(\zeta)d\zeta\\
&\quad -\int \partial_t\Gamma^{\xi}_{\xi_0}(\xi,\zeta)(a(\zeta)-P^1_{\xi_0}a(\zeta))^2(\partial_t v(\zeta)-\partial_t v(\xi_0))\phi(\zeta)d\zeta\\
&\quad +2\int \Gamma_{\xi_0}(\xi,\zeta)(X-X_{\xi_0})a(\zeta)(\partial_t v(\zeta)-\partial_t v(\xi_0))\phi(\zeta)d\zeta\\
&\quad +\int \Gamma_{\xi_0}(\xi,\zeta)(a(\zeta)-P^1_{\xi_0}a(\zeta))^2\partial_t v(\zeta)\partial_t\phi(\zeta)d\zeta\\
&\quad -\partial_t v(\xi_0)\int \Gamma_{\xi_0}(\xi,\zeta)(a(\zeta)-P^1_{\xi_0}a(\zeta))^2\partial_t\phi(\zeta)d\zeta
\end{aligned}
$$

$$
\begin{aligned}
&= 2\int \Gamma_{\xi_0}(\xi,\zeta)(Xa(\zeta) - Xa(\xi_0))(\partial_t v(\zeta) - \partial_t v(\xi_0))\phi(\zeta)d\zeta \\
&- 2\int \tilde{X}_{\xi_0}\Gamma_{\xi_0}(\xi,\zeta)(a(\zeta) - P^1_{\xi_0}a(\zeta))(\partial_t v(\zeta) - \partial_t v(\xi_0))\phi(\zeta)d\zeta \\
&- \int \partial_t\Gamma_{\xi_0}(\xi,\zeta)(a(\zeta) - P^1_{\xi_0}a(\zeta))^2(\partial_t v(\zeta) - \partial_t v(\xi_0))\phi(\zeta)d\zeta \\
&+ \int \Gamma_{\xi_0}(\xi,\zeta)(a(\zeta) - P^1_{\xi_0}a(\zeta))^2\partial_t v(\zeta)\partial_t\phi(\zeta)d\zeta \\
&+ 2\int \Gamma_{\xi_0}(\xi,\zeta)(X - X_{\xi_0})v(\zeta)X_{\xi_0}\phi(\zeta)d\zeta \\
&- 2\partial_t v(\xi_0)\int \Gamma_{\xi_0}(\xi,\zeta)(a(\zeta) - P^1_{\xi_0}a(\zeta))X_{\xi_0}\phi(\zeta)d\zeta \\
&- \partial_t v(\xi_0)\int \Gamma_{\xi_0}(\xi,\zeta)(a(\zeta) - P^1_{\xi_0}a(\zeta))^2\partial_t\phi(\zeta)d\zeta =
\end{aligned}
$$

Analogously

$$
\begin{aligned}
(27)\qquad &- 2\int \Gamma_{\xi_0}(\xi,\zeta)(b(\zeta) - P^1_{\xi_0}b(\zeta))\partial_t Y_{\xi_0}v(\zeta)\phi(\zeta)d\zeta \\
&- \int \Gamma_{\xi_0}(\xi,\zeta)(b(\zeta) - P^1_{\xi_0}b(\zeta))^2\partial^2_{tt}v(\zeta)\phi(\zeta)d\zeta \\
&= 2\int \Gamma_{\xi_0}(\xi,\zeta)(Yb(\zeta) - Yb(\xi_0))(\partial_t v(\zeta) - \partial_t v(\xi_0))\phi(\zeta)d\zeta \\
&- 2\int \tilde{Y}_{\xi_0}\Gamma_{\xi_0}(\xi,\zeta)(b(\zeta) - P^1_{\xi_0}b(\zeta))(\partial_t v(\zeta) - \partial_t v(\xi_0))\phi(\zeta)d\zeta \\
&- \int \partial_t\Gamma_{\xi_0}(\xi,\zeta)(b(\zeta) - P^1_{\xi_0}b(\zeta))^2(\partial_t v(\zeta) - \partial_t v(\xi_0))\phi(\zeta)d\zeta \\
&+ \int \Gamma_{\xi_0}(\xi,\zeta)(b(\zeta) - P^1_{\xi_0}b(\zeta))^2\partial_t v(\zeta)\partial_t\phi(\zeta)d\zeta \\
&+ 2\int \Gamma_{\xi_0}(\xi,\zeta)(Y - Y_{\xi_0})v(\zeta)Y_{\xi_0}\phi(\zeta)d\zeta \\
&- 2\partial_t v(\xi_0)\int \Gamma_{\xi_0}(\xi,\zeta)(b(\zeta) - P^1_{\xi_0}b(\zeta))Y_{\xi_0}\phi(\zeta)d\zeta \\
&- \partial_t v(\xi_0)\int \Gamma_{\xi_0}(\xi,\zeta)(b(\zeta) - P^1_{\xi_0}b(\zeta))^2\partial_t\phi(\zeta)d\zeta\,.
\end{aligned}
$$

From (24), the expression of $L - L_{\xi_0}$, (26) and (27), the thesis follows.

THEOREM 3.2. *Assume that* $v \in C^{2,\beta}_L(\Omega)$ *and* $\partial_t v \in C^{1,\beta}_L(\Omega)$. *Then for every* $\phi \in C^\infty_0(\Omega)$ *we have:*

$$\phi v(\xi) = A_1(\xi,\xi_0) + B_1(\xi,\xi_0) + \partial_t v(\xi_0)C(\xi,\xi_0) + D_1(\xi,\xi_0) + E_1(\xi,\xi_0)\,,$$

where we have denoted

$$A_1(\xi,\xi_0)=\int \Gamma_{\xi_0}(\xi,\zeta)(Lv(\zeta)-2aX\partial_t v(\xi_0)-2bY\partial_t v(\xi_0))\phi(\zeta)d\zeta\,,$$

$$\begin{aligned}B_1(\xi,\xi_0)=&\,2\int \Gamma_{\xi_0}(\xi,\zeta)(Xa(\zeta)-Xa(\xi_0))\phi(\zeta)(\partial_t v(\zeta)-P^1_{\xi_0}\partial_t v(\zeta))d\zeta\\
&-2\int \tilde{X}_{\xi_0}\Gamma_{\xi_0}(\xi,\zeta)(a(\zeta)-P^1_{\xi_0}a(\zeta))\phi(\zeta)(\partial_t v(\zeta)-P^1_{\xi_0}\partial_t v(\zeta))(\zeta)d\zeta\\
&-\int \partial_t\Gamma_{\xi_0}(\xi,\zeta)(a(\zeta)-P^1_{\xi_0}a(\zeta))^2\phi(\zeta)(\partial_t v(\zeta)-P^1_{\xi_0}\partial_t v(\zeta))d\zeta\\
&+2\int \Gamma_{\xi_0}(\xi,\zeta)(Yb(\zeta)-Yb(\xi_0))\phi(\zeta)(\partial_t v(\zeta)-P^1_{\xi_0}\partial_t v(\zeta))d\zeta\\
&-2\int \tilde{Y}_{\xi_0}\Gamma_{\xi_0}(\xi,\zeta)(b(\zeta)-P^1_{\xi_0}b(\zeta))\phi(\zeta)(\partial_t v(\zeta)-P^1_{\xi_0}\partial_t v(\zeta))d\zeta\\
&-\int \partial_t\Gamma_{\xi_0}(\xi,\zeta)(b(\zeta)-P^1_{\xi_0}b(\zeta))\phi(\zeta)(\partial_t v(\zeta)-P^1_{\xi_0}\partial_t v(\zeta)d\zeta\,,\end{aligned}$$

$$\begin{aligned}C(\xi,\xi_0)=&-2\int \Gamma_{\xi_0}(\xi,\zeta)(a(\zeta)-P^1_{\xi_0}a(\zeta))X_{\xi_0}\phi(\zeta)d\zeta\\
&-2\int \Gamma_{\xi_0}(\xi,\zeta)(b(\zeta)-P^1_{\xi_0}b(\zeta))Y_{\xi_0}\phi(\zeta)d\zeta\\
&-\int \Gamma_{\xi_0}(\xi,\zeta)(a(\zeta)-P^1_{\xi_0}a(\zeta))^2\partial_t\phi(\zeta)d\zeta\\
&-\int \Gamma_{\xi_0}(\xi,\zeta)(b(\zeta)-P^1_{\xi_0}b(\zeta))^2\partial_t\phi(\zeta)d\zeta\,,\end{aligned}$$

$$D_1(\xi,\xi_0)=t(x-x_0)X\partial_t v(\xi_0)-t(y-y_0)Y\partial_t v(\xi_0)\,,$$

$$\begin{aligned}E_1(\xi,\xi_0)=&\,2\int \Gamma_{\xi_0}(\xi,\zeta)(Xv(\zeta)-tX\partial_t v(\xi_0)-aX\partial_t v(\xi_0)(x-x_0))X_{\xi_0}\phi(\zeta)d\zeta\\
&+2\int \Gamma_{\xi_0}(\xi,\zeta)(Yv(\zeta)-tY\partial_t v(\xi_0)-bY\partial_t v(\xi_0)(y-y_0))Y_{\xi_0}\phi(\zeta)d\zeta\\
&+\int \Gamma_{\xi_0}(\xi,\zeta)(v(\zeta)-X\partial_t v(\xi_0)t(x-x_0)-Y\partial_t v(\xi_0)t(y-y_0))L_{\xi_0}\phi(\zeta)d\zeta\\
&+\int \Gamma_{\xi_0}(\xi,\zeta)(a(\zeta)-P^1_{\xi_0}a(\zeta))^2(\partial_t v(\zeta)-P^1_{\xi_0}\partial_t v(\zeta))\partial_t\phi(\zeta)d\zeta\\
&+\int \Gamma_{\xi_0}(\xi,\zeta)(b(\zeta)-P^1_{\xi_0}b(\zeta))^2(\partial_t v(\zeta)-P^1_{\xi_0}\partial_t v(\zeta))\partial_t\phi(\zeta)d\zeta\,.\end{aligned}$$

PROOF. The assertion can be proved applying Theorem 3.1 to

$$w(\zeta)=v(\zeta)-t(X\partial_t v(\xi_0)(x-x_0)-Y\partial_t v(\xi_0)(y-y_0))$$

and using the fact that

$$\begin{aligned} Lw(\zeta) &= Lv(\zeta) - 2a\partial_t v(\xi_0) - bY\partial_t v(\xi_0) \\ &\quad - (Xa + Yb)\big(X\partial_t v(\xi_0)(x - x_0) + Y\partial_t v(\xi_0)(y - y_0)\big)\,, \\ Xw(\zeta) &= Xv(\zeta) - tX\partial_t v(\xi_0) - aX\partial_t v(\xi_0)(x - x_0)\,, \\ Yw(\zeta) &= Yv(\zeta) - tY\partial_t v(\xi_0) - aY\partial_t v(\xi_0)(y - y_0)\,, \\ \partial_t w(\xi_0) &= \partial_t v(\xi_0)\,, \\ \partial_t w(\zeta) - \partial_t w(\xi_0) &= \partial_t v(\zeta) - P^1_{\xi_0}\partial_t v(\zeta)\,. \end{aligned}$$

□

4. – Regularity in the directions X_{ξ_0} and Y_{ξ_0}

We will now differentiate all terms which appear in the representation formula. To this end we introduce some notations: $D = (X, Y)$ is the subelliptic gradient, and

$$\Delta_i f(h)(\xi) = f(e^{hD_i}(\xi)) - f(\xi)\,,$$

is the difference in the D_i direction. Hence, by definition of derivative of f in the direction D_i, we have

$$D_i f(\xi) = \lim_{h\to 0} \frac{\Delta_i f(h)(\xi)}{h}\,.$$

Moreover we will denote the second difference in the i and j directions

$$\Delta^2_{i,j} f(h,k)(\xi) = \Delta_i(\Delta_i f(h))(k)(\xi)$$

and, if it exists, we will call

$$\delta^2_{i,j} f(\xi) = \lim_{h,k\to 0} \frac{\Delta^2_{i,j} f(h,k)(\xi)}{hk}\,.$$

Obviously $\delta^2_{i,j} f(\xi)$ could exist even if f has first derivative only at the point ξ; if however f is differentiable in all directions in a neighborhood of ξ and there exists $\delta^2_{i,j} f(\xi)$, then we have

$$\delta^2_{i,j} f(\xi) = D^2_{i,j} f(\xi)\,.$$

In the same way we define the difference quotient of higher order, and the derivative in the directions X_{ξ_0}. In particular we will call $D_{\xi_0} = (X_{\xi_0}, Y_{\xi_0})$, so that $D_{\xi_0;1} = X_{\xi_0}$ and $D_{\xi_0;2} = Y_{\xi_0}$, $\Delta_{\xi_0;i} f(h)(\xi)$ the difference of f in the direction $D_{\xi_0;i}$, and $\Delta^k_{\xi_0;i} f(h)(\xi)$ the k-th difference in the same direction, $\delta^k_{\xi_0;i} f(\xi)$ will denote the limit of the difference quotient, when the derivative does not exist. Finally we call

$$\delta_i \delta^2_{\xi_0;j,k} f(\xi) = \lim_{h,l,s\to 0} \Delta_i(\Delta^2_{\xi_0;j,k} f(h,l))(s)(\xi)\,,$$

if it exists.

With these notations the main theorem in the section is the following:

THEOREM 4.1. *Assume that $f \in C_L^{1,\alpha}(\Omega)$ and that the coefficients a and b of L are of class $C_L^{1,\alpha}(\Omega)$. Then for any solution $v \in C_L^{2,\alpha}(\Omega)$ of $Lv = f$, there exists $\delta_i \delta^2_{\xi_0;j,k} v(\xi_0)$ for all $\xi_0 \in \Omega$ and for all i, j, k.*

Since this is a local result, we can fix three open sets Ω, Ω_1 and Ω_2 such that $\Omega_2 \subset\subset \Omega_1 \subset\subset \Omega$, a function $\phi \in C_0^\infty(\Omega)$ such that $\phi = 1$ in Ω_1, and we study only $v_{|\Omega_2} = v\phi_{|\Omega_2}$. Then, from Theorem 3.1 we get

$$v(\xi) = A(\xi, \xi_0) + B(\xi, \xi_0) + \partial_t v(\xi_0) C(\xi, \xi_0) + E(\xi, \xi_0)$$

for every $\xi, \xi_0 \in \Omega_2$. Let us differentiate this formula.

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THEOREM 4.2. *Let $f \in C_L^{1,\alpha}(\Omega)$, $\xi_0 \in \Omega_2$ and for every ξ in Ω_2*

$$A(\xi, \xi_0) = \int \Gamma_{\xi_0}(\xi, \zeta) f(\zeta)\phi(\zeta) d\zeta \, .$$

Then

$$\begin{aligned}\delta_i \delta^2_{\xi_0;j,k} A(\xi_0, \xi_0) &= \int D^3_{\xi_0;i,j,k} \Gamma_{\xi_0}(\xi_0, \zeta)(f - P^1_{\xi_0} f)(\zeta)\phi(\zeta) d\zeta \\ &+ \int D_{\xi_0;i} \Gamma_{\xi_0}(\xi_0, \zeta) \tilde{D}^2_{\xi_0;j,k}(P^1_{\xi_0} f \phi)(\zeta) d\zeta \, ,\end{aligned}$$

where the difference quotient $\delta_i \delta^2_{\xi_0;j,k} A(\xi_0, \xi_0)$ is taken with respect to the first variable, and $\tilde{D}_{\xi_0;1} = \tilde{X}_{\xi_0}$, $\tilde{D}_{\xi_0;2} = \tilde{Y}_{\xi_0}$ are the operators defined in Remark 3.3, and $\tilde{D}^2_{\xi_0;j,k}$ is the second order derivative in the same direction. Finally $P^1_{\xi_0} f$ is the Taylor polynomial of order 1 of f, defined in Remark 2.3, and ν_i is the outer normal.

PROOF. It is standard to see that

$$D_{\xi_0;k} A(\xi, \xi_0) = \int D^\xi_{\xi_0;k} \Gamma_{\xi_0}(\xi, \zeta) f(\zeta)\phi(\zeta) d\zeta =$$

($D^\xi_{\xi_0;k} \Gamma_{\xi_0}(\xi, \zeta)$ denotes the derivative of Γ_{ξ_0} with respect to the variable ξ. Also recall that $D^\xi_{\xi_0} \Gamma_{\xi_0}(\xi, \zeta) = -\tilde{D}^\xi_{\xi_0} \Gamma_{\xi_0}(\xi, \zeta)$)

$$= \int D^\xi_{\xi_0;k} \Gamma_{\xi_0}(\xi, \zeta)(f - P^1_{\xi_0} f)(\zeta)\phi(\zeta) d\zeta - \int \tilde{D}^\zeta_{\xi_0;k} \Gamma_{\xi_0}(\xi, \zeta) P^1_{\xi_0} f(\zeta)\phi(\zeta) d\zeta =$$

(integrating by parts the second term)

$$\begin{aligned}&= \int D^\xi_{\xi_0;k} \Gamma_{\xi_0}(\xi, \zeta)(f - P^1_{\xi_0} f)(\zeta)\phi(\zeta) d\zeta \\ &+ \int \Gamma_{\xi_0}(\xi, \zeta) \tilde{D}_{\xi_0;k}(P^1_{\xi_0} f \, \phi(\zeta)) d\zeta = v_1(\xi) + v_2(\xi) \, .\end{aligned}$$

Differentiating v_2 we obtain

$$D_{\xi_0;j}v_2(\xi) = \int \Gamma_{\xi_0}(\xi,\zeta)\tilde{D}^2_{\xi_0;j,k}(P^1_{\xi_0}f\phi)(\zeta)d\zeta\,,$$

hence

$$D_iD_{\xi_0;j}v_2(\xi) = \int D_i\Gamma_{\xi_0}(\xi,\zeta)\tilde{D}^2_{\xi_0;j,k}(P^1_{\xi_0}f\,\phi)(\zeta)d\zeta\,,$$

and, since $D_i\Gamma_{\xi_0}(\xi_0,\zeta) = D_{\xi_0;i}\Gamma_{\xi_0}(\xi_0,\zeta)$

$$D_iD_{\xi_0;j}v_2(\xi_0) = \int D_{\xi_0;i}\Gamma_{\xi_0}(\xi_0,\zeta)\tilde{D}^2_{\xi_0;j,k}(P^1_{\xi_0}f\,\phi)(\zeta)d\zeta\,.$$

Next

$$\begin{aligned}D_{\xi_0;j}v_1(\xi) &= \int D^2_{\xi_0;j.k}\Gamma_{\xi_0}(\xi,\zeta)(f-P^1_{\xi_0}f)(\zeta)\phi(\zeta)d\zeta\\ &-(f-P^1_{\xi_0}f)(\xi)\phi(\xi)\int_{\{\zeta:d_{\xi_0}(\zeta,0)=1\}} D_{\xi_0;k}\Gamma_{\xi_0}(0,\zeta)\frac{\tilde{D}_{\xi_0;j}d_{\xi_0}(0,\zeta)}{|\nabla d_{\xi_0}(0,\zeta)|}d\sigma(\zeta)\,.\end{aligned}$$

In order to take another derivative of $D_{\xi_0;j}v_1(\xi)$, we first fix a function θ in $C^\infty(\mathbb{R})$ satisfying $0\le\theta\le 1$, $\theta(\tau)=0\,\forall\,\tau\le 1$ and $\theta(\tau)=1\,\forall\,\tau\ge 1$, and define for every $\epsilon>0$

$$v_{\epsilon,j}(\xi) = \int D^2_{\xi_0;j,k}\Gamma_{\xi_0}(\xi,\zeta)\theta\left(\frac{d_{\xi_0}(\xi,\zeta)}{\epsilon}\right)(f-P^1_{\xi_0}f)(\zeta)\phi(\zeta)d\zeta\,.$$

Now we show that

$$\sup_{d(\xi,\xi_0)<\epsilon/2}|v_{\epsilon,j}(\xi)-D_{\xi_0;j}v_1(\xi)| = O(e^{1+\alpha})\quad\text{as}\quad \epsilon\to 0\,. \tag{28}$$

indeed

$$\begin{aligned}&v_{\epsilon,j}(\xi)-D_{\xi_0,j}v_1(\xi)\\ &=-\lim_{h\to 0}\int_{\{\zeta:h<d_{\xi_0}(\xi,\zeta)<\epsilon\}} D^2_{\xi;j,k}\Gamma_{\xi_0}(\xi,\zeta)\\ &\quad\cdot\big((f-P^1_{\xi_0}f)(\zeta)\phi(\zeta)-(f-P^1_{\xi_0}f)(\xi)\phi(\xi)\big)d\zeta\\ &\quad+\int_{\{\zeta:\epsilon<d_{\xi_0}(\xi,\zeta)<2\epsilon\}} D^2_{\xi_0;j,k}\Gamma_{\xi_0}(\xi,\zeta)\left(\theta\left(\frac{d_{\xi_0}(\xi,\zeta)}{\epsilon}\right)-1\right)\\ &\quad\cdot(f-P^1_{\xi_0}f)(\zeta)\phi(\zeta)d\zeta\\ &\quad+(f-P^1_{\xi_0}f)(\xi)\phi(\xi)\int_{\{\zeta:d_{\xi_0}(\zeta,0)=1\}} X_{\xi_0}\Gamma_{\xi_0}(\zeta,0)\frac{\tilde{D}_{\xi_0;j}d_{\xi_0}(0,\zeta)}{|\nabla d_{\xi_0}(0,\zeta)|}d\sigma(\zeta)\,.\end{aligned}$$

The integrand in the first term can be evaluated as follows

$$(f(\zeta) - P^1_{\xi_0} f(\zeta))\phi(\zeta) - (f(\xi) - P^1_{\xi_0} a(\xi))\phi(\xi)$$
$$= \big((f - P^1_{\xi_0} f)(\zeta) - (f - P^1_{\xi_0} f)(\xi)\big)\phi(\xi) + (f - P^1_{\xi_0} f)(\zeta)(\phi(\zeta) - \phi(\xi)) \leq$$

(by Remark 3.1)

$$\leq d^{1+\alpha}_{\xi_0}(\xi,\zeta) + d^{\frac{1+\alpha}{2}}_{\xi_0}(\xi,\zeta) d^{\frac{1+\alpha}{2}}_{\xi_0}(\xi,\xi_0) + d_{\xi_0}(\xi,\zeta) d^{\alpha}_{\xi_0}(\xi,\xi_0)$$
$$+ d^{1+\alpha}_{\xi_0}(\zeta,\xi_0) d^{\alpha}_{\xi_0}(\xi,\zeta) \leq$$

(on the set $\{\zeta : d_{\xi_0}(\xi,\zeta) < \epsilon\}$ we have $d_{\xi_0}(\xi_0), \zeta) \leq d_{\xi_0}(\xi_0,\xi) + d_{\xi_0}(\xi,\zeta) \leq Cd(\xi_0,\xi) + d_{\xi_0}(\xi,\zeta) \leq C\epsilon$, where C is a constant dependent only on the coefficients a and b of L).

$$\leq d^{1+\alpha}_{\xi_0}(\xi,\zeta) + \epsilon^{\frac{1+\alpha}{2}} d^{\frac{1+\alpha}{2}}_{\xi_0}(\xi,\zeta) + \epsilon^{\alpha} d_{\xi_0}(\xi,\zeta) + \epsilon^{1+\alpha} d^{\alpha}_{\xi_0}(\xi,\zeta).$$

The second integrand can be evaluated:

$$|f(\zeta) - P^1_{\xi_0} f(\zeta)| \leq d^{1+\alpha}_{\xi_0}(\zeta,\xi_0) \leq \epsilon^{1+\alpha}.$$

Hence

$$|v_{\epsilon,j}(\xi) - D_{\xi_0;j} v_1(\xi)| \leq \epsilon^{\frac{1+\alpha}{2}} \int_0^{\epsilon} \rho^{\frac{\alpha-1}{2}} + \epsilon^{1+\alpha} \int_{\epsilon}^{2\epsilon} \rho^{-1} + \epsilon^{1+\alpha} \leq C^{1+\alpha}_{\epsilon}.$$

Next, if

$$w_1(\xi_0) = \int D^3_{\xi_0;i,j,k} \Gamma_{\xi_0}(\xi_0,\zeta)(f(\zeta) - P^1_{\xi_0} f(\zeta))\phi(\zeta) d\zeta,$$

will show that

$$\sup_{d(\xi,\xi_0)<\epsilon/2} |D_i v_{\epsilon,j}(\xi) - w_1(\xi_0)| = o(1) \quad \text{as} \quad \epsilon \to 0. \tag{29}$$

Indeed

$$D_{\xi_0;i} v_{\epsilon,j}(\xi) - w_1(\xi_0)$$
$$= \int D_i D^2_{\xi_0;j,k} \Gamma_{\xi_0}(\xi,\zeta) \theta\left(\frac{d_{\xi_0}(\xi,\zeta)}{\epsilon}\right)(f(\zeta) - P^1_{\xi_0} f(\zeta))\phi(\zeta) d\zeta$$
$$- \int D^3_{\xi_0;i,j,k} \Gamma_{\xi_0}(\xi_0,\zeta)(f(\zeta) - P^1_{\xi_0} f(\zeta))\phi(\zeta) d\zeta$$
$$+ \frac{1}{\epsilon} \int D^2_{\xi_0;j,k} \Gamma_{\xi_0}(\xi,\zeta) \theta'\left(\frac{d_{\xi_0}(\xi,\zeta)}{\epsilon}\right) D_i d_{\xi_0}(\xi,\zeta)(f(\zeta) - P^1_{\xi_0} f(\zeta))\phi(\zeta) d\zeta$$
$$- D_i\big((f - P^1_{\xi_0} f)\phi\big)(\xi) \int_{\{\zeta : d_{\xi_0}(\zeta,0)=1\}} D_{\xi_0;k} \Gamma_{\xi_0}(0,\zeta) \frac{\tilde{D}_{\xi_0;j} d_{\xi_0}(0,\zeta)}{|\nabla d_{\xi_0}(0,\zeta)|} d\sigma(\zeta)$$
$$= I_1(\xi,\xi_0) + I_2(\xi,\xi_0) + I_3(\xi,\xi_0) + I_4(\xi,\xi_0) + I_5(\xi,\xi_0) + I_6(\xi,\xi_0),$$

where the integrals I_s, are given by

$$I_1 = \int (D^3_{\xi_0;i,j,k}\Gamma_{\xi_0}(\xi,\zeta) - D^3_{\xi_0;i,j,k}\Gamma_{\xi_0}(\xi_0,\zeta)) \cdot \theta\left(\frac{d_{\xi_0}(\xi,\zeta)}{\epsilon}\right)(f(\zeta) - P^1_{\xi_0}f(\zeta))\phi(\zeta)d\zeta\,,$$

$$I_2 = \int D^3_{\xi_0;i,j,k}\Gamma_{\xi_0}(\xi_0,\zeta)\left(\theta\Big(\frac{d_{\xi_0}(\xi,\zeta)}{\epsilon}\Big) - 1\right)(f(\zeta) - P^1_{\xi_0}f(\zeta))\phi(\zeta)d\zeta\,,$$

$$I_3 = \frac{1}{\epsilon}\int D^2_{\xi_0;j,k}\Gamma_{\xi_0}(\xi_0,\zeta)\theta'\left(\frac{d_{\xi_0}(\xi,\zeta)}{\epsilon}\right)D_{\xi_0;i}d_0(\xi,\zeta)(f - P^1_{\xi_0}f)(\zeta)\phi(\zeta)d\zeta\,,$$

$$I_4 = (a(\xi) - P^1_{\xi_0}a(\xi))\int \partial_t D^2_{\xi_0;j,k}\Gamma_{\xi_0}(\xi_0,\zeta) \cdot \theta\left(\frac{d_{\xi_0}(\xi,\zeta)}{\epsilon}\right)(f(\zeta) - P^1_{\xi_0}f(\zeta))\phi(\zeta)d\zeta\,,$$

$$I_5 = \frac{1}{\epsilon}(a(\xi) - P^1_{\xi_0}a(\xi))\int D_{\xi_0;j}\Gamma_{\xi_0}(\xi_0,\zeta)\theta'\left(\frac{d_{\xi_0}(\xi,\zeta)}{\epsilon}\right) \cdot \partial_t d_0(\xi,\zeta)(f - P^1_{\xi_0}f)(\zeta)\phi(\zeta)d\zeta\,,$$

$$I_6 = -D_i\big((f - P^1_{\xi_0}f)\phi\big)(\xi)\int_{\{\zeta:d_{\xi_0}(\zeta,0)=1\}} D_{\xi_0;k}\Gamma_{\xi_0}(0,\zeta)\frac{\tilde{D}_{\xi_0;j}d_{\xi_0}(0,\zeta)}{|\nabla d_{\xi_0}(0,\zeta)|}d\sigma(\zeta)\,.$$

The integral I_1 is made on the set $\{\zeta : d_{\xi_0}(\xi,\zeta) > \epsilon\}$; but $2d_{\xi_0}(\xi,\xi_0) < \epsilon$, hence $d_{\xi_0}(\xi_0,\zeta) > d_{\xi_0}(\xi,\zeta) - d_{\xi_0}(\xi,\xi_0) > \frac{1}{2}d_{\xi_0}(\xi,\zeta) > \epsilon/2$. This implies that $D^2_{\xi_0;i,j}X_{\xi_0}\Gamma_{\xi_0}(\xi,\zeta) - D^2_{\xi_0;i,j}X_{\xi_0}\Gamma_{\xi_0}(\xi_0,\zeta)$ is integrable and for a suitable $\tilde{\xi} \in [\xi_0,\xi]$

$$I_1 \le d_{\xi_0}(\xi_0,\xi)\int_{\{\zeta:d_{\xi_0}(\xi,\zeta)>\epsilon\}} d_{\xi_0}^{-N-2}(\tilde{\xi},\zeta)d_{\xi_0}(\xi_0,\zeta)^{1+\alpha}d\zeta \le$$

(since $d_{\xi_0}(\tilde{\xi},\zeta) \ge d_{\xi_0}(\xi,\zeta) - d_{\xi_0}(\tilde{\xi},\xi) \ge d_{\xi_0}(\xi,\zeta) - d_{\xi_0}(\xi_0,\xi) > \frac{1}{2}d_{\xi_0}(\xi,\zeta)$)

$$\le d_{\xi_0}(\xi_0,\xi)\int_{\substack{\{\zeta:d_{\xi_0}(\xi_0,\zeta)<2d_{\xi_0}(\xi,\zeta),\\ d_{\xi_0}(\xi,\zeta)>\epsilon\}}} d_{\xi_0}^{-N-2}(\xi,\zeta)d_{\xi_0}(\xi_0,\zeta)^{1+\alpha}d\zeta + d_{\xi_0}(\xi_0,\xi)\int_{\substack{\{\zeta:d_{\xi_0}(\xi_0,\zeta)>2d_{\xi_0}(\xi,\zeta)\\ d_{\xi_0}(\xi,\zeta)>\epsilon\}}} d_{\xi_0}^{-N-2}(\xi,\zeta)d_{\xi_0}(\xi_0,\zeta)^{1+\alpha}d\zeta \le$$

(if $d_{\xi_0}(\xi_0,\zeta) > 2d_{\xi_0}(\xi,\zeta)$ then $d_{\xi_0}(\xi_0,\xi) > d_{\xi_0}(\xi_0,\zeta) - d_{\xi_0}(\xi,\zeta) > \frac{1}{2}d_{\xi_0}(\xi_0,\zeta)$)

$$\le d_{\xi_0}(\xi_0,\xi)\int_{\{\zeta:d_{\xi_0}(\xi,\zeta)>\epsilon\}} d_{\xi_0}(\xi,\zeta)^{-N-1+\alpha}d\zeta + d_{\xi_0}(\xi_0,\xi)^{2+\alpha}\int_{\{\zeta:d_{\xi_0}(\xi,\zeta)>\epsilon\}} d_{\xi_0}(\xi,\zeta)^{-N-2}d\zeta \le C\epsilon^\alpha\,.$$

On the set $\{\zeta : d_{\xi_0}(\xi, \zeta) < 2\epsilon\}$ we have $d_{\xi_0}(\xi_0, \zeta) < 5/2\epsilon$, and I_2 can be evaluated

$$|I_2| \leq \int_{\{\zeta : d_{\xi_0}(\xi_0,\zeta) < 5/2\epsilon\}} d_{\xi_0}(\xi_0, \zeta)^{-N+\alpha} d\zeta = C\epsilon^{\alpha} .$$

On $\{\zeta : \epsilon < d(\xi, \zeta) < 2\epsilon\}$ we get $d_{\xi_0}(\xi, \zeta) > \epsilon > 2d_{\xi_0}(\xi_0, \xi)$ and $\frac{1}{2}d_{\xi_0}(\xi, \zeta) < d_{\xi_0}(\xi_0, \zeta) < \frac{3}{2}d_{\xi_0}(\xi, \zeta)$. Hence

$$|I_3| < \frac{1}{\epsilon} \int_{\{\zeta : \epsilon/2 < d_{\xi_0}(\xi_0,\zeta) < 3\epsilon\}} d_{\xi_0}(\xi_0, \zeta)^{-N+1+\alpha} d\zeta = C\epsilon^{\alpha} .$$

$$|I_4| \leq d_{\xi_0}(\xi_0, \xi)^{1+\alpha} \int_{\{\zeta : \epsilon < d_{\xi_0}(\xi_0,\zeta)\}} d_{\xi_0}(\xi_0, \zeta)^{-N-1+\alpha} d\zeta \leq C\epsilon^{2+\alpha} .$$

Analogously $|I_5| \leq \epsilon^{2\alpha}$, and, for the Hölder continuity of f, $|I_6| \leq \epsilon^{\alpha}$.

Collecting all terms, we have assertion (29).

Finally, we can compute $D_i D_{\xi_0;j} v_1$, applying the definition

$$\frac{\Delta_i (D_{\xi_0;j} v_1)(h)(\xi_0)}{h} =$$

(by (28))

$$= \frac{\Delta_i v_{h,j}(h)(\xi_0) + O(h^{1+\alpha})}{h} =$$

(for a suitable $\tilde{\xi}$)

$$= D_i v_{h,j}(\tilde{\xi}) + O(h^{\alpha}) = w_1(\xi_0) + O(h^{\alpha}) .$$

Letting $h \to 0$, we obtain $D_i D_{\xi_0;j} v_1(\xi_0) = w_1(\xi_0)$, and Theorem 4.2 is proved. □

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LEMMA 4.1. *Let $v \in C_L^{2,\alpha}(\Omega)$, and for all ξ_0, $\xi \in \Omega_2$ we denote*

$$v_3(\xi) = \int \Gamma_{\xi_0}(\xi, \zeta)(Xa(\zeta) - Xa(\xi_0))\phi(\zeta)(\partial_t v(\zeta) - \partial_t v(\xi_0)) d\zeta .$$

Then there exists $D_i D^2_{\xi_0;j,k} v_3(\xi_0)$ and

$$D_i D^2_{\xi_0;j,k} v_3(\xi_0) = \int D^3_{\xi_0;i,j,k} \Gamma_{\xi_0}(\xi_0, \zeta)(Xa(\zeta) - Xa(\xi_0))\phi(\zeta)(\partial_t v(\zeta) - \partial_t v(\xi_0)) d\zeta .$$

PROOF. First note that

$$D^2_{\xi_0;i,j}v_3(\xi) = \int D^2_{\xi_0;i,j}\Gamma_{\xi_0}(\xi,\zeta)(Xa(\zeta) - Xa(\xi_0))\phi(\zeta)(\partial_t v(\zeta) - \partial_t v(\xi_0))d\zeta$$
$$- (Xa(\xi) - Xa(\xi_0))\phi(\xi)(\partial_t v(\xi) - \partial_t v(\xi_0)) \int_{\{\zeta : d_{\xi_0}(0,\zeta)=1\}}$$
$$\cdot D_{\xi_0;j}\Gamma_{\xi_0}(0,\zeta)\frac{\tilde{D}_{\xi_0;i}d_{\xi_0}(0,\zeta)}{|\nabla d_{\xi_0}(0,\zeta)|}d\sigma(\zeta),$$

then we can follow the proof of Theorem 4.2 to compute $D_k D^2_{\xi_0;i,j}v_3$. Indeed, if θ is the same as before, we set

$$v_{\epsilon,i,j}(\xi) = \int D^2_{\xi_0;i,j}\Gamma_{\xi_0}(\xi,\zeta)\theta\left(\frac{d_{\xi_0}(\xi,\zeta)}{\epsilon}\right)$$
$$\cdot (Xa(\zeta) - Xa(\xi_0))\phi(\zeta)(\partial_t v(\zeta) - \partial_t v(\xi_0))d\zeta\,,$$

and we have

$$\text{(30)}\qquad \sup_{d_{\xi_0}(\xi,\xi_0)<\epsilon/2} |v_{\epsilon,i,j}(\xi) - D^2_{\xi_o;i,j}v_3(\xi)| = O(\epsilon^{2\alpha}) \quad \text{as} \quad \epsilon \to 0\,,$$

where $2\alpha > 1$ by hypothesis. Clearly $v_{\epsilon,i,j}$ is of class $C_L^{1+\alpha}(\Omega)$, and, it satisfies

$$\text{(31)}\qquad \sup_{d_{\xi_0}(\xi,\xi_0)<\epsilon/2} |D_k v_{\epsilon,i,j}(\xi) - w_3(\xi_0)| = o(1) \quad \text{as} \quad \epsilon \to 0\,,$$

where

$$w_3(\xi_0) = \int D^3_{\xi_0;k,i,j}\Gamma_{\xi_0}(\xi_0,\zeta)(Xa(\zeta) - Xa(\xi_0))\phi(\zeta)(\partial_t v(\zeta) - \partial_t v(\xi_0))d\zeta\,.$$

Now $D_k D^2_{\xi_0;i,j}v_3(\xi_0,\xi_0)$ can be computed using the definition:

$$\frac{\Delta_k(D^2_{\xi_0;i,j}v_3)(h)(\xi_0)}{h} =$$

(by (30))

$$\frac{\Delta_k(v_{h,i,j})(h)(\xi_0) + O(h^{2\alpha})}{h}$$
$$= D_k v_{h,i,j}(\tilde{\xi}) + O(h^{2\alpha-1}) = w_3(\xi_0) + o(1)$$

(for a suitable $\tilde{\xi} \in [\xi_0, \gamma_k(1,\xi_0)]$ and by (31)). Letting $h \to 0$ we get the desired result. □

LEMMA 4.2. *Let* $v \in C_L^{2,\alpha}(\Omega)$, $\xi, \xi_0 \in \Omega$ *and*

$$v_4(\xi) = \int X_{\xi_0}\Gamma_{\xi_0}(\xi, \zeta)(a(\zeta) - P^1_{\xi_0}(\zeta))\phi(\zeta)(\partial_t v(\zeta) - \partial_t v(\xi_0))d\zeta \,.$$

There exists $\delta_i \delta^2_{\xi_0;j,k} v_4(\xi_0)$ *and*

$$\delta_i \delta^2_{\xi_0;j,k} v_4(\xi_0) = \int D^3_{\xi_0;i,j,k} X_{\xi_0}\Gamma_{\xi_0}(\xi_0, \zeta)(a(\zeta) - P^1_{\xi_0}a(\zeta))\phi(\zeta)(\partial_t v(\zeta) - \partial_t v(\xi_0))d\zeta \,.$$

PROOF. The proof is a simple modification of Theorem 4.2. Indeed it is standard to show that

$$D_{\xi_0;k} v_4(\xi) = \int D_{\xi_0;k} X_{\xi_0}\Gamma_{\xi_0}(\xi, \zeta)(a(\zeta) - P^1_{\xi_0}a(\zeta))\phi(\zeta)(\partial_t v(\zeta) - \partial_t v(\xi_0))d\zeta$$

$$-(a(\xi) - P^1_{\xi_0}a(\xi))\phi(\xi)(\partial_t v(\xi) - \partial_t v(\xi_0)) \int_{\{\zeta : d_{\xi_0}(\zeta)=1\}} X_{\xi_0;k}\Gamma_{\xi_0} \frac{\tilde{D}_{\xi_0;j} d_{\xi_0}(0, \zeta)}{|\nabla d_{\xi_0}(0, \zeta)|} d\sigma(\zeta).$$

Then, if for all $\epsilon > 0$ we put

$$v_{\epsilon,k}(\xi) = \int D_{\xi_0;k} X_{\xi_0}\Gamma_{\xi_0}(\xi, \zeta)\theta\left(\frac{d_{\xi_0}(\xi, \zeta)}{\epsilon}\right) \cdot (a(\zeta) - P^1_{\xi_0}a(\zeta))\phi(\zeta)(\partial_t v(\zeta) - \partial_t v(\xi_0))d\zeta \,,$$

and

$$w_4(\xi_0) = \int D^3_{\xi_0;i,j,k} X_{\xi_0}\Gamma_{\xi_0}(\xi_0, \zeta)(a(\zeta) - P^1_{\xi_0}a(\zeta))\phi(\zeta)(\partial_t v(\zeta) - \partial_t v(\xi_0))d\zeta \,,$$

we get

$$\sup_{d_{\xi_0}(\xi,\xi_0)<\epsilon/2} |v_{\epsilon,k}(\xi_0) - D_{\xi_0;k} v_4(\xi)| = O(\epsilon^{1+2\alpha}) \quad \text{as} \quad \epsilon \to 0 \tag{32}$$

and

$$\sup_{d_{\xi_0}(\xi,\xi_0)<\epsilon/2} |D_i D_{\xi_0;j} v_{\epsilon,k}(\xi) - w_4(\xi_0)| = o(1) \quad \text{as} \quad \epsilon \to 0. \tag{33}$$

Finally $\delta_i \delta^2_{\xi_0;j,k} v_4(\xi_0)$ can be computed as follows:

$$\frac{\Delta_i \Delta_{\xi_0;j}(D_{\xi_0;k} v_4)(h, l)(\xi_0)}{hk} =$$

$(\epsilon = \min(h, l))$

$$= \frac{\Delta_i \Delta_{\xi_0;j}(v_{\epsilon,k})(h, l)(\xi_0) + \epsilon^{1+2\alpha}}{hl}$$

$$= D_i D_{\xi_0;j} v_{\epsilon;j} v_{\epsilon k}(\xi_0) + \epsilon^{2\alpha-1} = w_4(\xi_0) + o(1)\,.$$

Since $2\alpha - 1 > 0$, there exists $\delta_i \delta^2_{\xi_0;j,k} v_4(\xi_0) = w_4(\xi_0)$. □

LEMMA 4.3. *Let* $v \in C_L^{2,\alpha}(\Omega)$, ξ, $\xi_0 \in \Omega$ *and*

$$v_5(\xi) = \int \partial_t \Gamma_{\xi_0}(\xi, \zeta)(a(\zeta) - P^1_{\xi_0} a(\zeta))^2 \phi(\zeta)(\partial_t v(\zeta) - \partial_t v(\xi_0)) d\zeta \, .$$

Then there exists $\delta_i \delta^2_{\xi_0; j,k} v_5(\xi_0)$ *and*

$$\delta_i \delta^2_{\xi_0; j,k} v_5(\xi_0) = \int D^3_{\xi_0; i,j,k} \partial_t \Gamma_{\xi_0}(\xi_0, \zeta)(a(\zeta) - P^1_{\xi_0} a(\zeta))^2 \phi(\zeta)(\partial_t v(\zeta) - \partial_t v(\xi_0)) d\zeta \, .$$

PROOF. If

$$v_\epsilon(\xi) = \int \partial_t \Gamma_{\xi_0}(\xi, \zeta) \theta \left(\frac{d_{\xi_0}(\xi, \zeta)}{\epsilon} \right) (a(\zeta) - P^1_{\xi_0} a(\zeta))^2 \phi(\zeta)(\partial_t v(\zeta) - \partial_t v(\xi_0)) d\zeta$$

and

$$w_5(\xi_0) = \int D^3_{\xi_0; i,j,k} \partial_t \Gamma_{\xi_0}(\xi_0, \zeta)(a(\zeta) - P^1_{\xi_0} a(\zeta))^2 \phi(\zeta)(\partial_t v(\zeta) - \partial_t v(\xi_0)) d\zeta \, ,$$

it can be shown that

$$\sup_{d_{\xi_0}(\xi, \xi_0) < \epsilon/2} |v_\epsilon(\xi_0) - v_5(\xi)| = O(\epsilon^{2+2\alpha}) \quad \text{as} \quad \epsilon \to 0 \, , \tag{34}$$

and that

$$\sup_{d_{\xi_0}(\xi, \xi_0) < \epsilon/2} |D_i D^2_{\xi_0; j,k} v_\epsilon(\xi) - w_5(\xi_0)| = o(1) \quad \text{as} \quad \epsilon \to 0 \, . \tag{35}$$

Hence

$$\frac{\Delta_i \Delta^2_{\xi_0; j,k}(v_5)(h, s, l)(\xi_0)}{hsl} =$$

$(\epsilon = \min(h, s, l))$

$$= \frac{\Delta_i \Delta^2_{\xi_0; j,k}(v_\epsilon)(h, s, l)(\xi_0) + \epsilon^{2+2\alpha}}{hsl}$$
$$= D_i D^2_{\xi_0; j,k} v_\epsilon(\tilde{\xi}) + \epsilon^{2\alpha - 1} = w_5(\xi_0) + o(1) \, .$$

Then there exists $\delta_i \delta^2_{\xi_0; j,k} v_5(\xi_0) = w_5(\xi_0)$. □

Arguing the same way with all terms in $B(\xi,\xi_0)$ we get

THEOREM 4.3. *There exists* $\delta_i\delta^2_{\xi_0;j,k}B(\xi_0,\xi_0)$ *for all* $\xi_0\in\Omega_2$ *and*

$$
\begin{aligned}
&\delta_i\delta^2_{\xi_0;j,k}B(\xi_0,\xi_0)\\
&=2\int D^3_{\xi_0;i,j,k}\Gamma_{\xi_0}(\xi_0,\zeta)(Xa(\zeta)-Xa(\xi_0))\phi(\zeta)(\partial_t v(\zeta)-\partial_t v(\xi_0))d\zeta\\
&\quad-2\int D^3_{\xi_0;i,j,k}\tilde{X}_{\xi_0}\Gamma_{\xi_0}(\xi_0,\zeta)(a(\zeta)-P^1_{\xi_0}a(\zeta))\phi(\zeta)(\partial_t v(\zeta)-\partial_t v(\xi_0))d\zeta\\
&\quad-\int D^3_{\xi_0;i,j,k}\partial_t\Gamma_{\xi_0}(\xi_0,\zeta)(a(\zeta)-P^1_{\xi_0}a(\zeta))^2\phi(\zeta)(\partial_t v(\zeta)-\partial_t v(\xi_0))d\zeta\\
&\quad+2\int D^3_{\xi_0;i,j,k}\Gamma_{\xi_0}(\xi_0,\zeta)(Yb(\zeta)-Yb(\xi_0))\phi(\zeta)(\partial_t v(\zeta)-\partial_t v(\xi_0))d\zeta\\
&\quad-2\int D^3_{\xi_0;i,j,k}\tilde{Y}_{\xi_0}\Gamma_{\xi_0}(\xi_0,\zeta)(b(\zeta)-P^1_{\xi_0}b(\zeta))\phi(\zeta)(\partial_t v(\zeta)-\partial_t v(\xi_0))d\zeta\\
&\quad-\int D^3_{\xi_0;i,j,k}\partial_t\Gamma_{\xi_0}(\xi_0,\zeta)(b(\zeta)-P^1_{\xi_0}b(\zeta))^2\phi(\zeta)(\partial_t v(\zeta)-\partial_t v(\xi_0))d\zeta\,.
\end{aligned}
$$

DERIVATIVES OF C AND E

LEMMA 4.4. *Let* $u\in C(\Omega)$, $\xi_0\in\Omega_2$ *and* $\xi\in\Omega$

$$v_6(\xi)=\int\Gamma_{\xi_0}(\xi,\zeta)u(\zeta)X_{\xi_0}\phi(\zeta)d\zeta\,.$$

Then $v_6\in C^\infty(\Omega_2)$.

For the choice of ϕ, $X_{\xi_0}\phi=0$ in a neighborhood of ξ, hence the integrand has no singularities, and the result yields.

THEOREM 4.4. *If* v *is* $C^{2,\alpha}_L(\Omega)$, *from Lemma 4.4 it immediately follows that* $C(\ ,\xi_0)$ *and* $E(\ ,\xi_0)$ *are of class* $C^\infty(\Omega_2)$, *in particular*

$$
\begin{aligned}
D_iD^2_{\xi_0;j,k}C(\xi_0,\xi)&=-2\int D^3_{\xi_0;i,j,k}\Gamma_{\xi_0}(\xi_0,\zeta)(a(\zeta)-P^1_{\xi_0}a(\zeta))X_{\xi_0}\phi(\zeta)d\zeta\\
&\quad-2\int D^3_{\xi_0;i,j,k}\Gamma_{\xi_0}(\xi_0,\zeta)(b(\zeta)-P^1_{\xi_0}b(\zeta))Y_{\xi_0}\phi(\zeta)d\zeta\\
&\quad-\int D^3_{\xi_0;i,j,k}\Gamma_{\xi_0}(\xi_0,\zeta)(a(\zeta)-P^1_{\xi_0}a(\zeta))^2\partial_t\phi(\zeta)d\zeta\\
&\quad-\int D^3_{\xi_0;i,j,k}\Gamma_{\xi_0}(\xi_0,\zeta)(b(\zeta)-P^1_{\xi_0}b(\zeta))^2\partial_t\phi(\zeta)d\zeta\,,
\end{aligned}
$$

and

$$\begin{aligned}
D_i D^2_{\xi_0;j,k} E(\xi_0,\xi_0) = {} & 2\int D^3_{\xi_0;i,j,k}\Gamma_{\xi_0}(\xi_0,\zeta)Xv(\zeta)X_{\xi_0}\phi(\zeta)d\zeta \\
& + 2\int D^3_{\xi_0;i,j,k}\Gamma_{\xi_0}(\xi_0,\zeta)Yv(\zeta)Y_{\xi_0}\phi(\zeta)d\zeta \\
& + \int D^3_{\xi_0;i,j,k}\Gamma_{\xi_0}(\xi_0,\zeta)v(\zeta)L_{\xi_0}\phi(\zeta)d\zeta \\
& + \int D^3_{\xi_0;i,j,k}\Gamma_{\xi_0}(\xi_0,\zeta)(a(\zeta)-P^1_{\xi_0}a(\zeta))^2\partial_t v(\zeta)\partial_t\phi(\zeta)d\zeta \\
& + \int D^3_{\xi_0;i,j,k}\Gamma_{\xi_0}(\xi_0,\zeta)(b(\zeta)-P^1_{\xi_0}b(\zeta))^2\partial_t v(\zeta)\partial_t\phi(\zeta)d\zeta\,.
\end{aligned}$$

PROOF OF THEOREM 4.1. From the representation formula and Theorems 4.2, 4.4 and 4.3 we can now simply deduce Theorem 4.1. □

5. – Hölder estimates

In this section we prove the following Hölder estimate of the solution:

THEOREM 5.1. *Assume that the coefficients a and b of L are of class $C^{1,\alpha}_L(\Omega)$ and let v be a solution of class $C^{2,\alpha}_L(\Omega)$, of $Lv = f$ where $f \in C^{1,\alpha}_L(\Omega)$. Then the function $\xi_0 \to \delta_i\delta^2_{\xi_0;j,k}v(\xi_0)$ is of class $C^\alpha_{L,\mathrm{loc}}(\Omega)$.*

As in the previous section we fix two open sets Ω_1 and Ω_2 such that $\Omega_2 \subset\subset \Omega_1 \subset\subset \Omega$, and a function ϕ such that $\phi = 1$ on Ω_1. With these notations we have proved an explicit formula for $\delta_i\delta^2_{\xi_0;j,k}v(\xi_0)$, and will study it here.

ESTIMATE OF $\delta_i\delta^2_{\xi_0;j,k}A(\xi_0,\xi_0)$

LEMMA 5.1. *If $f \in C^{1,\alpha}_L(\Omega)$, and $\forall\,\xi_0 \in \Omega$*

$$w_1(\xi_0) = \int D^3_{\xi_0;i,j,k}\Gamma_{\xi_0}(\xi_0,\zeta)(f(\zeta)-P^1_{\xi_0}f(\zeta))\phi(\zeta)d\zeta\,,$$

then w_1 is of class $C^\alpha_L(\Omega_2)$.

PROOF. If $\epsilon = 2d(\xi,\xi_0)$, we can write

$$w_1(\xi) - w_1(\xi_0) = I_1(\xi,\xi_0) + I_2(\xi,\xi_0) + I_3(\xi,\xi_0) + I_4(\xi,\xi_0)\,,$$

where

$$I_1(\xi,\xi_0) = \int_{\Omega\setminus B(\xi,\epsilon)} \Big(D^3_{\xi;i,j,k}\Gamma_\xi(\xi,\zeta) - D^3_{\xi_0;i,j,k}\Gamma_{\xi_0}(\xi_0,\zeta)\Big) \cdot (f(\zeta) - P^1_{\xi_0} f(\zeta))\phi(\zeta)d\zeta\,,$$

$$I_2(\xi,\xi_0) = \int_{\Omega\setminus B(\xi,\epsilon)} D^3_{\xi;i,j,k}\Gamma_\xi(\xi,\zeta)(P^1_{\xi_0} f(\zeta) - P^1_\xi f(\zeta))\phi(\zeta)d\zeta\,,$$

$$I_3(\xi,\xi_0) = \int_{B(\xi,\epsilon)} D^3_{\xi;i,j,k}\Gamma_\xi(\xi,\zeta)(f(\zeta) - P^1_\xi f(\zeta))\phi(\zeta)d\zeta\,,$$

$$I_4(\xi,\xi_0) = \int_{B(\xi,\epsilon)} D^3_{\xi_0;i,j,k}\Gamma_{\xi_0}(\xi_0,\zeta)(f(\zeta) - P^1_{\xi_0} f(\zeta))\phi(\zeta)d\zeta\,.$$

A simple computation ensures that there exists $\tilde d \in [d_{\xi_0}(\xi_0,\zeta), d_\xi(\xi,\zeta)]$ such that

$$|D^3_{\xi;i,j,k}\Gamma_\xi(\xi,\zeta) - D^3_{\xi_0;i,j,k}\Gamma_{\xi_0}(\xi_0,\zeta) \le \tilde d^{-N-3}|d^2_\xi(\xi,\zeta) - d^2_{\xi_0}(\xi_0,\zeta)| \le$$

(by (23) and (21))

$$\le \tilde d^{-N-3}(d(\xi,\xi_0)(d(\xi,\zeta) + d(\xi_0,\zeta)) + d^\alpha(\xi,\xi_0)d^2(\xi,\zeta))$$
$$\le d(\xi,\xi_0)d^{-N-2}(\xi,\zeta) + d^\alpha(\xi,\xi_0)d^{-N-1}(\xi,\zeta)\,,$$

since $d(\xi_0,\zeta) \le d(\xi,\zeta) + d(\xi,\xi_0) \le \frac{3}{2}d(\xi,\zeta))$ and $d(\xi_0,\zeta) \ge d(\xi,\zeta) - d(\xi,\xi_0) \ge \frac{1}{2}d(\xi,\zeta)$, then $\tilde d \ge \frac{1}{2}d(\xi,\zeta)$. Hence I_1 can be evaluated

$$|I_1(\xi,\xi_0)| \le d(\xi,\xi_0)\int_{\Omega\setminus B(\xi,\epsilon)} d^{-N-1+\alpha}(\xi,\zeta)d\zeta + d^\alpha(\xi,\xi_0)\int_{\Omega\setminus B(\xi,\epsilon)} d^{-N+\alpha}(\xi,\zeta)d\zeta \le Cd^\alpha(\xi,\xi_0)\,.$$

Then we have

$$I_2(\xi,\xi_0) = -(f(\xi) - P^1_{\xi_0} f(\xi))\int_{\Omega\setminus B(\xi,\epsilon)} D^3_{\xi;i,j,k}\Gamma_\xi(\xi,\zeta)\phi(\zeta)d\zeta$$
$$+ (Xf(\xi) - Xf(\xi_0))\int_{\Omega\setminus B(\xi,\epsilon)} D^3_{\xi;i,j,k}\Gamma_\xi(\xi,\zeta)(x_\xi - x_\zeta)\phi(\zeta)d\zeta$$
$$+ (Yf(\xi) - Yf(\xi_0))\int_{\Omega\setminus B(\xi,\epsilon)} D^3_{\xi;i,j,k}\Gamma_\xi(\xi,\zeta)(y_\xi - y_\zeta)\phi(\zeta)d\zeta =$$

(integrating by parts the second and third terms, and using the fact that $D^3_{\xi;i,j,k}\Gamma_\xi(\xi,\zeta) = -\tilde D^\zeta_{\xi;i} D^2_{\xi;j,k}\Gamma_\xi(\xi,\zeta))$

$$= (f(\xi) - P^1_{\xi_0} f(\xi)) \int_{\Omega\setminus B(\xi,\epsilon)} D^3_{\xi;i,j,k}\Gamma_\xi(\xi,\zeta)\phi(\zeta)d\zeta$$
$$- (Xf(\xi) - Xf(\xi_0)) \int_{\partial B(\xi,\epsilon)} D^2_{\xi;j,k}\Gamma_\xi(\xi,\zeta)(x_\xi - x_\zeta)\phi(\zeta)\frac{\tilde{D}_{\xi_0;i}d_{\xi_0}(\xi,\zeta)}{|\nabla d_{\xi_0}(\xi,\zeta)|}d\sigma(\zeta)$$
$$+ (Xf(\xi) - Xf(\xi_0)) \int_{\Omega\setminus B(\xi,\epsilon)} D^2_{\xi;j,k}\Gamma_\xi(\xi,\zeta)\tilde{D}_{\xi;i}((x_\xi - x_\zeta)\phi(\zeta))d\zeta$$
$$- (Yf(\xi) - Yf(\xi_0)) \int_{\partial B(\xi,\epsilon)} D^2_{\xi;j,k}\Gamma_\xi(\xi,\zeta)(y_\xi - y_\zeta)\phi(\zeta)\frac{\tilde{D}_{\xi_0;i}d_{\xi_0}(\xi,\zeta)}{|\nabla d_{\xi_0}(\xi,\zeta)|}d\sigma(\zeta)$$
$$+ (Yf(\xi) - Yf(\xi_0) \int_{\Omega\setminus B(\xi,\epsilon)} D^2_{\xi;j,k}\Gamma_\xi(\xi,\zeta)\tilde{D}_{\xi;i}((y_\xi - y_\zeta)\phi(\zeta))d\zeta =$$

(again using the equality $D^2_{\xi;j,k}\Gamma_\xi(\xi,\zeta) = -\tilde{D}^\zeta_{\xi;j}D_{\xi;k}\Gamma_\xi(\xi,\zeta)$, and integrating by parts)

$$= (f(\xi) - P^1_{\xi_0} f(\xi)) \int_{\Omega\setminus B(\xi,\epsilon)} D^3_{\xi;i,j,k}\Gamma_\xi(\xi,\zeta)\phi(\zeta)d\zeta$$
$$- (Xf(\xi) - Xf(\xi_0)) \int_{\partial B(\xi,\epsilon)} D^2_{\xi;j,k}\Gamma_\xi(\xi,\zeta)(x_\xi - x_\zeta)\phi(\zeta)\frac{\tilde{D}_{\xi_0;i}d_{\xi_0}(\xi,\zeta)}{|\nabla d_{\xi_0}(\xi,\zeta)|}d\sigma(\zeta)$$
$$- (Xf(\xi) - Xf(\xi_0)) \int_{\partial B(\xi,\epsilon)} D_{\xi;k}\Gamma_\xi(\xi,\zeta)\tilde{D}_{\xi;i}((x_\xi - x_\zeta)\phi(\zeta))\frac{\tilde{D}_{\xi_0;j}d_{\xi_0}(\xi,\zeta)}{|\nabla d_{\xi_0}(\xi,\zeta)|}d\sigma(\zeta)$$
$$+ (Xf(\xi) - Xf(\xi_0) \int_{\Omega\setminus B(\xi,\epsilon)} D_{\xi;k}\Gamma_\xi(\xi,\zeta)\tilde{D}^2_{\xi;j,i}((x_\xi - x_\zeta)\phi(\zeta))d\zeta$$
$$- (Yf(\xi) - Yf(\xi_0)) \int_{\partial B(\xi,\epsilon)} D^2_{\xi;j,k}\Gamma_\xi(\xi,\zeta)(y_\xi - y_\zeta)\phi(\zeta)\frac{\tilde{D}_{\xi_0;i}d_{\xi_0}(\xi,\zeta)}{|\nabla d_{\xi_0}(\xi,\zeta)|}d\sigma(\zeta)$$
$$- (Yf(\xi) - Y(\xi_0)) \int_{\partial B(\xi,\epsilon)} D_{\xi;k}\Gamma_\xi(\xi,\zeta)\tilde{D}_{\xi;i}((y_\xi - y_\zeta)\phi(\zeta))\frac{\tilde{D}_{\xi_0;j}d_{\xi_0}(\xi,\zeta)}{|\nabla d_{\xi_0}(\xi,\zeta)|}d\sigma(\zeta)$$
$$+ (Yf(\xi) - Yf(\xi_0)) \int_{\Omega\setminus B(\xi,\epsilon)} D_{\xi;k}\Gamma_\xi(\xi,\zeta)\tilde{D}^2_{\xi;j,i}((y_\xi - y_\zeta)\phi(\zeta))d\zeta.$$

Hence

$$|I_2(\xi,\xi_0)| \le C\, d^\alpha(\xi,\xi_0)\,.$$

Finally

$$|I_3(\xi,\xi_0)| + |I_4(\xi,\xi_0)| \le C\, d^\alpha(\xi,\xi_0)\,. \qquad \square$$

Analogously we get:

LEMMA 5.2. *If $f \in C^{1,\alpha}_L(\Omega)$, and*

$$w_2(\xi_0) = \int D^2_{\xi_0;i,j}\Gamma_{\xi_0}(\xi_0,\zeta)D_{\xi_0;k}P^1_{\xi_0}f(\zeta)\phi(\zeta)d\zeta\,,$$

then $w_2 \in C^\alpha_L(\Omega_2)$.

REMARK 5.1. From Lemmas 5.1 and 5.2, and from Theorem 4.2 it follows that $\delta_i\delta^2_{\xi_0;j,k}A(\xi_0,\xi_0) \in C^\alpha_L(\Omega_2)$.

ESTIMATE OF $\delta_i\delta^2_{\xi_0;j,k}B(\xi_0,\xi_0)$:

LEMMA 5.3. *If $v \in C^{2,\alpha}_L(\Omega)$, then the function*

$$w_3(\xi_0) = \int D^3_{\xi_0;i,j,k}\Gamma_{\xi_0}(\xi_0,\zeta)(Xa(\zeta) - Xa(\xi_0))\phi(\zeta)(\partial_t v(\zeta) - \partial_t v(\xi_0))d\zeta$$

is of class $C^{2\alpha-1}_L(\Omega_2)$.

PROOF.

$$\begin{aligned}
w_3(\xi) - w_3(\xi_0) &= \int D^3_{\xi;i,j,k}\Gamma_\xi(\xi,\zeta)(Xa(\zeta) - Xa(\xi))\phi(\zeta)(\partial_t v(\zeta) - \partial_t v(\xi))d\zeta \\
&\quad - \int D^3_{\xi_0;i,j,k}\Gamma_{\xi_0}(\xi_0,\zeta)(Xa(\zeta) \\
&\quad - Xa(\xi_0))\phi(\zeta)(\partial_t v(\zeta) - \partial_t v(\xi_0))d\zeta \\
&= I_1(\xi,\xi_0) + I_2(\xi,\xi_0) + I_3(\xi,\xi_0) + I_4(\xi,\xi_0)\,,
\end{aligned}$$

where $\epsilon = 2d(\xi,\xi_0)$ and we have defined:

$$\begin{aligned}
I_1(\xi,\xi_0) &= \int_{\Omega\setminus B(\xi,\epsilon)} \left(D^3_{\xi;i,j,k}\Gamma_\xi(\xi,\zeta) - D^3_{\xi_0;i,j,k}\Gamma_{\xi_0}(\xi_0,\zeta)\right) \\
&\quad \cdot (Xa(\zeta) - Xa(\xi_0))\phi(\zeta)(\partial_t v(\zeta) - \partial_t v(\xi_0))d\zeta\,, \\
I_2(\xi,\xi_0) &= \int_{\Omega\setminus B(\xi,\epsilon)} D^3_{\xi\ i,j,k}\Gamma_\xi(\xi,\zeta)\big((Xa(\zeta) - Xa(\xi))(\partial_t v(\zeta) - \partial_t v(\xi))d\zeta \\
&\quad - (Xa(\zeta) - Xa(\xi_0))(\partial_t v(\zeta) - \partial_t v(\xi_0))\big)\phi(\zeta)d\zeta\,, \\
I_3(\xi,\xi_0) &= \int_{B(\xi,\epsilon)} D^3_{\xi;i,j,k}\Gamma_\xi(\xi,\zeta)(Xa(\zeta) - Xa(\xi))\phi(\zeta)(\partial_t v(\zeta) - \partial_t v(\xi))d\zeta\,, \\
I_4(\xi,\xi_0) &= -\int_{B(\xi,\epsilon)} D^3_{\xi_0;i,j,k}\Gamma_{\xi_0}(\xi_0,\zeta)(Xa(\zeta) \\
&\quad - Xa(\xi_0))\phi(\zeta)(\partial_t v(\zeta) - \partial_t v(\xi_0))d\zeta\,.
\end{aligned}$$

Now arguing as in the estimate of $\delta_i\delta^2_{\xi;j,k}A(\xi,\xi_0)$, we have

$$\begin{aligned}
|I_1(\xi,\xi_0)| &\le d(\xi,\xi_0)\int_{\Omega\setminus B(\xi,\epsilon)} d^{-N-2+2\alpha}(\xi,\zeta)d\zeta \\
&\quad + d^\alpha(\xi,\xi_0)\int_{\Omega\setminus B(\xi,\epsilon)} d^{-N-1+2\alpha}(\xi,\zeta)d\zeta \le Cd^{2\alpha-1}(\xi,\xi_0)\,.
\end{aligned}$$

$$|I_2(\xi,\xi_0)| \le |Xa(\xi_0) - Xa(\xi)| \int_{\Omega\setminus B(\xi,\epsilon)} D^3_{\xi;i,j,k}|\Gamma_\xi(\xi,\zeta)$$
$$\cdot (\partial_t v(\zeta) - \partial_t v(\xi_0))\phi(\zeta)|d\zeta$$
$$+ |\partial_t v(\xi) - \partial_t v(\xi_0)| \int_{\Omega\setminus B(\xi,\epsilon)} |D^3_{\xi;i,j,k}\Gamma_\xi(\xi,\zeta)(Xa(\xi_0) - Xa(\zeta))\phi(\zeta)|d\zeta$$
$$\le d^\alpha(\xi,\xi_0) \int_\epsilon^\infty \rho^{-N-1+\alpha+N-1} d\rho = Cd^{2\alpha-1}(\xi,\xi_0)\,.$$

And

$$|I_3(\xi,\xi_0)| + |I_4(\xi,\xi_0)| \le d^{2\alpha-1}(\xi,\xi_0)\,. \qquad \square$$

REMARK 5.2. Exactly in the same way we can estimate all terms in $\delta_i\delta^2_{\xi_0;j,k}B(\xi_0,\xi_0)$, which turns out to be of class $C_L^{2\alpha-1}(\Omega_2)$.

ESTIMATE OF $D_iD^2_{\xi_0;j,k}C(\xi_0,\xi_0)$:
Also in this case we study only one term, all the others being analogous.

LEMMA 5.4. *If $v \in C_L^{2,\alpha}(\Omega)$, then the function*

$$w_3(\xi_0) = \int D^3_{\xi_0;i,j,k}\Gamma_{\xi_0}(\xi_0,\zeta)(a(\zeta) - P^1_{\xi_0}a(\zeta))X_{\xi_0}\phi(\zeta)d\zeta\,,$$

is of class $C_L^{1,\alpha}(\Omega_2)$.

PROOF.

$$w_3(\xi_0) - w_3(\xi) = \int D^3_{\xi_0;i,j,k}\Gamma_{\xi_0}(\xi_0,\zeta)(a(\zeta) - P^1_{\xi_0}a(\zeta))X_{\xi_0}\phi(\zeta)d\zeta$$
$$- \int D^3_{\xi;i,j,k}\Gamma_\xi(\xi,\zeta)(a(\zeta) - P^1_\xi a(\zeta))X_\xi\phi(\zeta)d\zeta =$$

since ϕ is constant on Ω_2

$$= \int_{\Omega\setminus\Omega_2} (D^3_{\xi;i,j,k}\Gamma_\xi(\xi,\zeta) - D^3_{\xi_0;i,j,k}\Gamma_{\xi_0}(\xi_0,\zeta))(a(\zeta) - P^1_{\xi_0}a(\zeta))X_{\xi_0}\phi(\zeta)d\zeta$$
$$+ \int_{\Omega\setminus\Omega_2} D^3_{\xi_0;i,j,k}\Gamma_\xi(\xi,\zeta)\Big((a(\zeta) - P^1_{\xi_0}a(\zeta))X_{\xi_0}\phi(\zeta)$$
$$- (a(\zeta) - P^1_\xi a(\zeta))X_\xi\phi(\zeta)\Big)d\zeta$$

Finally it is not difficult to see that the absolute value of each of these terms is bounded by $Cd^\alpha(\xi,\xi_0)$. $\square$

Collecting the estimate of A, B, C, and E we immediately get the following:

LEMMA 5.5. *If a , b and f are of class $C_L^{1,\alpha}(\Omega)$ and v is a solution of class $C_L^{2,\alpha}(\Omega)$ of $Lv = f$, then the function $\delta_i\delta^2_{\xi;j,k}v(\xi)$ is of class $C_L^{2\alpha-1}(\Omega_2)$. In particular, since $\partial_t = \frac{[X_{\xi_0},Y_{\xi_0}]}{(Xb-Ya)(\xi_0)}$, we have $\partial_t v \in C_L^{1,2\alpha-1}(\Omega_2)$.*

ANOTHER ESTIMATE OF $\delta_i \delta^2_{\xi_0;j,k} B(\xi_0, \xi_0)$:

Will now make a better estimate of $\delta_i \delta^2_{\xi_0;j,k} B(\xi_0, \xi_0)$, using the extra regularity of $\partial_t v$, proved in Lemma 5.5. As usual we study in details only the first term.

LEMMA 5.6. *If $v \in C^{2,\alpha}_L(\Omega)$, then the function*

$$w_3(\xi_0) = \int D^3_{\xi_0;i,j,k}\Gamma_{\xi_0}(\xi_0,\zeta)(Xa(\zeta) - Xa(\xi_0))\phi(\zeta)(\partial_t v(\zeta) - \partial_t v(\xi_0))d\zeta\,.$$

is of class $C^\alpha_L(\Omega)$.

PROOF. Since $\partial_t v \in C^{1,2\alpha-1}_L(\Omega)$, by Remark 2.3

$$\begin{aligned}&\partial_t v(\zeta) - \partial_t v(\xi_0)\\ &= X_{\xi_0}\partial_t v(\xi_0)(x_\zeta - x_{\xi_0}) + Y_{\xi_0}\partial_t v(\xi_0)(y_\zeta - y_{\xi_0}) + O(d^{2\alpha}(\zeta,\xi_0))\,.\end{aligned} \tag{36}$$

Now we can choose $\epsilon = 2d(\xi,\xi_0)$ and write $w_3(\xi) - w_3(\xi_0)$ as in Lemma 5.3:

$$w_3(\xi) - w_3(\xi_0) = I_1(\xi,\xi_0) + I_2(\xi,\xi_0) + I_3(\xi,\xi_0) + I_4(\xi,\xi_0)\,.$$

Recall that

$$\begin{aligned}I_1(\xi,\xi_0) = \int_{\Omega\setminus B(\xi,\epsilon)} &(D^3_{\xi;i,j,k}\Gamma_\xi(\xi,\zeta) - D^3_{\xi_0;i,j,k}\Gamma_{\xi_0}(\xi_0,\zeta))\\ &\cdot(Xa(\zeta) - Xa(\xi_0))\phi(\zeta)(\partial_t v(\zeta) - \partial_t v(\xi_0))d\zeta\,,\end{aligned}$$

where, from (36),

$$|\partial_t v(\zeta) - \partial_t v(\xi_0)| \le d(\zeta,\xi_0) \le d(\zeta,\xi) + d(\xi,\xi_0) \le Cd(\zeta,\xi)\,,$$

by hypothesis

$$|(Xa(\zeta) - Xa(\xi_0))\phi(\zeta)| \le d^\alpha(\zeta,\xi_0)\,,$$

and, as we have noted,

$$|D^3_{\xi;i,j,k}\Gamma_\xi(\xi,\zeta) - D^3_{\xi;i,j,k}\Gamma_{\xi_0}(\xi_0,\zeta)| \le d(\xi,\xi_0)d^{-N-2}(\xi,\zeta) + d^\alpha(\xi,\xi_0)d^{-N-1}(\xi,\zeta).$$

Hence

$$\begin{aligned}|I_1(\xi,\xi_0)| \le Cd(\xi,\xi_0)&\int_{\Omega\setminus B(\xi,\epsilon)} d^{-N-1+\alpha}(\xi,\zeta)d\zeta\\ &+ Cd^\alpha(\xi,\xi_0)\int_{\Omega\setminus B(\xi,\epsilon)} d^{-N+\alpha}(\xi,\zeta)d\zeta \le Cd^\alpha(\xi,\xi_0)\,.\end{aligned}$$

Let us estimate I_2

$$I_2(\xi,\xi_0)=\int_{\Omega\setminus B(\xi,\epsilon)} D^3_{\xi;i,j,k}\Gamma_\xi(\xi,\zeta)\big((Xa(\zeta)-Xa(\xi))(\partial_t v(\zeta)-\partial_t v(\xi))$$
$$-(Xa(\zeta)-Xa(\xi_0))(\partial_t v(\zeta)-\partial_t v(\xi_0))\big)\phi(\zeta)d\zeta$$
$$=(Xa(\xi_0)-Xa(\xi))\int_{\Omega\setminus B(\xi,\epsilon)} D^3_{\xi;i,j,k}\Gamma_\xi(\xi,\zeta)$$
$$\cdot(\partial_t v(\zeta)-\partial_t v(\xi))\phi(\zeta)d\zeta$$
$$+(\partial_t v(\xi)-\partial_t v(\xi_0))\int_{\Omega\setminus B(\xi,\epsilon)} D^3_{\xi;i,j,k}\Gamma_\xi(\xi,\zeta)$$
$$\cdot(Xa(\xi_0)-Xa(\zeta))\phi(\zeta)d\zeta=$$

by (36))

$$=(Xa(\xi_0)-Xa(\xi))X\partial_t v(\xi)\int_{\Omega\setminus B(\xi,\epsilon)} D^3_{\xi;i,j,k}\Gamma_\xi(\xi,\zeta)(x_\zeta-x_\xi)\phi(\zeta)d\zeta$$
$$+(Xa(\xi_0)-Xa(\xi))Y\partial_t v(\xi)\int_{\Omega\setminus B(\xi,\epsilon)} D^3_{\xi;i,j,k}\Gamma_\xi(\xi,\zeta)(y_\zeta-y_\xi)\phi(\zeta)d\zeta$$
$$+(Xa(\xi_0)-Xa(\xi))\int_{\Omega\setminus B(\xi,\epsilon)} D^3_{\xi;i,j,k}\Gamma_\xi(\xi,\zeta)d^{2\alpha}(\zeta,\xi)\phi(\zeta)d\zeta$$
$$+(\partial_t v(\xi)-\partial_t v(\xi_0))\int_{\Omega\setminus B(\xi,\epsilon)} D^3_{\xi;i,j,k}\Gamma_\xi(\xi,\zeta)(Xa(\xi_0)-Xa(\zeta))\phi(\zeta)d\zeta=$$

(integrating by parts the first and the second term, and using the fact that $D^3_{\xi;i,j,k}\Gamma_\xi(\xi,\zeta)=-\tilde D^\zeta_{\xi,i}D^2_{\xi;i,j,k}\Gamma_\xi(\xi,\zeta)$)

$$=-(Xa(\xi_0)-Xa(\xi))X\partial_t v(\xi)\int_{\partial B(\xi,\epsilon)} D^2_{\xi;j,k}\Gamma_\xi(\xi,\zeta)$$
$$\cdot(x_\zeta-x_\xi)\phi(\zeta)\frac{\tilde D_{\xi_0;i}d_{\xi_0}(\xi,\zeta)}{|\nabla d_{\xi_0}(\xi,\zeta)|}d\sigma(\zeta)$$
$$+(Xa(\xi_0)-Xa(\xi))X\partial_t v(\xi)\int_{\Omega\setminus B(\xi,\epsilon)} D^2_{\xi;j,k}\Gamma_\xi(\xi,\zeta)\tilde D_{\xi;i}((x_\zeta-x_\xi)\phi)(\zeta)d\zeta$$
$$-(Xa(\xi_0)-Xa(\xi))Y\partial_t v(\xi)\int_{\partial B(\xi,\epsilon)} D^2_{\xi;j,k}\Gamma_\xi(\xi,\zeta)(y_\zeta-y_\xi)\phi(\zeta)\frac{\tilde D_{\xi_0;i}d_{\xi_0}(\xi,\zeta)}{|\nabla d_{\xi_0}(\xi,\zeta)|}d\sigma(\zeta)$$
$$+(Xa(\xi_0)-Xa(\xi))Y\partial_t v(\xi)\int_{\Omega\setminus B(\xi,\epsilon)} D^2_{\xi;j,k}\Gamma_\xi(\xi,\zeta)\tilde D_{\xi;i}(y_\zeta-y_\xi)\phi)(\zeta)d\zeta$$
$$+O\big(d^\alpha(\xi,\xi_0)\big)\int_\epsilon^{\mathrm{diam}(\Omega)}\rho^{2\alpha-2}d\rho+O\big(d(\xi,\xi_0)\big)\int_\epsilon^{\mathrm{diam}(\Omega)}\rho^{\alpha-2}d\rho$$
$$=O\big(d^\alpha(\xi,\xi_0)\big).$$

Finally, always using (36), we get

$$|I_3(\xi, \xi_0)| + |I_4(\xi, \xi_0)| \leq d^\alpha(\xi, \xi_0),$$

and the result is proved. □

REMARK 5.3. Arguing in the same way with all terms in $\delta_i \delta^2_{\xi_0; j,k} B(\xi_0, \xi_0)$, we obtain that it is of class $C^\alpha_L(\Omega)$.

PROOF OF THEOREM 5.1. The claim immediately follows from Remarks 5.1 and 5.3, and from the representation formula.

REMARK 5.4. In particular, since $\partial_t v = \frac{[X_{\xi_0}, Y_{\xi_0}]v}{(Xb - Ya)(\xi_0)}$ we have proved that, under the hypotheses of Theorem 5.1, $\partial_t v \in C^\alpha_{L\,\mathrm{loc}}(\Omega)$.

6. – Derivatives in the directions X and Y

REMARK 6.1. Until now we have always assumed that the coefficients of L were of class $C^{1,\alpha}_L$. However, as we have already noted in the introduction, the derivatives in the X and Y direction, even of a C^∞ function, exist only if a and b are of class C^2_L, and for all ξ_0 we have for example

$$X^3 v(\xi_0) = X^2_{\xi_0} a(\xi_0) \partial_t v(\xi_0) + X^3_{\xi_0} v(\xi_0).$$

Hence we will assume that a and b satisfy a stronger regularity hypothesis.

The main result in the section is the following one:

THEOREM 6.1. *Let $k \in N$, $k \geq 2$, and assume that $a, b \in C^{k,\alpha}_L(\Omega)$, $f \in C^{k-1,\alpha}_L(\Omega)$, and L is the linear operator with coefficients a and b. If v is a solution of class $C^{2,\alpha}_L(\Omega)$ of*

$$Lv = f \quad in \quad \Omega,$$

then $v \in C^{k+1,\alpha}_{L,\mathrm{loc}}(\Omega)$.

Let us start with the case $k = 2$.

THEOREM 6.2. *Assume that a and b belong to $C^{2,\alpha}_L(\Omega)$, $f \in C^{1,\alpha}_L(\Omega)$, and let $v \in C^{2,\alpha}_L(\Omega)$ be a solution of $Lv = f$. Then $v \in C^{3,\alpha}_{L\,\mathrm{loc}}(\Omega)$.*

We will use the representation formula proved in Theorem 3.1 and differentiate each term.

DERIVATIVES OF A:

THEOREM 6.3. *Let* $f \in C_L^{1,\alpha}(\Omega)$, $\xi_0 \in \Omega_2$ *and*

$$A(\xi, \xi_0) = \int \Gamma_{\xi_0}(\xi, \zeta) f(\zeta)\phi(\zeta) d\zeta .$$

Then there exists $\delta^3_{i,j,k} A(\xi_0, \xi_0)$ *for all* i, j, k. *In particular*

$$\begin{aligned}\delta^2_{i,j} X A(\xi_0, \xi_0) = &\int D^2_{i,j} X \Gamma_{\xi_0}(\xi_0, \zeta)(f - P^1_{\xi_0} f)(\zeta)\phi(\zeta) d\zeta \\ &+ \int D_i \Gamma_{\xi_0}(\xi_0, \zeta) \tilde{D}_{\xi_0; j} \tilde{X}^\zeta_{\xi_0} (P^1_{\xi_0} f \phi)(\zeta) d\zeta \\ &+ D^2_{i,j} a(\xi_0) \int \Gamma_{\xi_0}(\xi_0, \zeta) \partial_t (P^1_{\xi_0} f \phi)(\zeta) d\zeta .\end{aligned}$$

PROOF. It is not difficult to see that there exists $XA(\xi, \xi_0)$ and

$$\begin{aligned}XA(\xi, \xi_0) = \int X\Gamma_{\xi_0}(\xi, \zeta) f(\zeta)\phi(\zeta) d\zeta = &\int X\Gamma_{\xi_0}(\xi, \zeta)(f - P^1_{\xi_0} f)(\zeta)\phi(\zeta) d\zeta \\ &+ \int X^\xi_{\xi_0} \Gamma_{\xi_0}(\xi, \zeta) P^1_{\xi_0} f(\zeta)\phi(\zeta) d\zeta \\ &+ (a - P^1_{\xi_0} a)(\xi) \int \partial^\xi_t \Gamma_{\xi_0}(\xi, \zeta) P^1_{\xi_0} f(\zeta)\phi(\zeta) d\zeta =\end{aligned}$$

(integrating by parts the second and the third term)

$$\begin{aligned}= &\int X\Gamma_{\xi_0}(\xi, \zeta)(f - P^1_{\xi_0} f)(\zeta)\phi(\zeta) d\zeta \\ &+ \int \Gamma_{\xi_0}(\xi, \zeta) \tilde{X}^\zeta_{\xi_0} (P^1_{\xi_0} f \phi)(\zeta) d\zeta \\ &+ (a - P^1_{\xi_0} a)(\xi) \int \Gamma_{\xi_0}(\xi, \zeta) \partial_t (P^1_{\xi_0} f \phi)(\zeta) d\zeta .\end{aligned}$$

Now we can conclude, arguing exactly as in Section 3. □

DERIVATIVES OF B:

Arguing as in Theorem 4.3 we get:

THEOREM 6.4. *Let ξ_0 be fixed, $v \in C_L^{2,\alpha}(\Omega)$, and let B be the function defined in Theorem 3.1. Then $\forall \xi_0 \in \Omega$ there exist $\delta^3_{i,j,k}B(\xi_0,\xi_0)$, and*

$$\begin{aligned}
\delta^3_{i,j,k}B(\xi_0,\xi_0) = {} & 2\int D^3_{i,j,k}\Gamma_{\xi_0}(\xi_0,\zeta)(Xa(\zeta)-Xa(\xi_0))\phi(\zeta)(\partial_t v(\zeta)-\partial_t v(\xi_0))d\zeta \\
& - 2\int D^3_{i,j,k}\tilde{X}_{\xi_0}\Gamma_{\xi_0}(\xi_0,\zeta)(a(\zeta)-P^1_{\xi_0}a(\zeta))\phi(\zeta)(\partial_t v(\zeta)-\partial_t v(\xi_0))d\zeta \\
& - \int D^3_{i,j,k}\partial_t\Gamma_{\xi_0}(\xi_0,\zeta)(a(\zeta)-P^1_{\xi_0}a(\zeta))^2\phi(\zeta)(\partial_t v(\zeta)-\partial_t v(\xi_0))d\zeta \\
& + 2\int D^3_{i,j,k}\Gamma_{\xi_0}(\xi_0,\zeta)(Yb(\zeta)-Yb(\xi_0))\phi(\zeta)(\partial_t v(\zeta)-\partial_t v(\xi_0))d\zeta \\
& - 2\int D^3_{i,j,k}\tilde{Y}_{\xi_0}\Gamma_{\xi_0}(\xi_0,\zeta)(b(\zeta)-P^1_{\xi_0}b(\zeta))\phi(\zeta)(\partial_t v(\zeta)-\partial_t v(\xi_0))d\zeta \\
& - \int D^3_{i,j,k}\partial_t\Gamma_{\xi_0}(\xi_0,\zeta)(b(\zeta)-P^1_{\xi_0}b(\zeta))^2\phi(\zeta)(\partial_t v(\zeta)-\partial_t v(\xi_0))d\zeta\,.
\end{aligned}$$

PROOF OF THEOREM 6.2. In Theorems 6.3 and 6,4 we showed that for every ξ_0 and for every i, j, k there exists $\delta^3_{i,j,k}A(\xi_0,\xi_0)$ and $\delta^3_{i,j,k}B(\xi_0,\xi_0)$. On the other side Theorem 4.4 ensures that C and E are of class $C^\infty(\Omega)$, hence there exists at every point $\delta^3_{i,j,k}v(\xi_0)$. By hypothesis $v \in C_L^{2,\alpha}$, then there exists $D^3_{i,j,k}v(\xi_0)$ at every point. The Hölder regularity of $D^3_{i,j,k}v(\xi_0)$ can be proved exactly as in Section 3.

Differentiating the representation formula of v proved in Theorem 3.2 we infer the following:

THEOREM 6.5. *Assume that f and the coefficients a and b of L are of class $C_L^{1,\alpha}(\Omega)$ and partially differentiable with respect to t, with derivative of class $C_L^\alpha(\Omega)$. Let v be a solution of $Lv = f$, such that $v \in C_L^{2,\alpha}(\Omega)$ and $\partial_t v \in C_L^{1,\alpha}(\Omega)$. Then $\partial_t v \in C^{2,\alpha}_{L,\mathrm{loc}}(\Omega)$. In particular for every ξ_0 we have*

$$\begin{aligned}
D^2_{i,j}\partial_t v(\xi_0) = {} & \delta^2_{i,j}\partial_t A_1(\xi_0,\xi_0) + \delta^2_{i,j}\partial_t B_1(\xi_0,\xi_0) \\
& + \partial_t v(\xi_0)D^2_{i,j}\partial_t C(\xi_0,\xi_0) + D^2_{i,j}\partial_t D_1(\xi_0,\xi_0) + D^2_{i,j}\partial_t E_1(\xi_0,\xi_0)
\end{aligned}$$

where

$$\delta^2_{i,j}\partial_t A_1(\xi_0,\xi_0) = \int D^2_{i,j}\Gamma_{\xi_0}(\xi_0,\zeta)(\partial_t Lv(\zeta) - 2\partial_t aX\partial_t v(\xi_0) - 2\partial_t bY\partial_t v(\xi_0))\phi(\zeta)d\zeta.$$

$$\delta^2_{i,j}\partial_t B_1(\xi_0,\xi_0) = 2\int D^2_{i,j}\partial_t\Gamma_{\xi_0}(\xi,\zeta)$$
$$\cdot (Xa(\zeta) - Xa(\xi_0))\phi(\zeta)(\partial_t v(\zeta) - P^1_{\xi_0}\partial_t v(\zeta))d\zeta$$
$$-2\int D^2_{i,j}\partial_t\tilde{X}_{\xi_0}\Gamma_{\xi_0}(\xi_0,\zeta)(a(\zeta) - P^1_{\xi_0}a(\zeta))\phi(\zeta)(\partial_t v(\zeta) - P^1_{\xi_0}\partial_t v(\zeta))d\zeta$$
$$-\int D^2_{i,j}\partial^2_{t,t}\Gamma_{\xi_0}(\xi_0,\zeta)(a(\zeta) - P^1_{\xi_0}a(\zeta))^2\phi(\zeta)(\partial_t v(\zeta) - P^1_{\xi_0}\partial_t v(\zeta))d\zeta$$
$$+2\int D^2_{i,j}\partial_t\Gamma_{\xi_0}(\xi_0,\zeta)(Yb(\zeta) - Yb(\xi_0))\phi(\zeta)(\partial_t v(\zeta) - P^1_{\xi_0}\partial_t v(\zeta))d\zeta$$
$$-2\int D^2_{i,j}\partial_t\tilde{Y}_{\xi_0}\Gamma_{\xi_0}(\xi_0,\zeta)(b(\zeta) - P^1_{\xi_0}b(\zeta))\phi(\zeta)(\partial_t v(\zeta) - P^1_{\xi_0}\partial_t v(\zeta))d\zeta$$
$$-\int D^2_{i,j}\partial^2_{t,t}\Gamma_{\xi_0}(\xi_0,\zeta)(b(\zeta) - P^1_{\xi_0}b(\zeta))\phi(\zeta)(\partial_t v(\zeta) - P^1_{\xi_0}\partial_t v(\zeta))d\zeta\,,$$

$$D^2_{i,j}\partial_t E_1(\xi_0,\xi_0) = 2\int D^2_{i,j}\partial_t\Gamma_{\xi_0}(\xi_0,\zeta)$$
$$\cdot (Xv(\zeta) - tX\partial_t v(\xi_0) - aX\partial_t v(\xi_0)(x-x_0))X_{\xi_0}\phi(\zeta)d\zeta$$
$$+2\int D^2_{i,j}\partial_t\Gamma_{\xi_0}(\xi_0,\zeta)(Yv(\zeta) - tY\partial_t v(\xi_0) - bY\partial_t v(\xi_0)(y-y_0))$$
$$\cdot Y_{\xi_0}\phi(\zeta)d\zeta$$
$$+\int D^2_{i,j}\partial_t\Gamma_{\xi_0}(\xi_0,\zeta)(v(\zeta) - X\partial_t v(\xi_0)t(x-x_0) - Y\partial_t v(\xi_0)t(y-y_0))$$
$$\cdot L_{\xi_0}\phi(\zeta)d\zeta$$
$$+\int D^2_{i,j}\partial_t\Gamma_{\xi_0}(\xi_0,\zeta)(a(\zeta) - P^1_{\xi_0}a(\zeta))^2(\partial_t v(\zeta) - P^1_{\xi_0}\partial_t v(\zeta))\partial_t\phi(\zeta)d\zeta$$
$$+\int D^2_{i,j}\partial_t\Gamma_{\xi_0}(\xi_0,\zeta)(b(\zeta) - P^1_{\xi_0}b(\zeta))^2(\partial_t v(\zeta) - P^1_{\xi_0}\partial_t v(\zeta))\partial_t\phi(\zeta)d\zeta\,.$$

PROOF OF THEOREM 6.1. Assume now that $k = 3$. Then, by definition (9), v is a solution of

$$X^2 v + Y^2 v - (Xa + Yb)\partial_t v = f \tag{37}$$

and, by Theorems 6.2 and 6.5, $v \in C^{3,\alpha}_{L,\mathrm{loc}}(\Omega)$ and $\partial_t v \in C^{2,\alpha}_{L,\mathrm{loc}}(\Omega)$. Let us differentiate equation (37), with respect to X:

$$X^3 v + XY^2 v - X((Xa + Yb)\partial_t v) = Xf\,, \tag{38}$$

and consider one term at a time.

$$XY^2 v = [X,Y]Yv + YXYv =$$

(by Remark 2.4)

$$= (Xb - Ya)\partial_t Yv + Y([X,Y]v) + Y^2Xv$$
$$= (Xb - Ya)\partial_t Yv + Y(Xb - Ya)\partial_t v + (Xb - Ya)Y\partial_t v + Y^2Xv.$$

But, by definition of Y, $\partial_t Yv = \partial_t b\partial_t v + Y\partial_t v$, hence

$$XY^2v = (Xb - Ya)\partial_t b\partial_t v + 2(Xb - Ya)Y\partial_t v + Y(Xb - Ya)\partial_t v + Y^2Xv.$$

Now we compute

$$X((Xa + Yb)\partial_t v) = X(Xa + Yb)\partial_t v + (Xa + Yb)X\partial_t v =$$

(since $X\partial_t v = \partial_t Xv - \partial_t a\partial_t v$)

$$= X(Xa + Yb)\partial_t v + (Xa + Yb)\partial_t Xv - (Xa + Yb)\partial_t a\partial_t v.$$

Hence if $v_1 = Xv$, we get from (38)

$$\begin{aligned} Lv_1 &= X^2v_1 + Y^2v_1 - (Xa + Yb)\partial_t v_1 \\ &= Xf - (Xb - Ya)\partial_t b\partial_t v - 2(Xb - Ya)Y\partial_t v - Y(Xb - Ya)\partial_t v \\ &\quad - X(Xa + Yb)\partial_t v + (Xa + Yb)\partial_t a\partial_t v, \end{aligned}$$

where the second member is of class $C_L^{1,\alpha}(\Omega)$. If follows, from Theorem 6.2 that $Xv \in C_L^{3,\alpha}(\Omega)$. Analogously $Yu \in C_L^{3,\alpha}(\Omega)$, and $v \in C_L^{4,\alpha}(\Omega)$.

Now we conclude the proof by induction. Under the states hypotheses, we can assume that

$$v \in C_L^{k,\alpha}(\Omega) \quad \text{and} \quad \partial_t v \in C_L^{k-2,\alpha}(\Omega).$$

Differentiating equation (37) with respect to t we obtain

$$\begin{aligned} \partial_t(X^2v) &= X\partial_t Xv + \partial_t a\partial_t Xv \\ &= X^2\partial_t v + X(\partial_t a\partial_t v) + \partial_t a\partial_t Xv \\ &= X^2\partial_t v + X\partial_t a\partial_t v + \partial_t aX\partial_t v + \partial_t a\partial_t Xv = \end{aligned}$$

(since $X\partial_t a\partial_t v = \partial_t Xa\partial_t v - (\partial_t a)^2\partial_t v$ and $\partial_t a\partial_t Xv = (\partial_t a)^2\partial_t v + \partial_t aX\partial_t v$)

$$= X^2\partial_t v + \partial_t Xa\partial_t v + 2\partial_t aX\partial_t v.$$

Analogously

$$\partial_t(Y^2v) = Y^2\partial_t v + \partial_t Yb\partial_t v + 2\partial_t bY\partial_t v,$$

and

$$\partial_t((Xa + Yb)\partial_t v) = (Xa + Yb)\partial_{tt}^2 v + \partial_t Xa\partial_t v + \partial_t Yb\partial_t v.$$

Summing up and using (37) we get

$$X^2(\partial_t v) + Y^2(\partial_t v) - (Xa + Yb)\partial_{tt}^2 v = -2\partial_t aX\partial_t v - 2\partial_t bY\partial_t v - \partial_t f,$$

where the right member is of class $C_L^{k-3,\alpha}(\Omega)$. Thus, by inductive hypothesis, $\partial_t v \in C_{L,\mathrm{loc}}^{k-1,\alpha}(\Omega)$. This implies that the right member of (39) is of class $C_{L,\mathrm{loc}}^{k-2,\alpha}(\Omega)$. Hence $Xv \in C_{L,\mathrm{loc}}^{k,\alpha}(\Omega)$ and, analogously, $Yv \in C_{L,\mathrm{loc}}^{k,\alpha}(\Omega)$, so that $v \in C_{L,\mathrm{loc}}^{k+1,\alpha}(\Omega)$.

7. – Regularity of solutions of (1)

We will study here the regularity of solutions of (1), which will always write in divergence form (7). First of all we give the definition of Lipschitz classes in this non linear situation. Indeed, if $u \in C^1(\Omega)$ (in euclidean sense) we can define, according to (6) and (8),

$$a(u) = \frac{u_y - u_x u_t}{1 + u_t^2}, \quad b(u) = -\frac{u_x - u_y u_t}{1 + u_t^2},$$

if these are Lipschitz we can apply to $\mathcal{L}$ the theory we have developed in the previous sections. In particular the class $C^1_{\mathcal{L}}(\Omega)$ is well defined, and we can assume that a and b satisfy (H1) (will see that (H2) is always satisfied). Then we say that $u \in C^1(\Omega) \cap C^2_{\mathcal{L}}(\Omega)$, is a solution of (1), if

$$X^2 u + Y^2 u - (Xa + Yb)\partial_t u = q(1 + |Xu|^2 + |Yu|^2). \tag{40}$$

One of the main properties of the vector fields we have chosen is the following: If Xu and Yu are the derivative of u in the X and Y directions, we have

$$\begin{aligned} Xu &= u_x + \frac{u_y - u_x u_t}{1 + u_t^2} u_t = \frac{u_x + u_y u_t}{1 + u_t^2} = -b \\ Yu &= u_y - \frac{u_x + u_y u_t}{1 + u_t^2} u_t = \frac{u_y - u_x u_t}{1 + u_t^2} = a \end{aligned} \tag{41}$$

hence

$$X = \begin{pmatrix} 1 \\ 0 \\ Yu \end{pmatrix} \quad Y = \begin{pmatrix} 0 \\ 1 \\ -Xu \end{pmatrix},$$

so that we also get

$$X^2 u + Y^2 u = Ya - Xb,$$

and

$$Xa + Yb = XYu - YXu = [X, Y]u = \tag{42}$$

(by Remark 2.5)

$$= (Xb - Ya)\partial_t u = (by (41)) = (-X^2 u - Y^2 u)\partial_t u = -(Ya - Xb)\partial_t u.$$

Thus, by (40),

$$(Ya - Xb)(1 + (\partial_t u)^2) = q(1 + |Xu|^2 + |Yu|^2). \tag{43}$$

Since $q(\xi) \neq 0$ for all $\xi \in \Omega$, then (H2) is satisfied and we can introduce the distance associated to $\mathcal{L}$, and the Lipschitz classes $C^{1,\alpha}_{\mathcal{L}}(\Omega)$ and $C^{2,\alpha}_{\mathcal{L}}(\Omega)$.

Then we will prove the main theorem in this paper:

THEOREM 7.1. *If* $q \in C^\infty(\Omega)$, $q(\xi) \neq 0$ *for all* $\xi \in \Omega$, $\alpha > \frac{1}{2}$ *and* $u \in C^{2,\alpha}_{\mathcal{L}}(\Omega)$ *is a solution of (1), then* $u \in C^\infty(\Omega)$.

Since a solution u of (40) is fixed, we can consider $\mathcal{L}$ as a linear operator, whose coefficients depend on u.

Also note that we will deal with derivatives in the directions X and Y, and derivatives in the directions X_{ξ_0} and Y_{ξ_0} and the following relations hold: if $f \in C^{1,\alpha}_{\mathcal{L}}(\Omega)$

$$\forall \xi_0 \quad \exists X_{\xi_0} f(\xi_0) = Xf(\xi_0). \tag{44}$$

If $f \in C^{1,\alpha}_{\mathcal{L}}(\Omega)$ and $\exists\, \partial_t f \in C^{\alpha}_{\mathcal{L}}(\Omega)$

$$\forall \xi_0, \xi \quad X_{\xi_0} f(\xi) = Xf(\xi) - (a - P^1_{\xi_0} a)(\xi)\partial_t f. \tag{45}$$

REMARK 7.1. Since $u \in C^{2,\alpha}_{\mathcal{L}}(\Omega)$, by (41) a and $b \in C^{1,\alpha}_{\mathcal{L}}(\Omega)$, and applying Remark 5.4 we get that

$$\partial_t u \in C^{1,\alpha}_{\mathcal{L},\mathrm{loc}}(\Omega),$$

hence it is not difficult to see that there also exist $X_{\xi_0}\partial_t u(\xi_0) = X\partial_t u(\xi_0)$, and $Y_{\xi_0}\partial_t u(\xi_0) = Y\partial_t u(\xi_0)$ for all ξ_0 in Ω.

LEMMA 7.1. *If* $u \in C^{2,\alpha}_{\mathcal{L}}(\Omega)$ *is a solution of* (1) *then there exist* $\partial_t X u$ *and* $\partial_t Y u$ *and they are of class* $C^{\alpha}_{\mathcal{L},\mathrm{loc}}(\Omega)$.

PROOF. If e_3 denotes the third element in the canonical basis of $\mathbb{R}^3$, we have

$$\begin{aligned}
&\frac{Xu(\xi_0 + te_3) - Xu(\xi_0)}{t} \\
&= \frac{X_{\xi_0}u(\xi_0 + te_3) - X_{\xi_0}u(\xi_0)}{t} + \frac{(X - X_{\xi_0})u(\xi_0 + te_3)}{t} =
\end{aligned}$$

(using (45) and the fact that $P^1_{\xi_0} a(\xi_0 + te_3) = a(\xi_0)$)

$$\begin{aligned}
&= \frac{X_{\xi_0}u(\xi_0 + te_3) - X_{\xi_0}u(\xi_0)}{t} + \frac{a(\xi_0 + te_3) - a(\xi_0)}{t}\partial_t u(\xi_0 + te_3) \\
&= \frac{X_{\xi_0}u(\xi_0 + te_3) - X_{\xi_0}u(\xi_0)}{t} + \frac{Yu(\xi_0 + te_3) - Yu(\xi_0)}{t}\partial_t u(\xi_0 + te_3) \\
&= \frac{X_{\xi_0}u(\xi_0 + te_3) - X_{\xi_0}u(\xi_0)}{t} + \frac{Y_{\xi_0}u(\xi_0 + te_3) - Y_{\xi_0}u(\xi_0)}{t}\partial_t u(\xi_0 + te_3) \\
&\quad + \frac{(Y - Y_{\xi_0})u(\xi_0 + te_3)}{t}\partial_t u(\xi_0 + te_3)
\end{aligned}$$

$$= \frac{X_{\xi_0}u(\xi_0 + te_3) - X_{\xi_0}u(\xi_0)}{t} + \frac{Y_{\xi_0}u(\xi_0 + te_3) - Y_{\xi_0}u(\xi_0)}{t}\partial_t u(\xi_0 + te_3)$$
$$+ \frac{b(\xi_0 + te_3) - b(\xi_0)}{t}(\partial_t u(\xi_0 + te_3))^2$$
$$= \frac{X_{\xi_0}u(\xi_0 + te_3) - X_{\xi_0}u(\xi_0)}{t} + \frac{Y_{\xi_0}u(\xi_0 + te_3) - Y_{\xi_0}u(\xi_0)}{t}\partial_t u(\xi_0 + te_3)$$
$$- \frac{Xu(\xi_0 + te_3) - Xu(\xi_0)}{t}(\partial_t u(\xi_0 + te_3))^2 .$$

Hence

$$\frac{Xu(\xi_0 + te_3) - Xu(\xi_0)}{t}(1 + (\partial_t u(\xi_0 + te_3))^2)$$
$$= \frac{X_{\xi_0}u(\xi_0 + te_3) - X_{\xi_0}u(\xi_0)}{t} + \frac{Y_{\xi_0}u(\xi_0 + te_3) - Y_{\xi_0}u(\xi_0)}{t}\partial_t u(\xi_0 + te_3) .$$

By Theorem 4.1 and Remark 7.1 there exists

$$\partial_t Xu(\xi_0) = \frac{1}{1 + (\partial_t u(\xi_0))^2}(\partial_t X_{\xi_0}u(\xi_0) + \partial_t Y_{\xi_0}u(\xi_0)\partial_t u(\xi_0)) .$$

Analogously there exists

$$\partial_t Yu(\xi_0) = \frac{1}{1 + (\partial_t u(\xi_0))^2}(\partial_t Y_{\xi_0}u(\xi_0) - \partial_t X_{\xi_0}u(\xi_0)\partial_t u(\xi_0)) .$$

Finally $\partial_t Xu(\xi_0)$ and $\partial_t Yu(\xi_0)$ are of class $C^\alpha_{\mathcal{L},\mathrm{loc}}(\Omega)$, since $\partial_t X_{\xi_0}u(\xi_0)$, $\partial_t Y_{\xi_0}u(\xi_0)$ and $\partial_t u(\xi_0)$ are.

In the hypotheses of Lemma 7.1 the coefficients of $\mathcal{L}$, $a = Yu$ and $b = -Xu$ are of class $C^{1,\alpha}_{\mathcal{L}}(\Omega)$, and differentiable with respect to t, with derivative of class $C^\alpha_{\mathcal{L}}(\Omega)$, hence, by Theorem 6.5

$$\partial_t u \in C^{2,\alpha}_{\mathcal{L},\mathrm{loc}}(\Omega) . \tag{46}$$

□

LEMMA 7.2. *If $u \in C^{2,\alpha}_{\mathcal{L}}(\Omega)$ is a solution of* (1), *then $a(u)$ and $b(u)$ are of class $C^{2,\alpha}_{\mathcal{L},\mathrm{loc}}(\Omega)$.*

PROOF. Let us for example compute $X^2 b$. Since $u \in C^{2,\alpha}_{\mathcal{L}}(\Omega)$, then for every ξ and ξ_0 there exists

$$X_{\xi_0}u(\xi) = Xu(\xi) - (a - P^1_{\xi_0}a)(\xi)\partial_t u(\xi) .$$

The second member is of class $C^{1,\alpha}_{\mathcal{L}}(\Omega)$, hence there exists

$$XX_{\xi_0}u(\xi) = X^2u(\xi) - (a - P^1_{\xi_0}a)(\xi)X\partial_t u(\xi) - (Xa(\xi) - Xa(\xi_0))X\partial_t u(\xi) .$$

On the other side $b = -Xu$, and

$$\begin{aligned} Xb(\xi) &= -X^2u(\xi) \\ &= -XX_{\xi_0}u(\xi) - (a - P^1_{\xi_0}a)(\xi)X\partial_t u(\xi) - (Xa(\xi) - Xa(\xi_0))\partial_t u(\xi)\,. \end{aligned}$$

Will now compute $Xa(\xi)$, arguing as before:

$$\begin{aligned} Xa(\xi) &= XYu(\xi) \\ &= XY_{\xi_0}u(\xi) + (b - P^1_{\xi_0}b)(\xi)X\partial_t u(\xi) - (Xb(\xi) - Xb(\xi_0))\partial_t u(\xi)\,. \end{aligned}$$

Collecting these terms we get:

$$\begin{aligned} Xb(\xi) = &-XX_{\xi_0}u(\xi) - (a - P^1_{\xi_0}a)(\xi)X\partial_t u(\xi) + Xa(\xi_0)\partial_t u(\xi) \\ &- \Big(XY_{\xi_0}u(\xi) + (b - P^1_{\xi_0}b)(\xi)X\partial_t u(\xi) - (Xb(\xi) - Xb(\xi_0))\partial_t u(\xi)\Big)\,\partial_t u(\xi). \end{aligned}$$

Finally, bringing Xb to the first member,

$$\begin{aligned} Xb(\xi)(1 + (\partial_t u)^2) = &-XX_{\xi_0}u(\xi) - (a - P^1_{\xi_0}a)(\xi)X\partial_t u(\xi) + Xa(\xi_0)\partial_t u(\xi) \\ &- XY_{\xi_0}u(\xi)\partial_t u(\xi) - (b - P^1_{\xi_0}b)(\xi)X\partial_t u(\xi)\partial_t u(\xi) \\ &+ Xb(\xi_0)(\partial_t u(\xi))^2\,. \end{aligned}$$

From (45), (46) and Lemma 7.1 it follows that $X_{\xi_0}u \in C^{1,\alpha}_{\mathcal{L},\mathrm{loc}}(\Omega)$ and has derivative with respect to t in $C^\alpha_{\mathcal{L}}(\Omega)$. Hence by (45)

$$XX_{\xi_0}u(\xi) = X^2_{\xi_0}u(\xi) + (a - P^1_{\xi_0}a)(\xi)\partial_t X_{\xi_0}u(\xi)$$

and, in the same way,

$$XY_{\xi_0}u(\xi) = X_{\xi_0}Y_{\xi_0}u(\xi) + (a - P^1_{\xi_0}a)(\xi)\partial_t Y_{\xi_0}u(\xi)\,.$$

Then Xb can be represented

$$(1 + (\partial_t u)^2)Xb(\xi) = T_1(\xi, \xi_0) + T_2(\xi, \xi_0)\,,$$

where

$$T_1 = -X^2_{\xi_0}u(\xi) + Xa(\xi_0)\partial_t u(\xi) - X_{\xi_0}Y_{\xi_0}u(\xi)\partial_t u(\xi) - Xb(\xi_0)(\partial_t u(\xi))^2\,,$$

and

$$\begin{aligned} T_2(\xi) = &-(a - P^1_{\xi_0}a)(\xi)\partial_t X_{\xi_0}u(\xi) + (a - P^1_{\xi_0}a)(\xi)\partial_t Y_{\xi_0}u(\xi)\partial_t u(\xi) \\ &+ (a - P^1_{\xi_0}a)(\xi)X\partial_t u(\xi) + (b - P^1_{\xi_0}b)(\xi)X\partial_t u(\xi)\partial_t u(\xi)\,. \end{aligned}$$

By Theorem 4.1 there exists $XT_1(\xi_0) \in C^\alpha_{\mathcal{L},\mathrm{loc}}(\Omega)$, while $T_2(\xi) - T_2(\xi_0) = O(d(\xi, \xi_0)^{1+\alpha})$, hence $XT_2(\xi_0) = 0$.

REMARK 7.2. Since a and b are of class $C^{2,\alpha}_{\mathcal{L},\mathrm{loc}}(\Omega)$, we can consider the Lipschitz class $C^{3,\alpha}_{\mathcal{L}}(\Omega)$. Indeed $u \in C^{3,\alpha}_{\mathcal{L},\mathrm{loc}}(\Omega)$, by (41).

PROOF OF THEOREM 7.1. We first differentiate both members of equation (40), with respect to X, and we get

$$XY^2u = [X, Y]Yu + YXYu = [X, Y]Yu + Y([X, Y]u) + Y^2Xu =$$

(by Remark 2.4)

$$= (Xb - Ya)\partial_t Yu + Y((Xb - Ya)\partial_t u) + Y^2Xu =$$

by (41) and (43)

$$= -(X^2u + Y^2u)\partial_t Y_u - Y\left(\frac{q(1 + |Xu|^2 + |Yu|^2)}{1 + (\partial_t u)^2}\partial_t u\right) + Y^2Xu\,.$$

Moreover

$$X((Xa + Yb)\partial_t u) = X(Xa + Yb)\partial_t u + (Xa + Yb)X\partial_t u =$$

(since $X\partial_t u = \partial_t Xu - \partial_t a\partial_t u$, and by (42) and (43)),

$$= -X\left(\frac{\partial_t u}{1 + (\partial_t u)^2}q(1 + |Xu|^2 + |Yu|^2)\right)\partial_t u + (Xa + Yb)(\partial_t Xu - \partial_t a\partial_t u) =$$

(by (41))

$$\begin{aligned} &= -X\left(\frac{\partial_t u}{1 + (\partial_t u)^2}q(1 + |Xu|^2 + |Yu|^2)\right)\partial_t u \\ &\quad + (Xa + Yb)\partial_t Xu - (XYu - YXu)\partial_t Yu\partial_t u\,. \end{aligned}$$

Then

$$\begin{aligned} \mathcal{L}Xu = {} & X(q(1 + |Xu|^2 + |Yu|^2)) + (X^2 + Y^2)\partial_t Yu \\ & + Y\left(\frac{\partial_t u}{1 + (\partial_t u)^2}q(1 + |Xu|^2 + |Yu|^2)\right) \\ & - X\left(\frac{\partial_t u}{1 + (\partial_t u)^2}q(1 + |Xu|^2 + |Yu|^2)\right)\partial_t u - (XYu - YXu)\partial_t Yu\partial_t u\,. \end{aligned}$$

Finally note that

$$\partial_t Yu = \partial_t b\partial_t u + Y\partial_t u = -\partial_t Xu\partial_t u + Y\partial_t u = -\partial_t a(\partial_t u)^2 - X\partial_t u\partial_t u + Y\partial_t u\,,$$

so that

$$\partial_t Yu = \frac{-X\partial_t u\partial_t u + Y\partial_t u}{1 + (\partial_t u)^2}\,, \tag{47}$$

and the equation becomes:

$$
(48)\quad
\begin{aligned}
\mathcal{L}Xu = {} & X(q(1+|Xu|^2+|Yu|^2)) - (X^2+Y^2)\frac{X\partial_t u\partial_t u - Y\partial_t u}{1+(\partial_t u)^2} \\
& + Y\left(\frac{\partial_t u}{1+(\partial_t u)^2} q(1+|Xu|^2+|Yu|^2)\right) \\
& - X\left(\frac{\partial_t u}{1+(\partial_t u)^2} q(1+|Xu|^2+|Yu|^2)\right)\partial_t u \\
& + (XYu - YXu)\frac{X\partial_t u\partial_t u - Y\partial_t u}{1+(\partial_t u)^2}\partial_t u\,.
\end{aligned}
$$

Analogously

$$
(49)\quad
\begin{aligned}
\mathcal{L}Yu = {} & Y(q(1+|Xu|^2+|Yu|^2)) - (X^2+Y^2)\frac{X\partial_t u + Y\partial_t u\partial_t u}{1+(\partial_t u)^2} \\
& - X\left(\frac{\partial_t u}{1+(\partial_t u)^2} q(1+|Xu|^2+|Yu|^2)\right) \\
& + Y\left(\frac{\partial_t u}{1+(\partial_t u)^2} q(1+|Xu|^2+|Yu|^2)\right)\partial_t u \\
& - (XYu - YXu)\frac{X\partial_t u + Y\partial_t u\partial_t u}{1+(\partial_t u)^2}\partial_t u\,,
\end{aligned}
$$

while, as we proved in Theorem 6.1, $\partial_t u$ is solution of

$$
(50)\quad
\begin{aligned}
\mathcal{L}\partial_t u &= -2\partial_t a X\partial_t u - 2\partial_t b Y\partial_t u - \partial_t(q(1+|Xu|^2+|Yu|^2)) \\
&= -2\partial_t Yu X\partial_t u + 2\partial_t Xu Y\partial_t u - \partial_t(q(1+|Xu|^2+|Yu|^2))\,.
\end{aligned}
$$

Since $u \in C^{3,\alpha}_{\mathcal{L},\mathrm{loc}}(\Omega)$ and $\partial_t u \in C^{2,\alpha}_{\mathcal{L},\mathrm{loc}}(\Omega)$, the second members in (48) and (49) are of class $C^{1,\alpha}_{\mathcal{L},\mathrm{loc}}(\Omega)$, and the coefficients of the operator are $C^{2,\alpha}_{\mathcal{L},\mathrm{loc}}(\Omega)$ hence $Xu \in C^{3,\alpha}_{\mathcal{L},\mathrm{loc}}(\Omega)$, $Yu \in C^{3,\alpha}_{\mathcal{L},\mathrm{loc}}(\Omega)$. Hence the second member in (50) is of class $C^{1,\alpha}_{\mathcal{L},\mathrm{loc}}(\Omega)$, so that $\partial_t u \in C^{3,\alpha}_{\mathcal{L},\mathrm{loc}}(\Omega)$. Thus the second member of (48) and (49) is of class $C^{2,\alpha}_{\mathcal{L},\mathrm{loc}}(\Omega)$, and the coefficients are $C^{3,\alpha}_{\mathcal{L},\mathrm{loc}}(\Omega)$. Iterating this procedure we get the thesis. □

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