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The Maxwell Equation in a Periodic Medium: Homogenization of the Energy Density

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1. – Introduction

We consider the evolution of the electro-magnetic field quantities in a periodic medium with a small period. Therefore let $\alpha > 0$ be a small parameter such that the lattice spacing of the medium is in $0(\alpha)$, and denote by E^α the electric field and by H^α the magnetic field. Assuming that the medium has zero conductivity, the Maxwell equations for the fields then read:

$$\varepsilon\left(\frac{x}{\alpha}\right)E_t^\alpha = \operatorname{curl} H^\alpha \qquad x \in \mathbb{R}^3, t \in \mathbb{R}, \tag{1.1}$$

$$\mu\left(\frac{x}{\alpha}\right)H_t^\alpha = -\operatorname{curl} E^\alpha \qquad x \in \mathbb{R}^3, t \in \mathbb{R}. \tag{1.2}$$

We impose the initial condition

$$E^\alpha(t=0) = E_I^\alpha\,, \quad H^\alpha(t=0) = H_I^\alpha\,, \quad x \in \mathbb{R}^3\,. \tag{1.3}$$

The functions $\varepsilon(\frac{\cdot}{\alpha})$, $\mu(\frac{\cdot}{\alpha})$ stand for the permittivity and, respectively, permeability of the medium; $\varepsilon = \varepsilon(x)$, $\mu = \mu(x) > 0$ are assumed to be periodic on a lattice with $0(1)$-spacing, uniformly bounded away from 0 and in $L^\infty(\mathbb{R}^3;\mathbb{R})$. We also assume that the initial data E_I^α, H_I^α are in $L^2(\mathbb{R}^3;\mathbb{R})^3$ and satisfy the compatibility condition

$$\operatorname{div}\left(\varepsilon\left(\frac{x}{\alpha}\right)E_I^\alpha\right) = \operatorname{div}\left(\mu\left(\frac{x}{\alpha}\right)H_I^\alpha\right) = 0\,. \tag{1.4(a)}$$

The homogenization limit $\alpha \to 0$ of the field quantities E^α, H^α is well-known (see, e.g. [BLP]). In particular there exist functions E and H such that (maybe after selection of a subsequence):

$$E^\alpha \overset{\alpha\to 0}{\longrightarrow} E \quad \text{in} \quad L^\infty\left(\mathbb{R}_t; L^2(\mathbb{R}^3)\right)^3 w - * \tag{1.4(b)}$$

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$$(1.4)(c) \qquad H^\alpha \overset{\alpha\to 0}{\longrightarrow} H \quad \text{in} \quad L^\infty\Big(\mathbb{R}_t; L^2(\mathbb{R}^3)\Big)^3 w - * .$$

E and H are weak solutions of a homogenized version of the Maxwell equations with initial data E_I, H_I such that

$$(1.5) \qquad E_I^\alpha \overset{\alpha\to 0}{\longrightarrow} E_I, \quad H_I^\alpha \overset{\alpha\to 0}{\longrightarrow} H_I \qquad \text{in} \quad L^2(\mathbb{R}^3;\mathbb{R})^3 w .$$

For the precise form of the homogenized permittivity and permeability functions we refer to [BLP].

An important quantity is the electro-magnetic energy density:

$$(1.6) \qquad n^\alpha(x,t) := \varepsilon\Big(\frac{x}{\alpha}\Big)|E^\alpha(x,t)|^2 + \mu\Big(\frac{x}{\alpha}\Big)|H^\alpha(x,t)|^2 ,$$

whose $L^1(\mathbb{R}_x^3)$-norm (i.e. the total electro-magnetic energy) is preserved by the Maxwell flow:

$$(1.7) \qquad \int_{\mathbb{R}_x^3} n^\alpha(x,t)\,dx \equiv \int_{\mathbb{R}_x^3} n^\alpha(x,t=0)\,dx \qquad \forall t \in \mathbb{R} .$$

The goal of this paper is to analyze the homogenization limit $\alpha \to 0$ of the energy density n^α. Obviously the topology in which the limit (1.4) of the field quantities takes place is not strong enough to carry out the limit of n^α directly. Thus, an alternative approach has to be taken.

We remark that compensated compactness methods can be used to pass to the limit $\alpha \to 0$ in certain nonlinear expression, e.g. the limit of $\varepsilon(x/\alpha)|E^\alpha(x,t)|^2 - \mu(x/\alpha)|H^\alpha(x,t)|^2$ can be carried out directly [BLP]). However, these methods are not applicable to the energy density n^α.

In this paper we proceed in analogy to the homogenization limit for the Schrödinger equation in a crystal presented in [MMP]. We construct subspaces invariant under the Maxwell operator by the well-known Bloch decomposition [RS], set up the so called band-Wigner transforms of the projections of the fields onto these subspaces as introduced in [MMP] and pass to the limit $\alpha \to 0$ in the evolution equation for the Wigner-functions obtaining a denumerable set of kinetic equations. Finally, we show using an argument of [G] that the homogenization limit of the energy density is obtained as sum over all bands of the position densities of the limits of the band-Wigner functions. In this way we exploit a somewhat hidden kinetic structure of the Maxwell equations.

We remark that the homogenization limits of the band-Wigner functions are the so-called semi-classical measures introduced in [G]. Also, there are obvious analogies to the construction of the full-space Wigner transform and their limiting Wigner measures given in [LP].

Another approach based on H-measures has been proposed in [FM]. In this work the authors give the measure limit of the energy density for the wave equation when the coefficients do not depend on the small parameter.

Our main result is the following. We construct infinitely many non-negative measures $w_{I,l}(x,k), x \in \mathbb{R}^3, k \in \mathbb{R}^3, l \in \mathbb{Z}$, each of them corresponding to an eigenvalue $\omega_l(k)$ of an elliptic problem indexed by k. For initial data oscillating at the scale α we obtain that:

$$n^\alpha \to \sum_l \int_B w_{I,l}(x - \nabla_k \omega_l(k), dk)\,.$$

The bounded domain B is the Brillouin zone defined in the next section.

2. – Bloch decomposition of the Maxwell equations in a periodic medium

Let $a_{(1)}, a_{(2)}, a_{(3)}$ be a basis in $\mathbb{R}^3$. Then we define the lattice

(2.1) $$L = \{a_{(1)} j_1 + a_{(2)} j_2 + a_{(3)} j_3 \,|\, j_1, j_2, j_3 \in \mathbb{Z}\}$$

and the dual lattice

(2.2) $$L^* = \{a^{(1)} j_1 + a^{(2)} j_2 + a^{(3)} j_3 \,|\, j_1, j_2, j_3 \in \mathbb{Z}\}$$

where the dual basis vector $a^{(1)}, a^{(2)}, a^{(3)}$ are determined by the equations

(2.3) $$a_{(l)} \cdot a^{(m)} = 2\pi \delta_{lm}\,, \quad l, m = 1, 2, 3\,.$$

The basic period cell of the lattice L is denoted by

(2.4) $$C = \Big\{ \sum_{j=1}^{3} a_{(j)} t_j \,\Big|\, 0 < t_1, t_2, t_3 < 1 \Big\}$$

and the Brillouin zone B is the Wigner-Seitz cell of the dual lattice:

(2.5) $$B = \{k \in \mathbb{R}^3 \,|\, |k| < |k - \sigma| \ \ \forall \sigma \in L^*\}\,.$$

Note that $|C|\,|B| = (2\pi)^3$ holds ($|\cdot|$ denotes the volume).

For the following let $\varepsilon = \varepsilon(x), \mu = \mu(x)$ be the (real-valued) dielectric and, respectively, permeability functions on $\mathbb{R}^3$, with the properties:

$$\varepsilon, \mu \in W^{1,\infty}(\mathbb{R}^3; \mathbb{R})\,,$$

$$\exists\, \underline{\varepsilon}, \bar{\varepsilon}, \underline{\mu}, \bar{\mu} > 0 \text{ such that } \underline{\varepsilon} \le \varepsilon(x) \le \bar{\varepsilon}, \underline{\mu} \le \mu(x) \le \bar{\mu} \quad \text{on } \mathbb{R}^3\,,$$

(A1) $$\varepsilon, \mu \quad \text{are} \quad \frac{L}{2} - \text{periodic on } \mathbb{R}^3, \text{i.e.} \quad \forall \gamma \in L \quad \text{we have}$$

$$\left\{ \begin{aligned} \varepsilon\Big(x + \frac{\gamma}{2}\Big) &= \varepsilon(x) \\ \mu\Big(x + \frac{\gamma}{2}\Big) &= \mu(x) \end{aligned} \right\} \quad \text{on} \quad \mathbb{R}^3\,.$$

We set $\varepsilon^\alpha(x) = \varepsilon(x/\alpha)$, $\mu^\alpha(x) = \mu(x/\alpha)$.

For a domain $\Omega \subseteq \mathbb{R}^3$ and a function $a \in L^\infty(\mathbb{R}^3;\mathbb{R})$ we consider the Hilbert space $H(\Omega, a, \text{div } 0) := \{u \in L^2(\Omega)^3 |\, \text{div}\,(au) = 0\}$ which we equip with the L^2-scalar product on $\mathbb{R}^3$ with weight a. The local version $H_{\text{loc}}(\Omega, a, \text{div } 0)$ is defined in the obvious way.

Now, let $\alpha \in (0, \alpha_0)$ for some fixed $\alpha_0 > 0$. We introduce the Maxwell operator

$$L^\alpha = i\alpha \begin{pmatrix} \varepsilon^\alpha(x)^{-1} & 0 \\ 0 & \mu^\alpha(x)^{-1} \end{pmatrix} \begin{pmatrix} 0 & \text{curl}_x \\ -\,\text{curl}_x & 0 \end{pmatrix} \tag{2.7)(a}$$

with domain

$$D(L^\alpha) = (H(\mathbb{R}^3, \text{curl}))^2 \tag{2.7)(b}$$

where $H(\mathbb{R}^3, \text{curl}) := \{u \in (L^2(\mathbb{R}^3))^3;\, \text{curl}\, u \in L^2(\mathbb{R}^3)\}$.

Note that L^α is obtained from L^1 by the rescaling of the position variable $x \to x/\alpha$.

The Maxwell equations (1.1), (1.3) then can be written as

$$i\alpha \frac{\partial}{\partial t} \begin{pmatrix} E^\alpha \\ H^\alpha \end{pmatrix} = L^\alpha \begin{pmatrix} E^\alpha \\ H^\alpha \end{pmatrix}, \quad t \in \mathbb{R}, \tag{2.8)(a}$$

$$E^\alpha(t=0) = E_I^\alpha, \quad H^\alpha(t=0) = H_I^\alpha\,. \tag{2.8)(b}$$

Assuming that the (real-valued) initial data satisfy

$$E_I^\alpha \in H(\mathbb{R}^3, \varepsilon^\alpha, \text{div } 0),\, H_I^\alpha \in H(\mathbb{R}^3, \mu^\alpha, \text{div } 0)\,, \tag{2.9}$$

there exists a unique solution $(E^\alpha, H^\alpha) \in C_b(\mathbb{R}_t, \varepsilon^\alpha, \text{div } 0) \times H\big(\mathbb{R}^3, \mu^\alpha, \text{div } 0)\big)$, since L^α maps $D(L^\alpha) \cap \big(H(\mathbb{R}^3_, \varepsilon^\alpha, \text{div } 0) \times H\big(\mathbb{R}^3_, \mu^\alpha, \text{div } 0)\big)$ into $H(\mathbb{R}^3_, \varepsilon^\alpha, \text{div } 0) \times H(\mathbb{R}^3, \mu^\alpha, \text{div } 0)$.

As usual we start the Bloch-decomposition with the introduction of spaces of quasi-periodic functions on $\mathbb{R}^3 \times B$. We define:

$$\begin{aligned} L^2_{\sharp,\alpha} := \Big\{ u = u(x,k) \in \big(L^2_{\text{loc}}(\mathbb{R}^3 \times \mathbb{R}^3)\big)^3 \big| \\ \forall \gamma \in L : u(x + \alpha\gamma, k) = e^{ik\cdot\gamma} u(x,k)\,, \\ \forall \gamma^* \in L^* : u(x, k + \gamma^*) = u(x,k) \text{ a.e in } \mathbb{R}^3 \times \mathbb{R}^3 \Big\}, \end{aligned} \tag{2.10)(a}$$

$$H_{\sharp,\alpha}(\text{curl}) := \big\{u \in L^2_{\sharp,\alpha} |\, \text{curl}\, u \in L^2_{\sharp,\alpha}\big\}\,, \tag{2.10)(b}$$

with the norms

$$\|u\|_{0,\sharp,\alpha} := \Big(\frac{1}{|B|} \int_{\alpha C \times B} |u(x,k)|^2 dx\, dk\Big)^{1/2}, \tag{2.10)(c}$$

$$\|u\|_{\text{curl},\sharp,\alpha} := \big(\|u\|^2_{0,\sharp,\alpha} + \|\text{curl}\, u\|^2_{0,\sharp,\alpha}\big)^{1/2}. \tag{2.10)(d}$$

Also, for αL-periodic functions $\alpha \in L^\infty\left(\mathbb{R}^3;\mathbb{R}\right)$ we define the space

$$H_{\sharp,\alpha}(a, \operatorname{div} 0) = \left\{u \in L^2_{\sharp,\alpha} \mid \operatorname{div}(au) = 0\right\} \tag{2.11)(a}$$

which we equip with the scalar product

$$(u, v)_{\alpha C\times B, a} := \int\limits_{\alpha c \times B} a(x) u(x,k)\cdot \bar{v}(x,k)\, dx\, dk\,. \tag{2.11)(b}$$

where "$^-$" denotes complex conjugation.

The following proposition is straightforward:

PROPOSITION 2.1. *The space* $L^2_{\sharp,\alpha}$, $H_{\sharp,\alpha}(\mathrm{curl})$, $H_{\sharp,\alpha}(a, \operatorname{div} 0)$ *are isometric to* $L^2(\mathbb{R}^3)$, $H(\mathbb{R}^3, \mathrm{curl})$ *and, respectively,* $H(\mathbb{R}^3, a, \operatorname{div} 0)$. *The isometry* I^α *is given by:*

$$U = U(x) \overset{I^\alpha}{\longmapsto} u(x,k) = \sum_{\gamma\in L} U(x-\alpha\gamma) e^{ik\cdot\gamma}\,, \tag{2.12)(a}$$

$$u = u(x,k) \overset{(I^\alpha)^{-1}}{\longmapsto} U(x) = \frac{1}{|B|}\int\limits_B u(x,k)\, dk\,. \tag{2.12)(b}$$

The proof follows directly from Parseval's identity.

Thus, the problem of finding $(E^\alpha, H^\alpha) \in C(\mathbb{R}_t; \left(L^2(\mathbb{R}^3))^3\right)^2$, which solves (2.8) under the assumption (2.9), is equivalent to solving:

$$i\alpha\frac{\partial}{\partial t}\begin{pmatrix} e^\alpha \\ h^\alpha \end{pmatrix} = l^\alpha \begin{pmatrix} e^\alpha \\ h^\alpha \end{pmatrix}, \quad t \in \mathbb{R} \tag{2.13)(a}$$

$$e^\alpha(t=0) = I^\alpha E^\alpha_I =: e^\alpha_I, \quad h^\alpha(t=0) = I^\alpha H^\alpha_I =: h^\alpha_I \tag{2.13)(b}$$

in $C\left(\mathbb{R}_t; (L^2_{\sharp,\alpha})^3\right)^2$, where we set

$$l^\alpha = I^\alpha \circ L^\alpha \circ (I^\alpha)^{-1}, \quad D(l^\alpha) = (H_{\sharp,\alpha}(\mathrm{curl}))^2\,. \tag{2.14}$$

The assumption (2.9) then reads

$$e^\alpha_I \in H_{\sharp,\alpha}(\varepsilon^\alpha, \operatorname{div} 0), \quad h^\alpha_I \in H_{\sharp,\alpha}(\mu^\alpha, \operatorname{div} 0) \tag{2.15}$$

and E^α, H^α are recovered from:

$$E^\alpha(t) = (I^\alpha)^{-1} e^\alpha(t), \quad H^\alpha(t) = (I^\alpha)^{-1} h^\alpha(t)\,. \tag{2.16}$$

Obviously, we have

$$l^{\alpha}\begin{pmatrix} e \\ h \end{pmatrix} = i\alpha \begin{pmatrix} \varepsilon^{\alpha}(x)^{-1}\operatorname{curl} h \\ -\mu^{\alpha}(x)^{-1}\operatorname{curl} e \end{pmatrix} \tag{2.17}$$

for $e, h \in H_{\sharp,\alpha}(\text{curl})$.

For fixed $k \in \overline{B}$ we introduce a space of k-quasi-periodic functions of $\mathbb{R}^3$:

$$\begin{aligned} L_{\sharp,\alpha}(k) := \{u \in \left(L^2_{\text{loc}}(\mathbb{R}^3)\right)^3 | \\ \forall \gamma \in L : u(x+\alpha\gamma) = e^{ik\cdot\gamma}u(x) \text{ a.e. in } \mathbb{R}^3\}. \end{aligned} \tag{2.18}$$

We denote by $l^{\alpha}(k)$ the operator (2.17) with domain

$$\begin{aligned} &D(l^{\alpha}(k)) \\ &= H_{\text{loc}}\left(\mathbb{R}^3, \text{curl}\right)^2 \cap \left(H_{\text{loc}}(\mathbb{R}^3, \varepsilon^{\alpha}, \text{div } 0) \times H_{\text{loc}}(\mathbb{R}^3, \mu^{\alpha}, \text{div } 0)\right) \cap L_{\sharp,\alpha}(k)^2 . \end{aligned}$$

Note that $l^{\alpha}(k)$ is a densely defined unbounded operator on the Hilbert space

$$H_{\sharp,\alpha}(k) = \left(H_{\text{loc}}(\mathbb{R}^3, \varepsilon^{\alpha}, \text{div } 0) \times H_{\text{loc}}(\mathbb{R}^3, \mu^{\alpha}, \text{div } 0)\right) \cap L_{\sharp,\alpha}(k)^2 \tag{2.19(a)}$$

which we equip with the scalar product

$$\left((e_1, h_1), (e_2, h_2)\right)_{\alpha C} = \frac{1}{2}\int_{\alpha C} \varepsilon^{\alpha} e_1 \cdot \bar{e}_2 \, dx + \frac{1}{2}\int_{\alpha C} \mu^{\alpha} h_1 \cdot \bar{h}_2 \, dx , \tag{2.19(b)}$$

i.e.

$$l^{\alpha}(k) : D(l^{\alpha}(k)) \subseteq H_{\sharp,\alpha}(k) \to H_{\sharp,\alpha}(k) .$$

An application of Green's-theorem for the curl-operator on the Lipschitz-domain αC using the k-quasi-periodicity shows that $l^{\alpha}(k)$ is self-adjoint on $H_{\sharp,\alpha}(k)$.

We now consider the eigenvalue problem for $l^{\alpha}(k)$. Since $l^1(k)$ and $l^{\alpha}(k)$ have the same eigenvalues, it suffices to analyze

$$l^1(k)\begin{pmatrix} e \\ h \end{pmatrix} = \omega(k)\begin{pmatrix} e \\ h \end{pmatrix} \tag{2.20}$$

for $\omega(k) \in \mathbb{R}$ and $(e, h) \in D(l^1(k))$. The eigenfunctions of $l^{\alpha}(k)$ are obtained from the eigenfunctions of $l^1(k)$ by applying the rescaling $x \to x/\alpha$ (and vice versa).

The eigenvalue problem (2.20) reads:

$$i \operatorname{curl} h = \omega(k)\varepsilon(x)e , \qquad \operatorname{div}(\varepsilon(x)e) = 0 , \tag{2.21(a)}$$

$$-i \operatorname{curl} e = \omega(k)\mu(x)h , \qquad \operatorname{div}(\mu(x)h) = 0 , \tag{2.21(b)}$$

$$\forall \gamma \in L : h(x+\gamma) = e^{ik\cdot\gamma}h(x) , \ e(x+\gamma) = e^{ik\cdot\gamma}e(x) \ \text{ a.e. in } \mathbb{R}^3 . \tag{2.21(c)}$$

At first we prove:

LEMMA 2.1. *For $k \in \overline{B}$ assume that $\omega(k) = 0$ holds. Then $k = 0$.*

PROOF. We obtain $h = \nabla\varphi_1, e = \nabla\varphi_2$ from (2.21)(a),(b). Because of (2.21)(c) we can write

$$\nabla\varphi_j(x) = e^{ik\cdot x}u_j(x), \quad j = 1,2 \tag{2.22}$$

where u_1, u_2 are L-periodic, i.e.

$$u_j(x) = \sum_{\sigma\in L^*} \hat{u}_j(\sigma)e^{ix\cdot\sigma}, \quad \hat{u}_j(\sigma) = \frac{1}{|C|}\int_C u_j(x)e^{-ix\cdot\sigma}dx\,.$$

From (2.22) we conclude

$$0 = \operatorname{curl}(e^{ik\cdot x}u_j(x)) \Rightarrow (k+\sigma)\times\hat{u}_j(\sigma) = 0$$

and $\hat{u}_j(\sigma) = \alpha_j(\sigma,k)(k+\sigma)$ follows for all $\sigma \in L^*$ with scalar functions α_j if $k \neq 0$. We obtain $\varphi_j(x) = -ie^{ik\cdot x}\sum_{\sigma\in L^*}\alpha_j(\sigma,k)e^{ix\cdot\sigma} + C_j \in L_{\sharp,1}(k)$, $j = 1,2$ for $k \neq 0$. Note that no condition on $\hat{u}_j(0)$ is obtained if $k = 0$, which implies $\varphi_j \in L_{\sharp,1}(k) + x\cdot\hat{u}_j(0)$.

Multiplying the equation $\operatorname{div}(\varepsilon\nabla\varphi_1) = \operatorname{div}(\nu\nabla\varphi_2) = 0$ by φ_1 and φ_2, respectively, and integration over C implies $\varphi_1 \equiv \varphi_2 \equiv 0$ for $k \in \overline{B}$, $k \neq 0$. □

Now let $k \in \overline{B}$, $k \neq 0$. Then since $\omega(k) \neq 0$ we can eliminate h using (2.21)(b):

$$h = -i\omega(k)^{-1}\mu(x)^{-1}\operatorname{curl} e \tag{2.23}$$

and the eigenvalue problem to be solved reads

$$\varepsilon(x)^{-1}\operatorname{curl}\left(\mu(x)^{-1}\operatorname{curl} e\right) = \omega(k)^2 e\,, \quad \operatorname{div}(\varepsilon(x)e) = 0 \tag{2.24(a)}$$

subject to the k-quasi-periodicity condition

$$\forall\gamma \in L : e(x+\gamma) = e^{ik\cdot\gamma}e(x) \quad a.e. \quad \text{in} \quad \mathbb{R}^3\,. \tag{2.24(b)}$$

We now consider the unbounded operator

$$A(k) := \varepsilon^{-1}\operatorname{curl}(\mu^{-1}\operatorname{curl}(\cdot)) \tag{2.25}$$

on $H_{\mathrm{loc}}(\mathbb{R}^3, \varepsilon, \operatorname{div} 0) \cap L^{(1)}_{\sharp,1}(k) =: H^{(1)}_{\sharp,1}(k)$ equipped with the scalar-product

$$(e_1, e_2)_{C,\varepsilon} := \int_C \varepsilon(x)e_1\cdot\bar{e}_2dx\,. \tag{2.26}$$

$A(k)$ is defined on its form domain

$$D(A(k)) = \left\{e \in H^{(1)}_{\sharp,1}(k)\middle| \int_C \mu^{-1}|\operatorname{curl} e|^2dx < \infty\right\},$$

i.e. $D(A(k)) = H^{(1)}_{\sharp,1}(k) \cap H_{\mathrm{loc}}(\mathbb{R}^3, \operatorname{curl})$.

LEMMA 2.2. *Let $k \in \overline{B}$. Then $A(k)$ is self-adjoint and bounded below (by 0) on $H^{(1)}_{\sharp,1}(k)$. Its resolvant $(A(k)+\lambda)^{-1} : H^{(1)}_{\sharp}(k) \to H^{(1)}_{\sharp}(k)$ is Hilbert-Schmidt uniformly in $k \in \overline{B}$ for every $\lambda > 0$.*

PROOF. The self-adjointness as boundedness from below (with bound 0) follow immediately from an application of Green's theorem for the curl-operator.

Consider now the resolvant equation $(A(k)+\lambda)e = f$ for $f \in H^{(1)}_{\sharp,1}(k)$ and $\lambda > 0$ written in weak form

$$\int_C \mu^{-1} \operatorname{curl} e \cdot \operatorname{curl} \bar{\varphi}\, dx + \lambda \int_C \varepsilon e \cdot \bar{\varphi}\, dx = \int_C \varepsilon f \cdot \bar{\varphi}\, dx$$

for $\varphi \in D(A(k))$.

The Lax-Milgram lemma immediately implies the existence of a unique solution with

$$\|e\|_{L^2(C)} + \|\operatorname{curl} e\|_{L^2(C)} \leq K \|f\|_{L^2(C)} .$$

Since $H_{\text{loc}}(\mathbb{R}^3, \text{curl}) \cap H^{(1)}_{\sharp,1}(k)$ is compactly embedded in $H^{(1)}_{\sharp,1}(k)$ for all $k \in \overline{B}$ we conclude that $(A(k)+\lambda)^{-1}$ is compact. For $k \in \overline{B}$ denote by

$$0 \leq \delta_1(k) \leq \delta_2(k) \leq \ldots \leq \delta_m(k) \leq \ldots \to \infty \tag{2.27}$$

the sequence of eigenvalues of $A(k)$, here listed according to their finite multiplicities. The min-max principle for eigenvalues [RS] implies $\delta_m(h) \geq (1/\bar{\mu}) \cdot \gamma_m(k)$, where $\gamma_m(k)$ are the eigenvalues of the operator

$$B(k) := \frac{1}{\varepsilon} \operatorname{curl}\operatorname{curl}(\cdot), \quad D(B(k)) = D(A(k)) .$$

Thus, we have to analyze the eigenvalue problem

$$\frac{1}{\varepsilon} \operatorname{curl}\operatorname{curl} u = \gamma(k) u, \quad \operatorname{div}(\varepsilon u) = 0 .$$

We set $z = \sqrt{\varepsilon} u$ and obtain the eigenvalue problem

$$C(k) z = \gamma(k) z ,$$

where $C(k)$ is the operator

$$C(k) z := \frac{1}{\sqrt{\varepsilon}} \operatorname{curl}\operatorname{curl}\left(\frac{z}{\sqrt{\varepsilon}}\right)$$

on $H_{\text{loc}}(\mathbb{R}^3, \sqrt{\varepsilon}, \text{div } 0) \cap L_{\sharp,1}(k)$ equipped with the usual $L^2(C)$-scalar product. A simple calculation using the condition $\operatorname{div}(\sqrt{\varepsilon} z) = 0$ shows that for $\zeta = \zeta(\varepsilon)$ sufficiently large

$$\int_C C(k) z \cdot \bar{z} dx \geq (1/\bar{\varepsilon}) \cdot \int_C |\nabla z|^2 dx - \zeta \int |z|^2 dx$$

holds for all z in the form-domain of $C(k)$, Thus, again, by using the min-max principle we conclude that $\gamma_m(k) \geq (1/\bar{\varepsilon})\beta_m(k) - \zeta$, where $\beta_m(k)$ are the eigenvalues of $-\Delta$ on $L_{\sharp,1}(k)$. Fourier analysis gives (after re-indexing)

$$\beta_\sigma(k) = |k+\sigma|^2, \quad \sigma \in L^*$$

with three-dimensional eigenspaces.

Therefore we have

$$\sum_{m=1}^{\infty} \frac{1}{(\delta_m(k)+\lambda)^2} \leq \text{const} \sum_{\sigma \in L^*} \frac{1}{(|k+\sigma|^2+\lambda)^2}$$

which is uniformly bounded in $k \in \overline{B}$. □

Applying Lemma 4.1 of [G] we conclude that the functions $\delta_m = \delta_m(k)$ have uniformly Lipschitz continuous L^*-periodic from $\overline{B}$ to $\mathbb{R}^3_k$ for every $m \in \mathbb{N}$. Also, the methods of [W] developed for the analysis of the Hamilton operator with a periodic electric potential, can be adapted to the analysis of $A(k)$. They show that for every $m \in \mathbb{N}$ there exists a closed set $F_m \subseteq \overline{B}$ of Lebesgue measure zero such that the L^*-periodic extension of δ_m is analytic $\mathbb{R}^3_k - \bigcup_{\sigma \in L^*}(F_m + \sigma)$. Moreover in $\mathbb{R}^3_x \times \overline{B}$ and such that for all $k \in \overline{B} - \bigcup_{m=1}^{\infty} F_m$ they form a complete orthonormed system in the space $H(C, \varepsilon, \operatorname{div} 0)$ (equipped with the scalar product $(\cdot,\cdot)_{C,\varepsilon}$).

Since the positive and negative square-roots of the eigenvalues $\delta_m(k) \neq 0$ of $A(k)$ are eigenvalues of $l^1(k)$ (and vice versa) we conclude that for $k \neq 0$ the operator $l(k)$ has a sequence of eigenvalues

$$\begin{aligned} \ldots - \omega_m(k) \leq -\omega_{m-1}(k) \leq \ldots \leq -\omega_1(k) < 0 < \omega_1(k) \leq \ldots \\ \leq \omega_m(k) \leq \omega_m(k) \ldots, \end{aligned} \tag{2.28}$$

listed according to their (finite) multiplicities. The regularity properties of $\omega_m(k) = \delta_m(k)^{1/2}$ are as follows:

The L^*-periodic extension of $\omega_m = \omega_m(k)$ are uniformly Lipschitz-continuous on $\mathbb{R}^3$ if $\delta_m(0) \neq 0$ and, respectively, on every closed subset of $\mathbb{R}^3_k - L^*$ if $\delta_m(0) = 0$. It is analytic in $\mathbb{R}^3_k - \bigcup_{\sigma \in L^*}(F_m + \sigma)$ if $\delta_m(0) \neq 0$ and, respectively, in $\mathbb{R}^3_k - L^* - \bigcup_{\sigma \in L^*}(F_m + \sigma)$ if $\delta_m(0) = 0$.
Obviously, $\omega_m(k)$ is in $C^{0,1/2}(\overline{B})$ even if $\delta_m(0) = 0$.

The eigenfunction of $l^1(k)$ corresponding to the eigenvalue $\pm\omega_m(k) \neq 0$ is given by

$$(e_m, \pm h_m) = \Big(e_m(x,k), \pm i\omega_m(k)^{-1}\mu(x)^{-1} \operatorname{curl} e_m(x,k)\Big). \tag{2.29}$$

It is an easy exercise to show that $\{h_m(\cdot,k)\}_{m \in \mathbb{N}}$ is a complete orthonormed set in the space $H(C, \mu, \operatorname{div} 0)$ equipped with the scalar product $(\cdot,\cdot)_{C,\mu}$ if

$k \in \overline{B} - F - \{10\}$ (we set $F := \bigcup_{m-1}^{\infty} F_m$). The measurability of h_m is a direct consequence of the continuity of ω_m and of the measurability of e_m.

These facts imply that $\{(e_m(\cdot, k), \pm h_m(\cdot, k))\}_{m \in \mathbb{N}}$ is a complete orthonormed system in $H(C, \varepsilon, \operatorname{div} 0) \times H(C, \mu \operatorname{div} 0)$ equipped with the scalar product $(\cdot, \cdot)_C$ (see (2.19)(b)) for $k \in \overline{B} - F - \{0\}$.

The null-space of $l^1(0)$ can be easily computed. From Lemma 2.1 we conclude $(e, h) \in \operatorname{Null}(l^1(0))$ if and only if

$$e = \nabla \varphi_1, \quad \varphi_1 = a_1 \cdot x + u_1(x)$$

$$h = \nabla \varphi_2, \quad \varphi_2 = a_2 \cdot x + u_2(x)$$

where $a_1, a_2 \in C^3$ are arbitrary and u_1, u_2 are L-periodic solutions of

$$\operatorname{div}(\varepsilon(x) \nabla u_1) = -\operatorname{div}(\varepsilon(x) a_1)$$

$$\operatorname{div}(\mu(x) \nabla u_2) = -\operatorname{div}(\mu(x) a_2).$$

Thus $\dim \big(\operatorname{Null}(l^1(0))\big) = 6$.

The eigenfunctions $(e_m^\alpha \pm h_m^\alpha)$ of $l^\alpha(k)$ are obtained by the rescaling $x \to x/\alpha$ and by normalization with respect to the scalar product $(\cdot, \cdot)_{\alpha C}$. They are given by

$$e_m^\alpha(x, k) = \frac{1}{\alpha^{3/2}} e_m\left(\frac{x}{\alpha}, k\right), \quad h_m^\alpha(x, k) = \frac{1}{\alpha^{3/2}} h_m\left(\frac{x}{\alpha}, k\right). \tag{2.30}$$

The following decomposition theorem is a direct consequence of the spectral analysis of the operator $l^\alpha(k)$ and of Proposition 2.1 (see, e.g. [RS] for a proof):

THEOREM 2.1. *For* $(E, H) \in H(\mathbb{R}^3, \varepsilon^\alpha, \operatorname{div} 0) \times H(\mathbb{R}^3, \mu^\alpha, \operatorname{div} 0)$ *set*

$$\widetilde{E}^\alpha(k, m) = \int_{\mathbb{R}^3} \varepsilon\left(\frac{x}{\alpha}\right) E(x) \cdot \bar{e}_m^\alpha(x, k) dx, \qquad k \in B, m \in \mathbb{N}, \tag{2.31)(b}$$

$$\widetilde{\widetilde{H}}^\alpha(k, m) = \int_{\mathbb{R}^3} \mu\left(\frac{x}{\alpha}\right) H(x) \cdot \bar{h}_m^\alpha(x, k) dx, \qquad k \in B, m \in \mathbb{N}. \tag{2.31)(b}$$

Then the following statements hold:

(i) *The maps*

$$\widetilde{\ } : H(\mathbb{R}^3, \varepsilon^\alpha, \operatorname{div} 0) \to \bigoplus_{m \in \mathbb{N}} L^2(B)$$

$$\widetilde{\widetilde{\ }} : H(\mathbb{R}^3, \mu^\alpha, \operatorname{div} 0) \to \bigoplus_{m \in \mathbb{N}} L^2(B)$$

are isometries:

$$\int_{\mathbb{R}^3} \varepsilon\Big(\frac{x}{\alpha}\Big) E_1(x) \cdot \overline{E}_2(x) dx = \frac{1}{|B|} \sum_{m\in\mathbb{N}} \int_B \widetilde{E}_1^\alpha(k,m) \overline{\widetilde{E}_2^\alpha}(k.m) dk\,,$$

$$\int_{\mathbb{R}^3} \mu\Big(\frac{x}{\alpha}\Big) H_1(x) \cdot \overline{H}_2(x) dx = \frac{1}{|B|} \sum_{m\in\mathbb{N}} \int_B \widetilde{\widetilde{H}}_1^\alpha(k,m) \overline{\widetilde{\widetilde{H}}_2^\alpha}(k,m) dk$$

with inverses

$$E(x) = \frac{1}{|B|} \sum_{m\in\mathbb{N}} \int_B \widetilde{E}^\alpha(k,m) e_m^\alpha(x,k) dk, \qquad x \in \mathbb{R}^3\,,$$

$$H(x) = \frac{1}{|B|} \sum_{m\in\mathbb{N}} \int_B \widetilde{\widetilde{H}}^\alpha(k,m) h_m^\alpha(x,k) dk, \qquad x \in \mathbb{R}^3\,.$$

(ii) *Let* $(E,H) \in H(\mathbb{R}^3, \varepsilon^\alpha, \operatorname{div} 0) \times H(\mathbb{R}^3, \mu^\alpha, \operatorname{div} 0) \cap H(\mathbb{R}^3, \operatorname{rot})^2$ *and denote* $L^\alpha \begin{pmatrix} E \\ H \end{pmatrix} = \begin{pmatrix} E_0 \\ H_0 \end{pmatrix}$. *Then:*

$$\widetilde{E}_0^\alpha(k,m) = \omega_m(k) \widetilde{\widetilde{H}}^\alpha(k,m), \qquad k \in B, \quad m \in \mathbb{N}\,,$$

$$\widetilde{\widetilde{H}}_0^\alpha(k,m) = \omega_m(k) \widetilde{E}^\alpha(k,m), \qquad k \in B, \quad m \in \mathbb{N}\,.$$

We now define the (Floquet) subspaces of $H((\mathbb{R}^3, \varepsilon^\alpha, \operatorname{div} 0) \times H(\mathbb{R}^3, \varepsilon^\alpha, \operatorname{div} 0)$:

$$\text{(2.32)(a)} \qquad S_m^\alpha := \Big\{ \frac{1}{|B|} \Big(\int_B \sigma(k) e_m^\alpha(x,k) dx, \int_B \sigma(k) h_m^\alpha(x,h) dx \Big) \Big| \sigma \in L^2(B) \Big\}$$

$$\text{(2.32)(b)} \qquad S_{-m}^\alpha := \Big\{ \frac{1}{|B|} \Big(\int_B \sigma(k) e_m^\alpha(x,k) dx, -\int_B \sigma(k) h_m^\alpha(x,h) dk \Big) \Big| \sigma \in L^2(B) \Big\}$$

for $m \in \mathbb{N}$. A simple calculation using Theorem 2.1 show that $S_m^\alpha, S_{-m}^\alpha$ are invariant under the action of L^α, that S_m^α and $S_{m_2}^\alpha$ are orthogonal with respect to the scalar product $(\cdot,\cdot)_{\mathbb{R}^3}$ (given by (2.19)(b) with αC replaced by $\mathbb{R}^3$) for $m_1 \neq m_2, m_1, m_2 \in \mathbb{Z} - \{0\}$, and that

$$\text{(2.32)(c)} \qquad H\Big(\mathbb{R}^3, \varepsilon^\alpha, \operatorname{div} 0\Big) \times H\Big(\mathbb{R}^3, \mu^\alpha, \operatorname{div} 0\Big) = \bigoplus_{m\in\mathbb{N}} \Big(S_m^\alpha \oplus S_{-m}^\alpha \Big)\,.$$

Also the following result is obtained by a straight-forward calculation:

LEMMA 2.3. *Let $m \in \mathbb{N}$ and denote by $\hat{\omega}_m(\gamma)$ the Fourier-coefficient of $\omega_m(k)$: $\omega_m(k) = \sum_{\gamma \in L} \hat{\omega}_m(\gamma) e^{ik\cdot\gamma}$. Then:*

(i) $$L^\alpha \begin{pmatrix} E \\ H \end{pmatrix} = \sum_{\gamma \in L} \hat{\omega}_m(\gamma) \begin{pmatrix} E(x+\alpha\gamma) \\ H(x+\alpha\gamma) \end{pmatrix} \quad \text{for} \quad \begin{pmatrix} E \\ H \end{pmatrix} \in S_m^\alpha,$$

(ii) $$L^\alpha \begin{pmatrix} E \\ H \end{pmatrix} = -\sum_{\gamma \in L} \hat{\omega}_m(\gamma) \begin{pmatrix} E(x+\alpha\gamma) \\ H(x+\alpha\gamma) \end{pmatrix} \quad \text{for} \quad \begin{pmatrix} E \\ H \end{pmatrix} \in S_{-m}^\alpha.$$

More explicit calculation can be carried out in the case of a homogeneous medium:

LEMMA 2.4. *Assume that ε and μ are positive constants, Then the eigenvalues of the problem* (2.21) *are:*

(2.33)(a) $$\omega_\gamma^+(k) = c|\gamma + k|,$$

(2.33)(b) $$\omega_\gamma^-(k) = -c|\gamma + k|$$

where c : $(\varepsilon\mu)^{1/2}$ denotes the light velocity. The multiplicity of each eigenvalue is 2 *a.e. in $\overline{B}$.*

PROOF. For $\omega(k) \neq 0$ the problem (2.21) reduced to

$$h = -(\mu\omega(k))^{-1} i \operatorname{curl} e$$

$$\Delta e = \frac{\omega^2(k)}{c^2} e, \quad \operatorname{div} e = 0$$

$$e(x+\gamma) = e^{ik\cdot\gamma} e(x) \quad \forall \gamma \in L \quad \text{a.e. in} \quad \mathbb{R}^3.$$

Expanding e in the Fourier series

$$e(x) = \sum_{\gamma \in L} \hat{e}(\gamma) e^{i(k+\gamma)\cdot x}$$

leads to

$$\omega(k) = \pm c|\gamma + k|$$

$$e(x) = \hat{e}(\gamma) e^{i(k+\gamma)\cdot x}$$

$$(\gamma + k) \cdot \hat{e}(\gamma) = 0.$$

Moreover $|\gamma + k| = |\gamma' + k|$ with $\gamma \neq \gamma'$ if and only if $2k\cdot(\gamma - \gamma') = |\gamma'|^2 - |\gamma|^2$, which is the equation of a plane in $\mathbb{R}^3$ and therefore a closed set of Lebesgue measure 0. □

This shows that, in general, eigenvalues are not simple. Since the subsequent analysis we need a non-degeneracy hypothesis (which has to permit multiple eigenvalues), we assume:

(A2) For all $m \in \mathbb{N}$ the eigenvalue $\omega_m(k)$ has a constant multiplicity $z(m)$ a.e. in $\overline{B}$.

To simplify the notation we set $\mathbb{Z}^* := \mathbb{Z} - \{0\}$ and $\omega_{-m}(k) := -\omega_m(k)$ for $m \in \mathbb{N}$.

Now let $\{m_1\}_{l\in\mathbb{Z}^*}$ be a sequence of integers in $\mathbb{Z}^*$ with the property that $m_{-l} = -m_l$ and

$$\omega_{m_l}(k) = \omega_{m_l+1}(k) = \ldots = \omega_{m_l+z(m_l)-1}(k) \quad \text{a.e. in } \overline{B} \text{ for all } l \in \mathbb{N}, \tag{2.34)(a}$$

$$\omega_{m_{l+1}}(k) \neq \omega_{m_l+z(m_l)}(k), \quad k \in \overline{B} \quad \text{for all} \quad l \in \mathbb{N}, \tag{2.34)(b}$$

$$\{\omega_{m_l}(k) | l \in \mathbb{Z}^*\} = \{\omega_m(k) | m \in \mathbb{Z}^*\} \quad \text{a.e. in} \quad \overline{B}. \tag{2.34)(c}$$

We define:

$$\zeta_l^\alpha := \bigoplus_{m=m_l}^{m_{l+1}-1} S_m^\alpha, \, l \in \mathbb{Z}^*. \tag{2.35}$$

Since the Fourier-coefficients of the eigenvalues $\omega_m(k)$, $m = m_l, \ldots, m = m_{l+1} - 1$ are the same we obtain the following extension of lemma 2.3:

LEMMA 2.5. *Let* $(E, H) \in \zeta_l^\alpha, l \in \mathbb{Z}^*$. *Then*

$$L^\alpha \begin{pmatrix} E \\ H \end{pmatrix} = \sum_{\gamma \in L} \hat{\omega}_{m_l}(\gamma) \begin{pmatrix} E(x+\alpha\gamma) \\ H(x+\alpha\gamma) \end{pmatrix}.$$

Obviously, we set $\hat{\omega}_{-m}(\gamma) = -\hat{\omega}_m(\gamma)$ for $m \in \mathbb{N}$.

Thus, given an initial datum $(E_I^\alpha, H_I^\alpha) \in (\mathbb{R}^3, \varepsilon^\alpha, \text{div } 0) \times H(\mathbb{R}^3, \mu^\alpha, \text{div } 0)$ we compute its projection $(E_{I,l}^\alpha, H_{I,l}^\alpha) \in \zeta_l^\alpha$ and re-write the Maxwell equation as:

$$i\alpha \frac{\partial}{\partial t} E_l^\alpha(x,t) = \sum_{\gamma \in L} \hat{\omega}_{m_l}(\gamma) E_l^\alpha(x+\alpha\gamma, t) \quad x \in \mathbb{R}^3, \quad t \in \mathbb{R}, \tag{2.36)(a}$$

$$i\alpha \frac{\partial}{\partial t} H_l^\alpha(x,t) = \sum_{\gamma \in L} \hat{\omega}(\gamma) H_l^\alpha(x+\alpha\gamma, t) \quad x \in \mathbb{R}^3, \quad t \in \mathbb{R}, \tag{2.36)(b}$$

$$E_l^\alpha(t=0) = E_{I,l}^\alpha, \quad H_l^\alpha(t=0) = H_{I,l}^\alpha, \quad x \in \mathbb{R}^3 \tag{2.36)(c}$$

for all $l \in \mathbb{Z}^*$. The solution (E^α, H^α) is reconstructed from

$$\begin{pmatrix} E^\alpha \\ H^\alpha \end{pmatrix} = \sum_{l \in \mathbb{Z}^*} \begin{pmatrix} E_l^\alpha \\ H_l^\alpha \end{pmatrix}. \tag{2.36)(d}$$

3. – Wigner–Functions

We now define the l-th band Wigner-function:

$$
\begin{aligned}
w_l^\alpha(x,k,t) = \sum_{\gamma\in l} \Big[\frac{1}{2}\varepsilon\Big(\frac{x}{\alpha}\Big) E_l^\alpha\Big(x-\frac{\alpha}{2}\gamma,t\Big)\cdot \overline{E}_l^\alpha\Big(x+\frac{\alpha}{2}\gamma,t\Big) \\
+\frac{1}{2}\mu\Big(\frac{x}{\alpha}\Big) H_l^\alpha\Big(x-\frac{\alpha}{2}\gamma,t\Big)\cdot \overline{H}_l^\alpha\Big(x+\frac{\alpha}{2}\gamma,t\Big)\Big] e^{ik\cdot\gamma}
\end{aligned}
\tag{3.1}
$$

for $x\in\mathbb{R}^3, k\in B, t\in\mathbb{R}$ and $l\in\mathbb{Z}^*$. For the basic properties of w_l^α we refer to [MMP]. In particular we remark that

$$
\frac{1}{|B|}\int_B w_l^\alpha(x,k,t)dx = \frac{1}{2}\varepsilon\Big(\frac{x}{\alpha}\Big)|E_l^\alpha(x,t)|^2 + \frac{1}{2}\mu\Big(\frac{x}{\alpha}\Big)|H_l^\alpha(x,t)|^2 =: n_l^\alpha(x,t)\,,
$$

which we shall call the l-th band energy density.

A simple calculation using the $L/2$-periodicity of ε and μ gives:

LEMMA 3.1. *The function w_l^α satisfies the initial value problem*

$$
\frac{\partial}{\partial t} w_l^\alpha + i\sum_{\gamma\in L}\hat{\omega}_l(\gamma)e^{ik\cdot\gamma}\frac{w_l^\alpha(x+\alpha\frac{\gamma}{2},k,t)-w_l^\alpha(x-\alpha\frac{\gamma}{2},k,t)}{\alpha} = 0 \tag{3.2)(a}
$$

for $x\in\mathbb{R}^3, k\in B, t\in\mathbb{R}$

$$
w_l^\alpha(t=0) = w_{I,l}^\alpha,\ x\in\mathbb{R}^3, k\in B\,, \ \textit{where } w_{I,l}^\alpha \textit{ is given by:} \tag{3.2)(b}
$$

$$
\begin{aligned}
w_{I,l}^\alpha(x,k) = \sum_{\gamma\in L}\Big[\frac{1}{2}\varepsilon\Big(\frac{x}{\alpha}\Big) E_{I,l}^\alpha\Big(x-\frac{\alpha}{2}\gamma\Big)\cdot \overline{E}_{i,l}^\alpha\Big(x+\frac{\alpha}{2}\gamma\Big) \\
+\frac{1}{2}\mu\Big(\frac{x}{\alpha}\Big) H_{I,l}^\alpha\Big(x-\frac{\alpha}{2}\gamma\Big)\cdot \overline{H}_{I,l}^\alpha\Big(x+\frac{\alpha}{2}\gamma\Big)\Big] e^{ik\cdot\gamma}\,.
\end{aligned}
\tag{3.2)(c}
$$

Also, we set up the Wigner function:

$$
\begin{aligned}
w^\alpha(x,k,t) = \sum_{\gamma\in L}\Big[\frac{1}{2}\varepsilon\Big(\frac{x}{\alpha}\Big) E^\alpha\Big(x-\frac{\alpha}{2}\gamma,t\Big)\cdot E^\alpha\Big(x-\frac{\alpha}{2},t\Big) \\
+\frac{1}{2}\mu\Big(\frac{x}{\alpha}\Big) H^\alpha\Big(x-\frac{\alpha}{2}\gamma,t\Big)\cdot H^\alpha\Big(x+\frac{\alpha}{2}\gamma,t\Big)\Big] e^{ik\cdot\gamma}\,.
\end{aligned}
\tag{3.3}
$$

Note that the energy density satisfies

$$
\begin{aligned}
n^\alpha(x,t) &= \frac{1}{2}\varepsilon\Big(\frac{x}{\alpha}\Big)|E^\alpha(x,t)|^2 + \frac{1}{2}\mu\Big(\frac{x}{\alpha}\Big)|H^\alpha(x,t)|^2 \\
&= \frac{1}{|B|}\int_B w^\alpha(x,k,t)\,dx\,.
\end{aligned}
\tag{3.4}
$$

Later on we shall relate the limits of w^α, n^α to the limits of w_I^α, n_I^α.

The initial Wigner function is defined in the obvious way:

$$w_I^\alpha(x,k) = \sum_{\gamma\in L}\left[\frac{1}{2}\varepsilon\left(\frac{x}{\alpha}\right)E_I^\alpha\left(x-\frac{\alpha}{2}\gamma\right)\cdot E_I^\alpha\left(x+\frac{\alpha}{2}\gamma\right)\right. \tag{3.5}$$
$$\left. + \frac{1}{2}\mu\left(\frac{x}{\alpha}\right)H_I^\alpha\left(x-\frac{\alpha}{2}\gamma\right)\cdot H_I^\alpha\left(x+\frac{\alpha}{2}\gamma\right)\right]e^{ik\cdot\gamma}.$$

We denote the weighted norm on $L^2(\mathbb{R}^3)^2$ by

$$|||(E,H)|||_\alpha := \left(\frac{1}{2}\int_{\mathbb{R}^3}\left[\varepsilon\left(\frac{x}{\alpha}\right)|E|^2+\mu\left(\frac{x}{\alpha}\right)|H|^2\right]dx\right)^{1/2} \tag{3.6}$$

and impose the following conditions on the initial data

(A3)(i) $$|||(E_I^\alpha, H_I^\alpha)|||_\alpha^2 + |||(L^\alpha)^4(E_I^\alpha, H_I^\alpha)|||_\alpha^2 \le K,$$

where K is independent of $\alpha \in (0, \alpha_0]$,

(ii) $$\lim_{R\to\infty}\sup_{\alpha\in(0,\alpha_0]}\int_{|x|>R}\left[\varepsilon\left(\frac{x}{\alpha}\right)|E_I^\alpha|^2+\mu\left(\frac{x}{\alpha}\right)|H_I^\alpha|^2\right]dx = 0,$$

(iii) $$(E_I^\alpha, H_I^\alpha) \in H\left(\mathbb{R}^3, \varepsilon\left(\frac{\cdot}{\alpha}\right), \operatorname{div} 0\right)\times H\left(\mathbb{R}^3, \mu\left(\frac{\cdot}{\alpha}\right), \operatorname{div} 0\right).$$

The first term on the left-hand side of (A3) (i) is the initial energy $\int_{\mathbb{R}^3} n_I^\alpha(x)\,dx$, where n_I^α stands for the initial energy density

$$n_I^\alpha(x) = \frac{1}{2}\varepsilon\left(\frac{x}{\alpha}\right)|E_I^\alpha(x)|^2 + \frac{1}{2}\mu\left(\frac{x}{\alpha}\right)|H_I^\alpha(x)|^2 = \frac{1}{|B|}\int_B w_I^\alpha(x,k)\,dx. \tag{3.7}$$

Obviously, the energy and $|||(L^\alpha)^4(E^\alpha(t), H^\alpha(t))|||_\alpha^2$ are conserved by the motion generated by the Maxwell equations (2.8):

(3.8)(a)
$$\int_{\mathbb{R}^3} n^\alpha(x,t)\,dx = \int_{\mathbb{R}^3} n_I^\alpha(x)\,dx \le K \quad \forall t\in\mathbb{R}$$

(3.8)(b)
$$|||(L^\alpha)^4(E^\alpha(t), H^\alpha(t))|||_\alpha^2 = |||(L^\alpha)^4(E_I^\alpha, H_I^\alpha)|||_\alpha^2 \le K \quad \forall t\in\mathbb{R}.$$

Now take a function $0 \le \varphi \in C^\infty(\mathbb{R}^3)$ with $\varphi(x) = 0$ for $|x| \le 1/2$ and $\varphi(x) = 1$ for $|x| > 1$. Multiplication of (1.1) by $E^\alpha(t,x)\varphi(x/R)$, of (1.2) by $H^\alpha(t,x)\varphi(x/R)$, summation and integration by parts gives

$$\frac{d}{dt}\int_{\mathbb{R}^3} n^\alpha(x,t)\varphi\Big(\frac{x}{R}\Big)dx \le \frac{c}{R}\int_{\mathbb{R}^3} n^\alpha(x,t)\,dx = \frac{c}{R}\int_{\mathbb{R}^3} n_I^\alpha(x)\,dx\,.$$

Thus, (A3)(i), (ii) imply

$$\lim_{R\to\infty}\ \sup_{\alpha\in(0,\alpha_0]}\int_{|x|>R} n^\alpha(x,t)\,dx = 0 \quad \forall t\in\mathbb{R}\,. \tag{3.9}$$

We conclude that, if the assumption (A3) is imposed on the initial data E_I^α, H_I^α, then it holds true for the solution $(E^\alpha(t), H^\alpha(t))$ for all $t \in \mathbb{R}$.

We point out that the assumption (A3) is equivalent to the condition of α-oscillating and compact at infinity data of [G].

A simple calculation using Theorem 2.1 gives

$$\int_{\mathbb{R}^3} n_l^\alpha(x,t)\,dx = \sum_{m=m_l}^{m_{l+1}-1}\frac{1}{|B|}\int_B\Big[|\widetilde{E}^\alpha(k,|m|,t)|^2 + |\widetilde{\widetilde{H}}^\alpha(k,|m|,t)|^2\Big]dx$$

$$|||(L^\alpha)^4(E^\alpha(t),H^\alpha(t))|||_\alpha^2 =$$

$$= \sum_{m=m_l}^{m_{l+1}-1}\frac{1}{|B|}\int_B \omega_m(k)^8\Big[|\widetilde{E}^\alpha(k,|m|,t)|^2 + |\widetilde{\widetilde{H}}^\alpha(k,|m|,t)|^2\Big]dx,$$

where $\widetilde{E}^\alpha$, $\widetilde{\widetilde{H}}^\alpha$ are defined according to (2.31).
Thus, we have

$$\int_{\mathbb{R}^3} n_l^\alpha(x,t)\,dx \le \frac{1}{\inf\limits_{k\in B}\delta_{|l|}(k)^4}|||(L^\alpha)^4(E_I^\alpha,H_I^\alpha)|||^2 \le \frac{K}{\inf\limits_{k\in B}\delta_{|l|}(k)^4} \tag{3.10}$$

(from now on we denote by K not necessarily equal constant independent of $\alpha \in (0,\alpha_0)$).

The assumption (A3) is sufficient to guarantee the existence of a subsequence of $\alpha \to 0$ (which, by abuse of notation, we denote by the same symbol) and of non-negative measures $w_{I,l}$, w_I, $w_l(t)$, $w(t)$ such that for all $l \in \mathbb{Z}^*$

(3.11)(a) $w_{I,l}^\alpha \xrightarrow{\alpha\to 0} w_{I,l}$ and $w_I^\alpha \xrightarrow{\alpha\to 0} w_I$ in $\mathcal{B}^* w-*$,

(3.11)(b) $w_l^\alpha \xrightarrow{\alpha\to 0} w_l$ and $w^\alpha \xrightarrow{\alpha\to 0} w$ in $L^\infty(\mathbb{R}_t;\mathcal{B}^* w-*) w-*$,

(3.11)(c) $n_l^\alpha \xrightarrow{\alpha\to 0} n_l = \dfrac{1}{|B|}\displaystyle\int_B w_l(x,dk,t)$ in $L^\infty(\mathbb{R}_t; C_0(\mathbb{R}^3)^* w-*) w-*$,

(3.11)(d) $n^\alpha \longrightarrow n = \dfrac{1}{|B|}\displaystyle\int_B w(x,dk,t)$ in $L^\infty(\mathbb{R}_t, C_0(\mathbb{R}^3)^* w-*) w-*$.

Here we denote the separable Banach space

$$\text{(3.12)(a)} \qquad \mathcal{B} := \left\{ \varphi(x,t) = \sum_{\gamma \in L} \hat{\varphi}(x,\gamma) e^{i\gamma \cdot k} \middle| \hat{\varphi} \in l^1(L; C_0(\mathbb{R}_x^3)) \right\},$$

$$\text{(3.12)(b)} \qquad \|\varphi\|_{\mathcal{B}} := |B| \sum_{\gamma \in L} \|\hat{\varphi}(\cdot,\gamma)\|_{L^\infty(\mathbb{R}_x^3)} .$$

The arguments used to show (3.11) are given in [MMP]. In particular, we remark that the non-negativity of w_l and w is a consequence of the Husimi-regularization [MM, LP, MMP].

Multiplying (1.1) by $E^\alpha \sigma$, (1.2) by $H^\alpha \sigma$, where $\sigma \in \mathcal{S}(\mathbb{R}_x^3)$, adding the equations and integrating by parts gives

$$\left| \int_{\mathbb{R}^3} n^\alpha \sigma \, dx \right| \leq C \quad \forall t \in \mathbb{R} .$$

We recall that every bounded set of $\mathcal{S}'(\mathbb{R}_x^3)$ is precompact, therefore we obtain (up to a subsequence) the uniform convergence in t:

$$n^\alpha \to n \quad \text{in} \quad C_b(\mathbb{R}_t; \mathcal{S}'(\mathbb{R}_x^3)) .$$

Similarly we obtain by using (3.2):

$$w_l^\alpha \to w_l \quad \text{in} \quad C_b(\mathbb{R}_t; \mathcal{S}'_{\text{per}}(\mathbb{R}_x^3 \times B)) ,$$

$$n_l^\alpha \to n_l \quad \text{in} \quad C_b(\mathbb{R}_t; \mathcal{S}'(\mathbb{R}_x^3)) .$$

It was also shown in [MMP] that the limiting Wigner-measures w_l satisfy the transport equations

$$\text{(3.13)(a)} \qquad \frac{\partial}{\partial t} w_l + \nabla_k \omega_{m_l}(k) \cdot \nabla_x w_l = 0$$

$$\text{in} \quad \mathcal{D}'_{\text{per}}\left(\mathbb{R}_x^3 \times \left(\overline{B} - \left(F_{|m_l|} \cup \{0\} \times \mathbb{R}_t\right)\right)\right)$$

$$\text{(3.13)(b)} \qquad w_l(t=0) = w_{I,l}$$

(the subscript "per" refers to L^*-periodicity in k).

We remark that the point $k = 0$ and the closed set $F_{|m_l|}$ of measure 0 are exempt because the necessary $C^{0,1}_{\text{loc}}$-regularity of ω_{m_l} for passing to the limit $\alpha \to 0$ in (3.2)(a) cannot be guaranteed there. This problem will be remedied later on.

We prove:

LEMMA 3.2. *Let* (A1), (A2), (A3) *hold. Then*

$$w = \sum_{l\in\mathbb{Z}^*} w_l \quad a.e.\ in \quad \mathbb{R}_t . \tag{3.14}$$

PROOF. Let $a = a(t)$ be in $C_0^\infty(\mathbb{R}_t)$, $\Psi^\alpha = \Psi^\alpha(x,t)$ and $\Phi^\alpha = \Phi^\alpha(x,t)$ be uniformly bounded sequence in $L^\infty(\mathbb{R}_t; (L^2(\mathbb{R}^3))^3)$. Following an idea of Gérard [G] we introduce the (x,t)-Wigner transform

$$\begin{aligned}\widetilde{W}^\alpha[\Psi^\alpha, \Phi^\alpha](x,k,t,\tau) := \int_{\mathbb{R}_s}\sum_{\gamma\in L} \Big[\frac{1}{2}\varepsilon\Big(\frac{x}{\alpha}\Big)\Psi^\alpha\Big(x-\frac{\alpha}{2},t-\frac{\alpha}{2}s\Big)\\ \cdot\overline{\Psi}^\alpha\Big(x+\frac{\alpha}{2}\gamma,t+\frac{\alpha}{2}s\Big)a\Big(t-\frac{\varepsilon}{2}s\Big)\bar{a}\Big(t+\frac{\varepsilon}{2}s\Big)\\ +\frac{1}{2}\mu\Big(\frac{x}{\alpha}\Big)\Phi^\alpha\Big(x-\frac{\alpha}{2}\gamma,t-\frac{\alpha}{2}s\Big)\overline{\Phi}^\alpha\Big(x+\frac{\alpha}{2}\gamma,t+\frac{\alpha}{2}s\Big)\\ \cdot a\Big(t-\frac{\varepsilon}{2}s\Big)\bar{a}\Big(t+\frac{\varepsilon}{2}s\Big)\Big]e^{i(k\cdot\gamma+\tau s)}ds.\end{aligned} \tag{3.15}$$

We set also

$$\begin{aligned}\tilde{w}^\alpha[\Psi^\alpha, \Phi^\alpha](x,k,t) = \sum_{\gamma\in L}\Big[\frac{1}{2}\varepsilon\Big(\frac{x}{\alpha}\Big)\Psi^\alpha\Big(x-\frac{\alpha}{2}\gamma,t\Big)\cdot\bar{\Psi}^\alpha\Big(x+\frac{\alpha}{2}\gamma,t\Big)|a(t)|^2\\ +\frac{1}{2}\mu\Big(\frac{x}{\alpha}\Big)\Phi^\alpha\Big(x-\frac{\alpha}{2}\gamma,t\Big)\cdot\bar{\Phi}^\alpha\Big(x+\frac{\alpha}{2}\gamma\cdot t\Big)\Big]|a(t)|^2.\end{aligned} \tag{3.16}$$

such that

$$\int_{\mathbb{R}_\tau}\widetilde{W}^\alpha[\Psi^\alpha, \Phi^\alpha]d\tau = \tilde{w}^\alpha[\Psi^\alpha, \Phi^\alpha] \tag{3.17}$$

holds. We have (see [MMP], [LP]), maybe after extraction of a subsequence:

$$\tilde{w}^\alpha[\Psi^\alpha, \Phi^\alpha] \overset{\alpha\to 0}{\longrightarrow} \tilde{w}_0 \quad \text{in} \quad L^2(\mathbb{R}_t; \mathcal{B}^*) - w^* \tag{3.18)(a}$$

$$\widetilde{W}^\alpha[\Psi^\alpha, \Phi^\alpha] \overset{\alpha\to 0}{\longrightarrow} \widetilde{W}_0 \quad \text{in} \quad L^2(\mathbb{R}_t; \mathcal{C}^*) - w^* , \tag{3.18)(b}$$

where the separable Banach-space $\mathcal{C}$ is defined by

$$\mathcal{C} = \Big\{\xi(x,k,t,\tau) = \sum_{\gamma\in L}\hat{\xi}(x,\gamma,t,\tau)e^{i\gamma\cdot k}\Big|\hat{\xi}\in l^1\big(L; L^1(\mathbb{R}_\tau; C_0(\mathbb{R}_x^3\times\mathbb{R}_t))\big)\Big\}$$

$$\|\xi\|_{\mathcal{C}} = |B|\sum_{\gamma\in L}\int_{\mathbb{R}_\tau}\|\hat{\xi}(\cdot,\gamma,\cdot,\tau)\|_{L^\infty(\mathbb{R}_x^3\times\mathbb{R}_t)}d\tau .$$

The corresponding Husimi-functions read

(3.19)(a)
$$\widetilde{W}_H^\alpha[\Psi^\alpha, \Phi^\alpha] = \widetilde{W}^\alpha[\psi^\alpha, \Phi^\alpha] *_x G^\alpha *_k F^\alpha *_t G_1^\alpha *_\tau G_1^\alpha ,$$

(3.19)(b)
$$\tilde{w}_H^\alpha[\Psi^\alpha, \Phi^\alpha] = \tilde{w}^\alpha[\Psi^\alpha, \Phi^\alpha] *_x G^\alpha *_k F^\alpha$$

with

$$G^\alpha(x) = \frac{1}{(2\pi\alpha)^{3/2}} \exp\Big(-\frac{|x|^2}{2\alpha}\Big), \quad F^\alpha(k) = \sum_{\gamma\in L} \exp\Big(-\frac{\alpha}{8}|\gamma|^2 + ik.\mu\Big),$$

$$G_1^\alpha(s) = \frac{1}{(2\pi\alpha)^{1/2}} \exp\Big(-\frac{|s|^2}{2\alpha}\Big).$$

The x, t and τ convolutions are defined in the usual way and

$$(\sigma_1 *_k \sigma_2)(k) = \frac{1}{|B|} \int_B \sigma_1(k_1)\sigma(k-k_1)dk_1$$

for L^*-periodic functions σ_1, σ_2.

The Husimi-functions are non-negative

(3.20)
$$\widetilde{W}_H^\alpha \geq 0, \quad \tilde{w}_H^\alpha \geq 0$$

and converge (after selection of a subsequence) to $\widetilde{W}_0$ and $\tilde{w}_0$ respectively [MMP], [LP].

Now let E_l^α, H_l^α satisfy (2.36)(a)-(c). Then a somewhat tedious calculation gives

$$i\tau\widetilde{W}_1^\alpha[E_l^\alpha] = -\frac{\alpha}{2}\int_{\mathbb{R}_s} \Big[\bar{a}'\Big(t+\frac{\alpha}{2}s\Big)a\Big(t-\frac{\alpha}{2}s\Big) - \bar{a}\Big(t+\frac{\alpha}{2}s\Big)a'\Big(t-\frac{\alpha}{2}s\Big)\Big]$$
$$\cdot \sum_{\gamma\in L}\frac{1}{2}\varepsilon\Big(\frac{x}{\alpha}\Big)E_l^\alpha\Big(x-\frac{\alpha}{2}\gamma, t-\frac{\alpha}{2}s\Big)\cdot \overline{E_l^\alpha}\Big(x+\frac{\alpha}{2}\gamma, t+\frac{\alpha}{2}s\Big)e^{i(k\cdot\gamma+\tau s)}ds$$
$$- i\sum_{\gamma\in L}\hat{\omega}_l(\gamma)e^{ik\cdot\gamma}\widetilde{W}_1^\alpha[E_l^\alpha]\Big(x+\frac{\alpha}{2}\gamma, k, t, \tau\Big),$$

$$i\tau\widetilde{W}_2^\alpha[H_l^\alpha] = -\frac{\alpha}{2}\int_{\mathbb{R}_s} \Big[\bar{a}'\Big(t+\frac{\alpha}{2}s\Big)a\Big(t-\frac{\alpha}{2}s\Big) - \bar{a}\Big(t+\frac{\alpha}{2}s\Big)a'\Big(t-\frac{\alpha}{2}s\Big)\Big]$$
$$\cdot \sum_{\gamma\in L}\frac{1}{2}\mu\Big(\frac{x}{\alpha}\Big)H_l^\alpha\Big(x-\frac{\alpha}{2}\gamma, t-\frac{\alpha}{2}s\Big)\cdot \overline{H_l^\alpha}\Big(x+\frac{\alpha}{2}\gamma, t+\frac{\alpha}{2}s\Big)e^{i(k\cdot\gamma+\tau s)}ds$$
$$- i\sum_{\gamma\in L}\hat{\omega}_l(\gamma)e^{ik\cdot\gamma}\widetilde{W}_2^\alpha[H_l^\alpha]\Big(x+\frac{\alpha}{2}\gamma, k, t, \tau\Big)$$

where $\widetilde{W_1^\alpha}[E_l^\alpha]$, $\widetilde{W_2^\alpha}[H_l^\alpha]$ stand for the first and, respectively, second term on the right-hand side of (3.15) when Ψ^α is replaced by E_l^α and Φ^α by H_l^α. We can immediately pass to the limit $\alpha \to 0$ in the above equations obtain

$$(\tau + \operatorname{sgn}(l)\omega_l(k))\widetilde{W}_{1,l} = 0, \qquad (\tau + \operatorname{sgn}(l)\omega_l(k))\widetilde{W}_{2,l} = 0\,,$$

where we denote by $\widetilde{W}_{1,l}$, $\widetilde{W}_{2,l}$ the weak limits of subsequences of $\widetilde{W_1^\alpha}[E_l^\alpha]$ and, respectively, $\widetilde{W_2^\alpha}[H_l^\alpha]$. Since $\widetilde{W^\alpha}[E_l^\alpha, H_l^\alpha] = \widetilde{W_1^\alpha}[E_l^\alpha] + [\widetilde{W_2^\alpha}[h_l^\alpha]$ we have

$$(\tau + \operatorname{sgn}(l)\omega_l(k))\widetilde{W}_l = 0\,,$$

where we write $\widetilde{W}_l$ for the weak limit (of a subsequence) of $\widetilde{W^\alpha}[E_l^\alpha, H_l^\alpha]$.

It is easy to check that the assumption (A2) is sufficient to pass to the limit in (3.17):

$$\int_{\mathbb{R}_\tau} \widetilde{W}_l d\tau = |a(t)|^2 w_l$$

and conclude that

$$\widetilde{W}_l(x, k, t, \tau) = |a(t)|^2 w_l(x, k, t)\delta(\tau + \operatorname{sgn}(l)\omega_l(k))\,. \tag{3.21}$$

Thus, the assumption (A3) implies that the measures $\widetilde{W}_l$, $\widetilde{W}_j$ are mutually singular if $l \neq j \in \mathbb{Z}^*$.

now let E_l^α, H_l^α be given by (2.32)(d). Then, due to the quadratic nature of the Husimi transform $\widetilde{W_H^\alpha}[E^\alpha, H^\alpha]$ we obtain(see [LP])

$$\widetilde{W_H^\alpha}[E^\alpha, H^\alpha] = \sum_{l\in\mathbb{Z}^*} \widetilde{W_H^\alpha}[E_l^\alpha, H_l^\alpha] + \sum_{\substack{l_1,l_2\in\mathbb{Z}^* \\ l_1\neq l_2}} R_{l_1,l_2}^\alpha + \sum_{\substack{l_1,l_2\in\mathbb{Z}^* \\ l_1\neq l_2}} S_{l_1,l_2}^\alpha \tag{3.22}$$

where

$$R_{l_1,l_2}^\alpha| \leq \widetilde{W_{1,H}^\alpha}[E_{l_1}^\alpha]^{1/2}\widetilde{W_{1,H}^\alpha}[E_{l_2}^\alpha]^{1/2} \tag{3.23)(a}$$

$$S_{l_1,l_2}^\alpha| \leq \widetilde{W_{2,H}^\alpha}[H_{l_1}^\alpha]^{1/2}\widetilde{W_{2,H}^\alpha}[H_{l_2}^\alpha]^{1/2}\,. \tag{3.23)(b}$$

Here $\widetilde{W_{1,H}^\alpha}[E_l^\alpha]$, $\widetilde{W_{2,H}^\alpha}[H_l^\alpha]$ are the Husimi transforms of $\widetilde{W_1^\alpha}[E_l^\alpha]$ and $\widetilde{W_2^\alpha}[H_l]$. Taking $\varphi \in C_0^\infty(\mathbb{R}_x^3 \times \overline{B} \times \mathbb{R}_t \times \mathbb{R}_\tau)$ and L^*-periodic in k gives

$$\begin{aligned}\int \widetilde{W_H^\alpha}[E^\alpha, H^\alpha]\varphi\, dx\, dk\, dt\, d\tau &= \sum_{l\in\mathbb{Z}^*} \int \widetilde{W_H^\alpha}[E_l^\alpha, H_l^\alpha]\varphi\, dx\, dk\, dt\, d\tau \\ &+ \sum_{\substack{l_1,l_2\in\mathbb{Z}^* \\ l_1\neq l_2}} \int R_{l_1,l_2}^\alpha \varphi\, dx\, dk\, dt\, d\tau \\ &+ \sum_{\substack{l_1,l_2\in\mathbb{Z}^* \\ l_1\neq l_2}} \int S_{l_1,l_2}^\alpha \varphi\, dx\, dk\, dt\, d\tau\,.\end{aligned} \tag{3.24}$$

The methods [LP] show that

$$\int R^{\alpha}_{l_1,l_2}\varphi\, dx\, dk\, dt\, d\tau \xrightarrow{\alpha\to 0} 0, \qquad l_1 \neq l_2$$

$$\int S^{\alpha}_{l_1,l_2}\varphi\, dx\, dk\, dt\, d\tau \xrightarrow{\alpha\to 0} 0, \qquad l_1 \neq l_2$$

since $\widetilde{W_{l_1}}$, $\widetilde{W_{l_2}}$ are mutually singular. Also we have

$$\int \widetilde{W^{\alpha}_{H}}[E^{\alpha}_{l}, H^{\alpha}_{l}]\varphi\, dx\, dk\, dt\, d\tau \xrightarrow{\alpha\to 0} \int \varphi \widetilde{W_l}(dx, dk, dt, d\tau)$$

$$\int W^{\alpha}_{H}[E^{\alpha}, H^{\alpha}]\varphi\, dx\, dk\, dt\, d\tau \longrightarrow \int \varphi \widetilde{W}(dx, dk, dt, d\tau)$$

where $\widetilde{W}$ is the weak limit of the (x,t)-Wigner transforms $\widetilde{W}[E^{\alpha}, H^{\alpha}]$. We estimate using (3.23), (3.17) and (3.10)

$$\left|\int R^{\alpha}_{l_1,l_2}\varphi\, dx\, dk\, dt\, d\tau\right|$$

$$\leq \|\varphi\|_{L^\infty}\sqrt{\int \widetilde{W^{\alpha}_{1,H}}[E^{\alpha}_{l_1}]dx\, dk\, dt\, d\tau \int \widetilde{W^{\alpha}_{1,H}}[E^{\alpha}_{l_2}]dx\, dk\, dt\, d\tau}$$

$$\leq \|\varphi\|_{L^\infty}\sqrt{\int |a(t)|^2 n^{\alpha}_{l_1}(x,t)dt\, dx \int |a(t)|^2 n^{\alpha}_{l_2}(x,t)dt\, dx}$$

$$\leq \begin{cases} K/\inf\limits_{k\in B}\delta_{l_1}(k)^2 \inf\limits_{k\in B}\delta_{l_1}(k)^2 & \text{if } \delta_{l_1}(0)=0, \quad \delta_{l_2}(0)\neq 0, \\ K & \text{if } \delta_{l_1}(0)=\delta_{l_2}(0)=0, \\ K/\inf\limits_{k\in B}\delta_{l_1}(k)^2 & \text{if } \delta_{l_1}(0)\neq 0, \quad \delta_{l_2}(0)\neq 0. \end{cases}$$

The same bounds hold for the other terms on the right-hand side of (3.24). From the proof of Lemma 2.2 we obtain

$$\sum_{l_1,l_2\notin I}\frac{1}{\inf\limits_{k\in B}\delta_{|l_1|}(k)^2\inf\limits_{k\in B}\delta_{|l_2|}(k)^2} + \sum_{\substack{l_1\in I\\ l_2\notin I}}\frac{1}{\inf\limits_{k\in B}\delta_{|l_2|}(k)^2} < \infty,$$

where I denotes the finite set in $\mathbb{Z}^*$ such that $\delta_l(0)=0$ for $l\in I$.

The dominated convergence theorem applied to (3.24) and (3.17) gives

$$\widetilde{W} = \sum_{l\in\mathbb{Z}^*}\widetilde{W_l} \Rightarrow |a(t)|^2 w = |a(t)|^2 \sum_{l\in\mathbb{Z}^*} w_l\,. \qquad \square$$

Note that (3.14) implies for the energy density

$$n = \sum_{l \in \mathbb{Z}^*} n_l \quad \forall t \in \mathbb{R}_t \,. \tag{3.25}$$

To characterize the limiting energy density n completely we have to determine the Wigner measures w_l in the sets $F_{|l|} \cap \{0\}$. Therefore we make another assumption on the initial data:
(A4) $\mathbb{R}^3_x \times (F \cup \{0\})$ is a null-set of the limiting initial Wigner measure w_I.

We prove:

LEMMA 3.3. *Let* (A1)-(A4) *hold and* $l \in \mathbb{Z}^*$. *Then* w_l *is the* $\mathcal{D}'_{\mathrm{per}}(\mathbb{R}^3_x \times \overline{B} \times \mathbb{R}_t)$-*solution of* (3.13) *given by*

$$w_l(x,k,t) = w_{I,l}(x - \nabla_k \omega_{m_l}(k)t, k) \,.$$

PROOF. The energy-conservation properties hold

$$\int_{\mathbb{R}^3_x} n^\alpha(x,t)dx = \int_{\mathbb{R}^3_x} n^\alpha_I \, dx \qquad \forall t \in \mathbb{R}$$

$$\int_{\mathbb{R}^3_x} n^\alpha_l(x,t)dx = \int_{\mathbb{R}^3_x} n^\alpha_{Il} \, dx \qquad \forall t \in \mathbb{R}, \quad l \in \mathbb{Z}^*$$

where $n^\alpha_{I,l} = n^\alpha_l(t=0)$. Since (after selecting a subsequence)

$$n^\alpha \overset{\alpha \to 0}{\longrightarrow} n \,, \quad n^\alpha_l \overset{\alpha \to 0}{\longrightarrow} n_l \quad \text{ in } \quad L^\infty\big(\mathbb{R}_t C_0(\mathbb{R}^3_x)^w - *\big) w - *$$

$$n^\alpha \overset{\alpha \to 0}{\longrightarrow} n_I \,, \quad n^\alpha_{I,l} \longrightarrow n_{I,l} \quad \text{ in } \quad C_0(\mathbb{R}^3_x)^* w - * \,,$$

we conclude from (3.9) and from (3.25):

$$\int_{\mathbb{R}^3_x} n(dx,t) = \int_{\mathbb{R}^3_x} n_I(dx) \qquad \forall t \in \mathbb{R} \tag{3.26)(a}$$

$$\int_{\mathbb{R}^3_x} n_l(dx,t) = \int_{\mathbb{R}^3_x} n_{Il}(dx) \qquad \forall t \in \mathbb{R}, \quad l \in \mathbb{Z}^* \tag{3.26)(b}$$

Now we denote by $z_l = z_l(x,k,t)$ the solution of

$$\frac{\partial}{\partial t} z_l + \nabla_k \omega_{m_l}(k) \cdot \nabla_x z_l = 0$$

$$z_l(t=0) = w_{I,l}$$

in $\mathcal{D}'_{\mathrm{per}}(\mathbb{R}^3_x \times \overline{B} \times \mathbb{R}_t)$, i.e.

$$z_l(x,k,t) = w_{I,l}(x - \nabla_k \omega_l(k)t, k)\,. \tag{3.27}$$

Let Ω_δ be open in $\overline{B}$, of Lebesgue measure less or equal δ and $F \cup \{0\} \subseteq \Omega_\delta$. Then

$$\int_{\mathbb{R}^3_x}\int_{B-\Omega_\delta} w_l(dx,dk,t) = \int_{\mathbb{R}^3_x}\int_{B-\Omega_\delta} w_I(dx,dk) \quad \forall t \in \mathbb{R}\,. \tag{3.28)(a}$$

Passing to the limit $\delta \to 0$ gives

$$\int_{\mathbb{R}^3_x}\int_{B} z_l(dx,dk,t) = \int_{\mathbb{R}^3_x}\int_{B} w_{I,l}(dx,dk) \tag{3.28)(b}$$

since, by (3.14) and (A4), $\mathbb{R}^3_x \times (F \cup \{0\})$ is a null-set of $w_{I,l}$ and, by (3.27), also of $z_l(t)$.

Also, we have $w_l(t) = z_l(t)$ on $\mathbb{R}^3_x \times (\overline{B} - (F \cup \{0\}))$ and, thus $w_l(t) - z_l(t) \geq 0$ on $\mathbb{R}^3_x \times \overline{B}$. Then, (3.26)(b) and (3.28) give

$$\int_{\mathbb{R}^3_x}\int_{\Omega_\delta} (w_l - z_l)(dx,dk,t) = \int_{\mathbb{R}^3_x}\int_{\Omega_\delta} w_{I,l}(dx,dk) \xrightarrow{\alpha\to 0} 0$$

and the assertion of the lemma follows. □

THEOREM 3.1. *Let the assumption* (A1)-(A4) *hold. Then, maybe after selection of a subsequence, the energy densities n^α satisfy:*

$$n^\alpha \xrightarrow{\alpha\to 0} \sum_{l\in\mathbb{Z}^*} \frac{1}{|B|} \int_B w_l(x,dk,t) \in C_b\big(\mathbb{R}_t, C_0(\mathbb{R}^3_x)^*\big) w - *\,,$$

where the band-Wigner measures $w_l(t) \in \mathcal{M}^+_{\mathrm{per}}(\mathbb{R}^3_x \times \overline{B})$ are given by

$$w_l(x,k,t) = w_{I,l}(x - \nabla_k \omega_{m_l}(k)t, k), \qquad l \in \mathbb{Z}^*\,.$$

Here $w_{I,l}$ is the $\mathcal{B}^* - w^*$ limit (of a subsequence) of the initial band Wigner function $w^\alpha_{I,l}$ given in (3.2)(c).

In the case of homogeneous media, the above theorem and Lemma 2.4 imply:

COROLLARY 3.1. *Let ε and μ be positive constants and $c = 1/\sqrt{\varepsilon\mu}$ the light velocity. Then for any initial data sequence satisfying* (A3), (A4), *the energy densities*

n^α verify

$$n^\alpha \overset{\alpha\to 0}{\longrightarrow} \sum_{\gamma\in L} \frac{1}{|B|} \int_B \left(w_\gamma^+(x,dx,t) + w_\gamma^-(x,dx,t)\right) \in C_b\big(\mathbb{R}_+, C_0(\mathbb{R}_x^3)^*\big) w - *$$

where the band Wigner measures $w_\gamma^+(t), w_\gamma^-(t) \in \mathcal{M}_{\text{per}}^+\big(\mathbb{R}_x^3 \times \overline{B}\big)$ are given by

$$w_\gamma^+(x,k,t) = w_{I,\gamma}^+\Big(x - c\frac{\gamma+k}{|\gamma+k|}t, k\Big),$$

$$w_\gamma^-(x,k,t) = w_{I,\gamma}^-\Big(x + c\frac{\gamma+k}{|\gamma+k|}t, k\Big), \quad \gamma \in L.$$

The measures $w_{I\gamma}^+$ (respectively, $w_{I\gamma}^-$) are the $\mathcal{B}^* - w*$ limit of the initial band Wigner functions $w_{I,\gamma}^{+,\alpha}$ (respectively, $w_{I,\gamma}^{-,\alpha}$) given in (3.2)(c) *with $\begin{pmatrix} E_{I,\gamma}^{+\alpha} \\ H_{I,\gamma}^{+\alpha} \end{pmatrix}$ $\Big($respectively, $\begin{pmatrix} E_{I,\gamma}^{-\alpha} \\ H_{I,\gamma}^{-\alpha} \end{pmatrix}\Big)$ being the projection corresponding to the eigenvalue $c|\gamma+k|$ (respectively, $-c|\gamma+k|$).*

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