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Pseudoconvexity of Rigid Domains and Foliations of Hulls of Graphs

E.M. CHIRKA - N.V. SHCHERBINA

1. - Introduction

It was proved in the paper [Sh1] of one of the authors that the polynomial hull of a continuous graph $\Gamma(\varphi) : v = \varphi(z, u)$ in $\mathbb{C}^2_{z,w}$ over the boundary of a strictly convex domain $G \subset\subset \mathbb{C}_z \times \mathbb{R}_u$ is a graph over $\overline{G}$, which is foliated by a family of complex analytic discs. Moreover, these discs are graphs over correspondent domains in $\mathbb{C}_z$ of holomorphic functions with continuous boundary values, and the boundaries of these discs are contained in $\Gamma(\varphi)$. In this paper, we study the conditions on G (weaker than the strict convexity) which guarantee the same properties of hulls in $\overline{G} \times \mathbb{R}$ for continuous graphs over bG. This question appears to be closely related to the description of the domains $G \subset \mathbb{C} \times \mathbb{R}$ for which the rigid domains $G \times \mathbb{R} \subset \mathbb{C}^2$ are pseudoconvex. The problem of finding a characterization of such domains is interesting itself. That is a reason why we consider it in the general situation, with G a domain in $\mathcal{M} \times \mathbb{R}$, where $\mathcal{M}$ is a Stein manifold.

Denote by π the natural projection $(z, u) \mapsto z$ in $\mathcal{M} \times \mathbb{R}$, and introduce the notion of a *covering model* $\mathcal{G}$ of G over $\mathcal{M}$ as the factor of G by the following equivalence relation: $(z', u') \sim (z'', u'')$, if $z' = z''$ and all the points $(z', tu' + (1 - t)u'')$, $0 \leq t \leq 1$, are contained in G. We introduce in $\mathcal{G}$ the factor-topology induced from G (which is not Hausdorff in general). The projection π' of $\mathcal{G}$ onto $\pi(G)$ induced by π is open and has at most countable fibres. Assuming $\pi' : \mathcal{G} \to \pi(G)$ is a local homeomorphism (this is a condition on G), we can introduce in $\mathcal{G}$ the structure of a complex manifold, namely, the (Riemann) domain over $\mathcal{M}$ with the holomorphic projection π'.

The boundary of G with respect to the projection π has two distinguished parts: b^+G consists of the upper ends of maximal intervals in the u-direction (from $-\infty$ to $+\infty$ in $\mathbb{R}$) contained in G, and b^-G is constituted by lower ends of such intervals. If $\mathcal{G}$ is a domain over $\mathcal{M}$, the sets $b^{\pm}G$ are obviously represen-

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ted as graphs over $\mathcal{G}$ of lower and upper semicontinuous functions, respectively.

The following theorem gives a complete characterization of domains $G \subset \mathcal{M} \times \mathbb{R}$ such that $G \times \mathbb{R}$ are pseudoconvex domains in $\mathcal{M} \times \mathbb{C}$.

THEOREM 1. *Let G be a domain in $\mathcal{M} \times \mathbb{R}$, where $\mathcal{M}$ is a Stein manifold. The rigid domain $G \times \mathbb{R}$ in $\mathcal{M} \times \mathbb{C}$ is pseudoconvex if and only if the following conditions are satisfied:*

(a) *The covering model $\mathcal{G}$ of G is a domain over $\mathcal{M}$, and this domain is pseudoconvex,*

(b) *b^-G and b^+G are the graphs over $\mathcal{G}$ of a plurisubharmonic and a plurisuperharmonic function, respectively.*

The covering model $\mathcal{G}$ is a domain over $\mathcal{M}$, if, for instance, the closure of each maximal interval in G along u-direction is a maximal segment in $\overline{G}$. Moreover, in this case the covering model $\mathcal{G}$ can be geometrically represented as the set of centers of maximal intervals in the u-direction contained in G. For domains G with smooth boundaries the conditions (a)–(b) of Theorem 1 can be written in terms of standard defining functions.

Let $h^{\pm}(\varsigma)$ be, respectively, the upper and lower ends of the interval corresponding to a point $\varsigma \in \mathcal{G}$. It follows from Theorem 1 that the pseudoconvex rigid domain $G \times \mathbb{R}$ is biholomorphically equivalent to a rigid domain

$$\{(\varsigma, w) \in \mathcal{G} \times \mathbb{C} : h^-(\varsigma) < u < h^+(\varsigma)\}$$

where h^- and $-h^+$ are plurisubharmonic in $\mathcal{G}$ (see Sect. 2). This "straightened" model is much simpler for many purposes then the original domain $G \times \mathbb{R}$.

The topological structure of rigid pseudoconvex domains $G \times \mathbb{R}$ described above can be considerably complicated, even for $\mathcal{M} = \mathbb{C}^n$. We show, for instance, that an arbitrary finite 1-dimensional graph embedded in $\mathbb{C} \times \mathbb{R}$ (*e.g.*, an arbitrary knot in $\mathbb{R}^3$) is isotopic to the diffeomorphic retract of a rigid pseudoconvex domain $G \times \mathbb{R} \subset\subset \mathbb{C}^2$.

Note, that for the case (not so rich topologically), when the projection π' of $\mathcal{G}$ onto $\pi(G)$ is one-to-one, the circular version of Theorem 1 was proved by E.Casadio Tarabusi and S. Trapani (see Proposition 3.4 of [CT1]). Note also, that pseudoconvexity of the covering model $\mathcal{G}$ for domains G with pseudoconvex $G \times \mathbb{R}$ was proved in more general situation by C. Kiselman (see Proposition 2.1 of [K]).

As we mentioned above, the pseudoconvexity of $G \times \mathbb{R}$ is essentially related to the structure of hulls of graphs over bG with respect to the algebra $A(G \times \mathbb{R})$ of functions holomorphic in $G \times \mathbb{R}$ and continuous in $\overline{G} \times \mathbb{R}$. The situation with hulls for dim $\mathcal{M} > 1$ has proved to be much more complicated due to the example of Ahern and Rudin [AR], see also [An]. This is the reason why we consider in this paper 2-dimensional graphs only, so the manifold $\mathcal{M}$ considered is a noncompact Riemann surface (or simply the plane $\mathbb{C}$). We show that for any $G \subset\subset \mathbb{C} \times \mathbb{R}$ such that $G \times \mathbb{R}$ is *not* pseudoconvex, there is a smooth function φ on bG such that the hull $\widehat{\Gamma}(\varphi)$ in $\overline{G} \times \mathbb{R}$ of the graph

$\Gamma(\varphi): v = \varphi(z, u)$ over bG contains a Levi-flat hypersurface in $G \times \mathbb{R}$ which is not a graph over G (*i.e.*, is not schlicht). Moreover, there is a smooth φ such that $\widehat{\Gamma}(\varphi)$ contains a nonempty open subset of $G \times \mathbb{R}$. Thus, the condition of pseudoconvexity of $G \times \mathbb{R}$ in $\mathcal{M} \times \mathbb{C}$ (with $\dim \mathcal{M} = 1$) is a necessary assumption for the good structure of hulls of graphs over bG.

We have to assume also some regularity of the domain G. We say that G is a *regular* domain in $\mathcal{M} \times \mathbb{R}$ if the following two conditions are satisfied:

a) The covering model $\mathcal{G}$ of the domain G is a domain over $\mathcal{M}$ and, moreover, this domain is a relatively compact subdomain with locally Jordan boundary in a bigger domain over $\mathcal{M}$,

b) There is $\varepsilon > 0$ such that for each point $z \in \pi(G)$ the minimal distance between two different maximal intervals in $\pi^{-1}(z) \cap G$ is not less than ε.

The following theorem describes the structure of hulls $\widehat{\Gamma}(\varphi)$ for the case, when domains $G \times \mathbb{R}$ are pseudoconvex and functions φ are continuous.

THEOREM 2. *Let G be a regular domain in $\mathcal{M} \times \mathbb{R}$ where $\mathcal{M}$ is a noncompact Riemann surface. Suppose that the functions h^- and $-h^+$ are continuous in $\overline{\mathcal{G}}$, Hölder continuous and subharmonic but nowhere harmonic in $\mathcal{G}$.*
Let φ be a real continuous function on bG and $\Gamma(\varphi)$ is its graph in $bG \times \mathbb{R}$. Then

1) *The hull $\widehat{\Gamma}(\varphi)$ of $\Gamma(\varphi)$ with respect to the algebra $A(G \times \mathbb{R})$ is the graph $\Gamma(\Phi)$ of some continuous function Φ on the closed domain $\overline{G}$,*

2) *The set $\widehat{\Gamma}(\varphi) \setminus \Gamma(\varphi)$ is (locally) foliated by one-dimensional complex submanifolds.*

If, moreover, G is homeomorphic to a 3-ball, then

3) *The set $\widehat{\Gamma}(\varphi) \setminus \Gamma(\varphi)$ is the disjoint union of complex analytic discs S_α,*

4) *For each α, there is a simply connected domain $\Omega_\alpha \subset \mathcal{G}$ and a holomorphic function f_α in Ω_α such that the disc S_α is the graph of f_α over Ω_α.*

If, moreover, $h^- = h^+$ over the boundary of $\mathcal{G}$, then, for each α,

5) *The function f_α extends to a continuous function f^*_α on the closure $\overline{\Omega}_\alpha$ in $\overline{\mathcal{G}}$, and the graph of f^*_α over $b\Omega_\alpha$ is contained in $\Gamma(\varphi)$ and coincides with the boundary $bS_\alpha = \overline{S}_\alpha \setminus S_\alpha$ of S_α,*

6) *The set $\mathcal{G} \setminus \overline{\Omega}_\alpha$ contains no connected component relatively compact in $\mathcal{G}$.*

If, moreover, the functions $h^\pm \circ g$, where g is a conformal mapping of the unit disc $\Delta \subset \mathbb{C}$ onto $\mathcal{G}$, are Hölder continuous in $\overline{\Delta}$, then, for each α,

7) *The set $\overline{\Omega}_\alpha \subset \mathcal{G}$ does not bound any connected component of the set $\mathcal{G} \setminus \overline{\Omega}_\alpha$,*

8) *The set $b\Omega_\alpha \setminus b\overline{\Omega}_\alpha$ can not be a union of a finite or a countable family of connected components.*

This theorem has a natural corollary.

COROLLARY 1.1. *Let G be a bounded domain in $\mathbb{C} \times \mathbb{R}$ such that the domain $G \times \mathbb{R}$ is strictly pseudoconvex. Let φ be a real continuous function on bG and $\Gamma(\varphi)$ is its graph in $bG \times \mathbb{R}$. Then*

1) *The hull $\widehat{\Gamma}(\varphi)$ of $\Gamma(\varphi)$ with respect to the algebra $A(G \times \mathbb{R})$ is the graph $\Gamma(\Phi)$ of some continuous function Φ on the closed domain $\overline{G}$,*
2) *The set $\widehat{\Gamma}(\varphi) \setminus \Gamma(\varphi)$ is (locally) foliated by one-dimensional complex submanifolds.*

Note that the statement of Corollary 1.1 is nontrivial even for the case of smooth functions φ. In fact, if bG has a positive genus, then the surface $\Gamma(\varphi)$ can be without any elliptic points, and so Bishop's method of constructing the complex discs with boundaries on $\Gamma(\varphi)$ cannot be applied. Moreover in this case some complex submanifolds of $\widehat{\Gamma}(\varphi)$ can be even everywhere dense in $\widehat{\Gamma}(\varphi)$ (see Example 5 below).

The paper is organized as follows. The proof of Theorem 1 is contained in Section 2. In Sect.3 we consider some examples motivating the restrictions on the domain G in Theorem 2. The property 1) in Theorem 2 is proved in Section 4. In Sect.5 we collect some properties of a Levi-flat foliation, in particular, we prove that, at the conditions of Theorem 2, the maximal leaves of the foliation are closed in $G \times \mathbb{R}$. The property 2) in Theorem 2 is proved in Section 6. The proof presented here differs from the proof of this property in [Sh1], the main difference being that instead of the paper of Bedford and Klingenberg [BK] we use more transparent paper of Bedford and Gaveau [BG]. The properties 3)–8) in Theorem 2 are proved in Sect.7 by repeating essentially the proofs of correspondent properties in [Sh1, Sh3].

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2. - A characterization of rigid pseudoconvex domains

We prove here Theorem 1 and its natural corollaries.

Sufficiency of the conditions (a)–(b).

Let the covering model $\mathcal{G}$ of a domain G be a domain over $\mathcal{M}$ endowed with the complex structure induced by the locally one-to-one projection $\pi' : \mathcal{G} \to \pi(G)$. Moreover, let this complex manifold $\mathcal{G}$ be Stein.

The factor-mapping $G \to \mathcal{G}$ has the form $G \ni (z,u) \mapsto \varsigma(z,u) \in \mathcal{G}$, and the projection $\pi' : \varsigma(z,u) \mapsto z$ is locally biholomorphic. Thus, we can "straighten" the domain G with respect to the projection π, substituting the Stein manifold $\mathcal{M}$ by another Stein manifold $\mathcal{G}$. It is better to consider this transformation on the rigid domain $G \times \mathbb{R} \subset \mathcal{M} \times \mathbb{C}$, where it becomes a biholomorphic map $F : (z, w) \mapsto (\varsigma(z, u), w)$. The image $F(G \times \mathbb{R})$ is a rigid domain in $\mathcal{G} \times \mathbb{C}$ of the form $G' \times \mathbb{R}$, where G' is a domain in $\mathcal{G} \times \mathbb{R}$. The advantage of this new representation is that now the fibres of the projection $\pi'' : G' \to \mathcal{G}$ with $\pi'' : (\varsigma, u) \mapsto \varsigma$ are connected (intervals), and the domain G' itself is given by global inequalities $h^-(\varsigma) < u < h^+(\varsigma)$, $\varsigma \in \mathcal{G}$, where h^- is an upper semicontinuous, and h^+ is a lower semicontinuous functions on the Stein manifold $\mathcal{G}$. The domain $G' \times \mathbb{R}$ biholomorphic to $G \times \mathbb{R}$ is defined in $\mathcal{G} \times \mathbb{C}$ by the same inequalities

$$h^-(\varsigma) < u < h^+(\varsigma), \quad \varsigma \in \mathcal{G},$$

($u + iv = w$ is the complex variable in $\mathbb{C}$).

As h^- and $-h^+$ are plurisubharmonic functions in $\mathcal{G}$ by the condition (b), the domain $G' \times \mathbb{R}$ is pseudoconvex in $\mathcal{G} \times \mathbb{C}$. As $G \times \mathbb{R}$ is biholomorphically equivalent to $G' \times \mathbb{R}$, it is also pseudoconvex.

Necessity of the conditions (a)–(b).

Assume that the domain $G \times \mathbb{R}$ is pseudoconvex.

Step 1. We show firstly that *$\mathcal{G}$ is a domain over $\mathcal{M}$.*

For an arbitrary given point $(z^0, u^0) \in G$ we have the maximal interval through (z^0, u^0) in G in the u-direction, corresponding to the point $\varsigma^0 = \varsigma(z^0, u^0)$ in $\mathcal{G}$. Let U^0 be a neighbourhood of z^0 in $\pi(G) \subset \mathcal{M}$, which is a ball in local holomorphic coordinates z, and such that $U^0 \times \{u^0\}$ is contained in G. Then we consider two special domains over U^0 : the connected component V^0 of $\pi^{-1}(U^0) \cap G$ containing (z^0, u^0) and the union W^0 of all maximal intervals in G along u-direction intersecting $U^0 \times \{u^0\}$. Let $\Psi : G \to \mathcal{G}$ be the factor-mapping. Then $\pi' : \Psi(W^0) \to U^0$ is a homeomorphism. Since $\Psi(V^0)$ is a neighbourhood of ς^0 in $\mathcal{G}$, it is enough to show that $V^0 = W^0$.

We argue by contradiction and suppose that $V^0 \neq W^0$. Then there is a point $(z^1, u^1) \in V^0$ contained in the boundary of W^0. We can assume $u^1 > u^0$, by changing w onto $-w$, if it is necessary. Let $U^1 \subset U^0$ be a ball containing z^1 and such that $U^1 \times \{u^1\} \subset\subset V^0$. Then there is an interval $I \ni u^1$ in $\mathbb{R}$ such that $U^1 \times I \subset\subset V^0$. As (z^1, u^1) is a boundary point of W^0, it follows that there is a point $(z^2, u^1) \in W^0$ with $z^2 \in U^1$.

Since the domain $G \times \mathbb{R}$ is pseudoconvex and U^0 is a ball in $\mathbb{C}^n$, the domain $V^0 \times \mathbb{R}$ is also pseudoconvex in $U^0 \times \mathbb{C}_w \subset \mathbb{C}^{n+1}$. As $G \times \mathbb{R}$ is rigid and the pseudoconvexity is the local property in boundary points, the image D of $G \times \mathbb{R}$ with respect to the locally biholomorphic mapping $(z, w) \mapsto (z, \eta = e^w)$ is pseudoconvex in all points where the last coordinate does not vanish. The domain D is a Hartogs domain in $\mathbb{C}^n_z \times \mathbb{C}_\eta$ with the Hartogs diagram $\{(z, e^u) : (z, u) \in G\}$.

By the construction, D contains a neighbourhood of a compact set

$$K = (\{z^2\} \times \{e^{u^0} \le |\eta| \le e^{u^1}\}) \cup (\overline{U^1} \times \{|\eta| = e^{u^0}\}) \cup (\overline{U^1} \times \{|\eta| = e^{u^1}\}).$$

It follows by the Kontinuitätssatz that each function holomorphic in a neighbourhood of K extends holomorphically into the domain $U^1 \times \{e^{u^0} < |\eta| < e^{u^1}\}$. But by the construction, there is u', $u^0 < u' < u^1$, such that $(z^1, u') \notin W^0$ (it is because $(z^1, u^1) \notin W^0$), and thus $(z^1, e^{u'}) \notin D$. This contradicts the pseudoconvexity of D and shows that $W^0 = V^0$. Thus, $\mathcal{G}$ in a neighbourhood of ς^0 is parametrized by the ball U^0, which implies that $\mathcal{G}$ is a domain over $\mathcal{M}$.

Step 2. Let us show that the function $h^+ : \varsigma \mapsto$ (upper end of the interval corresponding to ς) is plurisuperharmonic (or $\equiv +\infty$) and the function $h^- : \varsigma \mapsto$ (lower end of the interval corresponding to ς) is plurisubharmonic (or $\equiv -\infty$) in $\mathcal{G}$. The statement is local, so it is enough to prove it on an arbitrary given coordinate chart (U, z) in $\mathcal{G}$, with U being a ball with respect to the holomorphic coordinates z. Let, as above,

$$V = \{(z, u) \in U \times \mathbb{R} : h^-(z) < u < h^+(z)\}$$

and let D be the image of $V \times \mathbb{R}$ under the mapping $(z, w) \mapsto (z, e^w)$. We have shown in Step 1 that $V \times \mathbb{R}$ is biholomorphic to a connected component of $(G \cap \pi^{-1}(U)) \times \mathbb{R}$. Thus,

$$D = \{(z, \eta) \in \mathbb{C}^{n+1} : z \in U, e^{h^-(z)} < |\eta| < e^{h^+(z)}\}$$

is a pseudoconvex Hartogs domain. But then it is well known (see, *e.g.*, [V]) that h^+ is plurisuperharmonic and h^- is plurisubharmonic in U.

Step 3. We show now that $\mathcal{G}$ is pseudoconvex.

For $n = 1$ it is true because in this case $\mathcal{G}$ is a domain over a noncompact Riemann surface $\mathcal{M}$, and thus it is itself Riemann and noncompact. Therefore, we can assume in what follows that $n = \dim_{\mathbb{C}} \mathcal{M} \ge 2$.

If $\mathcal{G}$ is not pseudoconvex, there is (by [DG]) a continuous family of mappings $f_t : \overline{\Delta} \to \mathcal{G}, 0 \le t < 1$, holomorphic in the unit disc Δ and of class C^∞ in $\overline{\Delta}$ such that

(1) $\bigcup_{0 \le t < 1} f_t(b\Delta) \subset K$ for some compact set $K \subset \mathcal{G}$,

(2) the family $f_t | b\Delta$ converges to a mapping $f_1 : b\Delta \to K$ uniformly on $b\Delta$ as $t \to 1$, but

(3) the points $f_t(0) \in \mathcal{G}$ leave an arbitrary compact subset of $\mathcal{G}$ as $t \to 1$ (go to the "boundary" of $\mathcal{G}$).

The function $h^+ \circ f_t - h^- \circ f_t$ is positive and lower semicontinuous on the compact set $b\Delta \times [0, 1]$, hence there is a constant $m > 0$ such that $h^+ \circ f_t > h^- \circ f_t + m$. It follows that there exists a smooth function u_t on

$b\Delta \times [0,1]$ such that $h^- \circ f_t < u_t < h^+ \circ f_t$. Solving the Dirichlet problem in Δ with the boundary data u_t for each t, we obtain a continuous function $\tilde{u}_t$ on $\overline{\Delta} \times [0,1]$, harmonic in Δ and smooth in $\overline{\Delta}$ for each fixed $t \in [0,1]$. As $h^- \circ f_t$ is subharmonic, $h^+ \circ f_t$ is superharmonic in Δ and $h^- \circ f_t < u_t < h^+ \circ f_t$ on $b\Delta$, we have $h^- \circ f_t < \tilde{u}_t < h^+ \circ f_t$ on $\overline{\Delta}$ for each $t \in [0,1]$. Let $\tilde{v}_t$ be a continuous function on $\overline{\Delta} \times [0,1]$ which is harmonically conjugate to $\tilde{u}_t$ for each fixed t (it exists evidently). Then

$$F_t : \Delta \ni \lambda \mapsto (f_t(\lambda), \tilde{u}_t(\lambda) + i\tilde{v}_t(\lambda)) \in G' \times \mathbb{R}, \quad 0 \le t < 1,$$

is a continuous family of analytic discs in the complex manifold $G' \times \mathbb{R}$ biholomorphic to $G \times \mathbb{R}$ and described in the first part of the proof. The boundaries of these discs are contained in a compact set $K' \subset G' \times \mathbb{R}$, $\lim_{t\to 1} F_t|b\Delta$ exists, but $F_t(0)$ has no limit in $G' \times \mathbb{R}$ as $t \to 1$. Thus, assuming that $\mathcal{G}$ is not pseudoconvex, we obtain, via the Kontinuitätssatz, a contradiction to the pseudoconvexity of $G \times \mathbb{R}$.

The proof of Theorem 1 is complete. □

An equivalent formulation of Theorem 1 is the following statement for Hartogs domains. Here the covering model for a Hartogs domain D is its factor with respect to the equivalence relation: $(z', w') \approx (z'', w'')$ with $|w'| \le |w''|$, if $z' = z''$ and the annulus $\{z'\} \times \{|w'| < |w| < |w''|\}$ is contained in D.

COROLLARY 2.1. *Let $D \subset \mathcal{M} \times \mathbb{C}_w$ be a Hartogs domain over a Stein manifold $\mathcal{M}$ and $\mathcal{D}$ is its covering model. The domain D is pseudoconvex if and only if the following conditions are satisfied:*

(a) *$\mathcal{D}$ is a domain over $\mathcal{M}$, and this domain is pseudoconvex,*

(b) *$D \setminus \{w = 0\}$ is biholomorphic to a Hartogs domain*

$$\{(\varsigma, w) : \varsigma \in \mathcal{D}, \psi^-(\varsigma) < |w| < \psi^+(\varsigma)\}$$

where $\pm \log \psi^{\mp}$ are plurisubharmonic functions (or $\equiv -\infty$) on $\mathcal{D}$.

PROOF. Note that the Hartogs domain $D \subset \mathcal{M} \times \mathbb{C}_w$ is pseudoconvex if and only if $D \setminus \{w = 0\}$ is pseudoconvex (see, *e.g.*, [D]), so we can assume that D does not intersect the hypersurface $\{w = 0\}$. Then D has a barrier $1/w$ at all boundary points of the form $(z, 0)$. In a neighbourhood of an arbitrary other boundary point, D is biholomorphic to the rigid domain $\widetilde{D} = \{(z, \omega) : (z, e^\omega) \in D\}$ with the "base" $G = \{(z, u) \in \mathcal{M} \times \mathbb{R} : (z, e^u) \in D\}$. The covering models $\mathcal{G}$ and $\mathcal{D}$ essentially coincide (the mapping $(\varsigma(z,u) \mapsto \eta(z, e^u)$ commutes with projections into $\mathcal{M}$ and thus it is biholomorphic). The rest follows from Theorem 1. □

It is interesting to show how the Bochner tube theorem follows from Theorem 1.

COROLLARY 2.2. *A tube domain $D + i\mathbb{R}^n_y$, where D is a domain in $\mathbb{R}^n_x \subset \mathbb{C}^n_z$, is pseudoconvex if and only if it is convex.*

PROOF. In one direction the statement is trivial, so we assume that $D+i\,\mathbb{R}^n_y$ is pseudoconvex and show that D is convex.

Consider firstly the case $n = 2$ (for $n = 1$ the statement is trivial). Represent $D+i\,\mathbb{R}^2_y$ in the form $G\times\mathbb{R}_{y_2}$, where $G = D\times\mathbb{R}_{y_1}\subset\mathbb{C}_{z_1}\times\mathbb{R}_{x_2}$ is as in Theorem 1. As $G\times\mathbb{R}_{y_2}$ is pseudoconvex, the covering model $\mathcal{G}$ of G is a domain over $\mathbb{C}$. But this model can be obviously represented in the form $\gamma\times\mathbb{R}_{y_1}$ where γ is the covering model of the domain $D\subset\mathbb{R}^2_x$ with respect to the projection $\tilde{\pi}:(x_1,x_2)\mapsto x_1$. It follows evidently that γ must be a graph over the interval $\tilde{\pi}(D)\subset\mathbb{R}_{x_1}$, that is, $D\cap\{x_1=c_1\}$ is connected (an interval) for each $c_1\in\tilde{\pi}(D)$. Using linear transformations of $\mathbb{C}^2$ with real coefficients we obtain that $D\cap L$ is connected for each real line $L\subset\mathbb{R}^2_x$. This means precisely that D is convex.

In a general case, let $a,\ b\in D$ and $l_1\cup\cdots\cup l_N$ be a polygon in D connecting a and b. By induction in N we show that the interval (a,b) is contained in D. Let c be the end of l_2 and $\Lambda\subset\mathbb{R}^n_x$ be a real 2- plane contained $l_1\cup l_2$. After a linear transformation of the coordinates (with real coefficients) we can assume Λ to be the coordinate plane $\mathbb{R}^2\subset\mathbb{R}^n$. By the first part of the proof, the interval $(a,c)=l'_2$ is contained in D. But then we can substitute the polygon $l_1\cup\cdots\cup l_N$ by $l'_2\cup\cdots\cup l_N$ with $N-1$ intervals only. By the induction, $(a,b)\subset D$. □

Theorem 1 admits the following improvement.

COROLLARY 2.3. *Let G be a domain in* $\mathcal{M}\times\mathbb{R}$ *where* $\mathcal{M}$ *is a Stein manifold and*

$$D=\{(z,u+iv):(z,u)\in G,\psi^-(z)<v<\psi^+(z)\}$$

where ψ^- *and* $-\psi^+$ *are plurisubharmonic functions in* $\pi(G)$ *such that* $\psi^+-\varepsilon>\psi>\psi^-+\varepsilon$ *for some constant* $\varepsilon>0$ *and some function* ψ *defined and continuous in* $\overline{\pi(G)}$. *The domain D is pseudoconvex if and only if the following conditions are satisfied:*

(a) *The covering model* $\mathcal{G}$ *of G is a domain over* $\mathcal{M}$, *and this domain is pseudoconvex (the last property is satisfied automatically, if* $\dim_{\mathbb{C}}\mathcal{M}=1$),

(b) b^-G *and* b^+G *are the graphs over* $\mathcal{G}$ *of a plurisubharmonic and a plurisuperharmonic function, respectively.*

PROOF. If the conditions (a)–(b) are satisfied, the domain $G\times\mathbb{R}$ is pseudoconvex by Theorem 1. As the functions $\psi^-(z)-v$ and $v-\psi^+(z)$ are plurisubharmonic in $G\times\mathbb{R}$, the domain D is also pseudoconvex.

Now let D be pseudoconvex. This property is a local property of boundary points. By assumption, D is pseudoconvex at each boundary point $(z,u+i\psi(z))$ with $(z,u)\in bG$. But then the domain $G\times\mathbb{R}$ is pseudoconvex at each boundary point (z^0,u^0+iv^0) because the translation $(z,w)\mapsto(z,u+i(v-v^0+\psi(z^0)))$ sends a neighbourhood of this point biholomorphically onto a neighbourhood of $(z^0,u^0+i\psi(z^0))$ and is itself an automorphism of $G\times\mathbb{R}$. By Theorem 1, it follows that conditions (a)–(b) are satisfied. □

As we mentioned in the introduction, the topology of a pseudoconvex rigid domain $G \times \mathbb{R}$, even for $G \subset \mathbb{C} \times \mathbb{R}$ can be very complicated. We use below the notion of a graph from another area of mathematics. By definition, a finite one-dimensional graph K piecewise smoothly imbedded in a smooth manifold M is a connected compact finite union of smooth Jordan arcs $\gamma_j \subset M$ such that the set $\gamma_i \cap \gamma_j$ for $i \neq j$ is either empty set or a common endpoint of γ_i and γ_j. In the second case the curves γ_i and γ_j have to be transversal at the common endpoint.

PROPOSITION 2.1. *Let K be a finite one-dimensional piecewise smoothly imbedded graph in $\mathcal{M} \times \mathbb{R}$, where $\mathcal{M}$ is a noncompact Riemann surface. Then there is a graph K' isotopic to K in $\mathcal{M} \times \mathbb{R}$ and a domain $G \subset \mathcal{M} \times \mathbb{R}$, such that $G \times \mathbb{R}$ is pseudoconvex and K' is a retract of G.*

PROOF. Let γ_j^0 be the set of inner (not end-) points of γ_j. Then there are neighbourhoods $U_j \supset V_j \supset \gamma_j^0$ such that $U_i \cap U_j = \emptyset$ for $i \neq j$, and $\overline{V_j} \subset U_j \cup \gamma_j$. There is a diffeomorphism g_j of $\overline{U_j}$ onto itself, smooth in $\overline{U_j}$, flat at the endpoints of γ_j, equal to the identity on $\overline{U_j} \setminus V_j$, and such that the projection of $\gamma_j' = g_j(\gamma_j)$ into $\mathcal{M}$ is a smooth immersion. (Such g_j obviously exists because $\dim_{\mathbb{R}} U_j \geq 3$.) The mapping g which is equal to g_j in U_j and the identity in $(\mathcal{M} \times \mathbb{R}) \setminus (\cup V_j)$, is a diffeomorphism of $\mathcal{M} \times \mathbb{R}$. Thus, the graph $K' = \cup \gamma_j'$ is isotopic to K in $\mathcal{M} \times \mathbb{R}$.

Let $\{a_\nu\}$ be the set of end-points of all γ_j and W_ν' be a neighbourhood of a_ν such that the projection of $K \cap \overline{W_\nu'}$ into $\mathcal{M}$ is a "star", that is, a finite union of Jordan arcs $\lambda_{\nu k}$ such that $\lambda_{\nu k} \cap \lambda_{\nu l} = \pi(a_\nu)$ for all $k \neq l$. As the set $\{a_\nu\}$ is finite, we can choose W_ν' with mutually disjoint closures. As $K \cap W_\nu'$ is the graph of a real function over the star $\cup_k \lambda_{\nu k}$, it extends to the graph of a continuous function over a neighbourhood of $\pi(a_\nu)$. This gives a surface $S_\nu \supset K' \cap W_\nu''$ for some neighbourhood $W_\nu'' \subset\subset W_\nu'$ of a_ν. Shrinking W_ν'' we can assume that $S_\nu \cap \overline{W_\nu''}$ is compact.

For j fixed, let a_k, a_l be the endpoints of γ_j'. As $\pi|\gamma_j'$ is an immersion, there is a smooth 2-dimensional surface $S_j' \subset V_j$ such that:

1. S_j' contains γ_j',
2. $S_j' \cap W_k''$ and $S_j' \cap W_l''$ are contained in $S_k \cap W_k''$ and $S_l \cap W_l''$, respectively,
3. $\pi|S_j'$ is an immersion.

Set $S = (\cup_\nu (S_\nu \cap W_\nu'')) \cup (\cup_j S_j')$. By the construction, S is a (Riemann) domain over $\mathcal{M}$ containing K', and there is a fundamental sequence of neighbourhoods of K' on S, each of which can be retracted onto K'.

Let $\delta = \inf\{|u' - u''| : (z, u') \in K', (z, u'') \in K', u' \neq u''\}$. As K' is compact and $\pi|K'$ is locally one-to-one, this number δ is positive. By the approximation theorem on noncompact Riemann surfaces, there is a harmonic function φ in a neighbourhood S' of K' on S such that $|\varphi(\varsigma) - u(\varsigma)| < \delta/6$ on S' (here $u(\varsigma)$ is the u-coordinate of a point $\varsigma \in S \subset \mathcal{M} \times \mathbb{R}_u$). Shrinking S' we can assume that K' is a retract of S'.

The imbedding of S into $\mathcal{M} \times \mathbb{R}$ gives (z, u) as the function of ς, so we can define a domain G in $\mathcal{M}\times\mathbb{R}$ as

$$G=\{(z(\varsigma),u):\varsigma\in S',|u-\varphi(\varsigma)|<\delta/3\}$$

(each $\varsigma\in S'$ defines an interval in $\mathcal{M}\times\mathbb{R}$ along u-direction, and these intervals constituting G are mutually disjoint). As $|\varphi(\varsigma)-u(\varsigma)|<\delta/6$ for $\varsigma\in S'$, the surface S' is contained in G. By the construction (and the definition of δ) S' is a covering model of G and a retract of G. As K' is a retract of S', it is a retract of G as well. As φ is a harmonic function on S', the domain $G\times\mathbb{R}$ is pseudoconvex in $\mathcal{M}\times\mathbb{C}$ by Theorem 1. □

REMARK 1. The topology of the domain G in Proposition 2.1 reflects the *imbedded* topology of the graph K which is substantial already for imbeddings of the circle into $\mathbb{R}^3$, where we have a beautiful and far advanced theory of knots.

REMARK 2. As we mentioned in the introduction, the covering model $\mathcal{G}$ of the domain $G\subset\subset\mathcal{M}\times\mathbb{R}$ can be represented geometrically as the set S of the centers of maximal intervals in u-direction contained in G, if the closure of each maximal interval in G along u-direction is a maximal segment in $\overline{G}$. In this case, the statement of Proposition 2.1 can be inversed. The surface S is obviously a retract of G (along u), and there is (for $n=1$) a one-dimensional graph K on S which is a retract of S and thus a retract of G.

REMARK 3. For $n>1$ the imbedded topology of the pseudoconvex domain $G\times\mathbb{R}\subset\mathcal{M}\times\mathbb{C}$ can be more complicated. Note firstly that G can be always retracted onto a real n-dimensional CW-complex imbedded into $\mathcal{M}\times\mathbb{R}$. If G satisfies the conditions of Remark 2, it can be done by a retraction of G onto the "middle segment" realization S of $\mathcal{G}$, and then by a retraction of S using a strictly plurisubharmonic Morse function on $\mathcal{G}$. As the indexes of all critical points of the Morse function on an n-dimensional complex manifold are not more than n, the resulting CW-complex will be not more than n-dimensional. We do not know, if the n-dimensional version of Proposition 2.1 is true, but we can prove a slightly weaker statement.

PROPOSITION 2.2. *Let $M\subset\mathcal{M}\times\mathbb{R}$ be a smooth compact manifold with $\dim_{\mathbb{R}} M=n=\dim_{\mathbb{C}}\mathcal{M}$ such that the projection $\pi|M$ is a totally real immersion of M into $\mathcal{M}$. Then there is a domain G in $\mathcal{M}\times\mathbb{R}$ which can be retracted onto M and such that the domain $G\times\mathbb{R}$ is pseudoconvex in $\mathcal{M}\times\mathbb{C}$.*

PROOF. As $\pi|M$ is an immersion, there is a neighbourhood $U\supset M$ in $\mathcal{M}\times\mathbb{R}$ and a real hypersurface S closed in U, containing M and such that $\pi|S$ is a local homeomorphism. Thus, S is a domain over $\mathcal{M}$ which can be endowed with the complex structure induced from $\mathcal{M}$ such that $\pi|S$ is a local biholomorphism.

As the immersion $\pi|M$ is totally real (*i.e.*, $\pi_*(T_aM)$ is a totally real subspace in $T_{\pi(a)}\mathcal{M}$ for each $a\in M$), the manifold M is totally real in S. Then, as is well known (see, *e.g.*, [HW], [C1]), there is a nonnegative function ρ defined and strictly plurisubharmonic in a neighbourhood V of M in S such that M coincides with the zero-set of ρ. For each $\delta > 0$, let $S_\delta = \{\varsigma \in S : \rho(\varsigma) < \delta\}$. As M is compact, there is $\delta_0 > 0$ such that S_{δ_0} is relatively compact in V. Moreover, since $\pi|M$ is an immersion and M is compact, we can choose δ_0 so small that

$$c_0 = \frac{1}{3} \inf\{|u' - u''| : (z, u') \in S_{\delta_0}, (z, u'') \in S_{\delta_0}, u' \neq u''\} > 0.$$

Then, for each $\delta < \delta_0$, the manifold S_δ is a strictly pseudoconvex domain over $\mathcal{M}$. The imbedding of S_{δ_0} into $\mathcal{M} \times \mathbb{R}$ defines the "coordinate functions" $z(\varsigma)$, $u(\varsigma)$, $\varsigma \in S_{\delta_0}$, so, for each $C > 0$ and δ, $0 < \delta < \delta_0$, we can define a corresponding domain G in $\mathcal{M} \times \mathbb{R}$ as

$$G = \{(z(\varsigma), u) : \varsigma \in S_\delta, C\rho(\varsigma) - c_0 < u - u(\varsigma) < c_0 - C\rho(\varsigma)\}.$$

Since the function $\rho(\varsigma)$ is strongly plurisubharmonic in S_{δ_0}, it follows that the functions $u(\varsigma) + C\rho(\varsigma)$ and $-(u(\varsigma) - C\rho(\varsigma))$ are also plurisubharmonic for sufficiently large values of the constant C. Then the corresponding domain $G \times \mathbb{R} \subset \mathcal{M} \times \mathbb{R}$ is pseudoconvex by Theorem 1. Moreover, by construction, the hypersurface S_δ in $\mathcal{M} \times \mathbb{R}$ is a retract of G, if δ is small enough. Hence, M is also a retract of G. □

REMARK 4. If $\iota : M \to \mathcal{M}$ is a totally real immersion, then there is evidently an imbedding $\iota' : M \to \mathcal{M} \times \mathbb{R}$ such that $\iota = \pi \circ \iota'$. Thus, the manifolds in Proposition 2.2 cover the class of manifolds admitting a totally real immersion into $\mathcal{M}$. For $\mathcal{M} = \mathbb{C}^n$ the class of compact smooth n-manifolds admitting a totally real immersion into $\mathbb{C}^n$ consists precisely of those manifolds M for which the complexified tangent bundle $TM \otimes \mathbb{C}$ is trivial (see, *e.g.*, [SZ], [C3]). Note however that the essential matter in Proposition 2.2 is the *imbedded* topology of $M \hookrightarrow \mathcal{M} \times \mathbb{R}$ (see Remark 1).

3. - Hulls of graphs: some examples

We study the problem of existence of a Levi-flat hypersurface in $\mathbb{C}^2$ with a prescribed boundary, which is in general a topological 2-manifold. We restrict ourselves to the case of boundaries which are graphs over some 2-manifold in $\mathbb{C} \times \mathbb{R}$. More precisely, we consider a relatively compact domain $G \subset\subset \mathbb{C} \times \mathbb{R}$, the graph $\Gamma(\varphi)$ of a continuous function φ on bG, and look for conditions which guarantee the existence of a Levi-flat hypersurface in $\mathbb{C}^2$ with the boundary $\Gamma(\varphi)$. We take into account the result from [Sh1]: if G is strictly convex, then such a surface exists, coincides with the polynomial hull of $\Gamma(\varphi)$, and is itself the graph of a continuous function over $\overline{G}$.

The following examples show that the situation in general case (even for real-analytic bG and φ) can be essentially more complicated.

EXAMPLE 1. Let G_1 be the domain in $\mathbb{C}_z \times \mathbb{R}_u$ defined by the inequalities

$$-\sqrt{1-|z|^2} < u < -\frac{1}{2}\cos\left(\frac{3}{2}\pi|z|\right), \quad |z| < 1,$$

(it is the unit ball squeezed from above to inside). Set $\varphi(z,u) = 0$ on the semisphere $u = -\sqrt{1-|z|^2} \le 0$ and on $bG_1 \cap \{|z| \ge 2/3\}$, but on the rest, "squeezed part" of the boundary, set $\varphi(z,u) = \left(\frac{1}{2} - u\right)^k$. (Note that the function φ is of class $C^{k-1}(bG_1)$ in the sense of Whitney.) Then there is obviously a Levi-flat hypersurface S in $\mathbb{C}^2$ with the boundary $\Gamma(\varphi)$ which is the union of two graphs, $S_0 : v = 0$ over the convex hull $co(G_1)$ of G_1, and $S_1 : v = \left(\frac{1}{2} - u\right)^k$ over $co(G_1)\setminus\overline{G_1}$, glueing together by the disc $\{|z| < 2/3, w = 1/2\}$. This hypersurface is foliated by analytic discs parallel to z-plane, but it is not C^1-smooth (near $w = 1/2$) and it is not a graph over a domain in $\mathbb{C} \times \mathbb{R}$, being two-sheeted over $co(G_1) \setminus \overline{G_1}$. We can take instead of $\left(\frac{1}{2} - u\right)^k$ an arbitrary function $\psi(u)$ with $\psi(1/2) = 0$. The graph over bG remains continuous, but the singularity at $w = 1/2$ can be very complicated, and the union $S_0 \cup \{S_1 = \Gamma(\psi)\}$ over $co(G_1)\setminus\overline{G_1}$ may not even be an imbedded topological hypersurface in $\mathbb{C}^2$. Approximating G_1 by a domain with a smooth algebraic boundary invariant with respect to the rotations $z \mapsto e^{it}z$, $t \in \mathbb{R}$, and approximating φ by a polynomial, we obtain the same effect with algebraic bG_1 and a polynomial function $\varphi(z,u)$.

In these examples there is *no* Levi-flat hypersurface over G_1 with the prescribed boundary: the surface $S_0 \cap (G_1 \times \mathbb{R})$ does not contain in its boundary whole the graph $\Gamma(\varphi)$. Thus, for the understanding of the nature of the surface S we must go ouside of $G_1 \times \mathbb{R}$, namely, into the hull of holomorphy of $G_1 \times \mathbb{R}$. But even assuming that this hull is schlicht (imbedded in $\mathbb{C}^2$, as in the case of G_1) we can not hope that the Levi-flat hypersurface with the boundary $\Gamma(\varphi)$ coincides with some hull of $\Gamma(\varphi)$, (*e.g.*, with respect to polynomials, to algebra $A(G_1 \times \mathbb{R})$, e.t.c.).

EXAMPLE 2. Let G_1 be as in Example 1, $\varphi = 0$ on $(bG_1 \cap \{|z| \ge 2/3\}) \cup \{u = -\sqrt{1-|z|^2}\}$ and

$$\varphi(z,u) = x\,\frac{u - 1/2}{1+y} \quad \text{on} \quad bG_1 \cap co(G_1).$$

Then $\Gamma(\varphi)$ is a border of a "Levi-flat hypersurface" $S = S_0 \cup S_1$ where $S_0 = co(G_1) \times \{0\}$ and S_1 is given over $co(G_1)\setminus G_1$ by the equation $v = \mathrm{Re}(z(w - 1/2))$. But here $\overline{S_0} \cap \overline{S_1}$ is the union of the disc $\{|z| \le 2/3, w = 1/2\}$ and a piece of the totally real plane $\{x = v = 0\} \cap ((co(G_1)\setminus G_1) \times \mathbb{R})$. By Kneser's theorem (see, *e.g.*,

[V]) the hull of holomorphy of S (hence, the hulls with respect to polynomials or $A(G_1 \times \mathbb{R})$) contains a neighbourhood of $\{x = v = 0\} \cap ((co(G_1) \setminus G_1) \times \mathbb{R})$ in $\mathbb{C}^2$. Thus, the hull of $\Gamma(\varphi)$ is far from being a hypersurface in any sense. The inner points of the hull can also be placed over $G \times \mathbb{R}$, as may be seen in the next variation of this example.

Let G_2 be a domain defined by the inequalities

$$-\sqrt{1-|z|^2} < u < \frac{1}{2}\cos\left(\frac{5}{2}\pi|z|\right), \quad |z| < 1,$$

and the function φ is defined to be zero on $\{u = -\sqrt{1-|z|^2}\} \cup (bG_2 \cap \{|z| \geq 4/5\})$ and

$$\varphi(z,u) = x\,\frac{u-1/2}{1+y} \quad \text{on} \quad bG_2 \cap co(G_2).$$

Then there is a "Levi-flat hypersurface" S with the boundary on $\Gamma(\varphi)$, $S = S_0 \cup S_1$, where $S_0 = G_2 \times \{0\}$ and S_1 is given over some part of $(co(G_2) \setminus G_2) \cup (G_2 \cap \{|z| < 2/5\})$ by the equation $v = \mathrm{Re}(z(w - 1/2))$. But $S_0 \cap S_1 \cap (G \times \mathbb{R})$ contains nonempty piece of a totally real plane $\{x = v = 0\}$ and thus, the holomorphic hull of $\Gamma(\varphi)$ contains inner points of $G \times \mathbb{R}$.

The constructed examples show that there are essential obstructions in the considered Plateau problem, and these obstructions are related with the additional hull of holomorphy of the rigid domain $G \times \mathbb{R}$. Using a Docquer – Grauert criterium of pseudoconvexity [DG], we can construct corresponding "counterexamples" for an arbitrary relatively compact domain $G \subset \mathcal{M} \times \mathbb{R}$ such that $G \times \mathbb{R}$ is not pseudoconvex. Thus, we assume in the rest part of the paper that $G \times \mathbb{R}$ is pseudoconvex, *i.e.*, conditions (a)–(b) of Theorem 1 are fulfiled. A lack of the strict convexity at boundary points can generate some additional difficulties in the construction of a Levi-flat hypersurface with the boundary on a continuous graph over bG.

EXAMPLE 3. Let G_3 be the cutted ball

$$|z|^2 + u^2 < 1, \quad u < 1/2.$$

Then $G_3 \times \mathbb{R}$ is convex (hence pseudoconvex) in $\mathbb{C}^2$, but the boundary of this domain contains the flat part over $u = 1/2$, $|z| < \sqrt{3}/2$, foliated by one-parametric family of analytic discs $\{|z| < \sqrt{3}/2, w = 1/2 + it\}$, $t \in \mathbb{R}$. Let φ be the function on bG_3 vanishing on $u < 1/2$ and equals $\sqrt{3}/2 - |z|$ on $bG_3 \cap \{u = 1/2\}$. Then the graph $\Gamma(\varphi)$ is the boundary of the Levi-flat hypersurface $S = S_0 \cup S_1$ where $S_0 = G_3 \times \{0\}$ and $S_1 = \{(z, u+iv) : u = 1/2, 0 \leq v < \sqrt{3}/2 - |z|\}$, but this hypersurface is not a graph over G_3. We can obviously modify G_3 and φ making them smooth, with the same phenomenon for S. To avoid this new obstruction we must choose the values of φ in some special way: either along a leaf of the foliation of Levi-flat part of $bG \times \mathbb{R}$, or in such a way that the intersection of $\Gamma(\varphi)$ with the Levi-flat part of $bG \times \mathbb{R}$ is totally real, e.t.c.

We will not specify the problem further. Note only that the described phenomenon can occur each time when bG has a piece of the form $u = h(z)$ where h is a harmonic function (in a domain in $\mathbb{C}$). Trying to avoid the details demanding additional technical complications we exclude from our consideration the domains G with such "harmonic" parts on the boundary.

4. - The hull of a graph is a graph

In the studying of the hulls we follow the general scheme of [Sh1], but due to the generality of the domain of definition we have to overcome some additional difficulties. They appear firstly in the proof of the graph-structure of the hull of a graph.

PROPOSITION 4.1. *Let $\mathcal{G}$ be a Riemann surface with nonempty locally Jordan boundary $b\mathcal{G}$ and compact $\mathcal{G} \cup b\mathcal{G}$. Let G be a domain in $\mathcal{G} \times \mathbb{R}$ of the form*

$$\{(z,u) : h^-(z) < u < h^+(z)\}$$

where h^- and $-h^+$ are continuous on $\overline{\mathcal{G}}$, Hölder continuous and subharmonic but nowhere harmonic functions with $h^- < h^+$ in $\mathcal{G}$. Let φ be an arbitrary continuous real function on bG and $\widehat{\Gamma}(\varphi)$ is the hull of its graph $\Gamma(\varphi)$ with respect to the algebra $A(G \times \mathbb{R})$ of functions holomorphic in $G \times \mathbb{R} \subset \mathcal{G} \times \mathbb{C}$ and continuous up to the boundary. Then $\widehat{\Gamma}(\varphi)$ is the graph of some continuous function over $\overline{G}$.

The special cases of the Proposition 4.1 were considered by H. Alexander [Al] and Slodkowski and Tomassini [ST].

The condition on $\mathcal{G}$ means that $\mathcal{G} \cup b\mathcal{G}$ is a compact subset of a bigger Riemann surface in which $\mathcal{G}$ is a subdomain with locally Jordan boundary. We formulate the Proposition for Riemann surfaces $\mathcal{G}$ not simply for generality. They appear naturally as covering models in consideration of domains in $\mathbb{C} \times \mathbb{R}$, and these models do not in general admit an imbedding into $\mathbb{C}$. On the other hand, the proof of the Proposition does not simplify, if we restrict ouselves on domains in $\mathbb{C} \times \mathbb{R}$ only.

PROOF. *Step 1: A construction.*

Let $\mathcal{F}_\varphi^0$ be the set of all lower semicontinuous functions F on $\overline{G}$ such that $F \geq \varphi$ on bG and the domain $(G \times \mathbb{R}) \cap \{v < F(z,u)\}$ is pseudoconvex. Let $\mathcal{F}_\varphi$ be the subset of $\mathcal{F}_\varphi^0$ consisting of functions F such that $F(P) = \lim\inf_{G \ni P' \to P} F(P')$ for each $P \in bG$. As φ is uniformly bounded on bG, this class of functions is nonempty (it contains at least the function $F(P) \equiv M = \max_{bG} \varphi$).

Using the Perron method of the construction of weak solutions (in our case – for nonlinear Levi equation with boundary data φ), we define on $\overline{G}$ the functions

$$\Phi_0(P) = \inf\{F(P) : F \in \mathcal{F}_\varphi\} \quad \text{and} \quad \Phi(P) = \liminf_{P' \to P} \Phi_0(P').$$

We prove eventually that the graph of Φ coincides with $\widehat{\Gamma}(\varphi)$.

Step 2. We show firstly that $\overline{\Gamma(\Phi)}$ is contained in the hull $\widehat{\Gamma}(\tilde{\varphi})$ of the graph $\Gamma(\tilde{\varphi})$ of an arbitrary continuous function $\tilde{\varphi}$ on $\overline{G}$ with $\tilde{\varphi}|bG = \varphi$.

Suppose not. Then there is a point $p_0 \in \Gamma(\Phi) \setminus \widehat{\Gamma}(\tilde{\varphi})$. The graph $\Gamma(\tilde{\varphi})$ divides $G \times \mathbb{R}$ in two disjoint domains $\tilde{D}^\pm$ where $v > \tilde{\varphi}$ and $v < \tilde{\varphi}$, respectively. Assume that $p_0 \in \tilde{D}^+$. By the definition of the hull, there is a function $f \in A(G \times \mathbb{R})$ such that $f(p_0) = 1 > m = \max_{\Gamma(\tilde{\varphi})} |f|$. Then the real hypersurface $\Sigma : |f| = (1+m)/2$ in $G \times \mathbb{R}$ is contained in $\tilde{D}^+$ and have nonempty intersection with $D^- : v < \Phi(z, u)$. Let D_1 be the component of $(G \times \mathbb{R}) \setminus \Sigma$ containing $\tilde{D}^-$. Then D_1 is pseudoconvex (because $f \in A(G \times \mathbb{R})$) and thus, the domain

$$\tilde{D}_1 = \cap_{t \geq 0} \{(z, w + it) : (z, w) \in D_1\} \cap \{v \leq M\}$$

is also pseudoconvex. By the construction, it has the form $\{v < F_1(z, u)\}$ for some $F_1 \in \mathcal{F}_\varphi$. On the other hand, $\tilde{D}_1$ contains some points where $v < \Phi(z, u)$. But this contradicts to the definition of Φ and thus, shows that $\overline{\Gamma(\Phi)} \subset \widehat{\Gamma}(\tilde{\varphi}) \cup \tilde{D}^-$.

Now we prove that $\Gamma(\Phi) \subset \widehat{\Gamma}(\tilde{\varphi})$. We argue by contradiction and suppose that there is $p_0 \in \Gamma(\Phi) \setminus \widehat{\Gamma}(\tilde{\varphi})$. Then $p_0 \in \tilde{D}^-$. By the construction of Φ, there is a function $F \in \mathcal{F}_\varphi$ and a point $p_1 \in (\Gamma(F) \cap \tilde{D}^-) \setminus \widehat{\Gamma}(\tilde{\varphi})$. It means that $f(p_1) = 1 > \max_{\widehat{\Gamma}(\tilde{\varphi})} |f|$ for some $f \in A(G \times \mathbb{R})$. Let S be an irreducible component containing p_1 of the one-dimensional analytic set $(G \times \mathbb{R}) \cap \{f = 1\}$. Then S is contained in $\tilde{D}^-$ and its boundary is placed on the fixed positive distance from $\widehat{\Gamma}(\tilde{\varphi})$ (in v-direction). Hence, the analytic sets $S_t = \{(z, w - it) : (z, w) \in S\}$, $t \geq 0$, have even bigger distances to $\widehat{\Gamma}(\tilde{\varphi})$, and $S_t \subset \{v < F(z, u)\}$, if t is sufficiently large. As $S \ni p_1$ and the domain $\{v < F(z, u)\}$ is pseudoconvex, we obtain the contradiction with the Kontinuitätssatz. Thus, we have proved the inclusion $\Gamma(\Phi) \subset \widehat{\Gamma}(\tilde{\varphi})$.

Step 3: $\Phi = \varphi$ along $bG \cap \{z \in b\mathcal{G}\}$.

Let $(z^0, u^0) \in bG$ and $z^0 \in b\mathcal{G}$. For proving the continuity of Φ at (z^0, u^0) and the equality $\Phi(z^0, u^0) = \varphi(z^0, u^0)$ it is enough to show, according to Step 2, that $\widehat{\Gamma}(\tilde{\varphi}) \cap \{z = z^0\}$ consists of one point $p^0 = (z^0, u^0 + i\varphi(z^0, u^0))$ only. Let $p^1 \neq p^0$ be an arbitrary point in $(bG \times \mathbb{R}) \cap \{z = z^0\}$. As $(\mathcal{G}, b\mathcal{G})$ is a domain with locally Jordan boundary in a bigger Riemann surface, there is a function $f(z) \in A(\mathcal{G}) \hookrightarrow A(G \times \mathbb{R})$ such that $f(z^0) = 1$ and $|f(z)| < 1$ on $\overline{\mathcal{G}} \setminus \{z^0\}$. The set $bG \cap \{z = z^0\}$ is a segment $I : h^-(z^0) < u < h^+(z^0)$ (possibly, a point), and $\Gamma(\varphi) \cap \{z = z^0\}$ is just the graph of φ over I. This arc is polynomially convex in the strip $I \times \mathbb{R}$ parallel to $\mathbb{C}_w$, and this arc does not contain $p^1 = (z^0, w^1)$. Thus, there is a polynomial $g(w)$ such that

$g(w^1) = 1 > m > \max\{|g(w)| : (z^0, w) \in \Gamma(\varphi)\}$. Let U be a neighbourhood of $\Gamma(\varphi) \cap \{z = z^0\}$ on which $|g|$ is still less then m. Then $\Gamma(\tilde{\varphi}) \setminus U$ is compact, and $|f| \le \theta < 1$ on this set. Thus, there is a positive integer N such that $|f^N g| < m$ on $\Gamma(\tilde{\varphi}) \setminus U$. As $|f| \le 1$ on $\Gamma(\tilde{\varphi})$, we have $|f^N g| < m < 1$ everywhere on $\Gamma(\tilde{\varphi})$, hence on $\widehat{\Gamma}(\tilde{\varphi})$. As $f^N g = 1$ at the point p^1, this point is not contained in $\widehat{\Gamma}(\tilde{\varphi})$.

Step 4: $\Phi = \varphi$ along $bG \cap \{u = h^{\pm}(z)\}$.

Let $(z^0, u^0) \in bG$ with $z^0 \in \mathcal{G}$ and $u^0 = h^+(z^0)$. Choose some holomorphic coordinate in a neighbourhood of z^0 in $\mathcal{G}$ and fix a disc $\Delta \subset\subset \mathcal{G}$ in this neighbourhood with the center $z^0 \cong 0$ and the radius $\delta > 0$. Let $h(z)$ be the harmonic function in Δ with boundary values $h^+(\varsigma)$, $\varsigma \in b\Delta$. As h^+ is superharmonic but not harmonic in Δ, we have the strong inequality $h^+(z) > h(z)$ in Δ. As h^+ is Hölder continuous, with an exponent, say, $\alpha \in (0, 1)$, there is a constant C_α depending on α only, such that the function $\tilde{h}$ harmonically conjugate to h in $\overline{\Delta}$ and vanishing at 0 does not exceed in modulus of the number $C_\alpha \min_{c \in \mathbb{R}} \|h^+ - c\|_\alpha$, where $\|\cdot\|_\alpha$ is the standard norm in the Hölder space $C^\alpha(b\Delta)$. We have also $\min_{c\in\mathbb{R}} \|h^+ - c\|_\alpha \le \min_{c\in\mathbb{R}} \|h^+ - c\|_0 + C'\delta^\alpha \le C\delta^\alpha$, where $\|\cdot\|_0$ is the uniform norm on $b\Delta$ and C is a constant depending on α and h^+ (but not on $\delta \le \delta_0$ for some $\delta_0 > 0$). It follows that the real hypersurface $\Sigma = \{z \in \Delta, u = h(z) + u^0 - h(0)\}$ in $\Delta \times \mathbb{C}$ through $p^0 = (0, u^0 + i\varphi(0, u^0))$ is foliated by analytic discs $S_t : w = f_t(z) \equiv h(z) + i\tilde{h}(z) + u^0 - h(0) + it$, $t \in \mathbb{R}$, and each this disc is placed between two real hypersurfaces, $-C\delta^\alpha < v - t < C\delta^\alpha$.

Fix again a continuous function $\tilde{\varphi}$ in $\overline{G}$ with $\tilde{\varphi}|bG = \varphi$ and denote by $\omega(\delta)$ its modulus of continuity. Then $\Gamma(\tilde{\varphi}) \cap \Sigma$ is contained in the strip $-\omega(\delta^\alpha) < v - \varphi(0, u^0) < \omega(\delta^\alpha)$. As $u^0 - h(0) = h^+(0) - h(0) > 0$, the boundary of Σ (containing the boundaries of all S_t) has the form $\gamma \times \mathbb{R}$ where γ is the curve $\{z \in b\Delta, u = h(z) + u^0 - h(0)\}$ which has no common point with $\overline{G}$. Thus, $\widehat{\Gamma}(\tilde{\varphi}) \subset \overline{G} \times \mathbb{R}$ does not intersect $b\Sigma = \cup_t bS_t$. It follows, by the local maximum modulus principle (see [R]) for functions $1/(w - f_t(z))$ holomorphic in $(\Delta \times \mathbb{C}) \setminus \{w = f_t(z)\}$, that $\widehat{\Gamma}(\tilde{\varphi}) \cap \Sigma$ is contained in the strip $|v - \varphi(0, u^0)| \le \omega(\delta^\alpha) + C\delta^\alpha$.

As $\delta \in (0, \delta_0)$ is arbitrary, it means that $\widehat{\Gamma}(\tilde{\varphi}) \cap \{(z^0, u^0) \times \mathbb{R}\} = p^0$. According to Step 2, Φ is continuous at (z^0, u^0) and $\Phi(z^0, u^0) = \varphi(z^0, u^0)$.

Step 5: The domains $D^- : v < \Phi(z, u)$ and $D^+ = (G \times \mathbb{R}) \setminus \overline{D^-}$ are pseudoconvex.

The domain D^- in $G \times \mathbb{R}$ is pseudoconvex as the interior of the intersection of pseudoconvex domains $(G \times \mathbb{R}) \cap \{v < F(z, u)\}$, $F \in \mathcal{F}_\varphi$. (It follows, by the way, that the function Φ itself is contained in the family $\mathcal{F}_\varphi$.)

Concerning the pseudoconvexity of D^+, it is enough to show that each analytic disc $S \subset G \times \mathbb{R}$ with boundary $bS \subset D^+$ also contained in $\overline{D^+}$. Assume the contrary, *i.e.*, $S \cap D^-$ is not empty. The domain $D^- \setminus S$ is pseudoconvex because $bS \subset D^+$. The same is true for domains $D^- \setminus \{(z, w + it) : (z, w) \in S\}$, $t \ge 0$. It follows that the intersection of these domains is pseudoconvex. But this intersection $D^- \setminus \bigcup_{t \ge 0} \{(z, w + it) : (z, w) \in S\}$ has the form $(G \times \mathbb{R}) \cap \{v < F(z, u)\}$

where F is lower semicontinuous in $\overline{G}$ and equals φ on bG, *i.e.*, $F \in \mathcal{F}_\varphi$. As this domain is a proper subset of D^-, we have $F < \Phi$ in some points of G, and this contradicts to the definition of Φ.

Step 6: Φ is continuous in $\overline{G}$ and $\Gamma(\Phi) = \widehat{\Gamma}(\varphi)$.

In the Step 5 we have proved that the common boundary $\Gamma_0(\Phi) = bD^- \cap (G \times \mathbb{R}) \supset \Gamma(\Phi) \cap (G \times \mathbb{R})$ of the domains $D^\pm$ in $G \times \mathbb{R}$ is pseudoconcave. Then, by the local maximum principle for plurisubharmonic functions (see [C1] or [Sl]), it follows that $\overline{\Gamma_0(\Phi)}$ coincides with the $A(G \times \mathbb{R})$-hull of the set $\overline{\Gamma_0(\Phi)} \cap (bG \times \mathbb{R})$. As Φ is continuous on bG (Steps 3 and 4), the last set coincides with $\Gamma(\varphi)$. Thus, we obtain that $\widehat{\Gamma}(\varphi) = \overline{\Gamma_0(\Phi)}$.

Suppose now on the contrary that Φ is not continuous, *i.e.*, $\Gamma(\Phi)$ is not closed.

Then there is a point $(z^0, u^0) \in G$ such that

$$\varepsilon = \max\{|v' - v''| : (z^0, u^0 + iv') \in \Gamma_0(\Phi), (z^0, u^0 + iv'') \in \Gamma_0(\Phi), v' \neq v''\},$$

the width along v-direction, is positive and maximally possible. (The point is inside G because Φ is continuous on bG.) It follows that $\Gamma_0(\Phi)$ is contained in the pseudoconvex domain $D_t = (G \times \mathbb{R}) \cap \{v < \Phi(z, u) + \varepsilon + t\}$ with an arbitrary $t > 0$. The function $-\log d_w(p)$ where $d_w(p)$ is the distance from p to $bD_t \cap \{z = z(p)\}$ (the boundary distance in D_t along w-direction) is plurisubharmonic in D_t. It is uniformly in $t > 0$ bounded on $\overline{\Gamma_0(\Phi)} \cap (bG \times \mathbb{R}) = \Gamma(\varphi)$ because $\varepsilon > 0$. But its maximum on $\overline{\Gamma_0(\Phi)}$ tends to $+\infty$ as $t \to 0$, and this contradicts to the maximum principle for plurisubharmonic functions (see [C1], [Sl]).

The proof of Proposition 4.1 is complete. □

5. - Some properties of Levi-flat foliations

Before the proving of the existence of a Levi-foliation for the hull $\Gamma(\Phi)$, we obtain some *a priori* estimates for maximal leaves of such foliations. We consider in this section only domains in $\mathbb{C} \times \mathbb{R}$ of the form

$$G = \{(z, u) : |z| < 1, h^-(z) < u < h^+(z)\}$$

where $h^\pm$ are continuous functions in $|z| \leq 1$ and $h^- < h^+$ in $\Delta = \{|z| < 1\}$.

LEMMA 5.1. *Let Φ be a real continuous function in $\overline{G}$ and A is a one-dimensional complex analytic set which is contained in the graph $\Gamma(\Phi)$. Then A has no singular points and it is locally represented as a graph over domains in $\mathbb{C}_z$.*

PROOF. Let $a \in A$. As A is contained in the graph $v = \Phi(z, u)$, it contains no disc on the plane $z = z(a)$. This implies that there is a neighbourhood

$U = U_1 \times U_2$ of a such that U_1 is a disc in $\mathbb{C}_z$, $A \cap U \cap \{z = z(a)\} = \{a\}$ and $A \cap U$ is an analytic cover over U_1. Shrinking U_1 we can assume also that $(A \cap U) \setminus \{a\}$ is a locally one-to-one covering over $U_1 \setminus \{z(a)\}$ (see, *e.g.*, [C2]). As $A \subset \Gamma(\Phi)$, the projection A' of $A \cap U$ into $\mathbb{C} \times \mathbb{R} \cong \mathbb{R}^3$ has the same property, $A' \setminus \{a'\}$ is a finite locally one-to-one covering over $U_1 \setminus \{z(a)\}$. It implies that each connected component A'_j of $A' \setminus \{a'\}$ is the graph of a harmonic function u_j in $U_1 \setminus \{z(a)\}$. By the removable singularity theorem, u_j extends to a harmonic function in U_1, and we keep the notation u_j for this extension. As $u_j(z(a)) = u_k(z(a))$, the real harmonic functions u_j, u_k coincide on the union of real analytic arcs passing through $z(a)$. As the projection $A \cap U \to A'$ is one-to-one, the corresponding irredusible components A_j, A_k of $A \cap U$ coincide by the uniqueness theorem for analytic sets (see, [C2]). Hence, $A'_j = A'_k$, $u_j = u_k$, and we obtain that A' is the graph $u = u(z)$ of some harmonic function $u(z)$ in U_1. But then $A \cap U$ is the graph of a holomorphic function $u(z) + iv(z)$, where $v(z)$ is a corresponding harmonically conjugate function to $u(z)$ in U_1. □

LEMMA 5.2. *Let Φ be a real continuous function in $\overline{G}$ such that its graph $\Gamma(\Phi)$ is foliated over G by one-dimensional complex submanifolds. Then each maximal leaf S of this foliation is closed (properly imbedded) in $G \times \mathbb{R}$ and it is represented globally as the graph of some holomorphic function, $S : w = f(z)$, over some domain $\Omega_S \subset \mathbb{C}_z$.*

PROOF. Let $\{h_j^\pm\}$ be two sequences of functions with the following properties: $h_j^\pm$ are defined and real analytic in a neighbourhood of $\overline{\Delta}_j = \{|z| \leq 1 - 1/j\}$, $h_j^+ > h_j^-$, $0 < h^+ - h_j^+ < 1/j$ and $0 < h_j^- - h^- < 1/j$ in $\overline{\Delta}_j$.

By the Lemma 5.1, S is a (Riemann) domain over Δ.

By the Sard's theorem for smooth functions $h_j^\pm | S$ the intersections of S with almost each level set of $h_j^\pm$ in $\Delta_j \times \mathbb{C}$ is transversal. Thus, substituting $h_j^\pm$, if it is necessarily, onto $h_j^\pm \mp t_j$ with sufficiently small constants $t_j > 0$, we can assume that the intersections of hypersurfaces $\{(z, u + iv) \in \overline{\Delta}_j \times \mathbb{C} : u = h_j^\pm(z)\}$ with S are transversal at common points. Set

$$G_j = \{(z, u) \in \Delta_j \times \mathbb{R} : h_j^-(z) < u < h_j^+(z)\}$$

and choose for each j a connected component S_j of the set $S \cap (G_j \times \mathbb{R})$ so that $S_i \subset S_j$ for $i \leq j$. Since $G = \bigcup_j G_j$, it follows that $S = \bigcup_j S_j$. Hence, it is enough to prove the statement of Lemma 5.2 with the domain G_j instead of G and with the leaf S_j instead of S.

By the construction, bS_j is contained in a disjoint and not more then countable union of smooth real analytic arcs γ_k which are defined over a neighbourhood of $\overline{\Delta}_j$. Each of these curves is contained either in $bG_j^+ \times \mathbb{R} : u = h_j^+(z)$ or in $bG_j^- \times \mathbb{R} : u = h_j^-(z)$ or in $b\Delta_j \times \mathbb{C}$. As S is transversal to all these hypersurfaces, the projections γ'_k of the arcs γ_k into $\mathbb{C}_z$ are smooth imbeddings.

The complex manifold S has the standard orientation, and we orient γ_k as the parts of the boundary of S_j. This induces the corresponding orientation on γ'_k. As the projection $\gamma_k \to \gamma'_k$ is one-to-one, each arc γ'_k is closed in Δ_j. Thus, some of the curves γ'_k divides Δ_j in two domains, and we denote by Δ^k the component of $\Delta_j \setminus \gamma'_k$ which induces on γ'_k the orientation described above.

As the projection $\gamma_k \to \gamma'_k$ is one-to-one, the arc γ_k is the graph of a continuous complex function w_k over γ'_k. We constract now the surface Σ by glueing to S_j the domains $\Delta_j \setminus \Delta^k$ along the arcs γ_k, respectively. This surface can be realised as follows. Let $\tilde{w}_k$ be a continuous extension of w_k into $\Delta_j \setminus \Delta^k$ and w'_k is a continuous function in Δ_j, $w'_k|\Delta^k = 0$, $w'_k|\Delta_j \setminus \Delta^k \neq 0$ and $\arg w'_k|\Delta_j \setminus \Delta^k = 1/k$. The surface

$$\{(z, w_1, 0) \in \mathbb{C}^3 : (z, w_1) \in S_j\} \cup \cup_k \{(z, \tilde{w}_k(z), w'_k(z)) : z \in \Delta_j \setminus \Delta^k\}$$

is a representation of Σ in $\mathbb{C}^3$. We have on Σ the natural projection onto Δ_j, and this projection is locally one-to-one covering. As Δ_j is simply connected, this projection is globally one-to-one. As S_j can be considered as a subdomain of Σ, the projection of S_j into Δ_j is also one-to-one, *i.e.*, S_j is the graph $w = f_j(z)$ of a continuous function f_j over a domain $\Omega_{S_j} \subset \Delta_j$. As S_j is a complex manifold, the function f_j is holomorphic in Ω_{S_j}. Since $S_i \subset S_j$ for $i \leq j$, it follows that $\Omega_{S_i} \subset \Omega_{S_j}$ for $i \leq j$. Therefore, the surface $S = \bigcup_j S_j$ is also the graph $w = f(z)$ of a holomorphic function f over the domain $\Omega_S = \bigcup_j \Omega_{S_j} \subset \Delta$. □

We assume further in this section that the continuous functions h^- and $-h^+$ are subharmonic in $\{|z| < 1\}$.

LEMMA 5.3. *Let Φ be a continuous function in $\overline{G}$ such that $\Gamma(\Phi)$ is foliated over G by one-dimensional complex submanifolds and let S be a maximal leaf of this foliation. Then*

1) *S (hence Ω_S) is simply-connected,*
2) *For each point $(z^0, w^0) \in S$ there is a number $r > 0$ depending only on the distance of (z^0, u^0) to bG and $\max_{bG} |\Phi|$ such that Ω_S contains the disc $\{|z - z^0| < r\}$.*

PROOF. For the proof of 1), we repeat the arguments of the proof of Lemma 3.3 in [Sh1]. We argue by contradiction, assuming that S is not simply connected. Then there is a constant $\delta > 0$ and a subdomain $G_0 \subset G$ of the same form $G_0 = \{(z, u) : |z| < 1 - \delta, h_0^-(z) < u < h_0^+(z)\}$ with smooth functions $h_0^\pm$ such that h_0^- and $-h_0^+$ are strictly subharmonic, $h_0^- < h_0^+$ in $\{|z| < 1 - \delta\}$, $bG_0 \times \mathbb{R}$ is transversal to S at all common points, and $S \cap (G_0 \times \mathbb{R})$ is not simply connected. Then the projection of $S \cap (G_0 \times \mathbb{R})$ into $\mathbb{C}_z$ contains a multiconnected component Ω_S^0, *i.e.*, the set $\{|z| < 1 - \delta\} \setminus \Omega_S^0$ contains a compact connected component E with smooth boundary. Let S be the graph (over Ω_S) of a holomorphic function $f = u + iv$ (see Lemma 5.2). Then there is a smooth closed curve $\gamma \subset S \cap (bG_0 \times \mathbb{R})$ which projection coincides with bE. As E is a compact subset of $\{|z| < 1 - \delta\}$, the curve γ is placed completely either on the hypersurface $\{u = h_0^+(z)\}$ or on the hypersurface $\{u = h_0^-(z)\}$. Assume the last for the definiteness (the first case is treated in the same way). Then the function $u - h_0^-(z)$ vanishing on γ is superharmonic and positive on S. It follows, by Hopf's lemma, that $\int_\gamma d^c u > \int_\gamma d^c h_0^-$ (where $d^c = i(\overline{\partial} - \partial)$), and this implies, via the Cauchy – Riemann equation, that $\int_\gamma dv > \int_\gamma d^c h_0^-$. As γ is closed, we have $\int_\gamma dv = 0$. On the other hand,

$$\int_\gamma dv > \int_\gamma d^c h_0^- = \int_{bE} d^c h_0^- = \int_E dd^c h_0^- > 0,$$

as the function h_0^- is strictly subharmonic in $\{|z| < 1 - \delta\}$. This contradiction proves the property 1).

The property 2) is just Lemma 3.5 in [Sh1] whose proof is based on some estimates of harmonic measures for the domain $\Omega_S \subset \{|z| < 1\}$. We need not repeat it here. □

The statement of the Lemma 5.2 is not true if the covering model of G is not simply connected.

EXAMPLE 4. Let G be the domain in $\mathbb{C} \times \mathbb{R}$ defined by the inequalities

$$1 < |z| < 2, \ (|z| - 1)(|z| - 2) < u < (|z| - 1)(2 - |z|).$$

As the function $(|z| - 1)(|z| - 2)$ is subharmonic for $|z| > 3/4$, the rigid domain $G \times \mathbb{R} \subset \mathbb{C}^2$ is pseudoconvex. Set $\Phi(z, u) \equiv \frac{1}{5\pi} \log |z|$ on $\overline{G}$ and $\varphi = \Phi|bG$. Then the hull of $\Gamma(\varphi)$ with respect to $A(G \times \mathbb{R})$ coincides with $\Gamma(\Phi)$. This hypersurface is foliated over G by complex surfaces $S_t = (G \times \mathbb{R}) \cap \{z = e^{-5\pi i(w+t)}\}$, $-\frac{1}{5} < t \leq \frac{1}{5}$, but S_0 is not a graph over a domain in $\mathbb{C}_z$.

If the covering model of G is not simply connected, the maximal leaves of the foliation of $\Gamma(\Phi) \cap (G \times \mathbb{R})$ are even not necessary closed in $G \times \mathbb{R}$.

EXAMPLE 5. Let G be the domain in $\mathbb{C} \times \mathbb{R}$ defined by the inequalities

$$\frac{5}{6} < |z^2 - 1| < \frac{6}{5}, \quad |u| < \left(|z^2 - 1| - \frac{5}{6}\right)\left(\frac{6}{5} - |z^2 - 1|\right).$$

As the function $\left(|z^2 - 1| - \frac{5}{6}\right)\left(|z^2 - 1| - \frac{6}{5}\right)$ is subharmonic for $|z^2 - 1| > \frac{5}{6}$, the rigid domain $G \times \mathbb{R} \subset \mathbb{C}^2$ is pseudoconvex. Let $\Phi(z, u) \equiv \varepsilon(\log |z - 1| + \sqrt{2} \log |z + 1|)$ on $\overline{G}$ and $\varphi = \Phi|bG$. Then $\Gamma(\Phi)$ is the hull of $\Gamma(\varphi)$ with respect to $A(G \times \mathbb{R})$. The Levi-flat hypersurface $\Gamma(\Phi) \cap (G \times \mathbb{R})$ is foliated by one-dimensional complex submanifolds. But, for $\varepsilon > 0$ sufficiently small, the maximal leaf of this foliation through the origin is not closed in $G \times \mathbb{R}$. It takes place due to the possibility of analytic extension of $(z - 1)^\varepsilon (z + 1)^{\varepsilon\sqrt{2}}$ along the cycles of the type $k^+\gamma^+ - k^-\gamma^-$ where $k^\pm$ are suitable positive integers and $\gamma^\pm = \{z : |z^2 - 1| = 1, \pm\mathrm{Re}\, z > 0\}$ are semilemniscates oriented as the boundary of the unbounded component of the complement to their union.

6. - The local foliation of the hull

In the same notations, as in Sect. 4, we prove here that the graph $\Gamma(\Phi) \cap (G \times \mathbb{R})$ is foliated (locally) by one-dimensional complex submanifolds.

Step 1: Localization.

Let G_0 be a ball in $G \subset \mathcal{G} \times \mathbb{R}$ with respect to some holomorphic coordinate z in $\mathcal{G}$ and u in $\mathbb{R}$. Then we can repeat the construction of Sect. 4 for the graph of the function $\Phi|bG_0$. By the Proposition 4.1, the hull of $\Gamma(\Phi|bG_0)$ with respect to the algebra $A(G_0 \times \mathbb{R}) \supset A(G \times \mathbb{R})$ is a continuous graph $\Gamma(\tilde{\Phi})$ over $\overline{G}_0$. As $\Phi|\overline{G}_0 \in \mathcal{F}_{\Phi|bG_0}$, we have $\tilde{\Phi} \leq \Phi$. On the other hand, set $F = \Phi$ on $G \setminus G_0$ and $F = \tilde{\Phi}$ on G_0. Then the domain $(G \times \mathbb{R}) \cap \{v < F(z,u)\}$ is pseudoconvex being pseudoconvex at each boundary point. This means that $F \in \mathcal{F}_\varphi$, hence $F \geq \Phi$ by the definition of Φ. Thus, we obtain the equality $\Phi = \tilde{\Phi}$ on G_0. This is just what we mean by a localization. By this property, we may assume to the end of this section that G is the unit ball B in $\mathbb{C} \times \mathbb{R}$.

Step 2: On the modulus of continuity of Φ.

The following Lipschitz continuity of a Levi-flat solution of Plateau problem was proved firstly by Slodkowski and Tomassini [ST] using methods of (nonlinear) partial differential equations. We present here a simple geometrical proof suggested by Bo Berndtsson.

LEMMA 6.1. *Let φ be a function of class C^2 on the boundary of the unit ball B in $\mathbb{C} \times \mathbb{R}$, and Φ be a continuous function in $\overline{B}$ such that $\Gamma(\Phi) = \widehat{\Gamma}(\varphi)$. Then Φ is Lipschitz continuous in $\overline{B}$: there is a constant C such that $|\Phi(P') - \Phi(P'')| \leq C|P' - P''|$ for all P', P'' in $\overline{B}$.*

PROOF. Let $\tilde{\varphi}$ be an arbitrary C^2-extension of φ into a neighbourhood of $\overline{B}$. Then there is a positive constant A such that the function

$$\Phi^-(z,u) = \tilde{\varphi}(z,u) + A(|z|^2 + u^2 - 1)$$

is plurisubharmonic, and the function

$$\Phi^+(z,u) = \tilde{\varphi}(z,u) - A(|z|^2 + u^2 - 1)$$

is plurisuperharmonic in a neighbourhood of $\overline{B} \times \mathbb{R}$ (we consider them there as independent in v). As the set $\Gamma(\Phi) \cap (B \times \mathbb{R})$ is pseudoconcave (see Sect. 4), the functions $\mp\Phi^{\pm}$ can not take their maximum on $\Gamma(\Phi)$ inside of $B \times \mathbb{R}$ by the local maximum principle (see [C1] or [Sl]). As $\Phi = \Phi^- = \Phi^+$ over bB, it implies that

$$\Phi^- \leq \Phi \leq \Phi^+ \quad \text{everywhere in } \overline{B}.$$

As $\Phi^{\pm}$ are of class C^2, there is a constant C_1 such that $|\Phi^{\pm}(P') - \Phi^{\pm}(P'')| \leq C_1|P' - P''|$ for all P', $P'' \in \overline{B}$.

Fix two arbitrary points P', P'' in $\overline{B}$ with $|P' - P''| \leq \delta$ and assume, for the definiteness, that $\Phi(P'') \geq \Phi(P')$. If $|P''| \geq 1 - \delta$, let P^0 be a nearest point to P'' on bB. Then

$$\begin{aligned}\Phi(P'') - \Phi(P') &\leq \Phi^+(P'') - \Phi^-(P') \\ &= (\Phi^+(P'') - \Phi^+(P^0)) - (\Phi^-(P'') - \Phi^-(P^0)) \leq 3C_1\delta.\end{aligned}$$

Assume now that $|P''| < 1 - \delta$. For each point P with $|P| = 1 - \delta$ denote by $\tilde{P}$ the nearest point to P on bB. Then

$$\begin{aligned}\Phi(P - P'' + P') &\geq \Phi^-(P - P'' + P') \geq \Phi^-(P) - C_1\delta \\ &\geq \Phi(\tilde{P}) - 2C_1\delta \geq \Phi^+(P) - 3C_1\delta.\end{aligned}$$

Hence, if we define the function

$$F(P) = \begin{cases} \Phi^+(P), & \text{if } \{1 - \delta \leq |P| \leq 1\}, \\ \min(\Phi^+(P),\ \Phi(P - P'' + P') + 3C_1\delta), & \text{if } |P| < 1 - \delta, \end{cases}$$

it will be continuous in $\overline{B}$, and the domain $(B \times \mathbb{R}) \cap \{v < F(z,u)\}$ will be pseudoconvex. Thus, $F \in \mathcal{F}_\varphi$ in $\overline{B}$, hence, $\Phi \leq F$ on $\overline{B}$ by the definition of Φ. As $|P''| < 1 - \delta$ by our assumption, it follows from the inequality $\Phi \leq F$ and from the definition of F that

$$\Phi(P'') \leq F(P'') \leq \Phi(P') + 3C_1\delta,$$

i.e., $0 \leq \Phi(P'') - \Phi(P') \leq 3C_1\delta$. Thus, we have proved that Φ satisfies in $\overline{B}$ the Lipschitz condition with the constant $C = 3C_1$. □

Step 3: Local foliation of $\Gamma(\Phi)$ *for smooth* φ.

Let φ be a C^2-smooth function on bB and Φ is as above, with $\Gamma(\Phi) = \widehat{\Gamma}(\varphi)$. Then Φ is Lipschitz continuous in $\overline{B}$ by Lemma 6.1, and we want to show that in this case the graph $\Gamma(\Phi)$ is locally foliated by complex submanifolds. Given

Step 1, it is enough to prove this in a neighbourhood of the origin assuming for simplicity that $\Phi(0,0)=0$.

Choose a sequence $\{\Phi_\nu\}$ of C^∞-smooth functions on $\overline{B}$ uniformly convergent to Φ as $\nu\to\infty$ and uniformly satisfying the same Lipschitz condition

$$|\Phi_\nu(P')-\Phi_\nu(P'')|\le C|P'-P''|,\quad \forall P',P''\in\overline{B},\ \nu=1,2,\dots,$$

with a constant $C\ge 1$. We assume also that $\Phi_\nu\equiv\Phi(0,1)$ in a neighbourhood U_ν^+ of the point $(0,1)$ in $\overline{B}$, $\Phi_\nu\equiv\Phi(0,-1)$ in a neighbourhood U_ν^- of the point $(0,-1)$, and $U_\nu^\pm\supset\overline{B}\cap\{\pm u>\sqrt{1-\delta_\nu^2}\}$ for some sequence $\delta_\nu\downarrow 0$.

Choose a positive number $R<1/(8C)$ and construct for each ν a strictly convex domain $D_\nu\subset B$ with smooth boundary and of the form $D_\nu: H_\nu^-(z)<u<H_\nu^+(z)$, $|z|<2R$, such that

1. $H_\nu^\pm(z)=\pm(1-|z|^2)$ for $|z|<\delta_\nu/2$,
2. $\left|\dfrac{\partial}{\partial r}H_\nu^\pm(re^{it})\right|>C$ for $|z|\ge\delta_\nu$, and for $\delta_\nu/2\le|z|<\delta_\nu$, if $|H_\nu^\pm(z)|<\sqrt{1-\delta_\nu^2}$,
3. D_ν contains the cylinder $\{|z|\le R,|u|\le R\}$.

Such D_ν evidently exist.

Let M_ν be the graph of the function Φ_ν over bD_ν. It is placed on the hypersurface $\Gamma(\Phi_\nu)$. The complex tangent space to $\Gamma(\Phi_\nu)$ at each point has the form $w=Az$ with $|A|\le C$ because of the uniform Lipschitz condition on Φ_ν. By the construction of D_ν, the projections of this planes into $\mathbb{C}\times\mathbb{R}$ are transversal to bD_ν at all points except of two extreme points $(0,\pm1)$. It follows that the manifold M_ν is totally real outside of two points $(0,\pm1+i\Phi(0,\pm1))$, and both these points are elliptic in the sense of Bishop [B].

By the Bedford – Gaveau theorem [BG], there is a Lipschitz function Ψ_ν in $\overline{D}_\nu$, smooth in D_ν, equals to Φ_ν on bD_ν and such that its graph $v=\Psi_\nu(z,u)$ over D_ν is foliated by one-parameter family of complex analytic discs S_ν^t. By Lemma 5.3, each disc S_ν^t is of the form $w=f_\nu^t(z)$ over a correspondent domain $\Omega_\nu^t\subset\mathbb{C}_z$. Moreover, by the same lemma, there is a positive number $r<R$ independent in ν, t such that each disc S_ν^t which intersects the set $\{|z|<r,|u|<r\}$ has a subdisc $\tilde{S}_\nu^t$ which is a graph over the disc $|z|<r$ and all these discs $\tilde{S}_\nu^t$ are contained in the set $\{|u|<R\}$.

By the maximum principle and the Proposition 4.1, $\Gamma(\Psi_\nu)$ over $\overline{D}_\nu$ coincides with the polynomial hull $\widehat{M}_\nu$ of the set $M_\nu\subset\Gamma(\Phi_\nu)$. As $\Phi_\nu\to\Phi$ uniformly on $\overline{B}$ and $\Gamma(\Phi)$ is polynomially convex, the functions Ψ_ν also tend to Φ uniformly on the cylinder $\{|z|\le R,|u|\le R\}\subset\cap_\nu D_\nu$. In particular, analytic discs $\tilde{S}_\nu^t$ constitute a normal family, in which all partial limits belong to $\Gamma(\Phi)$. Thus, $\Gamma(\Phi)\cap\{|z|<r,|u|<r\}$ is contained in the union of analytic discs $S_\alpha\subset\Gamma(\Phi)$ of the form $\{w=f_\alpha(z),|z|<r\}$.

If $S_\alpha\neq S_\beta$, the discs S_α and S_β have no common points. Indeed, the projections of S_α, S_β into $\mathbb{C}\times\mathbb{R}$ are the graphs $u=h_\alpha(z)$, $u=h_\beta(z)$ of harmonic functions in $\{|z|<r\}$. The intersection of these harmonic surfaces

being nonempty is at least one-dimensional. But S_α, $S_\beta \subset \Gamma(\Phi)$, and the projection of $\Gamma(\Phi)$ into $\mathbb{C} \times \mathbb{R}$ is one-to-one. It follows that $S_\alpha \cap S_\beta$ is either empty or at least one-dimensional. By the uniqueness theorem the last case can occur only if $S_\alpha = S_\beta$.

Thus, we have proved that $\Gamma(\Phi)$ is locally foliated by one-dimensional complex submanifolds, if it is a Lipschitz graph, in particular, if φ is C^2-smooth.

Step 4: Local foliation of $\Gamma(\Phi)$ for continuous φ.

Let $\{\varphi_\nu\}$ be a sequence of smooth functions on bB uniformly convergent to φ as $\nu \to \infty$. Let Φ_ν be the correspondent functions over $\overline{B}$, with $\Gamma(\Phi_\nu) = \widehat{\Gamma}(\varphi_\nu)$. As we have proved above, $\Gamma(\Phi_\nu)$ are locally foliated by one-dimensional complex submanifolds. By the Lemma 5.3, for each point $(z^0, u^0) \in B$ there is $r > 0$ independent in ν and a neighbourhood $U \ni (z^0, u^0)$ such that the maximal leaves of the foliations of $\Gamma(\Phi_\nu) \cap \{|z - z^0| < r\}$ intersecting $U \times \mathbb{R}$ are graphs of holomorphic functions over the disc $\{|z - z^0| < r\}$. All these functions are uniformly bounded, hence, their partial limits constitute a family of holomorphic graphs over $\{|z - z_0| < r\}$ which are contained in $\Gamma(\Phi)$ and which union contains $\Gamma(\Phi) \cap (U \times \mathbb{R})$. By the uniqueness theorem, as above, it follows that some neighbourhood of the point $\Gamma(\Phi) \cap ((z_0, u_0) \times \mathbb{R})$ in $\Gamma(\Phi)$ is foliated by analytic discs. Since (z_0, u_0) is an arbitrary point of B, the whole $\Gamma(\Phi)$ is locally foliated by analytic discs.

By the localization property (Step 1), the graph $\Gamma(\Phi) \cap (G \times \mathbb{R})$ over general domain G (as in Sect. 4) is also locally foliated by one-dimensional complex submanifolds. □

7. - Foliation of hulls of graphs over 2-spheres

We prove in this section the properties 3)–8) from Theorem 2 for the foliation of $\Gamma(\Phi)$.

Properties 3)–4). If G is homeomorphic to a 3-ball, the covering model $\mathcal{G}$ is simply connected, hence, conformally equivalent to the unit disc. Thus, $G \times \mathbb{R}$ is biholomorphic to a domain of the form $\{(z, u) : |z| < 1, h^-(z) < u < h^+(z)\}$ which we studied in Sect. 5. The properties 3)–4) are proved for such domains in Lemmas 5.2, 5.3. We can assume further that $\mathcal{G} = \Delta = \{z \in \mathbb{C} : |z| < 1\}$. □

Property 5). Each maximal disc S_α of the foliation of $\Gamma(\Phi)$ is a holomorphic graph $w = f_\alpha(z)$ over Ω_α, properly imbedded into $G \times \mathbb{R}$ by Lemma 5.2. Let E be a connected component of $b\Omega_\alpha \cap \Delta$. Then, from part 2 of Lemma 5.3 it follows (by the same argument as in the proof of Part iii) in [Sh1]) that the cluster set of the vector function $(z, \operatorname{Re} f(z))$ on E is also connected. As it is contained in $bG \cap (\Delta \times \mathbb{R})$, and this set is the disjoint union of hypersurfaces $\{u = h^\pm(z), |z| < 1\}$, this cluster set is placed on one of these hypersurfaces.

But the projection of each of them into Δ is one-to-one, which implies that the function $\operatorname{Re} f_\alpha$ extends continuously onto E and thus, onto the whole $b\Omega_\alpha \cap \Delta$. As the graph of f_α is contained in $\Gamma(\Phi)$, this implies that $\operatorname{Im} f_\alpha$ also extends continuously onto $b\Omega_\alpha \cap \Delta$ as $\Phi(z, \operatorname{Re} f_\alpha(z))$.

If $h^-(z) = h^+(z)$ for $|z| = 1$, the real part of f_α extends continuously on the whole $b\Omega_\alpha$ (with values $h^-(z)$ for $|z| = 1$), and then the imaginary part also extends continuously as $\Phi(z, \operatorname{Re} f_\alpha(z))$. □

Property 5) is not satisfied in general, if $h^+ \neq h^-$ on the boundary of the covering model.

EXAMPLE 6. Let G be the convex domain in $\mathbb{C} \times \mathbb{R}$ defined by the inequalities

$$|z| < 1, \quad |u| < 2 + \sqrt{1 - |z|^2}.$$

All the conditions of Theorem 2 are then satisfied except the last one because $h^+(z) - h^-(z) \equiv 4$ for $|z| = 1$. Let $D \subset \{|z| < 2\}$ be a simply connected domain whose boundary is the union of the segment $[-i, i]$ and a smooth arc in $\mathbb{C} \setminus [-i, i]$ coinsiding with the graph $y = \sin(1/x)$ in a neighbourhood of $[-i, i]$. Let g be a conformal mapping of the upper halfplane $\{\operatorname{Im} z > 0\}$ onto the domain D. Then there is a point $a \in \mathbb{R} \cup \{\infty\}$ such that the set of limiting values of g at a coincides with $[-i, i]$. We can choose g such that $a = 0$ and $\operatorname{Re} g(i) = 0$. Then the function $\Phi(z) = \operatorname{Re} g\left(i\frac{1-z}{1+z}\right)$ extends continuously into the disc $|z| \leq 1$, and we set $\varphi = \Phi|bG$ considering this extension as the function in $\overline{G}$ independent in u. Then $\Gamma(\Phi)$ is the polynomial hull of $\Gamma(\varphi)$, and $\Gamma(\Phi)$ contains the graph $S_0 : w = ig\left(i\frac{1-z}{1+z}\right)$ over the whole disc $\Omega_0 : |z| < 1$ because $|\operatorname{Re}\left(ig\left(i\frac{1-z}{1+z}\right)\right)| < 2$ for $|z| < 1$. But the defining function $f_0(z) = ig\left(i\frac{1-z}{1+z}\right)$ does not extend continuously at the point $1 \in b\Omega_0$.

Property 6). We argue by contradiction and suppose that for some maximal analytic disc $S_\alpha \subset \Gamma(\Phi)$ the corresponding set $\mathbb{C} \setminus \overline{\Omega}_\alpha$ has a connected component E relatively compact in Δ. Then the set E is also relatively compact in some smaller disc $\Delta_r = \{|z| < r\}$, $r < 1$. Let $h_j^- \downarrow h^-$ and $h_j^+ \uparrow h^+$ be two sequences of smooth sub- and super-harmonic functions in Δ, respectively, satisfying the conditions in the proof of Lemma 5.2. (We can take as $h_j^\pm$ the standard regularizations of $h^\pm$, *i.e.*, the convolutions of $h^\pm$ with suitable smooth cutting functions, then dilations $z \mapsto z/r_j$, plus-minus suitable small positive constants.) In particular, the smooth hypersurfaces $\{(z, w) : u = h_j^\pm(z), |z| < 1\}$ are transversal to S_α at all common points. As the functions $h^\pm$ satisfy a Hölder condition on the covering model $\mathcal{G}$ of the domain G, their preimages on the unit disc (by a conformal mapping of Δ onto $\mathcal{G}$) also satisfy a Hölder condition on each compact subset of Δ. Then we can choose the functions $h_j^\pm$ such that they satisfy a Hölder condition on the disc $\overline{\Delta}_r$ uniformly on j, *i.e.*,

$|h_j^{\pm}(z') - h_j^{\pm}(z'')| \leq C|z' - z''|^{\beta}$ for all z', $z'' \in \overline{\Delta}_r$ with constants $C > 1$ and $\beta > 0$ independent in j.

We can assume, for simplicity, that S_α contains the origin in $\mathbb{C}^2$.

Then we have the connected components S_α^j of $S_\alpha \cap \{h_j^-(z) < u < h_j^+(z), |z| < r\}$ containing the origin, and the projections Ω_α^j of these components into $\mathbb{C}_z$ which constitute an increasing sequence of domains with the limit (= union) Ω_α^r.

Fix a point $a \in E$. As Ω_α^r is simply connected by Lemma 5.3, there is a continuous branch $\arg(z-a)$ of the argument of $z - a$ in Ω_α^r. As the set E is also one of the connected components of $\mathbb{C} \setminus \overline{\Omega_\alpha^r}$ and as $\Omega_\alpha^j \uparrow \Omega_\alpha^r$, we have then two sequences of points a_j', $a_j'' \in b\Omega_\alpha^j$ and a point $b \in \overline{E}$ such that $a_j' \to b$, $a_j'' \to b$ as $j \to \infty$, but $\arg(a_j' - a) - \arg(a_j'' - a) \to 2\pi$. By the same reason, there is a sequence of arcs $\gamma_j' \subset b\Omega_\alpha^j$ connecting a_j' with a_j'' such that all limiting points of $\{\gamma_j'\}$ (*i.e.*, the points of the form $\lim b_j$ with $b_j \in \gamma_j'$) are contained in $\overline{E}$. Let I_j' be the interval (a_j', a_j''). Then there are points b_j', b_j'' in $\gamma_j' \cap I_j'$ such that the subarc $\gamma_j \subset \gamma_j'$ with the ends b_j', b_j'' does not intersect the interval $I_j = (b_j', b_j'')$, and their union $\gamma_j \cup I_j$ constitute the boundary of a domain E_j containing a, if j is sufficiently large. We orient γ_j and I_j as the parts of bE_j.

As bE is connected, the boundary values of the vector function $(z, f_\alpha(z))$ on bE are contained in one of hypersurfaces $\{u = h^{\pm}(z), |z| \leq 1\}$ (see property 4)). We can assume, for definiteness, that these values satisfy the condition $u = h^-(z)$ (the case $u = h^+(z)$ is treated in the same way). Then the arcs $\{(z,w) \in S_\alpha : z \in \gamma_j'\}$ are contained in the correspondent hypersurfaces $u = h_j^-(z)$, if j is sufficiently large. As $f_\alpha(b_j') - f_\alpha(b_j'') \to 0$ with $j \to \infty$, we have $|\int_{\gamma_j} d(\operatorname{Im} f_\alpha)| = |\operatorname{Im} f_\alpha(b_j') - \operatorname{Im} f_\alpha(b_j'')| \to 0$ as $j \to \infty$. On the other hand, $\int_{\gamma_j} d(\operatorname{Im} f_\alpha) = \int_{\gamma_j} d^c(\operatorname{Re} f_\alpha)$ by the Cauchy – Riemann equation. As the function $h_j^- - \operatorname{Re} f_\alpha$ is subharmonic and negative in Ω_α^j, and it vanishes on γ_j, we have by Hopf's lemma the inequality $\int_{\gamma_j} d^c(\operatorname{Re} f_\alpha) > \int_{\gamma_j} d^c(h_j^-)$. The last integral is represented by the Stokes theorem in the form

$$\int_{\gamma_j} d^c h_j^- = \int_{E_j} dd^c h_j^- - \int_{I_j} d^c h_j^- .$$

As $h_j^- \downarrow h^-$ and all E_j with j large enough contain a disc $U \subset E$ with the center a, there is a positive constant c such that $\int_{E_j} dd^c h_j^- \geq \int_U dd^c h_j^- > c$ (recall that $h^{\pm}$ are nowhere harmonic).

For the estimate of the integral over I_j, let L_j be the line in $\mathbb{C}$ containing I_j, and D_j be a connected component of $\{|z| < r\} \setminus L_j$ which is situated near the interval I_j on the other side of I_j than the domain E_j. Without any loss of generality, we can assume also (possibly after choosing a suitable subsequence) that the lines L_j converge to some limit line L containing b. Denote by $\tilde{h}_j$ the harmonic extension of $h_j^-|bD_j$ into D_j. Then $\tilde{h}_j$ is Hölder continuous in $\overline{D}_j$ and smooth on I_j. As $\tilde{h}_j > h_j^-$ in D_j by the maximum principle, we have, by Hopf's lemma, the inequality $\int_{I_j} d^c\tilde{h}_j > \int_{I_j} d^c h_j^-$. Let g_j be the function harmonically conjugate to $\tilde{h}_j$ in D_j. As the Hilbert transform is a bounded operator in Hölder classes, the function g_j is smooth on I_j and Hölder continuous on $\overline{D}_j$, *i.e.*, $|g_j(z') - g_j(z'')| \leq \tilde{C}|z' - z''|^{\tilde{\beta}}$ for all z', $z'' \in \overline{D}_j$. (The constants $\tilde{C}$ and $\tilde{\beta}$ here can be different from the corresponding constants for the function $\tilde{h}_j$, because before the Hilbert transform we have to use a conformal mapping of the unit disc Δ onto the domain D_j, and after the Hilbert transform we use the inverse mapping from D_j onto Δ.) Since the lines L_j converge to a limit line L, the corresponding conformal mappings from Δ onto D_j and back are uniformly Hölder continuous. Therefore, by uniform Hölder continuity of the functions $h_j^{\pm}$, the constants $\tilde{C}$ and $\tilde{\beta}$ can be choosen independent in j. Then, by the Cauchy – Riemann equation, we have $d^c\tilde{h}_j = dg_j$ in D_j, which implies that $|\int_{I_j} d^c\tilde{h}_j| = |\int_{I_j} dg_j| = |g_j(b_j'') - g_j(b_j')| \to 0$ as $j \to \infty$.

Thus, we have eventually that

$$\int_{\gamma_j} d(\operatorname{Im} f_\alpha) > \int_{\gamma_j} d^c h_j^- \geq c - \int_{I_j} d^c\tilde{h}_j \to c > 0$$

as $j \to \infty$. This contradiction shows that E can not be relatively compact. □

Property 7). We repeat the arguments used in the proof of Property 6).

Suppose on the contrary that for some maximal analytic disc $S_\alpha \subset \Gamma(\Phi)$ the corresponding set $\mathbb{C} \setminus \overline{\Omega}_\alpha$ has a relatively compact connected component E. Then, by Property 6), the set $bE \cap b\Delta$ is not empty. Fix a point $b \in bE \cap b\Delta$.

Let $h_j^- \downarrow h^-$ and $h_j^+ \uparrow h^+$ be, as above, two sequences of smooth sub- and super-harmonic functions in Δ, respectively, satisfying a Hölder condition in $\overline{\Delta}$ uniformly in j (the last property is obtained due to the corresponding property of $h^{\pm}$). Then we have an increasing sequence of connected components S_α^j of $S_\alpha \cap \{h_j^-(z) < u < h_j^+(z), |z| < 1\}$ and their projections $\Omega_\alpha^j \subset \Delta$ such that $\cup\Omega_\alpha^j = \Omega_\alpha$. Choose a sequence $\varepsilon_j \downarrow 0$ and points b_j', b_j'' in $b\Omega_\alpha^j \cap \{|z - b| = \varepsilon_j\}$ such that for a fixed point $a \in E$, the variation of $\arg(z - a)$ over the subcurve γ_j of $b\Omega_\alpha^j$ with the ends at b_j' and b_j'' tends to 2π as $j \to \infty$. We can also assume that the open subarc I_j in $\{|z - b| = \varepsilon_j, |z| < 1\}$ with the ends at b_j' and b_j'' does not intersect $b\Omega_\alpha^j$. Denote by E_j the domain bounded by $\gamma_j \cup I_j$ and orient γ_j and I_j as parts of bE_j.

We can assume, as above, that the function $\operatorname{Re} f_\alpha$ is equal to h_j^- on γ_j. Then, again as above, $\int_{\gamma_j} d^c h_j^- < \int_{\gamma_j} d^c(\operatorname{Re} f_\alpha) \to 0$ as $j \to \infty$, and $\int_{I_j} d^c h_j^- < \int_{I_j} d^c\tilde{h}_j$ where $\tilde{h}_j$ is a solution of Dirichlet problem in the domain $D_j = \Delta \cap \{|z - b| < \varepsilon_j\}$ with boundary data $h_j^-|bD_j$. As $h_j^-|bD_j$ satisfy a Hölder condition uniformly in j, it follows that the harmonically conjugate functions g_j for $\tilde{h}_j$ in D_j also satisfy a Hölder condition (with the twice less exponent and with a constant which tends to zero as $j \to \infty$), hence,

$\int_{I_j} d^c h_j^- < \int_{I_j} d^c \tilde{h}_j = \int_{I_j} dg_j = |g_j(b_j'') - g_j(b_j')| \to 0$ as $j \to \infty$. But $\int_{E_j} dd^c h_j^- > c > 0$, as above, and we again obtain a contradiction, which shows that there is no relatively compact component in $\mathbb{C} \setminus \overline{\Omega}_\alpha$. □

Property 8). We repeat here the arguments of the proof of a corresponding property in [Sh3].

Suppose on the contrary that the set $E = b\Omega_\alpha \setminus b\overline{\Omega}_\alpha$ is not empty and contains at most a countable family of components $E_1, E_2, \ldots$. By Property 5), the boundary values of $\operatorname{Re} f_\alpha$ on each E_i coincide identically with h^+ or with h^-. Hence, $E = E^+ \cup E^-$ where $E^\pm = b\Omega_\alpha \setminus b\overline{\Omega}_\alpha \cap \{\operatorname{Re} f_\alpha^* = h^\pm\}$ are some unions of components E_i. We can assume that E^- is not empty.

As h^- is subharmonic in Δ, the function $\operatorname{Re} f_\alpha^*$ is subharmonic in the domain $\Omega_\alpha^- = (\operatorname{int} \overline{\Omega}_\alpha) \setminus E^+$. As h^- is nowhere harmonic, the function $\operatorname{Re} f_\alpha^*$ is not harmonic in Ω_α^- by the maximum principle for $h^- - \operatorname{Re} f_\alpha^*$. But then the Riesz measure $\Delta(\operatorname{Re} f_\alpha^*)$ in Ω_α^- is positive on (some part of) E^-. As E^- is a union of components E_i, it follows that $\Delta(\operatorname{Re} f_\alpha^*)(E_i) > 0$ for some i. Then we can repeat the arguments used in the proofs of Properties 6) and 7) which lead to the same contradiction with the Stokes formula. □

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