C. Denson Hill

M. Nacinovich

Duality and distribution cohomology of CR manifolds

Duality and Distribution Cohomology
of CR Manifolds

C. DENSON HILL - M. NACINOVICH

Our aim here is to investigate the $\bar{\partial}_M$-complexes on currents, in the sense of de Rham [deR], associated to a CR manifold M of arbitrary CR dimension and codimension. This brings into play global distribution $\bar{\partial}_M$-cohomology on M, and involves its relationship to the smooth $\bar{\partial}_M$-cohomology on M, as well as its connection with the classical Dolbeault cohomology of an ambient complex manifold X, in the case where M is assumed to be embeddable. We also discuss compact abstract CR manifolds M, and achieve a duality formula which is in the spirit of de Rham [deR] and Serre [S] for real and complex manifolds respectively, and which is related to the work [AK], [AB].

Actually the bridge between the intrinsic and extrinsic notions of distribution $\bar{\partial}_M$-cohomology on M is inspired by the work of Martineau [M] on extendable distributions and boundary values of holomorphic functions, and is also related to the residues of Leray [L] and Grothendiek [Gr]. The first applications of these ideas to CR manifolds can be found in [AHLM], [NV].

To simplify the exposition we always use the term *abstract CR* manifold to indicate one that is not assumed to be locally embeddable. Otherwise, by the term *CR-manifold* we mean one that is assumed to be locally embeddable. In particular we use the term *pseudoconcave CR* manifold for one that is locally embeddable and at least 1-pseudoconcave.

We rely on the previous work [NV], [HN1], [HN2], [HN3].

Contents

1. - Preliminaries

Abstract CR manifolds.

Let M be a smooth real manifold of dimension m, countable at infinity. A *partial complex structure* of type (n, k) on M is the pair consisting of a vector subbundle HM of the tangent bundle TM and a smooth vector bundle isomorphism $J : HM \to HM$ such that

$$\begin{cases} J^2 = -\mathrm{Id} : HM \to HM \\ JH_xM = H_xM \quad \forall x \in M \\ [X, Y] - [JX, JY] \in \Gamma(M, HM) \quad \forall X, Y \in \Gamma(M, HM) \\ \dim_{\mathbb{R}} H_xM = 2n \\ m = \dim M = 2n + k. \end{cases}$$

We say that the partial complex structure (HM, J) is *formally integrable* if

$$[X, JY] + [JX, Y] = J([X, Y] - [JX, JY]) \quad \forall X, Y \in \Gamma(M, HM).$$

We denote respectively by

$$T^{1,0}M = \{X - iJX | X \in HM\} \quad \text{and} \quad T^{0,1}M = \{X + iJX | X \in HM\}$$

the complex vector subbundles of the complexification $\mathbb{C}HM$ of HM corresponding to the eigenvalues i and $-i$ of J.

Then the condition of formal integrability can also be expressed by

$$[\Gamma(M, T^{1,0}M), \Gamma(M, T^{1,0}M)] \subset \Gamma(M, T^{1,0}M)$$

or, equivalently, by

$$[\Gamma(M, T^{0,1}M), \Gamma(M, T^{0,1}M)] \subset \Gamma(M, T^{0,1}M).$$

We note that the following relations hold:

$$T^{0,1}M = \overline{T^{1,0}M}$$

$$T^{1,0}M \cap T^{0,1}M = \underline{0}$$

$$T^{1,0}M \oplus T^{0,1}M = \mathbb{C}HM.$$

An abstract *CR* manifold (M, HM, J) of type (n, k) is the triple consisting of a smooth paracompact manifold M of dimension $m = 2n + k$ and a formally integrable partial complex structure (HM, J) of type (n, k) on it. We call n the *CR dimension* and k the *CR codimension* of (M, HM, J). In the following, we shall write for simplicity M instead of (M, HM, J).

The characteristic bundle and the Levi form.

Let M be a *CR* manifold of type (n, k). The *characteristic bundle* H^0M is the annihilator of HM in T^*M: it is a rank k subbundle of T^*M:

$$H^0M = \{\alpha \in T^*M \,|\, \alpha(X) = 0 \quad \forall X \in \Gamma(M, HM)\}.$$

Given $\alpha \in H^0_xM$, $X, Y \in H_xM$, let us choose $\tilde{\alpha} \in \Gamma(M, H^0M)$, $\tilde{X}$, $\tilde{Y} \in \Gamma(M, HM)$ such that $\tilde{\alpha}(x) = \alpha$, $\tilde{X}(x) = X$, $\tilde{Y}(x) = Y$. Then we have

$$\mathrm{d}\tilde{\alpha}(X, Y) = -\alpha([\tilde{X}, \tilde{Y}])$$

and hence the two sides of this equation only depend on α, X, Y.

In this way we associate to $\alpha \in H^0_xM$ a quadratic form

$$L(\alpha, X) = \alpha([J\tilde{X}, \tilde{X}]) = \mathrm{d}\tilde{\alpha}(X, JX)$$

on H_xM. This form is hermitian for the complex structure of H_xM defined by J. Indeed, we have

$$L(\alpha, JX) = L(\alpha, X) \quad \forall X \in H_xM.$$

We denote by $\sigma(\alpha) = (\sigma^+(\alpha), \sigma^-(\alpha))$ the signature of $L(\alpha, \cdot)$ as a hermitian form for the complex structure of H_xM defined by J: the numbers $\sigma^+(\alpha)$ and $\sigma^-(\alpha)$ are respectively the number of positive and negative eigenvalues of $L(\alpha, \cdot)$.

We say that M is *q-pseudoconcave at* $x \in M$ if for every $\alpha \in H^0_xM$ with $\alpha \neq 0$ we have $\sigma^-(\alpha) \geq q$ and that M is *q-pseudoconcave* if it is q-pseudoconcave at all points.

The hermitian form $L(\alpha, \cdot)$ is called the *Levi form* of M at $\alpha \in H^0M$.

Local embeddability of abstract CR manifolds.

A *CR* map of a *CR* manifold (M_1, H_1, J_1) into a *CR* manifold (M_2, H_2, J_2) is a differentiable map $\phi : M_1 \to M_2$ such that $\phi_*(H_1) \subset H_2$ and $\phi_*(J_1 X) = J_2\phi_*(X)$ for every $X \in H_1$.

A *CR embedding* ϕ of an abstract *CR* manifold M into a complex manifold X is a *CR* map which is an embedding. We say that a *CR* embedding $\phi : M \to X$ is generic if the complex dimension of X is $n + k$, where (n, k) is the type of M. In this case we say that X is a *tubular neighborhood* of M.

We say that a *CR* manifold M is a *CR*-submanifold of a *CR*-manifold N if M is a differentiable submanifold of N and the inclusion map $\iota : M \hookrightarrow N$ is *CR*.

Pseudoconcavity and pseudoconvexity at infinity.

We recall the definiton of pseudoconvexity and pseudoconcavity given in [AG], [A] for the case of complex manifolds.

An N-dimensional complex manifold X is called *r-pseudoconvex* (*r-pseudoconcave*) if there is a real valued smooth function ϕ on X, a compact subset K of X and a constant $c_0 \in \mathbb{R} \cup \{\infty\}$, such that

(1) $\phi < c_0$ on M;
(2)   for every $c < c_0$, the set $\{x \in X | \phi(x) \leq c\}$ is compact in X;
(3)   the complex hessian of ϕ has at least $(N - r)$ positive $((r + 1)$ negative) eigenvalues at each point of $X - K$.

If, in the definition of r-pseudoconvexity, we can choose $K = \emptyset$, the manifold X is called *r-complete*.

In [HN1] these notions have been extended to *CR* manifolds, in the following way.

Let M be an abstract *CR* manifold of type (n, k). We denote by $\mathcal{A}$ $(\overline{\mathcal{A}})$ the sheaf of germs of complex valued smooth differential 1-forms on M, which vanish on $T^{0,1}$ $(T^{1,0})$. Let $\mathcal{E}^{(1)}$ be the sheaf of germs of complex valued smooth 1-forms on M. Then we can define a sheaf homomorphism (called a *CR gauge*: cf. [MN])

$$\mathcal{E}^{(1)} \ni \xi \to (\lambda_1(\xi), \lambda_2(\xi)) \in \mathcal{A} \oplus \overline{\mathcal{A}}$$

such that

$$\xi = \lambda_1(\xi) - \lambda_2(\xi) \quad \forall \xi \in \mathcal{E}^{(1)}.$$

Having fixed a *CR* gauge on M, a *real transversal 1-jet* ψ on M is the pair (ϕ, α) of a smooth real valued function $\phi : M \to \mathbb{R}$ and of a smooth section $\alpha \in \Gamma(M, HM)$. Its *complex hessian* at $x \in M$ is the hermitian form on $H_x M$ defined by

$$h_x(\psi)(X) = d(\alpha + i^{-1}\lambda_1(d\phi))(X, JX) \quad X \in H_x M.$$

In [MN] it was shown that the complex hessian of a transversal 1-jet is invariant with respect to the choice of a *CR*-gauge and that, when M is a generic *CR* submanifold of a complex manifold X, there is a natural correspondence between

real transversal 1-jets ψ on M and 1-jets on M of real valued smooth functions ρ defined in X, in such a way that the complex hessian of ψ at $x \in M$ is the restriction to $H_x M$ of the complex hessian of ρ in X.

A q-pseudoconcave *CR* manifold M is said to be *q-pseudoconcave at infinity* if there is a real valued transversal 1-jet $\psi = (\phi, \alpha)$ on M, a compact subset K of M and a constant $c_0 \in \mathbb{R} \cup \{\infty\}$ such that

(1) $\phi < c_0$ on $M - K$;
(2) $\{x \in M | \phi(x) \le c\}$ is compact in M for every $c < c_0$;
(3)   for every $x \in M - K$ and $\beta \in H_x M$, the hermitian form

$$h_x(\psi)(X) + L(\beta, X) \quad \text{on} \quad \{X \in H_x M | d\phi(X) = 0\}$$

has at least q negative eigenvalues.

A q-pseudoconcave *CR* manifold M is said to be $(n - q)$-*pseudoconvex at infinity* if there is a real valued transversal 1-jet $\psi = (\phi, \alpha)$ on M, a compact subset K of M and a constant $c_0 \in \mathbb{R} \cup \{\infty\}$, such that

(1) $\phi < c_0$ on M;
(2) $\{x \in M | \phi(x) \le c\}$ is compact in M for every $c < c_0$;
(3)   for every $x \in M - K$ and $\beta \in H_x M$, the hermitian form

$$h_x(\psi)(X) + L(\beta, X) \quad \text{on} \quad H_x M$$

has at least q positive eigenvalues.

If, in the definition above, we can take $K = \emptyset$, then M is said to be $(n - q)$-*complete*.

2. - Distribution cohomology and currents

First we discuss distribution cohomology in the embeddable case. Let X be a complex manifold of complex dimension N. On X we have the Dolbeault complexes on the sheaves of germs of smooth forms:

$$\mathcal{E}^{p,*} : 0 \to \mathcal{E}^{p,0} \xrightarrow{\bar{\partial}} \mathcal{E}^{p,1} \xrightarrow{\bar{\partial}} \cdots \to \bar{\partial}\, \mathcal{E}^{p,N} \to 0.$$

Here $\mathcal{E}^{p,j}$ denotes the sheaf of germs of complex valued C^∞ forms of bidegree (p, j) on X. If U is an open subset of X, then the cohomology groups of these complexes on U will be denoted by $H^{p,j}(\mathcal{E}(U))$.

We also have the Dolbeault complexes on sheaves of germs of currents

$$\mathcal{D}'^{p,*} : 0 \to \mathcal{D}'^{p,0} \xrightarrow{\bar{\partial}} \mathcal{D}'^{p,1} \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \mathcal{D}'^{p,N} \to 0.$$

The space $\mathcal{D}'^{p,j}(U)$ is the topoplogical dual of the space $\mathcal{D}^{N-p,N-j}(U)$ of smooth forms of bidegree $(N - p, N - j)$ having compact support in U, and with the

standard Schwartz topology. We denote by $\mathrm{H}^{p,j}(\mathcal{D}'(U))$ the cohomology groups of these complexes on U. Actually we have that

$$\mathrm{H}^j(U, \Omega^p) \simeq \mathrm{H}^{p,j}(\mathcal{E}(U)) \simeq \mathrm{H}^{p,j}(\mathcal{D}'(U))$$

by the abstract de Rham theorem. Here Ω^p denotes the sheaf of germs of holomorphic p-forms on X, and $\mathrm{H}^j(U, \Omega^p)$ denotes Čech cohomology on U with coefficients in the sheaf Ω^p. These are the standard cohomology groups.

We may also consider the complexes in U for smooth forms having compact support contained in U:

$$\mathcal{D}^{p,*}(U) : 0 \to \mathcal{D}^{p,0}(U) \xrightarrow{\bar{\partial}} \mathcal{D}^{p,1}(U) \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \mathcal{D}^{p,N}(U) \to 0.$$

The corresponding cohomology groups we denote by $\mathrm{H}^{p,j}(\mathcal{D}(U))$.

Likewise we have the Dolbeault complexes in U for currents having compact support in U:

$$\mathcal{E}'^{p,*}(U) : 0 \to \mathcal{E}'^{p,0}(U) \xrightarrow{\bar{\partial}} \mathcal{E}'^{p,1}(U) \xrightarrow{\bar{\partial}} \cdots \to \bar{\partial}\, \mathcal{E}'^{p,N}(U) \to 0,$$

where $\mathcal{E}'^{p,j}(U)$ denotes the space of currents of bidegree (p, j) with compact support contained in U. We denote the cohomology groups of these complexes by $\mathrm{H}^{p,j}(\mathcal{E}'(U))$. Actually

$$\mathrm{H}^{p,j}(\mathcal{D}(U)) \simeq \mathrm{H}^{p,j}(\mathcal{E}'(U)),$$

as they can be interpreted as Čech homology groups with coefficients in the same cosheaf (see [AHLM, pp. 82-84] and [AK]).

Next we consider a $\mathcal{C}^\infty$ CR submanifold M of type (n, k) which is generically embedded and closed in X (so $n + k = N$). Let $\mathcal{I}_M$ denote the ideal sheaf in the Grassmann algebra $\mathcal{E}$ of germs of complex valued $\mathcal{C}^\infty$ forms on X, that is locally generated by functions which vanish on M and by their antiholomorphic differentials. We set $\mathcal{I}_M^{p,j} = \mathcal{I}_M \cap \mathcal{E}^{p,j}$, and note that $\mathcal{I}_M = \bigoplus_{0 \le p,j \le N} \mathcal{I}_M^{p,j}$. Since $\bar{\partial}\mathcal{I}_M^{p,j} \subset \mathcal{I}_M^{p,j+1}$, we have subcomplexes, for each $0 \le p \le n + k = N$,

$$\mathcal{I}_M^{p,*} : 0 \to \mathcal{I}_M^{p,0} \xrightarrow{\bar{\partial}} \mathcal{I}_M^{p,1} \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \mathcal{I}_M^{p,N} \to 0$$

of the $\mathcal{E}^{p,*}$, and hence quotient complexes $[\mathcal{E}^{p,*}]$, defined by the exact sequences of fine sheaves complexes:

$$0 \to \mathcal{I}_M^{p,*} \to \mathcal{E}^{p,*} \to [\mathcal{E}^{p,*}] \to 0.$$

The induced differentials are denoted by $\bar{\partial}_M$. We write the quotient complex as

$$[\mathcal{E}^{p,*}] : 0 \to [\mathcal{E}^{p,0}] \xrightarrow{\bar{\partial}_M} [\mathcal{E}^{p,1}] \xrightarrow{\bar{\partial}_M} \cdots \xrightarrow{\bar{\partial}_M} [\mathcal{E}^{p,n}] \to 0,$$

denote its cohomology groups on $M \cap U$ by $H^{p,j}([\mathcal{E}](M \cap U))$, and call them the *smooth $\bar{\partial}_M$-cohomology* groups of $M \cap U$. Note that, for $j > n$, $I_M^{p,j} = \mathcal{E}^{p,j}$, so that $[\mathcal{E}^{p,j}] = 0$ when $j > n$. Moreover, if $x \in X - M$, then $I_{M(x)}^{p,j} = \mathcal{E}_{(x)}^{p,j}$, so that $[\mathcal{E}^{p,j}]_{(x)} = 0$ and the sheaves $[\mathcal{E}^{p,j}]$ are concentrated on M. Note that $[\mathcal{E}^{p,*}]$ is a complex of first order partial differential operators acting on smooth sections of complex vector bundles on M.

In order to define the *distribution $\bar{\partial}_M$-cohomology* groups on $M \cap U$, we first consider the spaces $[\mathcal{D}^{p,j}](M \cap U)$ of sections in $[\mathcal{E}^{p,j}](M \cap U)$ having compact support in $M \cap U$. There is the exact sequence

$$(2.1) \qquad 0 \to I_M^{p,j}(U) \cap \mathcal{D}^{p,j}(U) \to \mathcal{D}^{p,j}(U) \to [\mathcal{D}^{p,j}](M \cap U) \to 0.$$

We define

$$[\mathcal{D}'^{p,j}](M \cap U) = ([\mathcal{D}^{n+k-p,n-j}](M \cap U))'$$

as the topological dual. Because of the exact sequence above, this dual space is isomorphic to the annihilator $(I_M(U) \cap \mathcal{D}^{n+k-p,n-j}(U))^o$ in the space $\mathcal{D}'^{p,k+j}(U)$. Note that there is an index shift equal to the real codimension of M. Since the definition of the ideal I_M only involves the pull-back of smooth forms to M, it follows that the annihilator, and hence $[\mathcal{D}'^{p,j}](M \cap U)$ consists of currents whose coefficients are single layer distributions supported on M. In this way we obtain, for each $0 \le p \le n+k$, a complex of sheaves

$$[\mathcal{D}'^{p,*}] : 0 \to [\mathcal{D}'^{p,0}] \overset{\bar{\partial}_M}{\to} [\mathcal{D}'^{p,1}] \overset{\bar{\partial}_M}{\to} \cdots \overset{\bar{\partial}_M}{\to} [\mathcal{D}'^{p,n}] \to 0,$$

whose cohomology on $M \cap U$ we denote by $H^{p,j}([\mathcal{D}'](M \cap U))$ and refer to as the *distribution $\bar{\partial}_M$-cohomology* groups of $M \cap U$. The differential operators in this complex are easily described by their identification with

$$\bar{\partial} : (I_M(U) \cap \mathcal{D}^{n+k-p,n-j}(U))^o \to (I_M(U) \cap \mathcal{D}^{n+k-p,n-j-1}(U))^o,$$

which should be calculated in the sense of currents (see [NV, p. 139]). Note that $[\mathcal{E}^{p,*}]$ and $[\mathcal{D}'^{p,*}]$ are compexes of fine sheaves. We may therefore consider the corresponding complexes, and their cohomology groups on $M \cap U$ with compact supports. These are, for $0 \le p \le n+k$:

$$[\mathcal{D}^{p,*}](M \cap U) : 0 \to [\mathcal{D}^{p,0}](M \cap U) \overset{\bar{\partial}_M}{\to} [\mathcal{D}^{p,1}](M \cap U) \overset{\bar{\partial}_M}{\to} \cdots$$
$$\cdots \overset{\bar{\partial}_M}{\to} [\mathcal{D}^{p,n}](M \cap U) \to 0$$

and

$$[\mathcal{E}'^{p,*}](M \cap U) : 0 \to [\mathcal{E}'^{p,0}](M \cap U) \overset{\bar{\partial}_M}{\to} [\mathcal{E}'^{p,1}](M \cap U) \overset{\bar{\partial}_M}{\to} \cdots$$
$$\cdots \overset{\bar{\partial}_M}{\to} [\mathcal{E}'^{p,n}](M \cap U) \to 0,$$

with *smooth* and *distribution* cohomology groups $H^{p,j}([\mathcal{D}](M \cap U))$ and $H^{p,j}([\mathcal{E}'](M \cap U))$ respectively. Here $[\mathcal{E}'^{p,j}](M \cap U)$ denotes the space of continuous sections of the sheaf $[\mathcal{D}'^{p,j}]$ having compact support in $M \cap U$. Note that

$$(2.2) \qquad [\mathcal{E}'^{p,j}](M \cap U) = (I_M^{n+k-p,n-j}(U))^o.$$

3. - Interpretation of the distribution cohomology on $M \cap U$

Let $\mathcal{F}_M$ denote the ideal sheaf of germs of smooth complex valued differential forms on X that are flat on M, and set $\mathcal{F}_M^{p,j} = \mathcal{F}_M \cap \mathcal{E}^{p,j}$. Note that $\mathcal{F}_M = \bigoplus_{0 \leq p,j \leq n+k} \mathcal{F}_M^{p,j}$ and $\bar\partial \mathcal{F}_M^{p,j} \subset \mathcal{F}_M^{p,j+1}$. Therefore $\mathcal{F}_M^{p,*}$ is a subcomplex of $I_M^{p,*}$ and we have the quotient complexes defined by the short exact sequence of fine sheaf complexes:

$$0 \to \mathcal{F}_M^{p,*} \to I_M^{p,*} \to I_M^{p,*}/\mathcal{F}_M^{p,*} \to 0.$$

Since M is generic we know, via the formal Cauchy-Kowalewski construction (see [AH], [AFN], [AN]) that the sequences

$$0 \to I_M^{p,0}/\mathcal{F}_M^{p,0} \xrightarrow{\bar\partial} I_M^{p,1}/\mathcal{F}_M^{p,1} \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} I_M^{p,N}/\mathcal{F}_M^{p,N} \to 0$$

are exact sequences of fine sheaves and that the corresponding sequences for sections with closed or compact supports are exact for every open subset U of X. We have, for the annihilators of forms that are flat on M,

$$(3.1) \qquad (\mathcal{F}_M^{n+k-p,n+k-j}(U))^o = \mathcal{E}_M'^{p,j}(U)$$

and

$$(3.2) \qquad (\mathcal{F}_M(U) \cap \mathcal{D}^{n+k-p,n+k-j}(U))^o = \mathcal{D}_M'^{p,j}(U),$$

where the subscript M means that the supports are contained in $M \cap U$. Dualizing the exact sequence above (see §9) we obtain exact sequences

$$(3.3) \qquad 0 \leftarrow \frac{(\mathcal{F}_M^{p,0}(U))^o}{(I_M^{p,0}(U))^o} \xleftarrow{\bar\partial} \frac{(\mathcal{F}_M^{p,1}(U))^o}{(I_M^{p,1}(U))^o} \xleftarrow{\bar\partial} \cdots \xleftarrow{\bar\partial} \frac{(\mathcal{F}_M^{p,N}(U))^o}{(I_M^{p,N}(U))^o} \leftarrow 0$$

and

$$(3.4) \qquad 0 \leftarrow \frac{(\mathcal{F}_M(U) \cap \mathcal{D}^{p,0}(U))^o}{(I_M(U) \cap \mathcal{D}^{p,0}(U))^o} \xleftarrow{\bar\partial} \frac{(\mathcal{F}_M(U) \cap \mathcal{D}^{p,1}(U))^o}{(I_M(U) \cap \mathcal{D}^{p,1}(U))^o} \xleftarrow{\bar\partial} \cdots$$
$$\xleftarrow{\bar\partial} \frac{(\mathcal{F}_M(U) \cap \mathcal{D}^{p,N}(U))^o}{(I_M(U) \cap \mathcal{D}^{p,N}(U))^o} \leftarrow 0$$

Thus we arrive at

PROPOSITION 1. *For each open subset U of X, and $0 \leq p \leq n + k$, $0 \leq j \leq n$, there are natural isomorphisms:*

$$(3.5) \qquad H^{p,j}([\mathcal{D}'](M \cap U)) \simeq H^{p,j+k}(\mathcal{D}'_M(U))$$

and

$$(3.6) \qquad H^{p,j}([\mathcal{E}'](M \cap U)) \simeq H^{p,j+k}(\mathcal{E}'_M(U)).$$

PROOF. The cohomology groups occurring on the right in the proposition are just the cohomology groups of the complexes

$$\mathcal{D}'^{p,*}_M(U) : 0 \to \mathcal{D}'^{p,0}_M(U) \xrightarrow{\bar{\partial}} \mathcal{D}'^{p,1}_M(U) \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \mathcal{D}'^{p,N}_M(U) \to 0,$$

and

$$\mathcal{E}'^{p,*}_M(U) : 0 \to \mathcal{E}'^{p,0}_M(U) \xrightarrow{\bar{\partial}} \mathcal{E}'^{p,1}_M(U) \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \mathcal{E}'^{p,N}_M(U) \to 0,$$

respectively. The quotients occurring in (3.4) and (3.3) are associated to the short exact sequences of complexes

$$(3.7) \qquad 0 \to (\mathcal{I}_M(U) \cap \mathcal{D}^{n+k-p,n-*}(U))^o \to (\mathcal{F}_M(U) \cap \mathcal{D}^{n+k-p,n-*}(U))^o$$
$$\to \frac{(\mathcal{F}_M(U) \cap \mathcal{D}^{n+k-p,n-*}(U))^o}{(\mathcal{I}_M(U) \cap \mathcal{D}^{n+k-p,n-*}(U))^o} \to 0,$$

$$(3.8) \qquad 0 \to (\mathcal{I}_M^{n+k-p,n-*}(U))^o \to (\mathcal{F}_M^{n+k-p,n-*}(U))^o$$
$$\to \frac{(\mathcal{F}_M^{n+k-p,n-*}(U))^o}{(\mathcal{I}_M^{n+k-p,n-*}(U))^o} \to 0.$$

By passing to the long exact sequences associated to (3.7), and using (3.2), the remark following (2.1), and the exactness of (3.4), we arrive at (3.5). The result for (3.6) is obtained analogously, using the long exact cohomology sequences associated to (3.8), and employing (3.1), (2.2) and the exactness of (3.3).

4. - Residues and extendable distributions

The Dolbeault sheaf complexes $\check{\mathcal{D}}'^{p,*}_M$ of currents on $X - M$ which are extendable across M are defined by the exact sequences of complexes

$$(4.1) \qquad 0 \to \mathcal{D}'^{p,*}_M \to \mathcal{D}'^{p,*} \to \check{\mathcal{D}}'^{p,*}_M \to 0.$$

If U is an open set in X we have that the sequence

$$0 \to \mathcal{D}_M^{\prime p,j}(U) \to \mathcal{D}^{\prime p,j}(U) \to \check{\mathcal{D}}_M^{\prime p,j}(U) \to 0$$

is exact for every $0 \leq p$, $j \leq n+k$. The elements of $\check{\mathcal{D}}_M^{\prime p,j}(U)$ can be thought of as currents T in $\mathcal{D}^{\prime p,j}(U - M)$ which are restriction to $U - M$ of currents $\tilde{T}$ in $\mathcal{D}^{\prime p,j}(U)$. We also have the interpretation, in terms of the topological dual:

$$\check{\mathcal{D}}_M^{\prime p,j}(U) = (\mathcal{F}_M(U) \cap \mathcal{D}^{n+k-p,n+k-j}(U))'$$

(see for example [M], [AHLM], [NV]).

We denote the cohomology of $\check{\mathcal{D}}_M^{\prime p,*}$ in U by $H^{p,j}(\check{\mathcal{D}}_M'(U))$.

As the sheaves $\check{\mathcal{D}}_M^{\prime p,j}$ are fine, we may also consider the space $\check{\mathcal{E}}_M^{\prime p,j}(U)$ of compactly supported sections in U, which is the topological dual

$$\check{\mathcal{E}}_M^{\prime p,j}(U) = (\mathcal{F}_M^{n+k-p,n+k-j}(U))'.$$

It is clear that we have a complex

$$\check{\mathcal{E}}_M^{\prime p,*}(U) : 0 \to \check{\mathcal{E}}_M^{\prime p,0}(U) \xrightarrow{\bar{\partial}} \check{\mathcal{E}}_M^{\prime p,1}(U) \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \check{\mathcal{E}}_M^{\prime p,n+k}(U) \to 0,$$

whose cohomology groups will be denoted by $H^{p,j}(\check{\mathcal{E}}_M'(U))$.

PROPOSITION 2. *For $0 \leq p \leq n+k$, and for each open subset U of X, we have:*

(4.2) $$H^{p,j}(\mathcal{D}_M'(U)) = 0 \quad \text{for} \quad j < k,$$

(4.3) $$H^{p,j}(\mathcal{E}_M'(U)) = 0 \quad \text{for} \quad j < k.$$

Moreover there are exact sequences

(4.4)
$$0 \to H^{p,k-1}(\mathcal{D}'(U)) \to H^{p,k-1}(\check{\mathcal{D}}_M'(U)) \xrightarrow{Res} H^{p,0}([\mathcal{D}'](M \cap U))$$
$$\to H^{p,k}(\mathcal{D}'(U)) \to H^{p,k}(\check{\mathcal{D}}_M'(U)) \xrightarrow{Res} H^{p,1}([\mathcal{D}'](M \cap U) \to \ldots$$
$$\ldots \to H^{p,n+k-1}(\mathcal{D}'(U)) \to H^{p,n+k-1}(\check{\mathcal{D}}_M'(U)) \to H^{p,n}([\mathcal{D}'](M \cap U))$$
$$\to H^{p,n+k}(\mathcal{D}'(U)) \to H^{p,n+k}(\check{\mathcal{D}}_M'(U)) \to 0,$$

(4.5)
$$0 \to H^{p,k-1}(\mathcal{E}'(U)) \to H^{p,k-1}(\check{\mathcal{E}}_M'(U)) \xrightarrow{Res} H^{p,0}([\mathcal{E}'](M \cap U))$$
$$\to H^{p,k}(\mathcal{E}'(U)) \to H^{p,k}(\check{\mathcal{E}}_M'(U)) \xrightarrow{Res} H^{p,1}([\mathcal{E}'](M \cap U) \to \ldots$$
$$\ldots \to H^{p,n+k-1}(\mathcal{E}'(U)) \to H^{p,n+k-1}(\check{\mathcal{E}}_M'(U)) \to H^{p,n}([\mathcal{E}'](M \cap U))$$
$$\to H^{p,n+k}(\mathcal{E}'(U)) \to H^{p,n+k}(\check{\mathcal{E}}_M'(U)) \to 0.$$

PROOF. We obtain (4.2) via (3.2) by observing that, in the first terms on the right in (3.4), the denominators are 0. The proof of (4.3) follows likewise from (3.1) and (3.3). We obtain (4.4) from the long exact cohomology sequence associated to (4.1), upon making the substitution (3.5). The proof of (4.5) is analogous.

REMARK. The maps labelled *Res* in Proposition 2 we refer to as the *residue homomorphisms*.

5. - Some applications

THEOREM 1. *Let* U *be open in* X, *and* $0 \leq p \leq n + k$. *Then the natural restriction maps*

$$(5.1) \qquad\qquad \mathrm{H}^{p,j}(\mathcal{D}'(U)) \xrightarrow{r} \mathrm{H}^{p,j}(\check{\mathcal{D}}'_M(U))$$

and

$$(5.2) \qquad\qquad \mathrm{H}^{p,j}(\mathcal{E}'(U)) \xrightarrow{r} \mathrm{H}^{p,j}(\check{\mathcal{E}}'_M(U))$$

are isomorphisms for $0 \leq j < k - 1$, *and are injective for* $j = k - 1$.

PROOF. We have the long exact cohomology sequence

$$\dots \to \mathrm{H}^{p,j}(\mathcal{D}'_M(U)) \to \mathrm{H}^{p,j}(\mathcal{D}'(U)) \to \mathrm{H}^{p,j}(\check{\mathcal{D}}'_M(U)) \to \mathrm{H}^{p,j+1}(\mathcal{D}'_M(U)) \to \dots$$

so (5.1) follows from (4.2). The proof of (5.2) is the same.

REMARKS 1. The statement about (5.1) is a variant of the second Riemann theorem on removable singularities in several complex variables. For example, when $p = 0$, $j = 0$ and $k \geq 2$, we obtain the following: *If* f *is a holomorphic function defined in* $U - M$, *which extends across* M *as a distribution, then* f *extends across* M *as a holomorphic function.*

2. Assume that X is $(n - 2)$-complete (see [A]), that $U \subset\subset X$ is open and $X - U$ has no compact connected component. Then $\mathrm{H}^{0,1}(\mathcal{E}'(U)) = \mathrm{H}^{0,1}(\mathcal{D}(U)) = 0$, as is well known. Moreover suppose that either $k > 2$, or else $k = 2$ and M is 1-pseudoconcave. In this case $\mathrm{H}^{0,1}(\check{\mathcal{E}}'_M(U)) = 0$. Indeed, if $k > 2$, this follows from the isomorphism (5.2). If $k = 2$ and M is 1-pseudoconcave, it follows from the exact sequence

$$0 \to \mathrm{H}^{0,1}(\mathcal{E}'(U)) \to \mathrm{H}^{0,1}(\check{\mathcal{E}}'_M(U)) \xrightarrow{Res} \mathrm{H}^{0,0}([\mathcal{E}'](M \cap U)) \to \dots,$$

as $\mathrm{H}^{0,0}([\mathcal{E}'](M \cap U)) = 0$ by holomorphic extension plus unique continuation.

Now let f be a distribution given on a neighborhood of bU, which is holomorphic on $bU - M$. Then there is a unique holomorphic function $\tilde{f}$ on U which equals f on $bU - M$. Without loss of generality we may consider $f \in \mathcal{D}'(X)$. Then $\bar{\partial}f|_{U-M}$ defines an element of $\mathrm{H}^{0,1}(\check{\mathcal{E}}'_M(U))$. Hence there is a $u \in \check{\mathcal{E}}'^{0,0}_M(U)$ such that $\bar{\partial}u = \bar{\partial}f$. Then $f - u \in \mathrm{H}^{0,0}(\check{\mathcal{D}}'_M(U))$; so by 1 it extends to the desired $\tilde{f}$.

3. Now make the same assumptions on X, U, M as in 2, but add that bU is a smooth manifold. In this case we may consider an $f \in \mathcal{D}'(bU)$ which is a CR-distribution on $bU - M$. Then $f|_{bU-M}$ extends as a CR distribution on bU across $M \cap bU$. Following the same argument as above, we arrive at a holomorphic function $\tilde{f}$ in U which extends as a distribution to $\bar{U}$. Its Res is a CR distribution on bU (now bU has codimension 1) which is equal to f on $bU - M$.

The following theorems are valid for $0 \le p \le n + k$.

THEOREM 2. (A) *Assume that U is r-pseudoconvex (r-complete). Then for the residue homomorphisms we have that:*

(i)
$$\mathrm{H}^{p,j+k-1}(\check{\mathcal{D}}'_M(U)) \overset{Res}{\to} \mathrm{H}^{p,j}([\mathcal{D}'](M \cap U))$$

have finite dimensional kernels and cokernels (are isomorphisms) for $j > r-k+1$. For $j = r - k + 1$ the cokernel is finite dimensional (Res is surjective).

(ii)
$$\mathrm{H}^{p,j+k-1}(\check{\mathcal{E}}'_M(U)) \overset{Res}{\to} \mathrm{H}^{p,j}[\mathcal{E}'](M \cap U))$$

have finite dimensional kernels and cokernels (are isomorphisms) for $j < n - r$. For $j = n - r$ the kernel is finite dimensional (Res is injective).

(B) *Assume that U is r-pseudoconcave. Then for the residue homomorphism we have that* (i) *have finite dimensional kernels and cokernels for $j < r$. For $j = r - k$ the kernel is finite dimensional.*

The maps (ii) *have finite dimensional kernels and cokernels for $j > n-r+1$. For $j = n - r + 1$ the cokernel is finite dimensional.*

PROOF. Using the results of [AG], the theorem follows from the long exact cohomology sequences (4.4) and (4.5).

THEOREM 3. *Let M be a compact q-pseudoconcave CR manifold of type (n, k), with $q \ge 1$. Then there is an $(n + k)$-dimensional complex manifold X, in which M has an embedding as a generic closed CR manifold. In X there is a fundamental system $\{U\}$ of open tubular neighborhoods of M which are $(n - q)$-pseudoconvex and q-pseudoconcave (cf. [HN1]). Then for any such tubular neighborhood U, we obtain that* (i) *has finite dimensional kernel and cokernel for $j < q - k$ and $j > n - q - k + 1$. For $j = q - k$ the kernel is finite dimensional, and for $j = n - q - k + 1$ the cokernel is finite dimensional. The map* (ii) *has finite dimensional kernel and cokernel for $j < q$ and $j > n - q + 1$.*

For $j = q$ the cokernel is finite dimensional, and for $j = n - q + 1$ the kernel is finite dimensional.

PROOF. The result follows from [HN1] and the previous theorem.

Dropping the assumption that M is compact, we obtain:

THEOREM 4. *Let M be a q-pseudoconcave CR manifold of type (n, k), $q \geq 1$. Then there is an $(n + k)$ dimensional complex manifold X, in which M has an embedding as a generic closed CR submanifold.*

(A) *Assume that M is also $(n - q)$-pseudoconvex at infinity. Then in X there is a fundamental system $\{U\}$ of open tubular neighborhoods of M for which* (i) *has finite dimensional kernel and cokernel for $j > n - q - k + 1$, and finite dimensional cokernel when $j = n - q - k + 1$. Likewise* (ii) *has finite dimensional kernels and cokernels for $j < q$, and finite dimensional kernel when $j = q$.*

(B) *Assume that M is also q-pseudoconcave at infinity. Then in X there is a fundamental system $\{U\}$ of open tubular neighborhoods of M for which* (i) *has finite dimensional kernel and cokernel for $j < q - k$ and finite dimensional kernel when $j = q - k$. Likewise* (ii) *has finite dimensional kernel and cokernel for $j > n - q + 1$ and finite dimensional cokernel when $j = n - q + 1$.*

PROOF. The result follows again from [AG], (4.4), (4.5) using the proof of [Theorem 6.1 in HN1] for (A), and the proof of [Theorem 5.1 in HN1] for (B).

6. - Intrinsic definition of the $\bar{\partial}_M$-complexes on smooth forms

In the discussion above we have taken an extrinsic point of view, following [AH], [AHLM], [AFN], [NV]. In [HN1] the $\bar{\partial}_M$-complexes were defined in an intrinsic way, suitable for treating abstract CR manifolds. The starting point is to consider an alternate description of the Dolbeault complexes on an N-dimensional complex manifold X as quotients of de Rham complexes.

(a) We now use $\mathcal{E}_X$ to denote the sheaf of germs of C^∞ complex valued differential forms on X. Let

$$\hbar = \bigoplus_{\substack{0 < p \leq N \\ 0 \leq j \leq N}} \mathcal{E}_X^{p,j}$$

be the ideal subsheaf of $\mathcal{E}_X$ generated by germs of smooth forms of bidegree $(1, 0)$. Note that $d\,\hbar^m \subset \hbar^m$ for $m \geq 0$, since

$$\hbar^m = \bigoplus_{\substack{m \leq p \leq N \\ 0 \leq j \leq N}} \mathcal{E}_X^{p,j}.$$

We set $\hbar^{m,\ell} = \hbar^m \cap \mathcal{E}_X^{(\ell)}$, where $\mathcal{E}_X^{(\ell)}$ indicates forms of total degree ℓ. Then we have the sheaf subcomplexes:

$$\hbar^{m,*} : 0 \to \hbar^{m,0} \overset{d}{\to} \hbar^{m,1} \overset{d}{\to} \dots \overset{d}{\to} \hbar^{m,2N} \to 0$$

of the complexified de Rham complex. (Here $\hbar^{m,j} = 0$ for $j < m$). Clearly $\hbar^{m+1,*}$ is a subcomplex of $\hbar^{m,*}$, and hence there is the quotient complex

$$\hbar^{m,*}/\hbar^{m+1,*} : 0 \to \hbar^{m,0}/\hbar^{m+1,0} \overset{\hat{d}}{\to} \hbar^{m,1}/\hbar^{m+1,1} \overset{\hat{d}}{\to} \dots$$

$$\dots \overset{\hat{d}}{\to} \hbar^{m,2N}/\hbar^{m+1,2N} \to 0$$

defined by the exact sequence of fine sheaf complexes

$$0 \to \hbar^{m+1,*} \to \hbar^{m,*} \to \hbar^{m,*}/\hbar^{m+1,*} \to 0.$$

The obvious inclusion $\mathcal{E}_X^{m,j} \hookrightarrow \hbar^{m,m+j}$ induces an isomorphism on the quotient

$$\mathcal{E}_X^{m,j} \overset{\sim}{\to} \hbar^{m,m+j}/\hbar^{m+1,m+j}.$$

Then there is a commutative diagram

$$\begin{array}{ccc}
\mathcal{E}_X^{m,j} & \overset{\bar{\partial}}{\to} & \mathcal{E}_X^{m,j+1} \\
\simeq\downarrow & & \downarrow\simeq \\
\hbar^{m,m+j}/\hbar^{m+1,m+j} & \underset{\hat{d}}{\to} & \hbar^{m,m+j+1}/\hbar^{m+1,m+j+1},
\end{array}$$

which shows that the standard Dolbeault complex, with an index shift of m units, is isomorphic to the complex $\hbar^{m,*}/\hbar^{m+1,*}$.

(b) Next we consider an abstract CR manifold M of type (n,k), and denote by $\mathcal{E}_M$ the sheaf of germs of C^∞ complex valued differential forms on M. In [HN1] the ideal subsheaf $\mathfrak{A}$ of $\mathcal{E}_M$ was introduced, as the Cartan ideal which annihilates $T^{0,1}M$; i.e.

$$\mathfrak{A} = \left\{ \alpha \in \bigoplus_{\ell>0} \to \mathcal{E}_M^{(\ell)} \,|\, \alpha|_{T^{0,1}M} = 0 \right\},$$

where $\mathcal{E}_M^{(\ell)}$ indicates forms of total degree ℓ. The intrinsic $\bar{\partial}_M$-complexes on M where introduced in [HN1] as quotients of the de Rham subcomplexes associated to powers of the ideal $\mathfrak{A}$. Explicitly we have

$$d\,\mathfrak{A}^m \subset \mathfrak{A}^m \quad \text{for} \quad m = 0, 1, \dots, n+k,$$

and the sheaf subcomplexes

$$\mathfrak{A}^{m,*} : 0 \to \mathfrak{A}^{m,0} \overset{d}{\to} \mathfrak{A}^{m,1} \overset{d}{\to} \dots \overset{d}{\to} \mathfrak{A}^{m,n+k} \to 0$$

of the complexified de Rham complex on M. (Again $\mathcal{A}^{m,j} = 0$ for $j < m$).
Also $\mathcal{A}^{m+1,*}$ is a subcomplex of $\mathcal{A}^{m,*}$, and there is the quotient complex

$$\mathcal{A}^{m,*}/\mathcal{A}^{m+1,*} : 0 \to \mathcal{A}^{m,0}/\mathcal{A}^{m+1,0} \xrightarrow{\hat{d}} \mathcal{A}^{m,1}/\mathcal{A}^{m+1,1} \xrightarrow{\hat{d}} \dots$$

$$\dots \xrightarrow{\hat{d}} \mathcal{A}^{m,n+k}/\mathcal{A}^{m+1,n+k} \to 0$$

defined by the exact sequence of complexes of fine sheaves:

$$0 \to \mathcal{A}^{m+1,*} \to \mathcal{A}^{m,*} \to \mathcal{A}^{m,*}/\mathcal{A}^{m+1,*} \to 0.$$

Except for an index shift of m units, this is our definition of the intrinsic
$\bar{\partial}_M$-complexes on M.

If we set $\mathbb{Q}^{p,p+j} = \mathcal{A}^{p,p+j}/\mathcal{A}^{p+1,p+j}$ and denote by $\bar{\partial}_M$ the $\hat{d}$ above, we obtain
the intrinsic $\bar{\partial}_M$-complexes:

$$\mathbb{Q}^{p,*} : 0 \to \mathbb{Q}^{p,0} \xrightarrow{\bar{\partial}_M} \mathbb{Q}^{p,1} \xrightarrow{\bar{\partial}_M} \dots \xrightarrow{\bar{\partial}_M} \mathbb{Q}^{p,n} \to 0,$$

for $0 \le p \le n + k$, which were introduced in [HN1].

(c) To explain the equivalence between this intrinsic definition and the
extrinsic one introduced in §2, we proceed as follows.

Suppose that M is generically embedded in X as in §2. Recall that we
have the ideal I_M in $\mathcal{E}_X$ that was introduced there. Since M is generic in X,
it follows that

$$\iota^*(\hbar^m) = \mathcal{A}^m \quad \text{and} \quad \iota^*(\hbar^m \cap I_M) \subset \mathcal{A}^{m+1}.$$

Moreover there is the exact sequence of sheaf complexes

$$0 \to (\hbar^{m+1,*} + I_M^{m,*}) \to \hbar^{m,*} \xrightarrow{\pi \circ \iota^*} \mathcal{A}^{m,*}/\mathcal{A}^{m+1,*} \to 0.$$

Therefore we obtain the isomorphisms of fine sheaf complexes:

$$\mathbb{Q}^{p,*} = \mathcal{A}^{p,p+*}/\mathcal{A}^{p+1,p+*}$$

$$\simeq \hbar^{p,p+*}/(\hbar^{p+1,p+*} + I_M^{p,*})$$

$$\simeq \frac{\hbar^{p,p+*}/\hbar^{p+1,p+*}}{I_M^{p,*}/(\hbar^{p+1,p+*} \cap I_M^{p,*})})$$

$$\simeq \mathcal{E}^{p,*}/I_M^{p,*}$$

$$= [\mathcal{E}^{p,*}].$$

Thus we have the equivalence, but we prefer to stick with the notation $[\mathcal{E}^{p,*}]$ even
for the intrinsic $\bar{\partial}_M$-complex for abstract CR manifolds. Keeping a consistent
notation, when V is an open subset of M, we use $[\mathcal{D}^{p,*}](V)$ to denote the intrinsic

$\bar{\partial}_M$-complex on smooth forms on M having compact support contained in V. This now agrees with our previous notation for $[D^{p,*}](M \cap U)$ for $V = M \cap U$.

7. - Intrinsic definition of the $\bar{\partial}_M$-complexes on currents

In §2 the complexes $[D'^{p,*}](M \cap U)$ and $[\mathcal{E}^{p,*}](M \cap U)$ were defined as the topological duals $([D^{n+k-p,n-*}](M \cap U))'$ and $([\mathcal{E}^{n+k-p,n-*}](M \cap U))'$ respectively. Indeed the spaces $[D'^{p,j}](M \cap U)$ and $[\mathcal{E}^{p,j}](M \cap U)$ were defined as the topological duals of $[D^{n+k-p,n-j}](M \cap U)$ and $[\mathcal{E}^{n+k-p,n-j}](M \cap U)$, respectively, and the maps were the dual (transposed) maps of

$$\bar{\partial}_M : [D^{n+k-p,n-j-1}](M \cap U) \to [D^{n+k-p,n-j}](M \cap U),$$

and

$$\bar{\partial}_M : [\mathcal{E}^{n+k-p,n-j-1}](M \cap U) \to [\mathcal{E}^{n+k-p,n-j}](M \cap U).$$

These topological duals can also be computed intrinsecally in terms of distribution sections of vector bundles on M, as was done in [AHLM]. For every open set V in M, $[\mathcal{E}^{p,j}](V)$ is the space of C^∞ sections of a smooth vector bundle $F^{p,j}$ over m, of rank $\binom{n+k}{p} + \binom{n}{j}$. The fiber $F^{p,j}_x$ at x is given by

$$F^{p,j}_x = \frac{J^p_x \cap \Lambda^{p+j}\mathbb{C}T^*_x M}{J^{p+1}_x \cap \Lambda^{p+j}\mathbb{C}T^*_x M},$$

where

$$J_x = \left\{ \beta \in \bigoplus_{h>0} \Lambda^h\mathbb{C}T^*_x M \mid \beta|_{T^{0,1}_x M} = 0 \right\}.$$

The dual bundle of $F^{n+k-p,n-j}$ can be identified with $F^{p,j} \otimes \omega_M$, where ω_M is the orientation bundle of M. This follows because the de Rham complex is self-transpose (cf. [AHLM]). We denote the sheaf of germs of smooth sections of this bundle by $[\mathcal{E}^{p,j} \otimes \omega_M]$, and if V is an open subset of M, use $[D^{p,j} \otimes \omega_M](V)$ to denote the space of its continuous sections which have compact support in V. We have the complex of fine sheaves

$$[\mathcal{E}^{p,*} \otimes \omega_M] : 0 \to [\mathcal{E}^{p,0} \otimes \omega_M] \overset{\bar{\partial}_M \otimes id}{\to} [\mathcal{E}^{p,1} \otimes \omega_M] \overset{\bar{\partial}_M \otimes id}{\to} \dots$$

$$\dots \overset{\bar{\partial}_M \otimes id}{\to} [\mathcal{E}^{p,n} \otimes \omega_M] \to 0,$$

which we call the *twisted* $\bar{\partial}_M$-complex on M. Likewise for the twisted $\bar{\partial}_M$-complex $[D^{p,*} \otimes \omega_M](V)$ with compact supports in V. Keeping with our notations, the cohomology groups of these complexes are denoted by $H^{p,j}([\mathcal{E} \otimes \omega_M](V))$ and $H^{p,j}([D \otimes \omega_M](V))$ respectively.

If we now take distribution sections of the bundles $F^{p,j} \otimes \omega_M$ over V, we obtain spaces $[D'^{p,j}](V)$ which coincide exactly with the $[D'^{p,j}](M \cap U)$ introduced

in §2, when M is embedded in X and $V = M \cap U$ for an open subset U of X. Likewise for $[\mathcal{E}'^{p,j}](V)$ and $[\mathcal{E}'^{p,j}](M \cap U)$.

REMARK. If the manifold M is orientable, and if we choose an orientation ϵ for M, then we obtain a canonical isomorphism

$$F^{p,j} \xrightarrow{\sim} F^{p,j} \otimes \omega_M$$

given by $\xi \rightarrow \xi \otimes \epsilon$. In this case all $\bar{\partial}_M$-cohomology groups are the same, for twisted and untwisted $\bar{\partial}_M$-complexes.

8. - A remark on the $\bar{\partial}_M$-cohomology of nonorientable CR manifolds

Every nonorientable manifold M has a double covering $\pi : \tilde{M} \rightarrow M$ by an orientable manifold $\tilde{M}$. If M is an abstract CR manifold of type (n, k), then on any covering space of it there is a unique structure of an abstract CR manifold of type (n, k) such that the projection map is a local CR diffeomorphism. Somewhat more generally, we have:

PROPOSITION 3. *Let $\pi : \tilde{M} \rightarrow M$ be a finitely sheeted covering space of M. Then the induced homomorphisms:*

$$\pi^* : \mathrm{H}^{p,j}([\mathcal{E}](M)) \rightarrow \mathrm{H}^{p,j}([\mathcal{E}](\tilde{M})),$$

$$\pi^* : \mathrm{H}^{p,j}([\mathcal{D}](M)) \rightarrow \mathrm{H}^{p,j}([\mathcal{D}](\tilde{M}))$$

are injective, and the induced homomorphisms

$$\pi_* : \mathrm{H}^{p,j}([\mathcal{D}'](\tilde{M})) \rightarrow \mathrm{H}^{p,j}([\mathcal{D}'](M)),$$

$$\pi_* : \mathrm{H}^{p,j}([\mathcal{E}'](\tilde{M})) \rightarrow \mathrm{H}^{p,j}([\mathcal{E}'](M))$$

are surjective for $0 \leq p \leq n + k$ and $0 \leq j \leq n$.

PROOF. Suppose we have an m-fold cover $\tilde{M}$ of M. We define a map

$$\pi_\sharp : [\mathcal{E}^{p,j}](\tilde{M}) \rightarrow [\mathcal{E}^{p,j}](M)$$

by

$$(\pi_\sharp f)(x) = \frac{1}{m} \sum_{y \in \pi^{-1}(x)} f(y),$$

which satisfies $\bar{\partial}_M \circ \pi_\sharp = \pi_\sharp \circ \bar{\partial}_{\tilde{M}}$. Thus $\pi_\sharp$ induces maps $\pi_\sharp^*$ and $\pi_{\sharp*}$ on cohomology. Since $\pi_\sharp \circ \pi^* = id$ on smooth forms, we obtain that $\pi_{\sharp*} \circ \pi^* = id$ on cohomology

of smooth forms and $\pi_* \circ \pi_\sharp^* = id$ on distribution cohomology. Therefore π^* is injective and π_* is surjective.

REMARK. In particular, for a nonorientable CR manifold M, if a certain $\bar\partial_M$ cohomology group of its orientable double covering $\tilde M$ is finite dimensional (vanishes), then the corresponding $\bar\partial_M$-cohomology group of M is also finite dimensional (vanishes). This holds for either smooth or distribution cohomology, with or without compact supports.

9. - Duality on compact abstract CR manifolds

(a) Let

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C$$

be a complex of locally convex topological vector spaces and continuous linear maps, and let

$$A' \xleftarrow{\alpha'} B' \xleftarrow{\beta'} C'$$

be the dual complex. Here the primes denote the dual spaces (continuous linear functionals) and the dual (transposed) linear maps. We have $\alpha' \circ \beta' = 0$ since $\beta \circ \alpha = 0$. There is a natural map

$$\frac{\ker \alpha'}{\operatorname{im} \beta'} \xrightarrow{\sigma} \left(\frac{\ker \beta}{\operatorname{im} \alpha} \right)'$$

induced by $\sigma : b' \to \langle b', b \rangle$, where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between B' and B. The following proposition is well known as the *duality lemma* (see [S], [AK], [AB], [AHLM]).

PROPOSITION 4.
(1) *The map σ is always surjective.*

(2) *If β is a topological homomorphism, then σ is an isomorphism and $\operatorname{im} \beta'$ is weakly closed.*

(3) *Assume that A, B, C are either all Fréchet-Schwartz or else all duals of Fréchet-Schwartz. Then if $\ker \beta / \operatorname{im} \alpha$ is finite dimensional, it follows that $\operatorname{im} \alpha$ is closed. Hence*

$$\dim \left(\frac{\ker \beta}{\operatorname{im} \alpha} \right)' = \dim \left(\frac{\ker \beta}{\operatorname{im} \alpha} \right) < \infty.$$

(4) *With the same hypothesis as (3), assume moreover that $\operatorname{im} \beta$ is closed. Then $\ker \alpha' / \operatorname{im} \beta'$ is finite dimensional, and*

$$\dim \left(\frac{\ker \alpha'}{\operatorname{im} \beta'} \right) = \dim \left(\frac{\ker \beta}{\operatorname{im} \alpha} \right).$$

(b) Next we recall the results about the cohomology of compact abstract CR manifolds proved in [HN1]:

THEOREM 5. (i) *Let M be a compact q-pseudoconcave abstract CR manifold of type (n, k). Then for all $0 \leq p \leq n + k$ and for $j < q$ and $j > n - q$, the cohomology groups $H^{p,j}([\mathcal{E}](M)$ are finite dimensional. Moreover, $H^{p,q}([\mathcal{E}](M))$ is Hausdorff.*

(ii) *Let M be a compact abstract CR manifold of type $(n, 1)$, i.e. of hypersurface type. Assume that for every $\omega \in H^{\circ}M$, $\omega \neq 0$, the Levi form $L(\omega, \cdot)$ has either at least $j + 1$ negative or at least $n - j + 1$ positive eigenvalues. Then $H^{p,j}([\mathcal{E}](M))$ is finite dimensional for $0 \leq p \leq n + k$.*

Using the duality lemma above, we obtain:

THEOREM 6. (i) *Let M be as in (i) of Theorem 5. Then for all $0 \leq p \leq n+k$ and for $j < q$ and $j > n - q$, the cohomology groups $H^{p,j}([\mathcal{D}'](M))$ are finite dimensional and*

$$H^{p,j}([\mathcal{D}'](M)) = (H^{n+k-p,n-j}([\mathcal{E}](M)))'.$$

Moreover $H^{p,n-q}([\mathcal{D}'](M))$ is Hausdorff and

$$H^{p,n-q}([\mathcal{D}'](M)) = (H^{n+k-p,q}([\mathcal{E}](M)))',$$

and hence is the dual of a Fréchet-Schwartz space.

(ii) *Let M be of hypersurface type as in (ii) of Theorem 5, but assume that for each $x \in M$ there is a choice of $\xi \in H_x M$ such that the Levi form $L(\xi, \cdot)$ has at least r positive and q negative eigenvalues, with $q \leq r$. Then for all $0 \leq p \leq n + k$, the cohomology groups $H^{p,j}([\mathcal{D}'](M))$ are finite dimensional for $0 \leq j < q$, $n - r < j < r$, $j > n - q$. Moreover $H^{p,n-q}([\mathcal{D}'](M))$ is Hausdorff if $2r > n$, and*

$$H^{p,j}([\mathcal{D}'](M)) = (H^{n+k-p,n-j}([\mathcal{E}](M)))'$$

for $0 \leq j < q$, $n - r < j < r$, and $j \geq n - q$; and hence are duals of Fréchet-Schwartz spaces.

REMARK. We obtain the same results for the twisted $\bar{\partial}_M$-cohomology, substituting in the statements $\mathcal{E} \otimes \omega_M$ for $\mathcal{E}$ and $\mathcal{D}' \otimes \omega_M$ for $\mathcal{D}'$.

PROOF. We obtain Theorem 6 upon combining the results of Theorem 5 with the duality lemma.

10. - Distribution cohomology for pseudoconcave CR manifolds that are q-pseudoconcave at infinity

We have the following regularity theorem for distribution cohomology.

PROPOSITION 5. *Let M be a q-pseudoconcave manifold of type (n, k). Then for all $0 \leq p \leq n + k$ and $j < q$ we have the isomorphisms*

$$H^{p,j}([\mathcal{E}](M)) \simeq H^{p,j}([\mathcal{D}' \otimes \omega_M](M)),$$

$$H^{p,j}([\mathcal{E} \otimes \omega_M](M)) \simeq H^{p,j}([\mathcal{D}'](M)).$$

PROOF. Since the Poincaré lemma is valid for each of the sheaf complexes $[\mathcal{E}^{p,*}]$ and $[\mathcal{D}'^{p,*} \otimes \omega_M]$, up to level $q-1$, (see [AiHe], [N], [NV]), we obtain two fine resolutions, of lenght q, of the same sheaf Ω^p_M. Hence the first isomorphism follows from the abstract de Rham theorem. The second isomorphism is proved in the same way, using two fine resolutions, up to level q, of the sheaf $\Omega^p_M \otimes \omega_M$.

By the regularity theorem it is possible to translate finiteness theorems about smooth $\bar{\partial}_M$-cohomology into finiteness theorems about distribution $\bar{\partial}_M$-cohomology.

THEOREM 7. *Let M be a q-pseudoconcave CR manifold of type (n, k) which is also q-pseudoconcave at infinity. Then for all $0 \leq p \leq n + k$ and $j < q$ the cohomology groups $H^{p,j}([\mathcal{D}'](M))$ and $H^{p,j}([\mathcal{D}' \otimes \omega_M](M))$ are finite dimensional.*

PROOF. This is just a translation of [Theorem 5.1 from HN1].

The above theorem allows us to complement our Theorem 1 as follows:

THEOREM 8. *Let M be a q-pseudoconcave CR manifold of type (n, k) which is also q-pseudoconcave at infinity. Moreover assume that M is generically embedded as a closed CR submanifold of an $(n + k)$-dimensional complex manifold X. Then for all $0 \leq p \leq n + k$ the natural restriction maps*

$$H^{p,j}(\mathcal{D}'(X)) \overset{r}{\to} H^{p,j}(\check{\mathcal{D}}'_M(X))$$

have finite dimensional kernels and cokernels for $0 \leq j < q + k - 1$.

PROOF. We use Theorem 7 and the long exact sequence (4.4).

11. - By-passes for pseudoconcave CR manifolds and pseudoconvexity at infinity

In [HN1], [HN2] we were interested in relating the smooth $\bar{\partial}_M$-cohomology on M to the cohomology of an appropriate tubular neighborhood U of M, and we found it convenient to discuss the Cauchy problem for cohomology classes. In this section we solve a dual version of the Cauchy problem for distribution cohomology classes. While we used in [HN2] aneurysms of pseudoconcave CR manifolds, as analogous of the bumps technique in

[AG], here the main tool are *by-passes*. First we prove the by-pass lemma for distribution cohomology.

THEOREM 9. *Let M be a q-pseudoconcave CR manifold of type (n, k). Then we can find an embedding of M as a closed generic CR submanifold of an $(n+k)$-dimensional complex manifold X in such a way that for all $0 \le p \le n+k$ and $j \ge n - q$ the residue maps:*

$$\mathrm{H}^{p,j+k}(\check{\mathcal{D}}'_M(X)) \overset{Res}{\to} \mathrm{H}^{p,j+1}([\mathcal{D}'](M))$$

be surjective.

PROOF. First we note that we can assume that M is orientable. Indeed, if M is not orientable, we consider its orientable double covering $\tilde{M}$. Then we can find tubular neighborhoods $\tilde{U}$ of $\tilde{M}$ and U of M such that $\tilde{U}$ is a double covering of U and we have a commutative diagram with surjective vertical arrows:

$$\mathrm{H}^{p,j+k}(\check{\mathcal{D}}'_{\tilde{M}}(\tilde{U})) \overset{Res}{\to} \mathrm{H}^{p,j+1}([\mathcal{D}'](\tilde{M}))$$
$$\downarrow \qquad\qquad\qquad \downarrow$$
$$\mathrm{H}^{p,j+k}(\check{\mathcal{D}}'_M(U)) \overset{Res}{\to} \mathrm{H}^{p,j+1}([\mathcal{D}'](M)).$$

Therefore, the statement is true for M if it is true for $\tilde{M}$.

Hence we assume in the following that M is orientable and embedded as a generic closed CR submanifold in an $(n + k)$-dimensional complex manifold $\tilde{X}$. By substituting to $\tilde{X}$ a smaller tubular neighborhood of M in $\tilde{X}$, we can assume that M is the generic intersection in $\tilde{X}$ of k transversal hypersurfaces $\Sigma_1, \ldots, \Sigma_k$:

$$M = \Sigma_1 \cap \ldots \cap \Sigma_k.$$

Let

$$N = \Sigma_1 \cap \ldots \cap \Sigma_{k-1}.$$

This is a closed q-pseudoconcave CR-submanifold of type $(n + 1, k - 1)$ of a (maybe smaller) tubular neighborhood of $\tilde{X}$, that we still denote by $\tilde{X}$. Assume now that N is orientable. Then, after shrinking N if necessary, we can assume that M is defined in N by an equation

$$M = \{x \in N \mid \rho(x) = 0\}$$

for a smooth real-valued function ρ in N with $\bar{\partial}_N \rho \neq 0$ in N. Using the Poincaré Lemma for distribution cohomology in [NV], we obtain:

We can find a covering of M by a sequence of relatively compact open sets $\{W_\nu\}$ of $\tilde{X}$, such that $\{\bar{W}_\nu\}$ is locally finite, a partition of unity $\{\phi_\nu\}$ on M by positive real valued smooth functions subordinated to the covering $\{W_\nu\}$ and a sequence $\{\epsilon_\nu\}$ of positive real numbers such that the following holds true. For

$$M^{(\nu)} = \left\{ x \in N \mid \rho(x) + \sum_{\mu < \nu} \epsilon_\mu \phi_\mu(x) = 0 \right\},$$

(1) $M^{(\nu)}$ is a smooth q-pseudoconcave CR manifold of type (n, k);

(2) $H^{p,j}(\mathcal{D}'_{M^{(\nu)}}(\tilde{X})) \to H^{p,j}(\mathcal{D}'_{M^{(\nu)}}(W_\nu))$ is the zero map for all $0 \le p \le n + k$ and for $j > n + k - q$.

Let us fix $j > n + k - q$. Given $\xi \in \mathcal{D}^{\prime p,j}_M(\tilde{X})$ with $\bar\partial \xi = 0$, we can construct sequences

$$\{\xi_\nu\} \subset \mathcal{D}^{\prime p,j}(\tilde{X}), \quad \{\eta_\nu\} \subset \mathcal{D}^{\prime p,j-1}(\tilde{X})$$

such that

(1) $\xi_0 = \xi$,

(2) $\operatorname{supp} \xi_\nu \subset M^{(\nu)}$,

(3) $\operatorname{supp} \eta_\nu \subset M^{(\nu)} \cap \bar{W}_\nu$,

(4) $\xi_{\nu+1} = \xi_\nu - \bar\partial \eta_\nu$.

Since $\{\operatorname{supp} \eta_\nu\}$ is a locally finite family, the series $\sum_{\nu=0}^\infty \eta_\nu$ defines a distribution $\eta_\infty \in \mathcal{D}^{\prime p,j-1}_N(\tilde{X})$ and the distribution

$$\xi_\infty = \xi - \bar\partial \eta_\infty$$

has support contained in

$$M^{(\infty)} = \left\{ x \in N \mid \rho(x) + \sum_{\nu=0}^\infty \epsilon_\nu \phi_\nu(x) = 0 \right\}.$$

We note that $M^{(\infty)}$ is a closed q-pseudoconcave CR submanifold of type (n, k) of N and that

$$M^{(\infty)} \cap M = \emptyset.$$

If we set $X = \tilde{X} - M^{(\infty)}$ we therefore obtain an open neighborhood of M in $\tilde{X}$. The restriction of η_∞ to $X - M$ defines a cohomology class in $H^{p,j-1}(\check{\mathcal{D}}'_M(X))$ which maps into the cohomology class defined by ξ in $H^{p,j}(\mathcal{D}'_M(X))$.

When N is not orientable, we consider the orientable double covering $\tilde{N}$ of N and the lifting $\tilde{M}$ of M to $\tilde{N}$. By using Proposition 3, we obtain a commutative diagram:

$$
\begin{array}{ccc}
H^{p,j-1}([\check{\mathcal{D}}'_{\tilde{M}}](\tilde{N})) & \to & H^{p,j}(\mathcal{D}'_{\tilde{M}}(\tilde{N})) \\
\downarrow & & \downarrow \\
H^{p,j-1}([\check{\mathcal{D}}'_M](N)) & \to & H^{p,j}(\mathcal{D}'_M(N))
\end{array}
$$

in which the vertical arrows are surjective. Then the proof reduces to the case where N is orientable. The proof is complete.

We refer to $M^{(\nu+1)}$ in the proof as a *by-pass* of $M^{(\nu)}$.

We use Theorem 9 to prove a regularity theorem giving a comparison between smooth and distribution $\bar\partial_M$-cohomology in degree $> n - q$ for a q-pseudoconcave CR manifold of type (n, k) (note that the different

cohomologies were proved to coincide in degree $j < q$). This is suitable to obtain finiteness results in this range.

THEOREM 10. *Let M be a q-pseudoconcave CR manifold of type (n, k). Then the natural maps:*

$$H^{p,j}([\mathcal{E}](M)) \to H^{p,j}([\mathcal{D}' \otimes \omega_M](M))$$

and

$$H^{p,j}([\mathcal{E} \otimes \omega_M](M)) \to H^{p,j}([\mathcal{D}'](\dot{M}))$$

are surjective for all $0 \le p \le n + k$ and for $j > n - q$.

PROOF. We use induction on the CR codimension k. The statement is true when $k = 0$. Assume now that $k > 0$ and the theorem holds true for q-pseudoconcave CR manifolds of type (n, ℓ) for every n and for $\ell < k$. Let $j > n - q$ be fixed. By using the argument at the beginning of the proof of Theorem 9, we can assume that M is orientable. Again arguing as in the proof of Theorem 9 we can suppose that M is a hypersurface inside a q-pseudoconcave CR manifold N, of type $(n + 1, k - 1)$, which is orientable and divided by M into two disjoint open subsets N_+ and N_-, with

$$M \subset N, \quad N - M = N_+ \cup N_-, \quad \bar{N}_+ \cap \bar{N}_- = M.$$

The construction of Theorem 9 can now be carried out taking by-passes $M^{(\nu)}$ all contained in N_-. In this way we obtain a surjective map, for some tubular neighborhood X of M:

$$H^{p,j+k-1}(\check{\mathcal{D}}'_M \cap \mathcal{D}'_{N_-}(X)) \to H^{p,j}([\mathcal{D}'](M)).$$

By Proposition 1 we have an identification

$$H^{p,j+k-1}(\check{\mathcal{D}}'_M \cap \mathcal{D}'_{N_-}(X)) \simeq H^{p,j}([\check{\mathcal{D}}'_M](N_-)),$$

were we assumed, as we can, that $N_- = N_- \cap X$.

By applying the technique of [NV] (cf also [Proposition 9 of N2] [N1], [N3]) we obtain that that every point x_0 of M has an open neighborhood U_{x_0} in N such that the natural restriction map

$$H^{p,j}([\check{\mathcal{D}}'](N_-)) \to H^{p,j}([\check{\mathcal{D}}'](N_- \cap U_{x_0}))$$

is the zero map. Using the bump lemma of [AG] implemented by the arguments in [HN2] and [HN3], we obtain an open neighborhood V of $\bar{N}_-$ in N such that the restriction map

$$H^{p,j}([\mathcal{D}'](V)) \to H^{p,j}([\check{\mathcal{D}}'](N_-))$$

is onto. By the inductive assumption we obtain by composition a surjective map

$$H^{p,j}([\mathcal{E} \otimes \omega_M](V)) \to H^{p,j}([\check{\mathcal{D}}'](N_-)).$$

Since we have a commutative diagram

$$
\begin{array}{ccc}
H^{p,j}([\mathcal{E} \otimes \omega_N](V)) & \to & H^{p,j}([\check{\mathcal{D}}'](N_-)) \\
\downarrow & & \downarrow \\
H^{p,j}([\mathcal{E} \otimes_M](M)) & \to & H^{p,j}([\mathcal{D}'](M))
\end{array}
$$

where the map

$$H^{p,j}([\mathcal{E} \otimes \omega_N](V)) \to H^{p,j}([\mathcal{E} \otimes \omega_M](M))$$

is obtained by taking the quotient of the pull-back on M of a smooth form representative of the cohomology class on V, the statement follows.

As a corollary, we obtain from [HN2]:

THEOREM 11. *Let M be a q-pseudoconcave CR manifold of type (n, k) which is also $n - q$-pseudoconvex at infinity (complete). Then the cohomology groups*

$$H^{p,j}([\mathcal{D}'](M)) \quad \text{and} \quad H^{p,j}([\mathcal{D}' \otimes \omega_M](M))$$

are finite dimensional (vanish) for all $0 \leq p \leq n+k$ and for $j > n - q$. Moreover in the same range we have the inequalities:

$$\dim H^{p,j}([\mathcal{D}'](M)) \leq \dim H^{p,j}([\mathcal{E} \otimes \omega_M](M)) \leq \dim H^j(M, \Omega^p \otimes \omega_M),$$

$$\dim H^{p,j}([\mathcal{D}' \otimes \omega_M](M)) \leq \dim H^{p,j}([\mathcal{E}](M)) \leq \dim H^j(M, \Omega^p).$$

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Department of Mathematics
SUNY at Stony Brook
Stony Brook
NY 11794 - U.S.A.

Dipartimento di Matematica
Università di Pisa
Via F. Buonarroti, 2
56127 PISA - ITALY