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# A relation between the Riemann zeta-function and the hyperbolic laplacian

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# A Relation between the Riemann Zeta-Function and the Hyperbolic Laplacian

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## 1. - Introduction

The aim of the present paper is to show a meromorphic continuation of the function

$$Z_2(\xi) = \int_1^\infty \left|\zeta\left(\frac{1}{2}+it\right)\right|^4 t^{-\xi} dt.$$

to the entire complex plane, where $\zeta$ stands for the Riemann zeta-function. It will turn out in particular that there are infinitely many simple poles on the straight line $\Re(\xi) = \frac{1}{2}$, and that they are contained in the set of the complex zeros of the Selberg zeta-function for the full modular group. As an application of this fact we shall prove a new omega result for $E_2(T)$, the remainder term in the asymptotic formula for the fourth power mean of the Riemann zeta-function.

We start our discussion with a brief survey of results on $E_2(T)$, since its study was our original motivation. Thus we have, by definition,

$$\int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^4 dt = TP_4(\log T) + E_2(T)$$

with a certain polynomial $P_4$ of degree four. We also put

$$I(T,\Delta) = \int_{-\infty}^{\infty} \left|\zeta\left(\frac{1}{2}+i(T+t)\right)\right|^4 e^{-(t/\Delta)^2} dt,$$

where $T$ and $\Delta$ are arbitrary positive parameters. In our former paper [8] an explicit formula for $I(T,\Delta)$ has been proved, which yields, among other things,

$$E_2(T) \ll T^{\frac{2}{3}}(\log T)^c \tag{1.1}$$

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with an explicitly computable $c > 0$. Although this has some significance, it is far from what is generally believed to be true about the size of $E_2(T)$: It is conjectured that

$$E_2(T) \ll T^{\frac{1}{2}+\varepsilon} \tag{1.2}$$

would hold for any fixed $\varepsilon > 0$. This must be extremely difficult to prove, for it would imply a deep estimate for the Riemann zeta-function on the critical line. But it should be stressed that the mean square estimate

$$\int_0^V E_2(T)^2 dT \ll V^2(\log V)^c, \tag{1.3}$$

which supports the conjecture (1.2), has been proved in our joint paper [4] with Ivić as another consequence of the explicit formula for $I(T,\Delta)$.

On the other hand, as for the lower bound of $E_2(T)$ a result that appears to be essentially the best possible has already been established. In fact an assertion in [5] (see also [3, Theorem 5.7]) states that

$$E_2(T) = \Omega(\sqrt{T}). \tag{1.4}$$

We maintain that (1.4) is more significant than the upper bound (1.1). The reason for this is explained in our survey articles [9] [12]; here we say only that (1.4) seems to reflect a highly peculiar nature of the Riemann zeta-function that should be discussed with the Riemann hypothesis in the background.

In general, proofs of omega results require representations of the relevant number theoretical quantities that are *explicit* in a certain sense. For instance, without the explicit formula for the Tchebychev function in the theory of prime numbers or the Voronoï formula for the sum of the number of divisors it would not be possible to discuss the omega properties in the distribution of primes and divisors. This applies to (1.4) as well. For, its proof depends on the explicit formula for $I(T,\Delta)$.

Our proof of (1.4) contains, however, a drawback. It does not seem to be able to yield a two-sided omega result for $E_2(T)$ that is naturally inferred from the experience in the theory of the mean square of the Riemann zeta-function. A reason of this shortcoming lies in the fact that in [5] we employed the process of taking a multiple average of the explicit formula for $I(T,\Delta)$ with respect to the parameter $T$. This made it difficult for us to handle the problem of large deviations of the size of $E_2(T)$ while tracing its sign changes.

Now, the meromorphic continuation of $Z_2(\xi)$ yields a comparatively direct approach to the problem, and we are able to prove the following improvement upon (1.4):

THEOREM 1.

$$E_2(T) = \Omega_{\pm}(\sqrt{T}). \tag{1.5}$$

This confirms the conjecture expressed by several people, especially by Ivić [3, (5.183)]. We think that there is a possibility that our new argument might yield even

$$E_2(T) = \Omega_\pm(k(T)\sqrt{T})$$

with an explicit $k(T)$ that increases monotonically to infinity with $T$. But this will definitely require some deep facts about the distribution of discrete eigenvalues of the hyperbolic Laplacian that are not available currently.

The proof of (1.5) is achieved by the combination of a version of Landau's lemma (Lemma 1 below) and an explicit configuration of non-trivial poles of $Z_2(\xi)$. The latter is, in turn, implied by a spectral decomposition of $Z_2(\xi)$. More precisely, we have:

THEOREM 2. *The function $Z_2(\xi)$ is meromorphic over the entire complex plane. In particular, in the half-plane $\Re(\xi) > 0$ it has a pole of order five at $\xi = 1$ and infinitely many simple poles of the form $\frac{1}{2} \pm \kappa i$; all other poles are of the form $\rho/2$. Here $\kappa^2 + \frac{1}{4}$ is in the discrete spectrum of the hyperbolic Laplacian with respect to the full modular group, and $\rho$ is a complex zero of the Riemann zeta-function.*

It should be noted that this assertion depends on a non-vanishing theorem [7, Theorem 3] about special values of automorphic $L$-functions, as it is to be shown in our explicit computation of the residues at $\frac{1}{2} \pm \kappa i$ in the final section. We remark also that this theorem makes clearer the relation between the Riemann zeta-function and the hyperbolic Laplacian than the theorem of [8] does.

The proof of Theorem 1 is immediate, if once we prove Theorem 2. For, we have:

LEMMA 1. *Let $g(x)$ be a continuous function such that*

$$G(\xi) = \int_1^\infty g(x)x^{-\xi-1}dx$$

*converges absolutely for a $\xi$. Let us suppose that $G(\xi)$ admits an analytic continuation to a domain including the half line $[\sigma, \infty)$, while having a simple pole at $\sigma + i\delta$, $\delta \neq 0$, with the residue $\gamma$. Then we have*

$$\limsup_{x\to\infty} g(x)x^{-\sigma} \geq |\gamma|, \quad \liminf_{x\to\infty} g(x)x^{-\sigma} \leq -|\gamma|.$$

This is a version of Landau's lemma; for the proof see e.g., [1]. Theorem 2 implies that we may set $g(x) = E_2(x)$; hence (1.5) follows.

It is appropriate to remark here that there exists a close resemblance between the problems on $E_2(T)$ and those on the binary additive divisor problem

$$D(N,a)=\sum_{n=1}^{N} d(n)d(n+a),$$

where $d$ is the divisor function and $a>0$. A comprehensive account on both $D(N,a)$ and its dual version can be found in our recent work [10]. There, for example, an analog of (1.4) is proved. But, recently Szydło [14] replaced it with a two sided omega result. While our work [10] relied on Kuznetsov's trace formulas, Szydło took the approach that was devised by Takhtadjan and Vinogradov [13] (cf., Jutila [6]). They considered a spectral decomposition of the additive divisor zeta-function

$$D_a(\xi)=\sum_{n=1}^{\infty} d(n)d(n+a)n^{-\xi},$$

which corresponds to our $Z_2(\xi)$. Szydło observed that this spectral decomposition could be combined with Landau's lemma to produce a two sided omega result for $D(N,a)$. Our present paper is a result of the stimulus that we got from a talk of Szydło at the Meigaku Seminar in Tokyo. However, our argument is essentially a variant of our former work [8], and independent from Takhtadjan and Vinogradov's work. It appears to us that their argument may not easily be extended so as to be able to cope with the fourth power moment of the Riemann zeta-function. Also, we stress that our argument developed in [10] is able to yield a spectral decomposition of $D_a(\xi)$ that contains more explicit details than Takhtadjan and Vinogradov's work could show. In fact, we need only to set $W(x)=x^L(x+1)^{-\xi-L}$ in Theorem 3 of [10], where $L$ is to be sufficiently large. Although this weight does not have a compact support that is required there, it is easy to see that the argument of [10] is applicable to it as well.

It should also be remarked that in [8, p. 182] (cf. [9] [12]) we gave a tentative explanation why there is a connection between the Riemann zeta-function and the hyperbolic Laplacian, though we formulated it in terms of generic four-fold sums over integers since the fourth power mean problem is reduced to such a situation. There a four-fold sum is regarded as a sum over $2\times 2$ integral matrices, and the summands are classified according to the value of determinant, which amounts to a natural extension, to four-fold sums, of Atkinson's dissection argument [2] for double sums. Then we appeal to Hecke's classification of integral matrices of given determinant with respect to the full modular group. In this way a four-fold sum can be taken for a sum of sums over the full modular group; to each of the inner-sums we may apply the spectral decomposition, and the outer-sum may be analyzed with the theory of Hecke operators.

The above reveals, to some extent, a structure behind our Theorem 2. On the other hand it might give the impression that only the fourth, but no higher, power mean of the Riemann zeta-function should have to do with the spectral analysis of the full modular group. But this is not correct. For, we have recently shown in [11] that the eighth power mean

$$\int_{-\infty}^{\infty} \left|\zeta\left(\frac{1}{2}+i(T+t)\right)\right|^8 e^{-(t/\Delta)^2} dt$$

admits an expression in terms of the objects pertaining to automorphic forms over the full modular group.

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## 2. - Spectral decomposition

Now we shall show a meromorphic continuation of $Z_2(\xi)$ to the region $\Re(\xi) > 0$; further continuation may be left out, since it is just a matter of technicality.

To this end we consider the expression

$$Y(\xi, D) = \int_{-\infty}^{\infty} \left|\zeta\left(\frac{1}{2}+it\right)\right|^4 (t^2+D^2)^{-\xi} dt,$$

where both $\Re(\xi)$ and $D$ are assumed to be positive and large, at least initially. Obviously we have, for any $D > 0$,

$$Z_2(\xi) = \frac{1}{2} Y(\xi/2, D) + h(\xi, D),$$

where $h(\xi, D)$ is a function regular for $\Re(\xi) > 0$. Also we have, by partial integration,

$$Y(\xi, D) = p_5\left(\left(\xi - \frac{1}{2}\right)^{-1}\right) + h(\xi, D) + 4\xi \int_{0}^{\infty} tE_2(t)(t^2+D^2)^{-\xi-1} dt, \tag{2.1}$$

where $p_5$ is a polynomial of fifth degree with constant coefficients. The assertion (1.3) implies that the last integral is uniformly convergent for $\Re(\xi) > \frac{1}{4}$. Thus we see that $Z_2(\xi)$ is regular for $\Re(\xi) > \frac{1}{2}$ except for the pole at $\xi = 1$ of order five, though the argument below will yield the same in a different way.

To continue $Z_2(\xi)$ beyond the line $\Re(\xi) = \frac{1}{2}$, we need a spectral decomposition of $Y(\xi, D)$. One way to achieve it is to incorporate [8, Theorem] into the relation

$$Y(\xi, D) = \frac{2}{\Gamma(\xi)} \int_0^\infty I(0,\Delta) e^{-(D/\Delta)^2} \Delta^{-2\xi-1} d\Delta.$$

Here an obvious modification of the statement of [8, Theorem] is needed because there the parameters $T$ and $\Delta$ are to satisfy a constraint that is not fulfilled by the present supposition. But we shall not take this approach. For, we think that the spectral decomposition of $Y(\xi, D)$ is to be understood in a more general context.

Thus we introduce

$$Z_2(f) = \int_{-\infty}^{\infty} \left| \zeta\left(\frac{1}{2} + it\right)\right|^4 f(t) dt,$$

which may be termed the biquadratic zeta-transform of the function $f$. A careful analysis of the argument developed in sections 2 to 5 of [8] will reveal that it works verbatim for $Z_2(f)$ as well, provided $f$ satisfies the following conditions:

$C_0$ : $f(t)$ is even.

$C_1$ : There exists a *large* positive constant $L$ such that $f$ is regular and $O((1+|t|)^{-L})$ in the horizontal strip $|\Im(t)| < L$.

The first condition is introduced only for the sake of simplicity. As for the second, we note that it is possible to work with rather small values of $L$, but the stringent condition given above seems to be sufficient for most purposes.

To state the spectral decomposition of $Z_2(f)$ we have to introduce some notions. Thus, we first define briefly the standard symbols from the theory of automorphic forms; for the details see [8]: We denote by $\left\{\lambda_j = \kappa_j^2 + \frac{1}{4}; \kappa_j > 0, j = 1, 2, \ldots\right\} \cup \{0\}$ the discrete spectrum of the hyperbolic Laplacian acting on the space of all non-holomorphic automorphic functions with respect to the full modular group. Let $\varphi_j$ be the Maass wave form attached to the eigenvalue $\lambda_j$ so that $\{\varphi_j\}$ forms an orthonormal base of the space spanned by all cusp forms, and each $\varphi_j$ is an eigen-function of every Hecke operator $T(n)$ $(n \geq 1)$. The latter means that there exists a certain real number $t_j(n)$ such that $T(n)\varphi_j = t_j(n)\varphi_j$. With the first Fourier coefficient $\rho_j$ of $\varphi_j$ we put $\alpha_j = |\rho_j|^2 (\cosh \pi\kappa_j)^{-1}$. The Hecke $L$-series $H_j(s)$ attached to $\varphi_j$ is defined by

$$H_j(s) = \sum_{n=1}^{\infty} t_j(n) n^{-s}, \quad (\Re(s) > 1).$$

This is actually an entire function, and of polynomial order with respect to $\kappa_j$ if $s$ is bounded.

As for the holomorphic cusp forms, we let $\{\varphi_{j,k}; 1 \le j \le \vartheta(k)\}$ stand for the orthonormal base, consisting of eigen functions of all Hecke operators $T_k(n)$, of the Petersson unitary space of holomorphic cusp forms of weight $2k$ with respect to the full modular group. This means, in particular, that we have $T_k(n)\varphi_{j,k} = t_{j,k}(n)\varphi_{j,k}$ with a certain real number $t_{j,k}(n)$. With the first Fourier coefficients $\rho_{j,k}$ of $\varphi_{j,k}$ we put $\alpha_{j,k} = 16\Gamma(2k)(4\pi)^{-2k-1}|\rho_{j,k}|^2$. As before we define the Hecke $L$-series $H_{j,k}(s)$ by

$$H_{j,k}(s) = \sum_{n=1}^{\infty} t_{j,k}(n)n^{-s}, \qquad (\Re(s) > 1).$$

Again this is entire, and in any fixed vertical strip it is of polynomial order with respect to the weight, uniformly for the index $j$, if $s$ is bounded.

Further we need to define a transform of the function $f$ that satisfies the conditions $C_0$ and $C_1$. Thus we put, for $\Re(\eta) > -\frac{1}{2}$,

$$\Xi(\eta; f) = \frac{\Gamma\left(\frac{1}{2}+\eta\right)^2}{\Gamma(1+2\eta)} \int_0^{\infty} x^{\eta-\frac{1}{2}}(1+x)^{-\frac{1}{2}} \tag{2.2}$$

$$\times F\left(\frac{1}{2}+\eta, \frac{1}{2}+\eta; 1+2\eta; -x\right) \hat{f}(\log(1+x))dx,$$

where $F$ is the hypergeometric function, and $\hat{f}$ the Fourier transform of $f$:

$$\hat{f}(x) = \int_{-\infty}^{\infty} f(t)e^{-ixt}dt.$$

We then put, for real $r$,

$$\Theta(r; f) = \frac{1}{2}\left(1 + \frac{i}{\sinh(\pi r)}\right)\Xi(ir; f) + \frac{1}{2}\left(1 - \frac{i}{\sinh(\pi r)}\right)\Xi(-ir; f).$$

Now, the spectral decomposition of $Z_2(f)$ runs as

LEMMA 2. *If $f$ satisfies the conditions $C_0$ and $C_1$, then we have*

$$
\begin{aligned}
Z_2(f) = &\int_{-\infty}^{\infty} [q_4(\log(t^2+1)) + h(t)] f(t)dt \\
&+ \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\left|\zeta\left(\frac{1}{2}+it\right)\right|^6}{|\zeta(1+2it)|^2}\Theta(t;f)dt + \sum_{j=1}^{\infty} \alpha_j H_j\left(\frac{1}{2}\right)^3 \Theta(\kappa_j;f) \\
&+ \sum_{k=6}^{\infty}\sum_{j=1}^{\vartheta(k)} \alpha_{j,k} H_{j,k}\left(\frac{1}{2}\right)^3 \Xi\left(k-\frac{1}{2};f\right),
\end{aligned}
\tag{2.3}
$$

*where $q_4$ is a polynomial of order four, and $h(t)$ is regular and $O((1+|t|)^{-1}(\log(|t|+2))^2)$ in the strip $|\Im(t)| < \frac{1}{2}$; moreover both are independent of $f$.*

Here we should remark that the condition $C_1$ implies the existence of a $c > 0$ such that

$$
\Xi(ir;f) \ll r^{-cL}, \quad \Xi(r;f) \ll r^{-cL} \tag{2.4}
$$

for large positive $r$. Since $H_j\left(\frac{1}{2}\right)$, $H_{j,k}\left(\frac{1}{2}\right)$ are of polynomial order and $\alpha_j$, $\alpha_{j,k}$ are constant on average, the sums and the integrals in (2.3) are all absolutely convergent. To show the estimates (2.4) briefly, we put, for $\Re(s) < \frac{1}{2}$,

$$
f^*(s) = \int_{-\infty}^{\infty} \frac{\Gamma\left(\frac{1}{2}s+it\right)}{\Gamma\left(\frac{1}{2}+it\right)} f(t)dt.
$$

By shifting the path vertically and applying Stirling's formula it can be seen that $f^*(s)$ is, in fact, regular and $O(\exp(\pi|s|/2)|s|^{-L/2})$ for $|\Re(s)| < L/2$, where $L$ is as in $C_1$. Then we have, instead of (2.2),

$$
\Xi(\eta;f) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\Gamma\left(\frac{1}{2}+\eta-s\right)}{\Gamma\left(\frac{1}{2}+\eta+s\right)} \Gamma(s)^3 f^*(s)ds, \tag{2.5}
$$

where the path separates the poles of $\Gamma\left(\frac{1}{2}+\eta-s\right)$ and $\Gamma(s)$ to the right and the left, respectively. If $\Re(\eta) > -\frac{1}{2}$ then the path can obviously be drawn. The assertion (2.4) is now a result of an application to (2.5) the above estimate of $f^*(s)$ and Stirling's formula.

## 3. - Specialization

We may now return to the function $Y(\xi, D)$. Thus we set $f(t)$ to be

$$f_\xi(t) = (t^2 + D^2)^{-\xi},$$

where the parameter $D$ is omitted on the left side for the sake of notational simplicity. Obviously this satisfies the conditions $C_0$ and $C_1$, provided both $D$ and $\Re(\xi)$ are large, though the condition on $\xi$ will be relaxed later. We stress that throughout this section $D$ is assumed to be large, and all statements below are possibly dependent on it.

Now, the spectral decomposition (2.3) gives, instead of (2.1)

$$Y(\xi, D) = p_5\left(\left(\xi - \frac{1}{2}\right)^{-1}\right) + h(\xi, D) + Y_c(\xi, D) + Y_d(\xi, D) + Y_h(\xi, D), \tag{3.1}$$

where the last three terms stand for the contributions of the continuous spectrum, the discrete spectrum, and the holomorphic cusp forms, respectively. To get an analytic continuation of this decomposition we need to prove the following assertion on the analytic property of $\Xi(\eta; f_\xi)$ as a function of two complex variables $\eta$ and $\xi$:

LEMMA 3. *Let us assume either that $\eta$ is in a fixed vertical strip where $\Re(\eta) \geq -\frac{1}{8}$, or that $\eta$ is on the half line $[1, \infty)$. Then, for any bounded $\xi$ with $\Re(\xi) \geq -\frac{1}{8}$ the function*

$$\Xi_1(\eta; f_\xi) = \left(\eta + 2\xi - \frac{1}{2}\right) \Xi(\eta; f_\xi)$$

*is regular and*

$$\Xi_1(\eta, \xi) \ll |\eta|^{-D/4}, \tag{3.2}$$

*provided $D$ is sufficiently large.*

We consider first the case where $\eta$ is in a vertical strip; thus we may assume that there is a positive constant $c$ such that

$$-\frac{1}{8} \leq \Re(\eta) \leq c \leq \Im(\eta) \tag{3.3}$$

and

$$-\frac{1}{8} \leq \Re(\xi) \leq c, \quad |\Im(\xi)| \leq c. \tag{3.4}$$

We note that for $x > 0$

$$\hat{f}_\xi(x) = 2\sqrt{\pi}(2D/x)^{\frac{1}{2}-\xi}K_{\xi-\frac{1}{2}}(Dx)/\Gamma(\xi),$$

where $K_\nu$ is the $K$-Bessel function of order $\nu$. Then, invoking Euler's integral representation of hypergeometric functions, we have

$$\Xi(\eta; f_\xi) = \int_0^\infty x^{\eta+2\xi-\frac{3}{2}} A(x,\xi)G(x,\eta)dx,$$

where

$$\begin{aligned}A(x,\xi) = {} & 2\sqrt{\pi}(2D)^{\frac{1}{2}-\xi}x^{1-2\xi}(1+x)^{-\frac{1}{2}}(\log(1+x))^{\xi-\frac{1}{2}} \\ & \times K_{\xi-\frac{1}{2}}(D\log(1+x))/\Gamma(\xi)\end{aligned} \tag{3.5}$$

and

$$G(x,\eta) = \int_0^1 (y(1-y))^{\eta-\frac{1}{2}}(1+xy)^{-\eta-\frac{1}{2}}dy.$$

The asymptotic property of the $K$-Bessel functions implies that $A(x,\xi)$ is of rapid decay when $x$ tends to infinity. On the other hand the behavior of $A(x,\xi)$ when $x$ is close to 0 can be inferred from the defining relation

$$K_\nu(z) = \frac{\pi}{2\sin(\pi\nu)}(I_{-\nu}(z) - I_\nu(z)), \tag{3.6}$$

where

$$I_\nu(z) = \sum_{n=0}^\infty \frac{\left(\frac{1}{2}z\right)^{\nu+2n}}{n!\Gamma(\nu+n+1)}. \tag{3.7}$$

More precisely, we have, for each $p \geq 0$,

$$\left(\frac{\partial}{\partial x}\right)^p A(x,\xi) \ll \begin{cases} (1+x^{1-p-2\Re(\xi)})\log(1/x) & \text{as } x \to +0, \\ x^{-D/2} & \text{as } x \to +\infty. \end{cases} \tag{3.8}$$

Also we have

$$\left(\frac{\partial}{\partial x}\right)^p G(x,\eta) \ll |\eta|^p, \tag{3.9}$$

uniformly for $x \geq 0$. Then the expression

$$\Xi_1(\eta; f_\xi) = -\int_0^\infty x^{\eta+2\xi-\frac{1}{2}} \frac{\partial}{\partial x}[A(x,\xi)G(x,\eta)]dx, \tag{3.10}$$

yields an analytic continuation of $\Xi_1(\eta; f_\xi)$ to the domain defined by the conditions (3.3) and (3.4).

To show the estimate (3.2) in the present case, we turn the line of integration in (3.10) through a small positive angle $\theta$. Thus we have

$$\Xi_1(\eta; f_\xi) = -\exp\left(\left(\eta + 2\xi - \frac{1}{2}\right)\theta i\right) \int_0^\infty x^{\eta+2\xi-\frac{1}{2}} \times \frac{\partial}{\partial x}[A(xe^{\theta i}, \xi)G(xe^{\theta i}, \eta)]dx. \tag{3.11}$$

The estimate (3.8) holds for $A(xe^{\theta i}, \xi)$ too; and also we have, instead of (3.9),

$$\left(\frac{\partial}{\partial x}\right)^p G(xe^{\theta i}, \eta) \ll |\eta|^p \exp(\Im(\eta) \arg(1 + xe^{\theta i})).$$

We then divide the integral in (3.11) into two parts corresponding to $0 \leq x \leq 1$ and the rest; accordingly we get a decomposition of $\Xi_1(\eta; f_\xi)$. If $\theta$ is small, the first part is

$$\ll |\eta| \exp\left(-\left(\theta - \arctan\left(\frac{\sin\theta}{1 + \cos\theta}\right)\right)\Im(\eta)\right)$$
$$\ll \exp\left(-\frac{\theta}{3}\Im(\eta)\right);$$

and the second part is

$$\ll |\eta| \int_1^\infty x^{-D/3} \exp\left(-\left(\theta \arctan\left(\frac{x\sin\theta}{1 + x\cos\theta}\right)\right)\Im(\eta)\right)dx$$
$$\ll |\eta| \int_1^\infty x^{-D/3} \exp\left(-\frac{\theta}{2x}\Im(\eta)\right)dx$$
$$\ll |\eta|^{-D/4}.$$

This obviously ends the proof of (3.2) in the case where (3.3) and (3.4) hold.

Next, we consider the case where $\eta \geq 1$, and the condition (3.4) holds. Here the regularity of $\Xi_1(\eta; f_\xi)$ is easy to check; thus let us prove the decay property (3.2) only. To this end we note that we now have, for $x \geq 0$,

$$G(x, \eta) \leq \int_0^1 \left(\frac{y(1-y)}{1 + xy}\right)^{\eta - \frac{1}{2}} dy$$
$$\leq (1 + \sqrt{1 + x})^{1-2\eta},$$

which can be shown by computing the maximum of the integrand. We insert this into (3.10). The part corresponding to $0 \le x \le 1$ is easily seen to be $O(2^{-\eta})$. The remaining part is

$$\ll \int_1^\infty x^{-D/3} \left( \frac{\sqrt{x}}{1+\sqrt{1+x}} \right)^{2\eta} dx,$$

which is much smaller than $\eta^{-D/2}$. This ends the proof of Lemma 3.

## 4. - Analytic continuation

We are now ready to finish the proof of Theorem 2. To this end we note first that we have

$$Y_h(\xi, D) = \sum_{k=6}^{\infty} \sum_{j=1}^{\vartheta(k)} \alpha_{j,k} H_{j,k} \left(\frac{1}{2}\right)^3 \Xi_1 \left(k - \frac{1}{2}; f_\xi\right) /(2\xi + k - 1).$$

Lemma 3 implies immediately that this sum is regular for $\Re(\xi) \ge -\frac{1}{8}$. On the other hand the contribution of the discrete spectrum is

$$Y_d(\xi, D) = Y_d^+(\xi, D) + Y_d^-(\xi, D),$$

where

$$Y_d^+(\xi, D) = \frac{1}{2} \sum_{j=1}^{\infty} \alpha_j H_j \left(\frac{1}{2}\right)^3 \left(1 + \frac{i}{\sinh(\pi\kappa_j)}\right) \Xi_1(i\kappa_j; f_\xi) / \left(2\xi - \frac{1}{2} + i\kappa_j\right)$$

and $Y_d^-(\xi, D)$ is the result of changing the sign of all $\kappa_j$ in $Y_d^+(\xi, D)$. Then Lemma 3 implies that $Y_d^+(\xi, D)$ is regular for $\Re(\xi) \ge -\frac{1}{8}$ except for the simple poles at $\xi = \frac{1}{2}\left(\frac{1}{2} - i\kappa\right)$, where $\kappa$ runs over the set of the distinct elements of $\{\kappa_j\}$ such that

$$M(\kappa) = \sum_{\kappa_j = \kappa} \alpha_j H_j \left(\frac{1}{2}\right)^3 \neq 0.$$

In our former paper [7] we have shown that there are infinitely many $\kappa$ that satisfy this non-vanishing condition. Thus $Y_d(\xi, D)$ has indeed infinitely many simple poles on the straight line $\Re(\xi) = \frac{1}{4}$.

Let us compute the residue $R(\kappa)$ of $Y_d(\xi, D)$ at the pole $\xi_\kappa = \frac{1}{2}\left(\frac{1}{2} - i\kappa\right)$. We have

$$R(\kappa) = \frac{1}{4} M(\kappa) \left(1 + \frac{i}{\sinh(\pi\kappa)}\right) \Xi_1(i\kappa; f_{\xi_\kappa}).$$

By (3.10) we have

$$\Xi_1(i\kappa; f_{\xi_\kappa}) = A(0, \xi_\kappa) G(0, i\kappa)$$
$$= A(0, \xi_\kappa) \frac{\Gamma\left(\frac{1}{2} + i\kappa\right)^2}{\Gamma(1 + 2i\kappa)}$$

But we have, by (3.5)-(3.7),

$$A(0, \xi) = \frac{\pi^{\frac{3}{2}} 2^{1-2\xi}}{\Gamma(\xi)\Gamma\left(\xi + \frac{1}{2}\right) \cos(\pi\xi)}$$

provided $\Re(\xi) < \frac{1}{2}$. Hence we have, after a rearrangement,

$$R(\kappa) = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \left( 2^{i\kappa} \frac{\Gamma\left(\frac{1}{2}\left(\frac{1}{2} + i\kappa\right)\right)}{\Gamma\left(\frac{1}{2}\left(\frac{1}{2} - i\kappa\right)\right)} \right)^3 \Gamma(-2i\kappa) \cosh(\pi\kappa) M(\kappa).$$

Next, we consider the contribution of the continuous spectrum. We have

$$Y_c(\xi, D) = \frac{1}{\pi i} \int_{(0)} \frac{\zeta\left(\frac{1}{2} + r\right)^3 \zeta\left(\frac{1}{2} - r\right)^3}{\zeta(1 + 2r)\zeta(1 - 2r)} \left(1 - \frac{1}{\sin(\pi r)}\right)$$
$$\times \Xi_1(r; f_\xi) \left(r - \frac{1}{2} + 2\xi\right)^{-1} dr,$$

where the path is the straight line $\Re(r) = 0$; note that here it is assumed that $\Re(\xi)$ is sufficiently large. Let $P$ be an arbitrary positive number such that $\zeta(s) \neq 0$ on the lines $\Re(s) = \pm 2P$, and also $|\Im(\xi)| < P/4$. We then move the path in the last integral to the one that is the result of connecting the points $-i\infty$, $\frac{3}{4} - iP$, $\frac{3}{4} + iP$, $iP$, $i\infty$ with straight lines. The resulting integral is regular for $\Re(\xi) \geq -\frac{1}{8}$ by virtue of Lemma 3. In this procedure we encounter poles at $r = \frac{1}{2}$ and $(1 - \rho)/2$, where $\rho$ runs over complex zeros of the zeta- function such that $|\Im(\rho)| < 2P$. As a function of $\xi$ the residue at $r = \frac{1}{2}$ is regular for $\Re(\xi) \geq -\frac{1}{8}$, except for the point $\xi = 0$, which is at most a simple pole. Further, the residue at $r = (1-\rho)/2$ is regular for $\Re(\xi) \geq -\frac{1}{8}$, except for the pole at $\xi = \rho/4$. Since $P$ is arbitrary, this proves that $Y_c(\xi, D)$ admits a meromorphic continuation at least to the domain $\Re(\xi) \geq -\frac{1}{8}$.

Collecting the above discussion, we find that $Z_2(\xi)$ is meromorphic at least for $\xi > 0$, having simple poles possibly at $\frac{1}{2} \pm i\kappa_j$ $(j = 1, 2, \ldots)$, while all other poles in this region are of the form $\rho/2$ $(\zeta(\rho) = 0)$. This ends the proof of Theorem 2.

CONCLUDING REMARK. If $\kappa^2 + \frac{1}{4}$, $\kappa > 0$, belongs to the discrete spectrum, then we have

$$\lim_{\xi \to \tau} (\xi - \tau) Z_2(\xi) = 2R(\kappa),$$

where $\tau = \frac{1}{2} - i\kappa$, and $R(\kappa)$ is as above. Thus, providing that $\lambda_j$ is a simple eigenvalue, we may recover the value of $H_j\left(\frac{1}{2}\right)$ from those of the Riemann zeta-function. Actually we are able to extend this fact to $H_j(s)$, $\frac{1}{2} \leq \Re(s) < 1$, by considering, instead of $Z_2(\xi)$, the expression

$$\int_1^\infty \zeta^2\left(\frac{1}{2}\left(s + \frac{1}{2}\right) + it\right) \zeta^2\left(\frac{1}{2}\left(s + \frac{1}{2}\right) - it\right) t^{-\xi} dt.$$

In fact it has a simple pole at $\xi = 1 - s - i\kappa_j$ with the residue involving $H_j\left(\frac{1}{2}\right)^2 H_j(s)$ in the place of $H_j\left(\frac{1}{2}\right)^3$, providing an obvious non-vanishing condition. Thus it is possible to view the Riemann zeta-function as a generator of Maass wave form $L$-functions. Since the Riemann zeta-function corresponds to the Eisenstein series, the above fact suggests that there might be a way to generate Maass waves by integrating the Eisenstein series. To these topics we shall return elsewhere.

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