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# $L^2$ Estimates for Pseudodifferential Operators

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In this work we establish some new $L^2$ estimates for pseudodifferential operators in the spirit of R. Coifman-Y. Meyer [1], that is, for operators with non smooth symbols.

In the beginning of this work was the search for classes of paradifferential operators which are stable under $C^\infty$ canonical transformations. See [2] or [9]. This led us to conditions on the symbols $a(x, \xi)$ which were, for most of them, symmetric with respect to $x$ and $\xi$. Note that symmetric conditions are natural in the sense that if $a(x, D)$ is bounded in $L^2$, so is $a(D, x)$.

The other feature of our conditions is that the uniform local $L^2$ regularity is used instead of the $L^\infty$ one. This idea is implicit in the works by R. Coifman-Y. Meyer [1] and G. Bourdaud-Y. Meyer [3]. These authors preferred rather to emphasize the spaces of multipliers.

In the same direction, we note the paper by I.L. Hwang [7] who extended some of the results of [1] and mainly simplified their proofs.

We also remark that, in the case of Sobolev and Besov conditions, T. Muramatu [4] and M. Sugimoto [10] also obtained results extending those of [1].

Our proofs are often elementary, based on decompositions, Taylor formula, Cauchy-Schwarz inequality, Parseval or Plancherel formula..., except for that of Theorem 13 where we applied an interpolation argument. Furthermore, some of them can be used to discuss more general oscillating integrals as we shall do in a forthcoming paper.

**Some notations.** If $a(x, y, \ldots)$ is a function of several (vector) variables $x$, $y, \ldots$, then $\mathcal{F}_1(a)$, $\mathcal{F}_2(a), \ldots$ will denote the Fourier transforms of $a$ with respect to $x$, $y, \ldots$ respectively.

If $\alpha$, $\beta$, are multiindices, we will often write $\partial_1^\alpha a$, $\partial_2^\beta a$, $\partial_1^\alpha \partial_2^\beta a, \ldots$, instead of $\partial_x^\alpha a(x, y, \ldots)$, $\partial_y^\beta a(x, y, \ldots)$, $\partial_x^\alpha \partial_y^\beta a(x, y, \ldots), \ldots$

"cst" will always stand for some constant which may vary from an inequality to another one.

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$|\cdot|_s$, $s \in R$ (resp. $\|\cdot\|_p$, $1 \leq p \leq \infty$; $\|\cdot\|_E$), will denote the norm in the Sobolev space $H^s$ (resp. the $L^p$ space; the space $E$).

$\mathcal{L}(L^2)$ is the space of bounded operators in $L^2$.

We fix some dyadic partition of unity in $\mathbb{R}^n$,

$$1 = \sum_{j \in \mathbf{N}} \varphi_j(\xi) = \varphi_0(\xi) + \sum_{j \geq 1} \varphi(2^{-j}\xi) \quad \varphi_0 \in \mathcal{D}(\mathbb{R}^n), \ \ \varphi \in \mathcal{D}(\mathbb{R}^n \backslash 0),$$

and an $n$-dyadic partition of unity

$$1 = \sum_{j \in \mathbf{N}^n} \varphi_j(\xi), \ \ \varphi_j(\xi) = \varphi_{j_1}(\xi_1) \ldots \varphi_{j_n}(\xi_n),$$

if $1 = \sum_{k \in \mathbf{N}} \varphi_k$ is the fixed dyadic partition of unity in $\mathbb{R}$.

If $u$ is a tempered distribution in $\mathbb{R}^n$, $u_j$ often denotes the $j$-th dyadic (or $n$-dyadic) term $\varphi_j(D)u$ of $u$, so that, $\sum_j u_j$ is the dyadic (or $n$-dyadic) decomposition of $u$.

## 1. - Statements of the results

We consider only the usual quantification,

$$a(x, D)u(x) = \int e^{ix\xi} a(x, \xi)\hat{u}(\xi)d\xi,$$

and insist on saying that, for Weyl's quantification $a^w(x, D)$, the results may be different. To give an idea, we remark that, for example, if $a \in L^1(\mathbb{R}^{2n})$, one can easily prove that $a^w(x, D)$ is bounded in $L^2$, but this is not true for $a(x, D)$ as it is shown by the example given below after Corollary 7).

**Bounded symbols**. First, we shall prove the following statement which sharpens a result of Bourdaud-Meyer [3]:

THEOREM 1. *Let $\omega_1$, $\omega_2$ be non-negative functions on $\mathbb{R}^n$ such that $\frac{1}{\omega_1}$ and $\frac{1}{\omega_2}$ are integrable functions. Set $\omega = \omega_1 \otimes \omega_2$ and let $a$ be a complex function on $\mathbb{R}^n \times \mathbb{R}^n$.*

*If for some $\chi \in \mathcal{D}(\mathbb{R}^{2n})$ with non-zero integral,*

$$\sup_{k \in \mathbb{R}^{2n}} \|\chi(z-k)a(z)\|_{A_\omega} \tag{1}$$

*is finite, then, $a(x, D)$ is bounded in $L^2(\mathbb{R}^n)$ with an operator norm estimated by* (1).

We recall that if $\omega$ is a non-negative weight on $\mathbb{R}^n$ such that $\frac{1}{\omega}$ is integrable, then, $A_\omega$ denotes the translation-invariant Hilbert space of functions $u$ on $\mathbb{R}^n$ such that $\int \omega(\xi)|\hat{u}(\xi)|^2 d\xi < \infty$. Of course, the condition $\frac{1}{\omega} \in L^1$ implies that $A_\omega \subset \mathcal{F}L^1$.

We point out that if $\omega$ is too large at infinity, it may happen that Theorem 1 says no more than "0 is bounded in $L^2$", as in the case when $\omega(\xi, y) = e^{\xi^2+y^2}$, for example. This shows the importance of temperate weights (i.e. weights which do not exceed some polynomial). However, we preferred to state Theorem 1 with arbitrary weights to emphasize the fact that the proof needs only the condition $\frac{1}{\omega_1}, \frac{1}{\omega_2} \in L^1(\mathbb{R}^n)$. Compare with Bourdaud-Meyer's result.

In the case of temperate weights, we have the following corollary which somewhat extends Théorème 3 of [3]:

COROLLARY 2. *Under the hypothesis of Theorem* 1 *and assuming that* $\omega_1$ *and* $\omega_2$ *are temperate, for* $a(x, D)$ *to be bounded in* $L^2(\mathbb{R}^n)$ *it is sufficient that* $a$ *is a pointwise multiplier of* $A_\omega$.

REMARKS. 1) The results above show that the Beurling algebra setting (see [1]) actually is not essential for the $L^2$ continuity of pseudodifferential operators though it remains important for their symbolic calculus.

2) We do not know whether the above statements are true under the assumption $\frac{1}{\omega} \in L^1(\mathbb{R}^{2n})$ alone (i.e. $\omega$ is not necessarily of the form $\omega_1 \otimes \omega_2$). Nevertheless, our proof works for weights $\omega$ such that $1/\sqrt{\omega} = \sum_j f_j \otimes g_j$ is an absolutely convergent series in $L^2(\mathbb{R}^{2n})$.

3) The example $a(x, \eta) = \exp(-x^2 - ix\eta)(1 + \eta^2)^{-n/4}$ due to Coifman-Meyer [1] shows that in Theorem 1 the condition $\frac{1}{\omega_1}, \frac{1}{\omega_2} \in L^1(\mathbb{R}^n)$ is optimal.

We refer to [3] for corollaries of this type of results. Here, we only recall two of them, since we shall need them in the sequel. It concerns the cases

$$\omega(\xi, y) = \langle\xi\rangle^{2s}\langle y\rangle^{2s'}, \; s > \frac{n}{2}, \; s' > \frac{n}{2},$$

$$\text{and} \quad \omega(\xi, y) = \prod_{i=1}^{n} \langle\xi_i\rangle^{2s_i}\langle y_i\rangle^{2s_{n+i}}, \; s_i > \frac{1}{2}, \; 1 \le i \le 2n.$$

COROLLARY 3. *For some* $\chi \in \mathcal{D}(\mathbb{R}^{2n})$ *with non-zero integral, we have:*

(i) *If* $s > \frac{n}{2}$, $s' > \frac{n}{2}$ *then*

$$\|a(x,D)\|^2_{\mathcal{L}(L^2)} \le \text{cst} \sup_{(k,l)} \int |(1-\Delta_x)^{\frac{s}{2}}(1-\Delta_\eta)^{\frac{s'}{2}}$$

$$[\chi(x-k,\eta-l)a(x,\eta)]|^2 dx d\eta.$$

(ii) *If* $s_i > \frac{1}{2}$, $1 \le i \le 2n$ *then*

$$\|a(x,D)\|^2_{\mathcal{L}(L^2)} \le \text{cst} \sup_k \int \left| \prod_{i=1}^{2n} (1-\Delta_i)^{\frac{s_i}{2}} (a\tau_k\chi)(z) \right|^2 dz.$$

Next, we shall give another proof of the following result due to T. Muramatu [4] and M. Sugimoto [10]:

THEOREM 4. *If $a$ is a complex function on $\mathbb{R}^n \times \mathbb{R}^n$, for $a(x,D)$ to be bounded in $L^2(\mathbb{R}^n)$ it is enough that*

(i) $a \in B^{\frac{n}{2},\frac{n}{2}}_{\infty,1}(\mathbb{R}^{2n})$, *(Muramatu-Sugimoto);*

(ii) $a \in B^{(\frac{1}{2},\frac{1}{2},\ldots,\frac{1}{2})}_{\infty,1}(\mathbb{R}^{2n})$, *(Sugimoto).*

Recall here that, if $s$, $s' \in \mathbb{R}$, $\sigma \in \mathbb{R}^n$ and $p$, $q \in [1,\infty]$, then,

— $B^s_{p,q}(\mathbb{R}^n)$ denotes the Besov space defined by:

$$u \in B^s_{p,q}(\mathbb{R}^n) \text{ iff } u \in S' \text{ and } (2^{js}\|u_j\|_{L^p}) \in \ell^q(\mathbb{N}),$$

if $\sum_j u_j$ is a dyadic decomposition of $u$. See, for instance, Triebel [5] or Bergh-Löfström [8] for classical properties of Besov spaces.

— $B^{s,s'}_{p,q}(\mathbb{R}^n \times \mathbb{R}^{n'})$ denotes the double Besov space defined by:

$$u \in B^{s,s'}_{p,q}(\mathbb{R}^n \times \mathbb{R}^{n'}) \text{ iff } u \in S' \text{ and } (2^{(js+ks')}\|u_{jk}\|_{L^p}) \in \ell^q(\mathbb{N}\times\mathbb{N}),$$

$\sum_{j,k} u_{jk}$ being a double dyadic decomposition of $u$ (with respect to the "directions" $\mathbb{R}^n \times \{0\}$ and $\{0\} \times \mathbb{R}^{n'}$). As in the case of a simple Besov space, this does not depend on the dyadic decomposition and defines a Banach space.

— $B^\sigma_{p,q}(\mathbb{R}^n)$ denotes the multiple Besov space defined by:

$$u \in B^\sigma_{p,q}(\mathbb{R}^n) \text{ iff } u \in S' \text{ and } (2^{j\sigma}\|u_j\|_{L^p}) \in \ell^q(\mathbb{N}^n),$$

$\sum_{j\in\mathbb{N}^n} u_j$ being an $n$-dyadic decomposition of $u$ and $j\sigma = \sum_{i=1}^n j_i\sigma_i$. As in the case of a simple Besov space, this does not depend on the dyadic decomposition and defines a Banach space.

Actually, our proof of Theorem 4 relies on the following more general estimates:

THEOREM 5. *Let* $a : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ *be a bounded function.*

(i) *If* $\operatorname{supp}(\widehat{a})$ *is contained in a product of balls* $B(0, R_1) \times B(0, R_2)$, *with* $B(0, R_i) \subset \mathbb{R}^n$, $R_i \geq 1$, $i = 1, 2$, *then,*

$$\|a(x, D)\|_{\mathcal{L}(L^2)} \leq \mathrm{cst}(R_1 R_2)^{\frac{n}{2}} |a|_{0,ul}.$$

(ii) *If* $\operatorname{supp}(\widehat{a})$ *is contained in a product of intervals* $\prod_{i=1}^{2n} [-R_i, R_i]$, *with* $R_i \geq 1$, $i = 1, \ldots, 2n$, *then,*

$$\|a(x, D)\|_{\mathcal{L}(L^2)} \leq \mathrm{cst}\sqrt{R_1 R_2 \ldots R_{2n}}\, |a|_{0,ul}.$$

*The constants in these estimates are independant of* $a$ *and the* $R_i$.

Here, $|\cdot|_{s,ul}$ is the norm in the uniformly local Sobolev space $H^s_{ul}$. This is defined by: $u \in H^s_{ul} \iff \sup_y |\chi(x - y)u(x)|_s < \infty$ for some $\chi \in \mathcal{D}$ with non-zero integral. This is a complete normable space and we define its norm by setting $|u|_{s,ul} = \sup_y |\chi(x - y)u(x)|_s$ and by fixing the $\chi$, since another choice of $\chi$ gives an equivalent norm. See Kato [6] for simple properties of $H^s_{ul}$.

Note that a function in $L^2_{ul} = H^0_{ul}$ with compact spectrum is bounded (see the appendix A1), so that assuming $a$ bounded in Theorem 5 is not a restriction of generality.

Of course, since $|u|_{0,ul} \leq \mathrm{cst}\,\|u\|_\infty$, we also have:

COROLLARY 6. *Under the assumptions of Theorem 5, we have, in the first case,*

$$\|a(x, D)\|_{\mathcal{L}(L^2)} \leq \mathrm{cst}(R_1 R_2)^{\frac{n}{2}}\, \|a\|_\infty,$$

*and, in the second case,*

$$\|a(x, D)\|_{\mathcal{L}(L^2)} \leq \mathrm{cst}\sqrt{R_1 R_2 \ldots R_{2n}}\, \|a\|_\infty.$$

Theorem 4 is in fact a consequence of Corollary 6:

If $\sum_{j,k} a_{jk}$ (resp. $\sum_j a_j$) is a double (resp. $2n$-) dyadic decomposition of $a$, then,

$$\|a(x, D)\|_{\mathcal{L}(L^2)} \leq \sum_{j,k} \|a_{jk}(x, D)\|_{\mathcal{L}(L^2)} \leq \mathrm{cst} \sum_{j,k} 2^{(j+k)\frac{n}{2}} \|a_{jk}\|_\infty,$$

$$(\text{resp. } \|a(x, D)\|_{\mathcal{L}(L^2)} \leq \sum_j \|a_j(x, D)\|_{\mathcal{L}(L^2)} \leq \mathrm{cst} \sum_j 2^{|j|\frac{1}{2}} \|a_j\|_\infty.$$

By interpolation, we obtain the following intermediate estimates:

COROLLARY 7. *If* $1 \le p \le \infty$, $a \in L^p(\mathbb{R}^n \times \mathbb{R}^n)$ *and*

(i) $\operatorname{supp}(\hat{a}) \subset B(0,R_1) \times B(0,R_2)$, *with* $R_i \ge 1$, $i = 1,2$, *then,*

$$\|a(x,D)\|_{\mathcal{L}(L^2)} \le \operatorname{cst}(R_1R_2)^{n\left|\frac{1}{2}-\frac{1}{p}\right|}\|a\|_p.$$

(ii) $\operatorname{supp}(\hat{a}) \subset \prod_{i=1}^{2n}[-R_i,R_i]$, *with* $R_i \ge 1$, $i = 1,\dots,2n$, *then,*

$$\|a(x,D)\|_{\mathcal{L}(L^2)} \le \operatorname{cst}(R_1R_2\dots R_{2n})^{\left|\frac{1}{2}-\frac{1}{p}\right|}\|a\|_p.$$

Of course the condition $a \in L^1(\mathbb{R}^{2n})$ is not sufficient for $a(x,D)$ to be bounded in $L^2(\mathbb{R}^n)$. (Take for example $a(x,\eta) = x^{-\frac{1}{2}}\mathbf{1}_{]0,1]}(x)e^{-\eta^2}$ in $\mathbb{R}\times\mathbb{R}$ and consider $a(x,D)e^{-x^2}$.)

Now, define the spaces $\mathcal{B}$ and $\mathcal{E}$ by:

DEFINITION 8. *If* $\sum_{j,k} u_{jk}$ *is a double dyadic decomposition of* $u \in S'(\mathbb{R}^n \times \mathbb{R}^n)$, $u \in \mathcal{B}$ *iff* $u_{jk} \in L^2_{ul} = H^0_{ul}$ *and* $\sum_{j,k} 2^{(j+k)\frac{n}{2}}|u_{jk}|_{0,ul} < \infty$.

*If* $\sum_{j_1,\dots,j_{2n}} u_{j_1\dots j_{2n}}$ *is a 2n-dyadic decomposition of* $u \in S'(\mathbb{R}^n \times \mathbb{R}^n)$, $u \in \mathcal{E}$ *iff* $u_{j_1\dots j_{2n}} \in L^2_{ul}$ *and* $\sum_{j_1,\dots,j_{2n}} 2^{(j_1+\dots+j_{2n})\frac{1}{2}}|u_{j_1\dots j_{2n}}|_{0,ul} < \infty$.

One can easily show that this does not depend on the dyadic decompositions and that, equipped with the obvious norms, $\mathcal{B}$ and $\mathcal{E}$ are Banach spaces. Working a little more, one can show that $\mathcal{B} \subset \mathcal{E}$ and that $\mathcal{B}$ and $\mathcal{E}$ are even Banach subalgebras of $L^\infty$. See the appendix A2.

Obviously, $B^{\frac{n}{2},\frac{n}{2}}_{\infty,1}(\mathbb{R}^{2n}) \subset \mathcal{B}$.

Clearly, the following statement is equivalent to Theorem 5:

COROLLARY 9. (i) *If* $a \in \mathcal{B}$, *then,* $a(x,D)$ *is bounded in* $L^2(\mathbb{R}^n)$ *and*

$$\|a(x,D)\|_{\mathcal{L}(L^2)} \le \operatorname{cst}\sum_{j,k} 2^{(j+k)\frac{n}{2}}|a_{jk}|_{0,ul} = \operatorname{cst}\|a\|_{\mathcal{B}}.$$

(ii) *If* $a \in \mathcal{E}$, *then,* $a(x,D)$ *is bounded in* $L^2(\mathbb{R}^n)$ *and*

$$\|a(x,D)\|_{\mathcal{L}(L^2)} \le \operatorname{cst}\sum_{j_1,\dots,j_{2n}} 2^{(j_1+\dots+j_{2n})\frac{1}{2}}|u_{j_1\dots j_{2n}}|_{0,ul} = \operatorname{cst}\|a\|_{\mathcal{E}}.$$

In the same spirit, we can estimate the norm of $a(x,D)$ in $\mathcal{L}(L^2)$ using the norm of $a$ in the following space:

DEFINITION 10. *$\mathcal{A}(\mathbb{R}^n)$ is the space of $u \in S'(\mathbb{R}^n)$ such that*

$$\sum_{k\in\mathbf{Z}^n} \|\chi(D-k)u\|_\infty < \infty$$

*for some $\chi \in \mathcal{D}$ satisfying $\sum_{k\in\mathbf{Z}^n} \chi(\xi - k) = 1$.*

Of course, $\mathcal{A}(\mathbb{R}^n)$ does not depend on the used partition of unity and one can easily check that, provided with the natural norm (another choice of $\chi$ will give an equivalent norm), this is a Banach subalgebra of $L^\infty(\mathbb{R}^n)$. See the appendix A3.

Note also that, since $|\chi(D-k)u|_{0,ul} \sim \|\chi(D-k)u\|_\infty$ (see the appendix A1), if we replace the $L^\infty$ norm by the $L^2_{ul}$ one in the definition above, we obtain the same space.

Concerning $\mathcal{A}$, we shall prove the following:

THEOREM 11. *If $a \in \mathcal{A}(\mathbb{R}^{2n})$, then, $a(x,D)$ is bounded in $L^2(\mathbb{R}^n)$ and*

$$\|a(x,D)\|_{\mathcal{L}(L^2)} \leq \text{cst}\, \|a\|_{\mathcal{A}}.$$

Since $\mathcal{A}$ contains the Fourier transforms of finite Borel measures, as a consequence, we have the following result which we could not find in the literature in spite of its simplicity:

COROLLARY 12. (i) *If $\hat{a}$ is a finite Borel measure, then, $a(x,D)$ is bounded in $L^2(\mathbb{R}^n)$ and*

$$\|a(x,D)\|_{\mathcal{L}(L^2)} \leq \text{cst} \int d|\hat{a}|;$$

(ii) *If $\hat{a} \in L^p(\mathbb{R}^{2n})$, $1 \leq p \leq 2$, then, $a(x,D)$ is bounded in $L^2(\mathbb{R}^n)$ and*

$$\|a(x,D)\|_{\mathcal{L}(L^2)} \leq \text{cst}\, \|\hat{a}\|_p.$$

Of course, this can be proved directly and, for instance, one can obtain it using an argument with Wigner integrals.

**Non bounded symbols.** Let us recall some notations.

Let $p$, $q \in [1,\infty]$, $s$, $s' \in \mathbb{R}$, $\sigma \in \mathbb{R}^n$ and $u \in S'(\mathbb{R}^n)$. We shall write

— $u \in H^s_p(\mathbb{R}^n)$ iff $(1-\Delta)^{\frac{s}{2}} u \in L^p(\mathbb{R}^n)$;

— $u \in H^{s,s'}_p(\mathbb{R}^n \times \mathbb{R}^n)$ iff $(1-\Delta_x)^{\frac{s}{2}}(1-\Delta_y)^{\frac{s'}{2}} u(x,y) \in L^p(\mathbb{R}^n \times \mathbb{R}^n)$;

— $u \in H^\sigma_p(\mathbb{R}^n)$ iff $\prod_{i=1}^{n} (1-\Delta_i)^{\frac{\sigma_i}{2}} u \in L^p(\mathbb{R}^n)$;

In the beginning, we established the following result as an improvement of Théorème 4 of [1], in several directions. Later, we noticed that M. Sugimoto [10] already improved it. However, Theorem 13(vi) below is still better than that of Sugimoto.

THEOREM 13. *Let $s,\ s' \in \mathbb{R}$, $\sigma \in \mathbb{R}^{2n}$, and $p,\ q \in [1,\infty]$. Let $p'$ be the conjugate of $p$ when $2 \le p \le \infty$ and let it be equal to 2 otherwise.*

*For $a(x,D)$ to be bounded in $L^2(\mathbb{R}^n)$, it is sufficient that $a$ belongs to one of the following spaces:*

(i) $H_p^s(\mathbb{R}^{2n})$ *or* $B_{p,q}^s(\mathbb{R}^{2n})$ *with* $s > n\left|1 - \frac{2}{p}\right|$;

(ii) $H_p^{s,s'}(\mathbb{R}^{2n})$ *or* $B_{p,q}^{s,s'}(\mathbb{R}^{2n})$ *with* $s,\ s' > n\left|\frac{1}{2} - \frac{1}{p}\right|$;

(iii) $H_p^{\sigma}(\mathbb{R}^{2n})$ *or* $B_{p,q}^{\sigma}(\mathbb{R}^{2n})$ *with* $\sigma_i > \left|\frac{1}{2} - \frac{1}{p}\right|$, $1 \le i \le 2n$;

(iv) $B_{p,p'}^{s}(\mathbb{R}^{2n})$ *with* $s = n\left|1 - \frac{2}{p}\right|$;

(v) $B_{p,p'}^{s,s}(\mathbb{R}^{2n})$ *with* $s = n\left|\frac{1}{2} - \frac{1}{p}\right|$;

(vi) $B_{p,p'}^{\sigma}(\mathbb{R}^{2n})$ *with* $\sigma_i = \left|\frac{1}{2} - \frac{1}{p}\right|$, $1 \le i \le 2n$.

*Moreover, one can estimate the operator norm of $a(x,D)$ in $L^2(\mathbb{R}^n)$ by the norm of $a$ in each of these spaces.*

Here, in view of the usual inclusions (see the appendix A4), it suffices to establish the statement concerning (vi). We shall do it using an argument of interpolation. See the next section.

REMARK. When $p < 2$, Theorem 13(vi) is optimal. In fact, take $f(x) = \sum_{j\ge1} \epsilon_j 2^{j\frac{n}{2}}\psi(2^j x)$ where $\widehat{\psi} \in \mathcal{D}(\mathbb{R}^n\backslash 0)$ and $(\epsilon_j) \in \ell^q$, $q > 2$. Then, clearly, $f \in B_{p,q}^{n\left(\frac{1}{p}-\frac{1}{2}\right)}(\mathbb{R}^n)$ for all $p \in [1,2]$. Moreover, If we choose $\operatorname{supp}(\widehat{\psi})$ in order to have an orthogonal sum and if $(\epsilon_j) \notin \ell^2$, then, $f \notin L^2$. Now, set $a(x,\eta) = f(x)f(\eta)e^{-ix\eta} = \sum_{j,k} \epsilon_j\epsilon_k 2^{(j+k)\frac{n}{2}}\psi(2^j x)\psi(2^k\eta)e^{-ix\eta}$. It follows from the appendix A5 that $a \in B_{p,q}^{s,s}(\mathbb{R}^{2n})$, $s = n\left(\frac{1}{p} - \frac{1}{2}\right)$, for all $p \in [1,2]$. However, since $a(x,D)v(x) = f(x)\int f(\eta)\widehat{v}(\eta)d\eta$, if $f \notin L^2$, $a(x,D)$ is not bounded in $L^2$.

When $p > 2$, we do not know whether the subindex $p'$ is optimal.

Next, we give a result which extends Lemme 6 of [1], Chap. 1, to cases of non bounded symbols:

THEOREM 14. *Let $a$ be a complex function on $\mathbb{R}^n \times \mathbb{R}^n$.*

(i) *Let $\omega \geq 0$ be a weight such that $\frac{1}{\omega} \in L^1(\mathbb{R}^n)$ and $\omega(\xi + \eta)/\omega(\xi)$ is bounded when $\eta$ is in a compact set. If $\eta \mapsto a(x, \eta)$ (or $x \mapsto a(x, \eta)$) is in $L^2_{ul}(\mathbb{R}^n, A_\omega(\mathbb{R}^n))$, then, $a(x, D)$ is bounded in $L^2(\mathbb{R}^n)$ with an operator norm estimated by the norm of $a$ in $L^2_{ul}(\mathbb{R}^n, A_\omega(\mathbb{R}^n))$.*

(ii) *Assume that $\eta \mapsto a(x, \eta)$ is in $L^2_{ul}(\mathbb{R}^n, L^2(\mathbb{R}^n))$ (or $L^\infty(\mathbb{R}^n, L^2(\mathbb{R}^n))$) and $\mathcal{F}_1(a)(\xi, \eta) = 0$ when $|\xi| > R$ for some $R \geq 1$. Then, $a(x, D)$ is bounded in $L^2(\mathbb{R}^n)$ and $\|a(x, D)\|_{\mathcal{L}(L^2)} \leq \text{cst}\, R^{\frac{n}{2}} \|a\|_{L^2_{ul}(\mathbb{R}^n, L^2(\mathbb{R}^n))}$.*

*We have the same conclusion if $x \mapsto a(x, \eta)$ is in $L^2_{ul}(\mathbb{R}^n, L^2(\mathbb{R}^n))$ (or $L^\infty(\mathbb{R}^n, L^2(\mathbb{R}^n))$) and $\mathcal{F}_2(a)(x, y) = 0$ when $|y| > R$.*

This result already appeared in [9] in the case of $L^\infty(\mathbb{R}^n, L^2(\mathbb{R}^n))$.

We shall use Theorem 14 to establish a curious result on non bounded symbols where we mix an $S^0_{0,0}$ type condition with an $S^0_{1,0}$ type one:

THEOREM 15. *Assume that $a \in H^s_{ul}(\mathbb{R}^{2n})$, $s > \frac{n}{2}$ and that $\{a(x, \lambda\eta)\theta(\eta); \lambda \geq 1\}$ is a bounded set in $H^s_{ul}(\mathbb{R}^{2n})$ for any $\theta \in \mathcal{D}(\mathbb{R}^n \backslash 0)$. Then, $a(x, D)$ is bounded in $L^2(\mathbb{R}^n)$ with an operator norm estimated by*

$$|a|_{s,ul} + \sup_{\lambda \geq 1} |a(x, \lambda\eta)\theta(\eta)|_{s,ul},$$

*for some $\theta \in \mathcal{D}(\mathbb{R}^n \backslash 0)$.*

*In particular, we have the same conclusion if $a$ satisfies $S^0_{1,0}$ type estimates up to $s$, $s > \frac{n}{2}$.*

This statement is optimal in the sense that it is false when $s = \frac{n}{2}$. In fact, it suffices to consider Coifman-Meyer example

$$a(x, \eta) = \exp(-x^2 - ix\eta) < \eta >^{-n/2}$$

for which $\langle \eta \rangle^{|\beta|} \partial^\alpha_x \partial^\beta_\eta a(x, \eta)$ is bounded if $|\alpha| + |\beta| \leq \frac{n}{2}$.

**Conditions of Calderon-Vaillancourt type**. If $B$ is some Banach space of tempered distributions in $\mathbb{R}^n$, we shall denote by $B_{ul}$ the space of tempered distributions in $\mathbb{R}^n$ which are locally uniformly in $B$.

We have the following generalizations of Calderon-Vaillancourt theorem:

THEOREM 16. *Let $E$ stands for one of the spaces $H^s_{ul}(\mathbb{R}^{2n})$, $s > n$, or $H^{s,s'}_{ul}(\mathbb{R}^{2n})$, $s$, $s' > \frac{n}{2}$, or $H^\sigma_{ul}(\mathbb{R}^{2n})$, $\sigma \in \mathbb{R}^{2n}$, $\sigma_i > \frac{1}{2}$, $1 \le i \le 2n$.*

*Let $a : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ be a bounded function satisfying:*

(i) *$a(x,\eta)\chi(\eta)$ is in $E$, for all $\chi$ in $\mathcal{D}(\mathbb{R}^n)$.*

(ii) *For some $\delta \in [0,1[$, the set $\{a(\lambda^{-\delta}x, \lambda^\delta\eta)\chi(\lambda^{\delta-1}\eta); \lambda \ge 1\}$ is bounded in $E$, for all $\chi$ in $\mathcal{D}(\mathbb{R}^n\backslash 0)$.*

*Then, $a(x,D)$ is bounded in $L^2(\mathbb{R}^n)$ and its norm is estimated by*

$$\|a(x,\eta)\chi(\eta)\|_E + \sup_{\lambda\ge 1} \|a(\lambda^{-\delta}x, \lambda^\delta\eta)\theta(\lambda^{\delta-1}\eta)\|_E,$$

*for some $\chi \in \mathcal{D}(\mathbb{R}^n)$, $\theta \in \mathcal{D}(\mathbb{R}^n\backslash 0)$.*

Returning to the $L^\infty$ local regularity, we obtain the following result (compare with Theorem 3 and Theorem 5 of Hwang [7]):

COROLLARY 17. *If $0 \le \delta < 1$, and $\langle\eta\rangle^{\delta(|\beta|-|\alpha|)}\partial^\alpha_x\partial^\beta_\eta a(x,\eta)$ are bounded functions when $\alpha$, $\beta \in \{0,1\}^n$, then, $a(x,D)$ is bounded in $L^2(\mathbb{R}^n)$.*

REMARKS. 1) If $u(x) = \sum_{j\ge 0} 2^{-j\delta}e^{i2^jx}$ with some $0 < \delta \le \frac{1}{2}$, then, $u \in \mathcal{A}(\mathbb{R})\backslash\mathcal{E}(\mathbb{R})$.

We do not know whether we have $\mathcal{E} \subset \mathcal{A}$ or $\mathcal{B} \subset \mathcal{A}$.

2) Set $\sigma_0 = \left(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}\right)$. There are two other interesting spaces which may allow $L^2$ pseudodifferential estimates: The algebra $\mathcal{M}B^{\sigma_0}_{2,1}$ of pointwise multipliers of $B^{\sigma_0}_{2,1}$ and the algebra $B^{\sigma_0}_{2,1,ul}$ of functions which are locally uniformly in $B^{\sigma_0}_{2,1}$. We can show that $\mathcal{E} \subset \mathcal{M}B^{\sigma_0}_{2,1} \subset B^{\sigma_0}_{2,1,ul}$; we know that the second inclusion is strict, see [11], but we do not know whether the first is. Moreover, we do not know whether one can estimate $\|a(x,D)\|_{\mathcal{L}(L^2)}$ by $\|a\|_{\mathcal{M}B^{\sigma_0}_{2,1}}$ or $\|a\|_{B^{\sigma_0}_{2,1,ul}}$.

3) We do not know whether Theorem 16 holds when $E = \mathcal{E}$, $\mathcal{B}$ or $A_{\omega,ul}$. We think that it may be true for $A_{\omega,ul}$. However, if we inspect the first part of its proof, we can see that it holds for any translation invariant Banach space $E$ satisfying the estimate $\|a\|_E \ge \text{cst}\|a(x,D)\|_{\mathcal{L}(L^2)}$, if one replaces the pseudodifferential operator $a(x,D)$ by any paradifferential operator associated with $a$. In other words, for paradifferential operators, an $L^2$ estimate with $S^0_{0,0}E$ type symbols implies an $L^2$ estimate with $S^0_{\delta,\delta}E$ type symbols, $0 \le \delta \le 1$.

4) In [4], T. Muramatu established, by other techniques, a theorem of Calderon-Vaillancourt type using the space $B^{\frac{n}{2},\frac{n}{2}}_{\infty,1}(\mathbb{R}^{2n})$. However, there were some restriction in his conditions so that the space $S^m_{\delta,\delta}B^{\frac{n}{2},\frac{n}{2}}_{\infty,1}(\mathbb{R}^{2n})$ was no longer the natural one.

## 2. - Proofs

If $u,\ v \in \mathcal{S}(\mathbb{R}^n)$, set

$$I = \int_{\mathbb{R}^n} a(x, D)v(x).u(x).dx.$$

In what follows, the goal will be at each time to establish the following inequality:

$$|I| \leq \text{cst}\, \|a\|_E |u|_0 |v|_0,$$

where $E$ is some functional space and, of course, the constant is independant of $u,\ v,\ a$.

PROOF OF THEOREM 1. We can assume that $\int \chi(z)dz = 1$. Take $\theta \in \mathcal{D}(\mathbb{R}^n)$ such that $\theta \otimes \theta = 1$ on $\text{supp}(\chi)$. Write then:

$$\begin{aligned} I &= \int e^{ix\eta} \chi(x-k, \eta - l) a(x, \eta) u(x) \widehat{v}(\eta) dx d\eta dk dl \\ &= \int e^{ikl} e^{ix\eta} a_{kl}(x, \eta) e^{ixl} u_k(x) e^{ik\eta} \widehat{v}_l(\eta) dx d\eta dk dl, \end{aligned}$$

where $a_{kl}(x, \eta) = \chi(x, \eta) a(x+k, \eta+l)$, $u_k(x) = \theta(x) u(x+k)$ and $\widehat{v}_l(\eta) = \theta(\eta)\widehat{v}(\eta+l)$. Since

$$e^{ix\eta} = \sum_{m=0}^{\infty} \sum_{|\alpha|=m} \frac{1}{\alpha!} (ix)^\alpha \eta^\alpha,$$

we can also write

$$I = \sum_{m=0}^{\infty} \sum_{|\alpha|=m} \frac{1}{\alpha!} \int e^{ikl} a_{kl}(x, \eta) e^{ixl} u_{k\alpha}(x) e^{ik\eta} \widehat{v}_{l\alpha}(\eta) dx d\eta dk dl,$$

where $u_{k\alpha}(x) = u_k(x)(ix)^\alpha$ and $\widehat{v}_{l\alpha}(\eta) = \widehat{v}_l(\eta)\eta^\alpha$. Applying Parseval's formula with respect to $(x, \eta)$ and then estimating using Cauchy-Schwarz inequality, we obtain:

$$|I| \le \sum_{m=0}^{\infty} \sum_{|\alpha|=m} \frac{1}{\alpha!} \int \|a_{kl}\|_{A_\omega}$$

$$\left(\int \frac{|\hat{u}_{k\alpha}(\xi+l)|^2 |v_{l\alpha}(y+k)|^2}{\omega_1(\xi)\omega_2(y)} d\xi dy\right)^{\frac{1}{2}} dkdl$$

$$\le \sum_{m=0}^{\infty} \sum_{|\alpha|=m} \frac{\sup_{k,l} \|a_{kl}\|_{A_\omega}}{\alpha!} \sqrt{\int \frac{|\hat{u}_{k\alpha}(\xi+l)|^2}{\omega_1(\xi)} d\xi dkdl}$$

$$\sqrt{\int \frac{|v_{l\alpha}(y+k)|^2}{\omega_2(y)} dydkdl}$$

$$\le \sup_{k,l} \|a_{kl}\|_{A_\omega} |\omega^{-\frac{1}{2}}|_0 \sum_{m=0}^{\infty} \sum_{|\alpha|=m} \frac{1}{\alpha!} \sqrt{\int |\hat{u}_{k\alpha}(l)|^2 dkdl}$$

$$\sqrt{\int |v_{l\alpha}(k)|^2 dkdl}$$

$$\le \text{cst} \sup_{k,l} \|a_{kl}\|_{A_\omega} |\omega^{-\frac{1}{2}}|_0 |\theta|_0^2 |u|_0 |v|_0,$$

where the constant can be taken to be $e^{nd^2}$ with $d = \sup\{|x|; x \in \operatorname{supp}(\theta)\}$. Theorem 1 is so proved.

PROOF OF THEOREM 5. Clearly, (ii) $\Longrightarrow$ (i). Let us show (ii).

If $\chi \in \mathcal{D}(\mathbb{R}^n)$ and $\int \chi(x)dx = 1$, we can write:

$$I = \sum_{\alpha \in \mathbb{N}^n} \frac{i^{|\alpha|}}{\alpha!} \int e^{ikl} a(x+k, \eta+l) e^{ixl} u_{k\alpha}(x) e^{ik\eta} \hat{v}_{l\alpha}(\eta) dx d\eta dkdl,$$

where $u_{k\alpha}(x) = u(x+k)x^\alpha \chi(x)$ and $\hat{v}_{l\alpha}(\eta) = \hat{v}(\eta+l)\eta^\alpha \chi(\eta)$.

If $\theta_i \in \mathcal{S}(\mathbb{R}^n)$, $i = 1, 2$, are such that $\hat{\theta}_1 \otimes \hat{\theta}_2 = 1$ near $\operatorname{supp}(\hat{a})$, we can also write:

$$\begin{aligned} a(x,\eta) &= \int a(y,\xi)\theta_1(x-y)\theta_2(\eta-\xi)dyd\xi \\ &= \int a_{pq}(y,\xi)\theta_1(x-y-p)\theta_2(\eta-\xi-q)dyd\xi dpdq, \end{aligned} \tag{2}$$

where $a_{pq}(y,\xi) = \chi(y)\chi(\xi)a(y+p, \xi+q)$.

If $N$ is an even integer, using Taylor formula, we can write:

$$\langle k-p\rangle^N\langle l-q\rangle^N = \sum_{\beta,\gamma,r,s,\varrho,\sigma} c_{\beta\gamma rs\varrho\sigma}(x+k-y-p)^\beta(\eta+l-\xi-q)^\gamma x^r y^s \xi^\varrho \eta^\sigma,$$

with some coefficients $c_{\beta\gamma rs\varrho\sigma}$; $\beta$, $\gamma$, $r$, $s$, $\varrho$, $\sigma$ being multi-indices and the sum being of course finite.

Therefore, we can rewrite $I$ as follows:

$$I = \sum_{\alpha,\beta,\gamma,r,s,\varrho,\sigma} \frac{i^{|\alpha|}c_{\beta\gamma rs\varrho\sigma}}{\alpha!} \int e^{ikl}\tilde{a}_{pq}(y,\xi)\frac{\theta_{1\beta}(x+k-y-p)\theta_{2\gamma}(\eta+l-\xi-q)}{\langle k-p\rangle^N\langle l-q\rangle^N}$$

$$e^{ixl}u_{k,\alpha+r}(x)e^{ik\eta}\hat{v}_{l,\alpha+\sigma}(\eta)dx\,d\eta\,dk\,dl\,dy\,d\xi\,dp\,dq,$$

where $\tilde{a}_{pq}(y,\xi) = y^s\xi^\varrho a_{pq}(y,\xi)$ and $\theta_{i\beta}(z) = z^\beta\theta_i(z)$, $i = 1, 2$, if $\beta \in \mathbb{N}^n$.

Now, consider the following functions:

$$f(l,z,k) = \int e^{ixl}\theta_{1\beta}(x+z)u_{k,\alpha+r}(x)dx,$$

$$g(k,\varsigma,l) = \int e^{ik\eta}\theta_{2\gamma}(\eta+\varsigma)\hat{v}_{l,\alpha+\sigma}(\eta)d\eta.$$

These are integrals of Wigner type, already used by Hwang [7], and the main (and simple!) fact in this proof is that $f$ and $g$ are square-integrable in $\mathbb{R}^{3n}$ and that:

$$|f|_0 = (2\pi)^{\frac{n}{2}}\,|\theta_{1\beta}|_0|u|_0|x^{\alpha+r}\chi|_0,$$

$$|g|_0 = (2\pi)^{\frac{n}{2}}\,|\theta_{2\gamma}|_0|\hat{v}|_0|\eta^{\alpha+\sigma}\chi|_0.$$

We have, using $f$ and $g$:

$$I = \sum_{\alpha,\beta,\gamma,r,s,\varrho,\sigma} \frac{i^{|\alpha|}c_{\beta\gamma rs\varrho\sigma}}{\alpha!}$$

$$\int e^{ikl}\tilde{a}_{pq}(y,\xi)\frac{f(l,k-p-y,k)g(k,l-\xi-q,l)}{\langle k-p\rangle^N\langle l-q\rangle^N}\,dk\,dl\,dy\,d\xi\,dp\,dq,$$

Now, taking $N \geq n+1$, we can estimate as follows:

$$|I| \leq \text{cst} \sum_{\alpha,\beta,\gamma,r,\sigma} \frac{1}{\alpha!}$$

$$\int |a_{pq}|_0 \frac{(\int |f(l,y,k)|^2 dy)^{\frac{1}{2}} (\int |g(k,\xi,l)|^2 d\xi)^{\frac{1}{2}}}{\langle k-p\rangle^N \langle l-q\rangle^N} dk dl dp dq$$

$$\leq \text{cst} \sum_{\alpha,\beta,\gamma,r,\sigma} \frac{1}{\alpha!} \sup_{p,q} |a_{pq}|_0 |f|_0 |g|_0$$

$$\leq \text{cst} \sum_{\alpha,\beta,\gamma} \frac{d^{2|\alpha|}}{\alpha!} |a|_{0,ul} |u|_0 |v|_0 |\theta_{1\beta}|_0 |\theta_{2\gamma}|_0,$$

where $d = \sup\{|x|; x \in \text{supp}(\chi)\}$. It remains to take $\theta_1(x) = \prod_{i=1}^n R_i \theta(R_i x_i)$ and $\theta_2(\eta) = \prod_{i=1}^n R_{n+i} \theta(R_{n+i} \eta_i)$ with a fixed $\theta \in S(\mathbb{R})$ such that $\widehat{\theta} = 1$ near $[-1,1]$, and to note then that $|\theta_{1\beta}|_0 |\theta_{2\gamma}|_0 = \prod_{i=1}^n R_i^{\frac{1}{2}-\beta_i} |x_i^{\beta_i} \theta|_0 R_{n+i}^{\frac{1}{2}-\gamma_i} |\eta^{\gamma_i} \theta|_0$. Hence,

$$|I| \leq \text{cst} \sum_{\alpha} \frac{d^{2|\alpha|}}{\alpha!} (R_1 \ldots R_{2n})^{\frac{1}{2}} |a|_{0,ul} |u|_0 |v|_0 = \text{cst}(R_1 \ldots R_{2n})^{\frac{1}{2}} |a|_{0,ul} |u|_0 |v|_0,$$

so that Theorem 5 is proved.

Proof of Corollary 7. Since (ii) $\Longrightarrow$ (i), it suffices to show (ii).

By interpolation (between $L^1$ and $L^2$ and between $L^2$ and $L^\infty$), it is enough to treat the case $p = 1$.

Using the integral representation (2) for $a$, we can write:

$$I = \int a(y,\xi) e^{ix\eta} \theta_1(x-y) \theta_2(\eta-\xi) \widehat{v}(\eta) u(x) dx d\eta dy d\xi.$$

Hence,

$$|I| \leq \|a\|_1 |\theta_1 \otimes \theta_2|_0 |\widehat{v}|_0 |u|_0.$$

Now, as at the end of the proof of Theorem 5, we take $\theta_1(x) = \prod_{i=1}^n R_i \theta(R_i x_i)$ and $\theta_2(\eta) = \prod_{i=1}^n R_{n+i} \theta(R_{n+i} \eta_i)$ with $\theta \in S(\mathbb{R})$ such that $\widehat{\theta} = 1$ near $[-1,1]$, and observe that $|\theta_1 \otimes \theta_2|_0 = \sqrt{R_1 R_2 \ldots R_{2n}}\, |\theta|_0^{2n}$. This establishes the corollary.

PROOF OF THEOREM 11. If $\chi \in \mathcal{D}(\mathbb{R}^{2n})$ and $\sum_{k,l\in\mathbf{Z}^n} \chi(\xi - k, y - l) = 1$, we can write:

$$I = \sum_{k,l\in\mathbf{Z}^n} \int e^{ix\eta} a_{kl}(x,\eta) e^{ixk} u(x) e^{i\eta l} \widehat{v}(\eta) dx d\eta,$$

where $a_{kl}(x,\eta) = e^{-ixk} e^{-i\eta l} \chi(D_x - k, D_\eta - l)[a(x,\eta)]$.

Applying Cordes theorem or Corollary 3(i) with some integer $s = s' = N$, we obtain

$$|I| \leq \text{cst} \sum_{k,l\in\mathbf{Z}^n} \sum_{|\alpha|,|\beta|\leq N} \|\partial_1^\alpha \partial_2^\beta a_{kl}\|_\infty |u|_0 |v|_0.$$

Since $\text{supp}(\widehat{a}_{kl})$ is contained in a compact set which is independant of $k$, $l$, we can write: $\|\partial_1^\alpha \partial_2^\beta a_{kl}\|_\infty \leq \text{cst}(\alpha,\beta) \|a_{kl}\|_\infty$; so that,

$$|I| \leq \text{cst} \sum_{k,l\in\mathbf{Z}^n} \|a_{kl}\|_\infty |u|_0 |v|_0 = \text{cst}\, \|a\|_A |u|_0 |v|_0,$$

which establishes Theorem 11.

PROOF OF THEOREM 13. Since the space in (vi) contains all the others, we have just to prove the statement concerning it.

Of course, when $p = 2$, (vi) is obvious, and when $p = \infty$, it is an obvious consequence of Corollary 6. Now, the case $p = 1$ is not less obvious; we have just to observe that $B_{1,2}^{(\frac{1}{2},\frac{1}{2},\ldots,\frac{1}{2})} \subset L^2$. In fact, if $u_j$ is the $j$-th $N$-dyadic term of $u$, then, clearly, $|u_j|_0 \leq \text{cst}\, 2^{\frac{|j|}{2}} \|u_j\|_1$.

All the other cases are obtained by an argument of interpolation which we develop to some extent in what follows. The problem, of course, is that the interpolation for multiple Besov spaces is not available in the literature as it is that for simple Besov spaces.

Define the space

$$l_q^\sigma(L^p(\mathbb{R}^n)) = \left\{ (u_j)_{j\in\mathbf{N}^n} ; u_j \in L^p(\mathbb{R}^n) \text{ and } \sum_{j\in\mathbf{N}^n} (2^{j\sigma} \|u_j\|_p)^q < \infty \right\},$$

$\sigma \in \mathbb{R}^n$, $1 \leq p$, $q \leq \infty$, $j\sigma = \sum_{i=1}^n j_i \sigma_i$. Endowed with the obvious norm, this is a Banach space as one can check easily.

If $1 = \sum_{k\geq 0} \varphi_k$ is a dyadic partition of unity in $\mathbb{R}$, consider the linear map

$$\ell : (a_j) \to \sum_j [\varphi_{j_1}(D_1) \ldots \varphi_{j_{2n}}(D_{2n}) a_j](x, D).$$

Let $\sigma_0 \in \mathbb{R}^{2n}$ stand for $\left(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}\right)$.

Obviously, $\ell$ is bounded from $l_2^0(L^2(\mathbb{R}^{2n}))$ into $\mathcal{L}(L^2(\mathbb{R}^n))$. It follows from Corollary 9(ii) (resp. the argument above) that it is also bounded from $l_1^{\sigma_0}(L^\infty(\mathbb{R}^{2n}))$ (resp. $l_2^{\sigma_0}(L^1(\mathbb{R}^{2n}))$) into $\mathcal{L}(L^2(\mathbb{R}^n))$. Hence, it is bounded from the complex interpolated space

$$[l_2^0(L^2(\mathbb{R}^{2n})), l_1^{\sigma_0}(L^\infty(\mathbb{R}^{2n}))]_\theta, \text{ (resp. } [l_2^{\sigma_0}(L^1(\mathbb{R}^{2n})), l_2^0(L^2(\mathbb{R}^{2n}))]_\theta),$$

$0 < \theta < 1$, into $\mathcal{L}(L^2(\mathbb{R}^n))$.

Assume for a moment that

$$[l_2^0(L^2(\mathbb{R}^{2n})), l_1^{\sigma_0}(L^\infty(\mathbb{R}^{2n}))]_\theta = l_{p'}^\sigma(L^p(\mathbb{R}^{2n})),$$

$$\text{(resp. } [l_2^{\sigma_0}(L^1(\mathbb{R}^{2n})), l_2^0(L^2(\mathbb{R}^{2n}))]_\theta = l_2^\sigma(L^p(\mathbb{R}^{2n}))),$$

where $2 < p < \infty$, $2 = p(1-\theta)$, $p + p' = pp'$ (resp. $1 < p < 2$, $(2-\theta)p = 2$) and $\sigma_i = \frac{1}{2} - \frac{1}{p}$, $1 \leq i \leq 2n$. Then, if $a \in B_{p,p'}^\sigma(\mathbb{R}^{2n})$, one can write:

$$a(x, D) = \sum_j [\varphi_{j_1}(D_1) \dots \varphi_{j_{2n}}(D_{2n})a_j](x, D) = \ell[(a_j)],$$

where $a_j = \psi_{j_1}(D_1) \dots \psi_{j_{2n}}(D_{2n})a$ with $\psi_k \in \mathcal{D}(\mathbb{R})$, $k \geq 0$, and, if $k \geq 1$, $\psi_k(\xi) = \psi(2^{-k}\xi)$, $\psi \in \mathcal{D}(\mathbb{R}\backslash 0)$, $\psi = 1$ near $\operatorname{supp}(\varphi)$.

Of course, $(a_j) \in l_{p'}^\sigma(L^p(\mathbb{R}^{2n}))$; hence, $a(x, D) \in \mathcal{L}(L^2(\mathbb{R}^n))$ and

$$\|a(x, D)\|_{\mathcal{L}(L^2)} \leq \text{cst} \sum_{j \in \mathbb{N}^{2n}} (2^{j\sigma} \|a_j\|_p)^{p'} \leq \text{cst} \|a\|_{B_{p,p'}^\sigma(\mathbb{R}^{2n})},$$

so that Theorem 13 will be established once the following interpolation lemma is proved:

LEMMA 18. *If $\sigma_1, \sigma_2 \in \mathbb{R}^n$ and if $p_1, p_2, q_1, q_2, \theta$ are real numbers such that $0 < \theta < 1$ and $1 \leq p_1, p_2, q_1, q_2 \leq \infty$, then, (complex interpolation)*

$$[l_{q_1}^{\sigma_1}(L^{p_1}(\mathbb{R}^n)), l_{q_2}^{\sigma_2}(L^{p_2}(\mathbb{R}^n))]_\theta = l_q^\sigma(L^p(\mathbb{R}^n)),$$

*where* $\sigma = (1-\theta)\sigma_1 + \theta\sigma_2$, $\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$ *and* $\frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}$.

PROOF OF LEMMA 18. We follow the proof of Theorem 5.6.3 of Bergh-Löfström [8]. If $S = \{z \in \mathbb{C}; 0 \leq \operatorname{Re}(z) \leq 1\}$, consider the space $\mathcal{F}$ of bounded and continuous applications

$$f : S \to l_{q_1}^{\sigma_1}(L^{p_1}(\mathbb{R}^n)) + l_{q_2}^{\sigma_2}(L^{p_2}(\mathbb{R}^n))$$

which are analytic in $S^o$ and such that $t \to f(it)$ (resp. $t \to f(1+it)$) is continuous from $\mathbb{R}$ into $l_{q_1}^{\sigma_1}(L^{p_1}(\mathbb{R}^n))$ (resp. $l_{q_2}^{\sigma_2}(L^{p_2}(\mathbb{R}^n))$). This is a Banach space when provided with the norm

$$\|f\|_{\mathcal{F}} = \max\left\{\sup_{t\in\mathbb{R}} \|f(it)\|_{l^{\sigma_1}_{q_1}(L^{p_1}(\mathbb{R}^n))}, \sup_{t\in\mathbb{R}} \|f(it)\|_{l^{\sigma_2}_{q_2}(L^{p_2}(\mathbb{R}^n))}\right\}.$$

If $f = (f_j) \in \mathcal{F}$, define $\tilde{f} = (\tilde{f}_j)$ by

$$\tilde{f}_j(z) = 2^{j[\sigma_1(1-z)+\sigma_2 z]} f_j(z).$$

Clearly, $f \to \tilde{f}$ is an isometric isomorphism from $\mathcal{F}$ onto $\mathcal{F}_0$, $\mathcal{F}_0$ being the space $\mathcal{F}$ when $\sigma_1 = \sigma_2 = 0$. Hence, the problem is reduced to the interpolation between $l^0_{q_1}(L^{p_1}(\mathbb{R}^n))$ and $l^0_{q_2}(L^{p_2}(\mathbb{R}^n))$ and now the proof goes on exactly as that of Theorem 5.6.3 of [8]; so, we refer to it and omit here the remaining details.

PROOF OF THEOREM 14. (i) If $\chi \in \mathcal{D}(\mathbb{R}^n)$ and $\int \chi(\eta)^2 d\eta = 1$, we can write:

$$\begin{aligned} I &= \int e^{ix\eta}\chi(\eta - k)^2 a(x,\eta)u(x)\widehat{v}(\eta)\,dx\,d\eta\,dk \\ &= \int e^{ix(\eta+k)} a_k(x,\eta)u(x)\widehat{v}_k(\eta)\,dx\,d\eta\,dk \\ &= \int \mathcal{F}_1^{-1}(a_k)(\xi+\eta+k,\eta)\widehat{u}(\xi)\widehat{v}_k(\eta)\,d\xi\,d\eta\,dk, \end{aligned}$$

where $a_k(x,\eta) = a(x,\eta+k)\chi(\eta)$ and $\widehat{v}_k(\eta) = \widehat{v}(\eta+k)\chi(\eta)$. Hence, by Cauchy-Schwarz inequality,

$$\begin{aligned} |I| &\le \text{cst} \int \sqrt{\int \omega(\xi)|\mathcal{F}_1(a_k)(\xi,\eta)|^2 d\xi\,d\eta} \\ &\qquad \sqrt{\int \frac{|\widehat{u}(\xi)\widehat{v}_k(\eta)|^2}{\omega(-\xi-\eta-k)}\,d\xi\,d\eta}\,dk \\ &\le \text{cst}\,\|a\|_{L^2_{ul}(\mathbb{R}^n, A_\omega(\mathbb{R}^n))} \sup_{(\xi,\eta)\in\mathbb{R}^n\times\operatorname{supp}(\chi)} \frac{\omega(\xi+\eta)^{\frac12}}{\omega(\xi)^{\frac12}} \\ &\qquad \cdot \sqrt{\int \frac{|\widehat{u}(\xi)|^2}{\omega(-\xi-k)}\,d\xi\,dk}\sqrt{\int |\widehat{v}_k(\eta)|^2 d\eta\,dk} \\ &\le \text{cst}\,\|a\|_{L^2_{ul}(\mathbb{R}^n, A_\omega(\mathbb{R}^n))} \\ &\qquad \sup_{(\xi,\eta)\in\mathbb{R}^n\times\operatorname{supp}(\chi)} \frac{\omega(\xi+\eta)^{\frac12}}{\omega(\xi)^{\frac12}} |\omega^{-\frac12}|_0 |u|_0 |v|_0, \end{aligned}$$

which proves the first part of the theorem.

(ii) Write

$$I = \int e^{ix\eta} a(R^{-1}x, R\eta) u(R^{-1}x) \widehat{v}(R\eta) dx d\eta.$$

Applying the first part, with $\omega(\xi) = \langle \xi \rangle^{2N}$, $N \in \mathbb{N}, N > \frac{n}{2}$, we obtain

$$|I|^2 \leq \operatorname{cst} |u|_0^2 |v|_0^2 \sup_{k \in \mathbb{R}^n, |\alpha| \leq N} \int |\partial_x^\alpha [a(R^{-1}x, R\eta)\chi(\eta - k)]|^2 dx d\eta,$$

with some convenient $\chi \in \mathcal{D}(\mathbb{R}^n)$. Now, since the spectrum of $x \to a(R^{-1}x, \eta)$ is contained in the ball $B(0,1)$, we have

$$|I|^2 \leq \operatorname{cst} |u|_0^2 |v|_0^2 \sup_{k \in \mathbb{R}^n} \int |a(R^{-1}x, R\eta)\chi(\eta - k)|^2 dx d\eta$$

$$\leq \operatorname{cst} R^n |u|_0^2 |v|_0^2 \sup_{k \in \mathbb{R}^n} \int |a(x, \eta)\chi(\eta - k)|^2 dx d\eta.$$

Here, we used the following simple lemma concerning $L^2_{ul}$:

LEMMA 19. *If $u \in L^2_{ul}(\mathbb{R}^n)$, $t > 0$ and $u_t(x) = u(tx)$, then, $u_t \in L^2_{ul}(\mathbb{R}^n)$ and*

$$|u_t|_{0,ul} \leq \operatorname{cst} \left(\frac{1+t}{t}\right)^{\frac{n}{2}} |u|_{0,ul}.$$

PROOF OF LEMMA 19. If $\chi \in \mathcal{D}(\mathbb{R}^n)$ and $\int |\chi(x)|^2 dx = 1$, we can write, with some convenient $\theta \in \mathcal{D}(\mathbb{R}^n)$,

$$\int |u(tx)\chi(x-k)|^2 dx = t^{-n} \int |u(x)\chi(xt^{-1} - k)\chi(x - l)|^2 dx dl$$

$$= t^{-n} \int \theta\left(\frac{tk - l}{1+t}\right) |u(x)\chi(xt^{-1} - k)\chi(x - l)|^2 dx dl.$$

In fact, on the support of integration, $\frac{tk-l}{1+t}$ is bounded. Hence,

$$\int |u(tx)\chi(x-k)|^2 dx \leq \operatorname{cst} t^{-n} \|\chi\|_\infty^2 \int \theta\left(\frac{tk-l}{1+t}\right) |u(x)\chi(x-l)|^2 dx dl$$

$$\leq \operatorname{cst} \left(\frac{1+t}{t}\right)^n |u|_{0,ul}^2.$$

This proves the lemma and, at the same time, achieves the proof of Theorem 14.

PROOF OF THEOREM 15. If $\theta \in \mathcal{D}(\mathbb{R}^n)$ and $\theta = 1$ near 0, we set $b(x, \eta) = \theta(D_\eta)[a(x, \eta)]$ and $r = a - b$.

We first consider $r$. Let us show that $x \to r(x,\eta)$ is in $L^2_{ul}(\mathbb{R}^n, H^s(\mathbb{R}^n))$, which, in view of Theorem 14, implies that $r(x,D)$ is bounded in $L^2(\mathbb{R}^n)$. To this end, take a dyadic partition of unity $1 = \varphi_0(\eta) + \sum_j \varphi(2^{-j}\eta)$ and write $a(x,\eta) = a(x,\eta)\varphi_0(\eta) + \sum_j a_j(x, 2^{-j}\eta)$ with $a_j(x,\eta) = a(x, 2^j\eta)\varphi(\eta)$. The term $a(x,\eta)\varphi_0(\eta)$ is clearly in $L^2_{ul}(\mathbb{R}^n, H^s(\mathbb{R}^n))$, so that, applying again Theorem 14, we can forget it in what follows. We can write:

$$|y|^s \mathcal{F}_2(r)(x,y) = \sum_j 2^{j(n-s)} |2^j y|^s \mathcal{F}_2(a_j)(x, 2^j y)(1 - \theta(y)).$$

Hence, if $\chi \in \mathcal{D}(\mathbb{R}^n)$, we have

$$\begin{aligned}
&\sqrt{\iint |y|^{2s} |\chi(x-z)\mathcal{F}_2(r)(x,y)|^2 dx dy} \\
&\leq \sum_j 2^{j(\frac{n}{2}-s)} \sqrt{\iint |y|^{2s} |\chi(x-z)\mathcal{F}_2(a_j)(x,y)(1-\theta(2^{-j}y))|^2 dx dy} \\
&\leq \text{cst} \sum_j 2^{j(\frac{n}{2}-s)} \|1-\theta\|_\infty |a_j|_{s,ul} \\
&\leq \text{cst} \sup_j |a_j|_{s,ul}.
\end{aligned}$$

This proves our assertion concerning $r$.

Now, consider $b$. If $\chi \in \mathcal{D}(\mathbb{R}^{2n})$, we can write:

$$\chi(x-k, \eta-l) b(x,\eta) = \int \chi(x-k,\eta-l) a(x, \eta-\varsigma) \mathcal{F}^{-1}(\theta)(\varsigma) d\varsigma;$$

hence,

$$\begin{aligned}
|\tau_{(k,l)}\chi . b|_s &\leq \int |\tau_{(k,l)}\chi . \tau_{(0,\varsigma)} a|_s |\mathcal{F}^{-1}(\theta)(\varsigma)| d\varsigma \\
&\leq |a|_{s,ul} \|\mathcal{F}^{-1}(\theta)\|_1.
\end{aligned}$$

This proves that $b \in H^s_{ul}(\mathbb{R}^{2n})$. Now, clearly, we can estimate $\partial^\alpha_\eta[\chi(x-k,\eta - l)b(x,\eta)]$, for all $\alpha \in \mathbb{N}^n$, in the same manner, and obtain:

$$|\partial^\alpha_2(b.\tau_{(k,l)}\chi)|_s \leq \text{cst}(\alpha)|a|_{s,ul}.$$

This means that $b \in H^{s,s'}_{ul}(\mathbb{R}^{2n})$, $\forall s' \in \mathbb{R}$, and, in view of Corollary 3, implies that $b(x,D)$ is bounded in $L^2(\mathbb{R}^n)$. Theorem 15 is so proved.

PROOF OF THEOREM 16. Since, for all $\epsilon > 0$, we have

$$H_{ul}^{n+2n\epsilon}(\mathbb{R}^{2n}) \subset H_{ul}^{\frac{n}{2}+n\epsilon,\frac{n}{2}+n\epsilon}(\mathbb{R}^{2n}) \subset H_{ul}^{(\frac{1}{2}+\epsilon,\dots,\frac{1}{2}+\epsilon)}(\mathbb{R}^{2n}),$$

with continuous injections, we can assume that $E = H_{ul}^{\sigma}(\mathbb{R}^{2n})$.

We follow the idea of proof of Théorème 7 of [1].

Write

$$a(x,\eta) = a(x,\eta)\varphi_0(\eta) + \sum_{j\geq 1} a_j(2^{j\delta}x, 2^{-j\delta}\eta),$$

where $a_j(x,\eta) = a(2^{-j\delta}x, 2^{j\delta}\eta)\varphi(2^{j(\delta-1)}\eta)$ and $\varphi_0 \in \mathcal{D}(\mathbb{R}^n)$, $\varphi \in \mathcal{D}(\mathbb{R}^n\backslash 0)$ define the dyadic partition of unity $1 = \varphi_0(\eta) + \sum_j \varphi(2^{-j}\eta)$.

In view of Corollary 3, since the term $a(x,\eta)\varphi_0(\eta)$ is in $E$, we can neglect it. By assumption, $(a_j)_j$ is a bounded sequence in $E$.

Write $a_j = b_j + r_j$ where $b_j$ is defined by

$$\mathcal{F}_1(b_j)(\xi,\eta) = \hat{\chi}(2^{j(\delta-1)}\xi)\mathcal{F}_1(a_j)(\xi,\eta), \quad \chi \in \mathcal{S}(\mathbb{R}^n), \ \hat{\chi} = 1 \text{ near } 0.$$

The first observation is that $(b_j)_j$ is also a bounded sequence in $E$. In fact, we can write:

$$b_j(x,\eta) = \int a_j(x-y,\eta)\chi(2^{j(1-\delta)}y)2^{j(1-\delta)n}dy;$$

hence, $E$ being translation invariant, $\|b_j\|_E \leq \|\chi\|_1\|a_j\|_E$.

Now, set $B = \sum_j b_j(2^{j\delta}x, 2^{-j\delta}D)$. The other observation is that, if $\operatorname{supp}(\hat{\chi})$ is taken small enough, $B$ turns out to be a paradifferential operator (associated with $a$), in the sense that the spectrum of $b_j(2^{j\delta}x, 2^{-j\delta}D)v$ is contained in $2^j\Gamma$, $\Gamma$ being some compact set in $\mathbb{R}^n\backslash 0$. Therefore, if $\theta \in \mathcal{D}(\mathbb{R}^n\backslash 0)$ and $\theta = 1$ on $\operatorname{supp}(\varphi)$, we can estimate as follows:

$$\begin{aligned}
|Bv|_0^2 &\leq \text{cst} \sum_j |b_j(2^{j\delta}x, 2^{-j\delta}D)\theta(2^{-j}D)v|_0^2 \\
&\leq \text{cst} \sum_j \|b_j(2^{j\delta}x, 2^{-j\delta}D)\|^2_{\mathcal{L}(L^2)}|\theta(2^{-j}D)v|_0^2 \\
&\leq \text{cst} \sum_j \|b_j(x, D)\|^2_{\mathcal{L}(L^2)}|\theta(2^{-j}D)v|_0^2 \\
&\leq \text{cst} \sum_j \|b_j\|^2_E|\theta(2^{-j}D)v|_0^2 \qquad \text{(by Corollary 3)} \\
&\leq \text{cst} \sup_j \|b_j\|^2_E|v|_0^2.
\end{aligned}$$

Hence, $\|B\|_{\mathcal{L}(L^2)} \le \text{cst} \sup_j \|a_j\|_E$.

Now, let us show that $\|r_j(x,D)\|_{\mathcal{L}(L^2)} \le \text{cst}\,\epsilon_j \sup_j \|a_j\|_E$, with $(\epsilon_j)_j \in l^1$, which will establish the theorem.

Set $F = H^{\sigma'}_{ul}(\mathbb{R}^{2n})$, with $\frac{1}{2} < \sigma'_i < \sigma_i$, $1 \le i \le 2n$.

By Corollary 3, $\|r_j(x,D)\|_{\mathcal{L}(L^2)} \le \text{cst}\,\|r_j\|_F$. To estimate $\|r_j\|_F$, we first establish some technical formula. If $\chi \in \mathcal{D}(\mathbb{R}^n)$, $\int \chi(x)dx = 1$, $\theta \in \mathcal{S}(\mathbb{R}^n)$ and $r : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ is some bounded function, we can write:

$$\widehat{\theta}(D_x)r(x,\eta) = \int \theta(y)\chi(x-y-z)r(x-y,\eta)dydz.$$

If $N$ is an even integer, using Taylor formula, we obtain:

$$\langle p - z\rangle^N = \sum_{\alpha,\beta,\gamma} c_{\alpha\beta\gamma}(x+p-y-z)^\alpha x^\beta y^\gamma,$$

the sum being finite and $c_{\alpha\beta\gamma}$ being some coefficients.

Setting $\theta_\gamma(y) = y^\gamma\theta(y)$, $\chi_\alpha(x) = x^\alpha\chi(x)$, we can write:

$$\begin{aligned}
&\chi(x)\chi(\eta)[\widehat{\theta}(D_x)r](x+p,\eta+q)\\
(3)\qquad &= \sum_{\alpha,\beta,\gamma} c_{\alpha\beta\gamma}\chi_\beta(x)\int \frac{\theta_\gamma(y)\chi_\alpha(x+p-y-z)\chi(\eta)r(x+p-y,\eta+q)}{\langle p-z\rangle^N}dydz\\
&= \sum_{\alpha,\beta,\gamma} c_{\alpha\beta\gamma}\chi_\beta(x)\int \frac{\widehat{\theta_\gamma}(D_x)((\tau_z\chi_\alpha)r)(x+p,\eta+q)\chi(\eta)}{\langle p-z\rangle^N}dz.
\end{aligned}$$

Now, write the $n$-dyadic decomposition of $r_j(x,\eta)$ in $x$: $r_j = \sum_{k_1,\dots,k_n} r_{jk_1\dots k_n}$. Because of the spectrum of $r_j$ (in $x$), there exists some integer $j_0$ such that $r_{jk_1\dots k_n} = 0$ if $k_i < j - j_0$, $\forall i$. Hence,

$$r_j = \sum_{i=1}^n \sum_{k_1,\dots,k_{i-1}<j-j_0\le k_i} r_{jk_1\dots k_n}.$$

Applying (3) with $r = r_{jk_1\dots k_n}$ and $\widehat{\theta}(\xi) = \varphi_{k_1}(\xi_1)\dots\varphi_{k_n}(\xi_n)$, the $\varphi_k$ being the functions defining the dyadic partition of unity in $\mathbb{R}$, we can estimate as follows (of course $N$ is taken $\geq n+1$):

$$\begin{aligned}
&\|(\chi \otimes \chi)\tau_{(-p,-q)} r_{jk_1\ldots k_n}\|_{H^{\sigma'}} \\
&\leq \operatorname{cst} \sum_{\alpha,\beta,\gamma} \int \frac{\|\widehat{\theta}_\gamma(2^{-k_1}D_1,\ldots,2^{-k_n}D_n)((\chi_\alpha \otimes \chi)\tau_{(-z,q)} r_j)\|_{H^{\sigma'}}}{\langle z\rangle^N}\, dz \\
&\leq \operatorname{cst} \sum_{\alpha,\beta,\gamma} \prod_{i=1}^{n} 2^{-k_i(\sigma_i-\sigma_i')} \int \frac{\|(\chi_\alpha \otimes \chi)\tau_{(-z,q)} r_j\|_{H^{\sigma}}}{\langle z\rangle^N}\, dz \\
&\leq \operatorname{cst} \prod_{i=1}^{n} 2^{-k_i(\sigma_i-\sigma_i')}\|r_j\|_E \leq \operatorname{cst} \prod_{i=1}^{n} 2^{-k_i(\sigma_i-\sigma_i')}\|a_j\|_E.
\end{aligned}$$

Hence,

$$\|r_j\|_F \leq \operatorname{cst} \sum_{i=1}^{n} \sum_{j-j_0\leq k_i} \prod_{l=1}^{n} 2^{-k_l(\sigma_l-\sigma_l')}\|a_j\|_E \leq \operatorname{cst} \sum_{i=1}^{n} 2^{-j(\sigma_i-\sigma_i')}\|a_j\|_E,$$

which achieves the proof of Theorem 16.

## 3. - Appendix

A0. *The bilinear map $(u,v) \to u * v$ is continuous from $L^1 \times L^2_{ul}$ into $L^2_{ul}$.*

PROOF. We have $|u * v|_{0,ul} \leq \int |u(y)||\tau_y v|_{0,ul} dy \leq \|u\|_1 |v|_{0,ul}$, since $L^2_{ul}$ is translation invariant.

A1. *Let $u$ be a function in $L^2_{ul}(\mathbb{R}^n)$ whose spectrum is contained in a compact set $K$. Then, $u$ is bounded and $\|u\|_\infty \leq \gamma_K |u|_{0,ul}$, with some constant $\gamma_K$. Moreover, if $K = z + Q$, $z \in \mathbb{R}^n$ (resp. $K = B(0,R)$, $R \geq 1$ or $K = \prod_{i=1}^{n}[-R_i, R_i]$, $R_i \geq 1$), we can take $\gamma_K$ independant of $z$ (resp. $\gamma_K = \operatorname{cst} R^{\frac{n}{2}}$ or $\gamma_K = \operatorname{cst}\sqrt{R_1R_2\ldots R_n}$).*

PROOF. Take $\psi \in S(\mathbb{R}^n)$ such that $\widehat{\psi} = 1$ on $K$, $\operatorname{supp}(\widehat{\psi}) \subset K_\epsilon$ and $\chi \in \mathcal{D}(\mathbb{R}^n)$ with $\int \chi(x)dx = 1$. Then, we can write $u(x) = \int \psi(x-y)\chi(y-k)$

$u(y)dydk$ which allows us to estimate as follows:

$$|u(x)| \leq \text{cst} \int \langle x-k\rangle^{-N}\langle x-y\rangle^{N}\langle y-k\rangle^{N}|\psi(x-y)\chi(y-k)u(y)|dydk$$
$$\leq \text{cst} \int \langle x-k\rangle^{-N} \sqrt{\iint |\langle x-y\rangle^{N}\psi(x-y)|^2 dy}$$
$$\cdot \sqrt{\iint |\langle y-k\rangle^{N}\chi(y-k)u(y)|^2 dy\, dk}$$
$$\leq \text{cst}\, |\widehat{\psi}|_N |u|_{0,ul},$$

with $N = n+1$.

Now, it remains to take $\widehat{\psi}(\xi)$ of the form $\theta(\xi - z)$ (resp. $\theta\left(\frac{\xi}{R}\right)$ or $\prod_{i=1}^{n} \theta\left(\frac{\xi_i}{R_i}\right)$) with $\theta \in S$ and $\theta = 1$ near $Q$ (resp. $B(0,1)$; $[-1,1]$), to obtain the desired estimates. So A1 is proved.

A2. (i) *We have the inclusions* $\mathcal{B} \subset \mathcal{E} \subset L^\infty$.

(ii) $\mathcal{B}$ *and* $\mathcal{E}$ *are algebras.*

PROOF. (i) Write the $2n$-dyadic decomposition $u = \sum_{j\in\mathbf{N}^{2n}} u_j$. By A1, we have $\|u_j\|_\infty \leq \text{cst}\, 2^{\frac{|j|}{2}} |u_j|_{0,ul}$. Hence, if $u \in L^\infty$, the series $\sum_j u_j$ is absolutely convergent in $L^\infty$. So, $\mathcal{E} \subset L^\infty$. Write now the double dyadic decompositions of $u$ and $u_j$: $u = \sum_{k,l\geq 0} u_{kl}$ and $u_j = \sum_{k,l\geq 0} u_{jkl}$. Because of the support of $\widehat{u}_{jkl}$, we have $2^k \sim 2^{j_1} + \cdots + 2^{j_n}$ and $2^l \sim 2^{j_{n+1}} + \cdots + 2^{j_{2n}}$ for non zero $u_{jkl}$. Therefore, we can estimate as follows ($k_0$ and $l_0$ are some fixed convenient integers):

$$\sum_j 2^{\frac{|j|}{2}} |u_j|_{0,ul} \leq \sum_{k,l\geq 0} \sum_{j_1,\dots j_n \leq k+k_0} \sum_{j_{n+1},\dots j_{2n}\leq l+l_0} 2^{\frac{|j|}{2}} |u_j|_{0,ul}$$
$$\leq \text{cst} \sum_{k,l\geq 0} 2^{(k+l)\frac{n}{2}} |u_{jkl}|_{0,ul}$$
$$\leq \text{cst} \sum_{k,l\geq 0} 2^{(k+l)\frac{n}{2}} |u_{kl}|_{0,ul},$$

by A0. So, $\mathcal{B} \subset \mathcal{E}$.

(ii) Set $N = 2n$ and write the $N$-dyadic decompositions: $u = \sum_{k\in\mathbf{N}^N} u_k$, $v = \sum_{l\in\mathbf{N}^N} v_l$. Since $\operatorname{supp}(\widehat{u}_k\widehat{v}_l) \subset \prod_{i=1}^{N} [-\text{cst}(2^{k_i} + 2^{l_i}), \text{cst}(2^{k_i} + 2^{l_i})]$, we can write, if

$j \in \mathbb{N}^N$ and $\nu$ is some fixed convenient integer:

$$\varphi_j(D)(uv) = \sum_{2^{j_i} \le \text{cst}(2^{k_i}+2^{l_i})} \varphi_j(D)(u_k v_l) = \sum_{j_i \le k_i+\nu \ \text{or} \ j_i \le l_i+\nu} \varphi_j(D)(u_k v_l).$$

We need the following lemma:

LEMMA. *We have:* $|u_k v_l|_{0,ul} \le \text{cst}\, 2^{\frac{|m|}{2}} |u_k|_{0,ul} |u_l|_{0,ul}$, *where* $m \in \mathbb{N}^N$ *is such that* $m_i = k_i$ *or* $l_i$, $1 \le i \le N$.

Assume the lemma for a moment. Applying it with $m_i = l_i$ if $j_i \le k_i + \nu$ and $m_i = k_i$ if $j_i \le l_i + \nu$, we can write:

$$\sum_j 2^{\frac{|j|}{2}} |\varphi_j(D)(u_k v_l)|_{0,ul} \le \text{cst} \sum_{k,l} \sum_{j_i \le m'_i+\nu} 2^{\frac{|m|+|j|}{2}} |u_k|_{0,ul} |u_l|_{0,ul},$$

where $m' = k + l - m$; hence,

$$\begin{aligned} \|uv\|_{\mathcal{E}} &= \sum_j 2^{\frac{|j|}{2}} |\varphi_j(D)(u_k v_l)|_{0,ul} \le \text{cst} \sum_{k,l} 2^{\frac{|k|+|l|}{2}} |u_k|_{0,ul} |u_l|_{0,ul} \\ &= \text{cst}\, \|u\|_{\mathcal{E}} \|v\|_{\mathcal{E}}, \end{aligned}$$

which is the desired estimate in the case of $\mathcal{E}$.

The proof is similar (and even easier) in the case of $\mathcal{B}$ and is left to the reader.

PROOF OF THE LEMMA. To be simple, let us treat the case $m = (l', k'')$ if $k = (k', k'')$ and $l = (l', l'')$ with respect to the decomposition $\mathbb{R}^N = \mathbb{R}^{N'} \times \mathbb{R}^{N''}$. The general case can be treated in the same way.

If $\chi$, $\tilde{\chi} \in \mathcal{D}(\mathbb{R}^N)$ and $\chi = 1$ on supp($\chi$), let us estimate $|u_k v_l \tau_y \chi|_0 = |f_k g_l|_0$ where $f_k = u_k \tau_y \chi$ and $g_l = v_l \tau_y \tilde{\chi}$. We have:

$$\begin{aligned} (2\pi)^{\frac{N}{2}} |f_k g_l|_0 &= \left( \int \left| \int \widehat{f}_k(\xi' - \eta', \eta'') \widehat{g}_l(\eta', \xi'' - \eta'') d\eta' d\eta'' \right|^2 d\xi' d\xi'' \right)^{\frac{1}{2}} \\ &\le \int \left( \int \left| \widehat{f}_k(\xi', \eta'') \widehat{g}_l(\eta', \xi'') \right|^2 d\xi' d\xi'' \right)^{\frac{1}{2}} d\eta' d\eta'' \\ &\le \int \left( \int \left| \widehat{f}_k(\xi', \eta'') \right|^2 d\xi' \right)^{\frac{1}{2}} d\eta'' \int \left( \int |\widehat{g}_l(\eta', \xi'')|^2 d\xi'' \right)^{\frac{1}{2}} d\eta'. \end{aligned}$$

Set $I_k = \displaystyle\int \left( \int |\widehat{f}_k(\xi)|^2 d\xi' \right)^{\frac{1}{2}} d\xi''$ and let us show that $I_k \le \text{cst}\, 2^{\frac{|k''|}{2}} |u_k|_{0,ul}$. Using the notation $2^{\pm k} x = (2^{\pm k_1} x_1, \dots, 2^{\pm k_N} x_N)$, we can write $\widehat{f}_k(\xi) = 2^{-|k|} \widehat{h}_k(2^{-k}\xi)$

where $h_k(x) = \chi(2^{-k}x - y)u_k(2^{-k}x)$. Now, we can estimate as follows:

$$
\begin{aligned}
I_k &\le \operatorname{cst} 2^{-\frac{|k'|}{2}} \sqrt{\int\!\!\int |\langle \xi'' \rangle^s \hat{h}_k(\xi)|^2 d\xi}, \quad (\text{with some } s \in \mathbb{N}) \\
&\le \operatorname{cst} 2^{-\frac{|k'|}{2}} \sum_{|\alpha| \le s} |\partial_{x''}^{\alpha} h_k|_0 \le \operatorname{cst} 2^{-\frac{|k'|}{2}} \sum_{|\alpha| \le s} 2^{-\alpha k''} 2^{\frac{|k|}{2}} |\partial_{x''}^{\alpha} f_k|_0 \\
&\le \operatorname{cst} 2^{\frac{|k''|}{2}} \sum_{|\alpha| \le s, \alpha' \le \alpha} \binom{\alpha}{\alpha'} 2^{-\alpha k''} |\tau_y \partial_{x''}^{\alpha'} \partial_{x''}^{\alpha - \alpha'} u_k|_0 \\
&\le \operatorname{cst} 2^{\frac{|k''|}{2}} \sum_{|\alpha| \le s, \alpha' \le \alpha} 2^{-\alpha k''} |\partial_{x''}^{\alpha - \alpha'} u_k|_{0,ul} \\
&\le \operatorname{cst} 2^{\frac{|k''|}{2}} \sum_{|\alpha| \le s, \alpha' \le \alpha} 2^{-\alpha k''} 2^{k''(\alpha - \alpha')} |u_k|_{0,ul}, \quad (\text{by A0}) \\
&\le \operatorname{cst} 2^{\frac{|k''|}{2}} |u_k|_{0,ul}.
\end{aligned}
$$

By the same method, we obtain $\int \left( \int |\hat{g}_l(\xi)|^2 d\xi'' \right)^{\frac{1}{2}} d\xi' \le \operatorname{cst} 2^{\frac{|l'|}{2}} |v_l|_{0,ul}$ and this achieves the proof of the lemma.

A3. *$\mathcal{A}(\mathbb{R}^n)$ is a subalgebra of $L^\infty(\mathbb{R}^n)$.*

PROOF. If $u \in \mathcal{A}$, $\chi \in \mathcal{D}$ and $\sum_{k \in \mathbf{Z}^n} \tau_k \chi = 1$, then $u = \sum_{k \in \mathbf{Z}^n} \chi(D - k)u$ and $\|u\|_\infty \le \sum_{k \in \mathbf{Z}^n} \|\chi(D - k)u\|_\infty$. Hence, $\mathcal{A} \subset L^\infty$.

If $u$, $v \in \mathcal{A}$, set $u_k = \chi(D - k)u$ and $v_l = \chi(D - l)v$. Since

$$\operatorname{supp}(\hat{u}_k \hat{v}_l) \subset \operatorname{supp}(\hat{u}_k) + \operatorname{supp}(\hat{v}_l) \subset k + l + Q,$$

$Q$ being some compact set, there exists $\theta \in \mathcal{D}$ such that:

$$\chi(D - j)(uv) = \sum_{k,l} \theta(k + l - j)\chi(D - j)(u_k v_l).$$

$$
\begin{aligned}
\text{Hence,} \sum_{j \in \mathbf{Z}^n} \|\chi(D - j)(uv)\|_\infty &\le \sum_{j,k,l} |\theta(k + l - j)| \|\mathcal{F}^{-1}\chi\|_1 \|u_k\|_\infty \|v_l\|_\infty \\
&\le \|\mathcal{F}^{-1}\chi\|_1 \sum_j |\theta(j)| \sum_k \|u_k\|_\infty \sum_l \|v_l\|_\infty,
\end{aligned}
$$

i.e. $\|uv\|_{\mathcal{A}} \le \operatorname{cst} \|u\|_{\mathcal{A}} \|v\|_{\mathcal{A}}$, which is the desired estimate. A3 is so proved.

A4. *Let $s$, $s'$, $s'' \in \mathbb{R}$, $\sigma \in \mathbb{R}^{2n}$. The following inclusions hold:*

(i) $B^s_{p,1} \subset H^s_p \subset B^s_{p,\infty}$; $B^{s',s''}_{p,1} \subset H^{s',s''}_p \subset B^{s',s''}_{p,\infty}$; $B^\sigma_{p,1} \subset H^\sigma_p \subset B^\sigma_{p,\infty}$.

(ii) $B^s_{p,q}(\mathbb{R}^{2n}) \subset B^{s',s''}_{p,q}(\mathbb{R}^n \times \mathbb{R}^n) \subset B^\sigma_{p,q}(\mathbb{R}^{2n})$ *if* $\sigma_i > 0$, $1 \leq i \leq 2n$, $\sigma_1 + \cdots + \sigma_n \leq s'$, $\sigma_{n+1} + \cdots + \sigma_{2n} \leq s''$ *and* $s' + s'' \leq s$.

(iii) $H^s_p(\mathbb{R}^{2n}) \subset H^{s',s''}_p(\mathbb{R}^n \times \mathbb{R}^n) \subset H^\sigma_p(\mathbb{R}^{2n})$ *if* $1 < p < \infty$, $\sigma_i > 0$, $1 \leq i \leq 2n$, $\sigma_1 + \cdots + \sigma_n \leq s'$, $\sigma_{n+1} + \cdots + \sigma_{2n} \leq s''$ *and* $s' + s'' \leq s$.

PROOF. (i) The first inclusions are well known. We shall only prove the last ones, the argument being similar for the others.

We can write: $\varphi_j(D) = 2^{-j\sigma} f_j(2^{-j}D) \prod_{i=1}^{2n} (1 - \Delta_i)^{\frac{\sigma_i}{2}}$ where $f_j(\xi) = \prod_{i=1}^{2n} \varphi(\xi_i) \langle 2^{j_i} \xi_i \rangle^{-\sigma_i}$. Observe that $(f_j)$ is a bounded sequence in $\mathcal{D}$. Hence,

$$\|\varphi_j(D)u\|_p \leq 2^{-j\sigma} \|\mathcal{F}^{-1}(f_j)\|_1 \|u\|_{H^\sigma_p},$$

which implies that $\|u\|_{B^\sigma_{p,\infty}} \leq \mathrm{cst}\, \|u\|_{H^\sigma_p}$. So, $H^\sigma_p \subset B^\sigma_{p,\infty}$.

For the other inclusion, write:

$$\prod_{i=1}^{2n} (1 - \Delta_i)^{\frac{\sigma_i}{2}} = \sum_j \prod_{i=1}^{2n} (1 - \Delta_i)^{\frac{\sigma_i}{2}} \varphi_j(D) J.$$

Then, by an argument similar to the one above, we obtain

$$\left\| \prod_{i=1}^{2n} (1 - \Delta_i)^{\frac{\sigma_i}{2}} \varphi_j(D) u \right\|_p \leq \mathrm{cst}\, 2^{j\sigma} \|\varphi_j(D)u\|_p;$$

Hence, $\|u\|_{H^\sigma_p} \leq \mathrm{cst}\, \|u\|_{B^\sigma_{p,\infty}}$.

(ii) Here, the proof is similar to that of A2(i). We have:

$$\sum_j (2^{j\sigma} \|u_j\|_p)^q \leq \mathrm{cst} \sum_j 2^{j\sigma q} \sum_{k,l} \|\varphi(2^{-k}D_x)\varphi(2^{-l}D_\eta)u_j\|_p^q;$$

in fact, the number of indices $(k,l)$ for which $\varphi(2^{-k}D_x)\varphi(2^{-l}D_\eta)u_j$ does not vanish identically is finite and independant of $j$. Hence,

$$\sum_j (2^{j\sigma} \|u_j\|_p)^q \leq \mathrm{cst} \sum_{k,l} \sum_{j_1,\ldots,j_n \leq k+k_0} \sum_{j_{n+1},\ldots,j_{2n} \leq l+l_0} 2^{j\sigma q} \|\varphi(2^{-k}D_x)\varphi(2^{-l}D_\eta)u\|_p^q$$

$$\leq \mathrm{cst} \sum_{k,l} 2^{kqs'} 2^{lqs''} \|\varphi(2^{-k}D_x)\varphi(2^{-l}D_\eta)u\|_p^q,$$

so that, $B^{s',s''}_{p,q} \subset B^\sigma_{p,q}$.

The proof of the other inclusion is similar.

(iii) We have: $\prod_{i=1}^{2n}(1-\Delta_i)^{\frac{\sigma_i}{2}} = h(D)(1-\Delta')^{\frac{s'}{2}}(1-\Delta'')^{\frac{s''}{2}}$, where $h(\xi,y) = \langle\xi\rangle^{-s'}\langle y\rangle^{-s''}\prod_{i=1}^{n}\langle\xi_i\rangle^{\sigma_i}\langle y_i\rangle^{\sigma_{n+i}}$ and $(1-\Delta')^{\frac{s'}{2}}$ (resp. $(1-\Delta'')^{\frac{s''}{2}}$) is the pseudodifferential operator with symbol $\langle\xi\rangle^{s'}$ (resp. $\langle y\rangle^{s''}$). So, it is sufficient to prove that $h(D)$ is bounded in $L^p(\mathbb{R}^{2n})$, $1 < p < \infty$, and this follows from Lizorkin theorem which extends Mikhlin theorem on Fourier multipliers in $L^p$. See [5], page 166.

The proof of the other inclusion is similar.

A5. *Let $p$, $q \in [1,\infty]$, $s > 0$, $s' > 0$, $(\epsilon_{jk}) \in l^q(\mathbb{N}\times\mathbb{N})$ and $(u_{jk})$ be a sequence in $L^p(\mathbb{R}^n\times\mathbb{R}^n)$. Assume that there exist integers $N > s$ and $N' > s'$ for which we have $\|\partial_1^\alpha\partial_2^\beta u_{jk}\|_p \le \epsilon_{jk}2^{j(|\alpha|-s)+k(|\beta|-s')}$ if $0\le|\alpha|\le N$ and $0\le|\beta|\le N'$. Then, $\sum_{j,k} u_{jk} \in B^{s,s'}_{p,q}(\mathbb{R}^n\times\mathbb{R}^n)$.*

PROOF. We assume that $N$ and $N'$ are even; the general case need a little more refined proof and is not needed here.

Set $u = \sum_{jk} u_{jk}$. If $1 = \sum_l \varphi_l$ is the fixed dyadic partition of unity in $\mathbb{R}^n$, set $\varphi_{lm}(D) = \varphi_l(D_1)\varphi_m(D_2)$. We can write:

$$\varphi_{lm}(D)u = \sum_{j\ge l,k\ge m}\varphi_{lm}(D)u_{jk} + \sum_{j<l,k<m}\varphi_{lm}(D)u_{jk} + \sum_{j<l,k\ge m}\varphi_{lm}(D)u_{jk} + \sum_{j\ge l,k<m}\varphi_{lm}(D)u_{jk}.$$

1st sum: We have

$$\left\|\sum_{j\ge l,k\ge m}\varphi_{lm}(D)u_{jk}\right\|_p \le \text{cst}\sum_{j\ge l,k\ge m}\|u_{jk}\|_p \le \text{cst}\,2^{-ls-ms'}\sum_{j\ge l,k\ge m}\epsilon_{jk}2^{(l-j)s+(m-k)s'}.$$

Set $\epsilon'_{lm} = \sum_{j\ge l,k\ge m}\epsilon_{jk}2^{(l-j)s+(m-k)s'}$. Clearly, $(\epsilon'_{jk})$ is the convolution of an $l^1$ sequence by an $l^q$ sequence so that it is in $l^q$.

2nd sum: We can write $\varphi_{lm}(D)u_{jk} = 2^{-lN-mN'}\tilde{\varphi}_{lm}(D)\Delta_1^{\frac{N}{2}}\Delta_2^{\frac{N'}{2}}u_{jk}$ where $\tilde{\varphi}_{lm}(\xi,y) = |2^{-l}\xi|^{-N}|2^{-m}y|^{-N'}\varphi_{lm}(\xi,y)$ and $\Delta_1^{\frac{N}{2}}$ (resp. $\Delta_2^{\frac{N'}{2}}$) is the pseudodifferential operator with symbol $|\xi|^N$ (resp. $|y|^{N'}$). Hence, we can estimate as follows:

$$\left\|\sum_{j<l,k<m} \varphi_{lm}(D)u_{jk}\right\|_p \leq \text{cst} \sum_{j<l,k<m} 2^{-lN-mN'}\|\Delta_1^{\frac{N}{2}}\Delta_2^{\frac{N'}{2}}u_{jk}\|_p$$

$$\leq \text{cst}\, 2^{-ls-ms'} \sum_{j<l,k<m} \epsilon_{jk}2^{(l-j)(s-N)+(m-k)(s'-N')}.$$

We conclude as for the first sum.

3rd sum: Write $\varphi_{lm}(D)u_{jk} = 2^{-lN}\tilde{\varphi}_{lm}(D)\Delta_1^{\frac{N}{2}}u_{jk}$ where $\tilde{\varphi}_{lm}(\xi,y) = |2^{-l}\xi|^{-N}\varphi_{lm}(\xi,y)$ and estimate as follows:

$$\left\|\sum_{j<l,k\geq m} \varphi_{lm}(D)u_{jk}\right\|_p \leq \text{cst} \sum_{j<l,k\geq m} 2^{-lN}\|\Delta_1^{\frac{N}{2}}u_{jk}\|_p$$

$$\leq \text{cst}\, 2^{-ls-ms'} \sum_{j<l,k\geq m} \epsilon_{jk}2^{(l-j)(s-N)+(m-k)s'}.$$

We conclude as for the other sums.

The discussion of the 4th sum is, of course, similar to that of the 3rd one.

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