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1. - Introduction

This paper is concerned with the problem of continuous dependence for solutions of the $m \times m$ system of conservation laws

$$u_t + [F(u)]_x = 0. \tag{1.1}$$

We assume that the system is strictly hyperbolic and that each characteristic field is either genuinely nonlinear or linearly degenerate. If the total variation of u at the initial time $t = 0$ is suitably small, the global existence of weak solutions of (1.1) was established in a fundamental paper of Glimm [7]. Since then, the question of uniqueness has been investigated by several authors [6, 9, 11, 12, 13, 14], but no general result is yet known. A paper by Temple [17] shows that monotone semigroup techniques cannot be applied to the general problem (1.1).

We remark that a natural way to establish the stability of a solution $\tilde{u}(\cdot)$ of a nonlinear evolution problem

$$\dot{u}(t) = f(u(t)) \tag{1.2}$$

relies on the study of the linearized variational system

$$\dot{v}(t) = Df(\tilde{u}(t)) \cdot v(t). \tag{1.3}$$

If (1.3) is globally stable, then one can often prove that $\tilde{u}$ is a stable solution of (1.2). This program, in connection with discontinuous solutions of the system (1.1), was initiated in [3]. For a class of piecewise Lipschitz continuous functions $u : \mathbb{R} \to \mathbb{R}^m$, one can define a space T_u of "generalized tangent vectors" and derive a linear system of equations describing how first order variations evolve

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in time, along solutions of (1.1). If u has N points of jump, a tangent vector to u has the form $(v, \xi) \in T_u \doteq \mathbf{L}^1 \times \mathbb{R}^N$, where v describes changes in the values of u while $\xi = (\xi_1, \dots, \xi_N)$ accounts for shifts in the locations of the shocks. The present paper contains a detailed study of these linearized equations for v, ξ. Our main result is the existence of a family of norms $\|\cdot\|_u$, defined on suitable tangent spaces T_u for u piecewise Lipschitz continuous with small total variation, having the following properties.

i) If $u = u(t, x)$ is a piecewise Lipschitz continuous solution of (1.1) and $(v(t), \xi(t))$ is a corresponding tangent vector to $u(t, \cdot)$, then the norm $\|(v(t), \xi(t))\|_{u(t)}$ is a nonincreasing function of time, even in the presence of interacting shocks.

ii) The Riemann-type metric obtained from the norms $\|\cdot\|_u$ is uniformly equivalent to the $\mathbf{L}^1$ distance.

A precise definition of these weighted norms is given in (4.2). In Section 4 we study the behavior of tangent vectors to solutions whose discontinuities remain isolated. The case of interacting shocks is then analyzed in section 5.

As an application, given any two piecewise Lipschitz continuous functions u, u' with small total variation, consider the family $\Sigma_{u,u'}$ of all continuous paths $\gamma : [0, 1] \to \mathbf{L}^1$ with $\gamma(0) = u$, $\gamma(1) = u'$ such that, for all $0 < \theta < 1$, the differential $D\gamma(\theta)$ is a well defined element in $T_{\gamma(\theta)}$. The length of a path $\gamma \in \Sigma_{u,u'}$ can then be measured as

$$\|\gamma\| \doteq \int_0^1 \|D\gamma(\theta)\|_{\gamma(\theta)} d\theta, \tag{1.4}$$

while the Riemannian distance between u and u' is given by

$$d(u, u') = \inf\{\|\gamma\|; \quad \gamma \in \Sigma_{u,u'}\}. \tag{1.5}$$

Because of (i), the length of every regular path does not increase in time along the flow of (1.1). This suggests that the flow generated by (1.1) should be globally contractive w.r.t. the distance (1.5); hence by (ii) it should also be Lipschitz continuous w.r.t. the usual $\mathbf{L}^1$ distance. A rigorous proof of this conjecture was given in [1, 2] for systems with coinciding shock and rarefaction curves, and in [5] for all 2×2 systems. We believe that the present analysis can provide a guideline for a proof in the general $n \times n$ case.

Other examples of nonlinear evolution equations generating a flow which is contractive w.r.t. a suitable Riemann-type metric can be found in [4, 13]. For stability results in the presence of viscosity, see [10, 16].

2. - Preliminaries

In the following, the euclidean norm and inner product on $\mathbb{R}^m$ are denoted by $|\cdot|$ and $\langle\cdot,\cdot\rangle$ respectively; Ω is an open convex subset of $\mathbb{R}^m$ and F is a three times continuously differentiable vector field whose domain contains the closure of Ω. For $u, u' \in \Omega$, we call $A(u) = DF(u)$ the Jacobian matrix of F at u and define the matrix

$$A(u,u') \doteq \int_0^1 A(\theta u + (1-\theta)u')d\theta. \tag{2.1}$$

We assume that there exist m disjoint intervals $[\lambda_i^{\min}, \lambda_i^{\max}]$, with $\lambda_i^{\max} < \lambda_{i+1}^{\min}$, such that the i-th eigenvalue $\lambda_i(u,u')$ of $A(u,u')$ satisfies

$$\lambda_i(u,u') \in [\lambda_i^{\min}, \lambda_i^{\max}], \qquad i = 1, \dots, m, \quad u, u' \in \Omega. \tag{2.2}$$

This is certainly the case if $u_0 \in \mathbb{R}^m$, $A(u_0)$ has m real distinct eigenvalues and Ω is a suitably small neighborhood of u_0. Throughout the paper, we assume that each characteristic field is either linearly degenerate or genuinely nonlinear according to Lax [8]. For $i = 1, \dots, m$, one can then select a $\mathcal{C}^2$ family of right and left eigenvectors $r_i(u,u')$, $l_i(u,u')$ of $A(u,u')$, normalized as follows. If the i-th characteristic field is linearly degenerate, choose r_i such that

$$|r_i(u,u')| \equiv 1. \tag{2.3}$$

If the i-th characteristic field is genuinely nonlinear, normalize $r_i(u,u')$ so that

$$\lim_{\varepsilon\to 0} \varepsilon^{-1}[\lambda_i(u + \varepsilon r_i(u,u'),\ u' + \varepsilon r_i(u,u')) - \lambda_i(u,u')] = 1. \tag{2.4}$$

This is certainly possible whenever u, u' are sufficiently close. Then choose the families $l_j(u,u')$ so that

$$\langle l_j(u,u'),\ r_i(u,u')\rangle = \delta_{ij} \qquad \forall i,j, \tag{2.5}$$

where δ_{ij} is the Kronecker symbol. Since $A(u,u) = A(u)$, we write $\lambda_i(u)$ for $\lambda_i(u,u)$ and similarly for r_i and l_i. The differential of λ_i at (u,u') is written $D\lambda_i(u,u')$. We thus have

$$D\lambda_i(u,u')\cdot(v,v') = \lim_{\varepsilon\to 0} \varepsilon^{-1}[\lambda_i(u + \varepsilon v, u' + \varepsilon v') - \lambda_i(u,u')].$$

The same notation will be used for the differentials of the eigenvectors r_i and l_i.

Let $u = u(t,x)$ be piecewise Lipschitz continuous weak solution of (1.1), taking values inside Ω. Then u_x exists almost everywhere and one can define its components

$$u_x^i(t,x) \doteq \langle l_i(u(t,x)),\ u_x(t,x)\rangle. \tag{2.6}$$

With this notation, for a.e. t, x one has

$$u_x = \sum_i u_x^i r_i(u), \tag{2.7}$$

$$u_t + A(u)u_x = u_t + \sum_i \lambda_i(u) r_i(u) u_x^i = 0. \tag{2.8}$$

In the following, if φ is a function defined on Ω, its directional derivative at $w \in \Omega$ along the vector field r_i is written

$$r_i \bullet \varphi(w) = D\varphi(w) \cdot r_i(w) = \lim_{\varepsilon \to 0} \varepsilon^{-1}[\varphi(w + \varepsilon r_i(w)) - \varphi(w)].$$

Differentiating (2.7) w.r.t. t and (2.8) w.r.t. x, combining the results and taking the inner products with $l_i(u)$, $i = 1, \dots, m$, one obtains the system of m scalar equations

$$(u_x^i)_t + (\lambda_i(u) u_x^i)_x = \sum_{j<k} G_{ijk}(u) u_x^j u_x^k \tag{2.9}$$

(see (2.6) in [1]), where

$$G_{ijk}(u) = (\lambda_k(u) - \lambda_j(u)) \langle l_i(u), \ [r_k, r_j](u) \rangle \tag{2.10}$$

and $[r_k, r_j] = r_k \bullet r_j - r_j \bullet r_k$ is the usual Lie bracket. If u is piecewise Lipschitz, then the measurable functions u_x^i provide an integral solution to the system (2.9). More precisely, for every i one can redefine u_x^i on a set of measure zero so that, along almost all characteristic lines $x = y_i(t)$ of the i-th family, with $\dot{y}_i(t) = \lambda_i(u(t, y_i(t)))$, the following holds. The component u_x^i is absolutely continuous and satisfies

$$\begin{aligned} u_x^i(t, y_i(t)) = u_x^i(s, y_i(s)) &+ \sum_{j<k} \int_s^t (G_{ijk}(u) u_x^j u_x^k)(\tau, y_i(\tau))\, d\tau \\ &- \sum_j \int_s^t (r_j \bullet \lambda_i(u) u_x^j u_x^i)(\tau, y_i(\tau))\, d\tau \end{aligned} \tag{2.11}$$

for $s < t$, as long as the characteristic does not cross a line where u is discontinuous. Let now $x_\alpha(t)$, $\alpha = 1, \dots, N$, describe the position of the α-th discontinuity of u at time t. If the jump at x_α occurs along the k_α-th characteristic family, the Rankine-Hugoniot conditions imply

$$\langle l_i(u^-, u^+), \ u^+ - u^- \rangle = 0 \qquad \forall i \neq k_\alpha, \tag{2.12}$$

$$\dot{x}_\alpha(t) = \lambda_{k_\alpha}(u^-, u^+). \tag{2.13}$$

Moreover, one has

$$\lambda_{k_\alpha}(u^+) = \lambda_{k_\alpha}(u^-, u^+) = \lambda_{k_\alpha}(u^-) \tag{2.14}$$

if the k_α-th characteristic field is linearly degenerate, while

$$\lambda_{k_\alpha}(u^+) + \epsilon_0|u^+ - u^-| \le \lambda_{k_\alpha}(u^-, u^+) \le \lambda_{k_\alpha}(u^-) - \epsilon_0|u^+ - u^-|, \tag{2.15}$$

in the genuinenly nonlinear case. Here $u^- \doteq u(x_\alpha^-)$ and $u^+ \doteq u(x_\alpha^+)$ denote respectively the left and right limits of $u(t,x)$ as x tends to $x_\alpha(t)$, while $\epsilon_0 > 0$ is a suitably small constant. Notations such as $\lambda_j^- \doteq \lambda_j(u^-)$, $\lambda_j^+ \doteq \lambda_j(u^+)$ will also be used. For any α, because of (2.11) the limits

$$u_x^{i^+} = u_x^i(t, x_\alpha^+), \quad u_x^{i^-} = u_x^i(t, x_\alpha^-)$$

can be defined pointwise for almost every t; except for the case where $i = k_\alpha$ and the k_α-th characteristic field is linearly degenerate. Differentiating (2.12) and using (2.7), (2.8), one obtains:

$$\left(\frac{\partial}{\partial t} + \dot{x}_\alpha \frac{\partial}{\partial x}\right) \langle l_i(u^-, u^+),\ u^+ - u^-\rangle = 0, \tag{2.16}$$

$$\begin{aligned} &\sum_j \langle Dl_i(u^-, u^+) \cdot ((\dot{x}_\alpha - \lambda_j^-)u_x^{j^-} r_j^-,\ (\dot{x}_\alpha - \lambda_j^+)u_x^{j^+} r_j^+),\ u^+ - u^-\rangle \\ &\qquad + \sum_j \langle l_i(u^-, u^+),\ (\dot{x}_\alpha - \lambda_j^+)u_x^{j^+} r_j^+ - (\dot{x}_\alpha - \lambda_j^-)u_x^{j^-} r_j^-\rangle = 0, \end{aligned} \tag{2.17}$$

for almost every t and all $i \ne k_\alpha$. Equations of the form (2.17) will arise over again, so we study them in greater detail. For $u^-, u^+ \in \Omega$ consider the functions

$$\begin{aligned} \Phi_i(u^-, u^+, w^-, w^+) \doteq &\sum_j \langle Dl_i(u^-, u^+) \cdot (w_j^- r_j(u^-),\ w_j^+ r_j(u^+)),\ u^+ - u^-\rangle \\ &+ \sum_j \langle l_i(u^-, u^+),\ w_j^+ r_j(u^+) - w_j^- r_j(u^-)\rangle. \end{aligned} \tag{2.18}$$

As in [3], define the sets $\mathcal{I}$ and $\mathcal{O}$ (incoming and outgoing) of signed indices:

$$\begin{aligned} \mathcal{I} &= \{i^+; i \le k_\alpha\} \cup \{i^-; i \ge k_\alpha\}, \\ \mathcal{O} &= \{j^-; j < k_\alpha\} \cup \{j^+; j > k_\alpha\}. \end{aligned} \tag{2.19}$$

For a fixed k_α, the system of $m-1$ equations

$$\Phi_i(u^-, u^+, w^-, w^+) = 0 \qquad (i \neq k_\alpha) \tag{2.20}$$

is linear homogeneous w.r.t. w^-, w^+, with coefficients which depend continuously on u^-, u^+. When $u^- = u^+$ one has

$$\frac{\partial \Phi_i}{\partial w_j^\pm} = \pm\delta_{ij}.$$

Therefore, whenever u^+, u^- are sufficiently close to each other we have

$$\det\left(\frac{\partial \Phi_i(u^-, u^+, w^-, w^+)}{\partial w_j^\pm}\right) \neq 0 \qquad (i \neq k_\alpha,\ j^\pm \in \mathcal{O}). \tag{2.21}$$

In turn, when the $(m-1)\times(m-1)$ determinant in (2.21) does not vanish, one can solve (2.20) for the $m-1$ outgoing variables $w_j^\pm, j^\pm \in \mathcal{O}$:

$$w_j^\pm = W^j(u^-, u^+)(w^I) \qquad j \neq k_\alpha, \tag{2.22}$$

where w^I denotes the set of $m+1$ incoming variables $\{w_i^\pm; i^\pm \in I\}$. In the special case where the k_α-th characteristic field is linearly degenerate, we have

$$\frac{\partial \Phi_i}{\partial w_{k_\alpha}^\pm} \equiv 0,$$

hence all functions W^j do not depend on $w_{k_\alpha}^+, w_{k_\alpha}^-$. Comparing (2.18) with (2.17), it is clear that the outgoing waves $u_x^{j^\pm}$ can be obtained as functions of the incoming waves $u_x^{i^\pm}, i^\pm \in I$:

$$u_x^{j^\pm} = U^j(u^-, u^+)(u_x^I) \qquad j \neq k_\alpha. \tag{2.23}$$

The linear homogeneous functions U^j, W^j are related by

$$\frac{\partial U^j}{\partial u_x^{k^\pm}} = \frac{(\dot{x}_\alpha - \lambda_k^\pm)}{(\dot{x}_\alpha - \lambda_j^\pm)} \cdot \frac{\partial W^j}{\partial w_k^\pm}. \tag{2.24}$$

Observe that in the linearly degenerate case, the components $u_x^{k_\alpha^+}, u_x^{k_\alpha^-}$ may not be defined. Yet, the functions U^j are well defined because they do not depend on $u_x^{k_\alpha^\pm}$.

Following [3], in connection with the system (1.1) we say that a function $u : \mathbb{R} \mapsto \Omega$ is in the class of functions $PLSD$ (*Piecewise Lipschitz with Simple Discontinuities*) if u is piecewise Lipschitz continuous with finitely many jumps, each jump consisting of a single admissible shock or a contact discontinuity. If u is in $PLSD$ and has N discontinuities at the points $x_1 < \cdots < x_N$, the space of *generalized tangent vectors* at u is defined as $T_u \doteq \mathbf{L}^1 \times \mathbb{R}^N$. Elements in T_u can be interpreted as first order tangent vectors as follows. On the family Σ_u of all continuous paths $\gamma : [0, \varepsilon_0] \to \mathbf{L}^1$ with $\gamma(0) = u$, define the equivalence relation

$$\gamma \sim \gamma' \qquad \text{iff} \qquad \lim_{\varepsilon \to 0+} \frac{1}{\varepsilon} \|\gamma(\varepsilon) - \gamma'(\varepsilon)\|_{\mathbf{L}^1} = 0. \tag{2.25}$$

We say that a continuous path $\gamma \in \Sigma_u$ *generates the tangent vector* $(v, \xi) \in T_u$ if γ is equivalent to the path $\gamma_{(v,\xi;u)}$ defined as

$$\begin{aligned} \gamma_{(v,\xi;u)}(\varepsilon) \doteq u + \varepsilon v + \sum_{\xi_\alpha < 0} [u(x_\alpha^+) - u(x_\alpha^-)]\, \chi_{[x_\alpha + \varepsilon \xi_\alpha, x_\alpha]} \\ - \sum_{\xi_\alpha > 0} [u(x_\alpha^+) - u(x_\alpha^-)]\, \chi_{[x_\alpha, x_\alpha + \varepsilon \xi_\alpha]}, \end{aligned} \tag{2.26}$$

where χ_I denotes the characteristic function of the interval I. Up to higher order terms, $\gamma(\varepsilon)$ is thus obtained from u by adding εv and by shifting the points x_α, where the discontinuities of u occur, by $\varepsilon \xi_\alpha$.

The main purpose for introducing this space of tangent vectors is to understand the behavior of "first order perturbations" of a given solution of (1.1). More precisely, let $\gamma : \varepsilon \mapsto \overline{u}^\varepsilon$ be a parametrized family of initial conditions, with $\overline{u}^\varepsilon \in PLSD$ for all $\varepsilon \in [0, \varepsilon_0]$. Following [3], we say that γ is a *Regular first order Variation* (R.V.) of $\overline{u}^0$ if the functions $\overline{u}^\varepsilon$ suffer jumps at points $x_1^\varepsilon < \cdots < x_N^\varepsilon$ depending continuously on ε, and have a uniform Lipschitz constant outside these discontinuities.

If γ is a R.V. of $\overline{u}^0$ generating a tangent vector $(\overline{v}, \overline{\xi})$, then one can consider the family of solutions $u^\varepsilon(t, \cdot)$ of (1.1) corresponding to the initial values $\overline{u}^\varepsilon$. As long as the shocks do not interact and the Lipschitz constants remain uniformly bounded, it was proved in [3] that the family $u^\varepsilon(t, \cdot)$ is still a R.V. of $u^0(t, \cdot)$, generating some tangent vector $(v(t), \xi(t))$. This vector provides an integral solution to the following linear system of equations:

$$v_t + A(u) v_x + [DA(u) \cdot v] u_x = 0 \tag{2.27}$$

outside the lines of discontinuity, together with the conditions

$$\begin{aligned} \langle Dl_i(u^-, u^+) \cdot (\xi_\alpha u_x^- + v^-,\ \xi_\alpha u_x^+ + v^+),\ u^+ - u^- \rangle \\ + \langle l_i(u^-, u^+),\ \xi_\alpha u_x^+ + v^+ - \xi_\alpha u_x^- - v^- \rangle = 0 \qquad \forall i \neq k_\alpha, \end{aligned} \tag{2.28}$$

$$\dot{\xi}_\alpha = D\lambda_{k_\alpha}(u^-, u^+) \cdot (\xi_\alpha u_x^- + v^-,\ \xi_\alpha u_x^+ + v^+), \tag{2.29}$$

on each line $x = x_\alpha(t)$ where u suffers a discontinuity in the k_α-th characteristic field. We remark that (2.27) is formally obtained by differentiating (2.8), while (2.28), (2.29) are derived from the Rankine-Hugoniot equations (2.12), (2.13).

Under generic conditions, the existence of a generalized tangent vector can be proved also beyond the time where two shocks interact. By studying how these first-order perturbations evolve in time, one can gain useful informations on the stability of the original system (1.1).

3. Some Basic Estimates

Let u be a function in the class $PLSD$ and assume that, for $\alpha = 1, \ldots, N$, u has a jump of the k_α-th characteristic family at x_α. The strength $J_\alpha \doteq J_{x_\alpha}^{k_\alpha}$ of the α-th discontinuity is then measured as follows. If the jump at x_α is a genuine shock, set

$$J_\alpha \doteq \lambda_{k_\alpha}(u(x_\alpha^-)) - \lambda_{k_\alpha}(u(x_\alpha^+)). \tag{3.1}$$

If the jump at x_α is a contact discontinuity, then there exists a unique integral curve of $\dot{u} = r_{k_\alpha}(u)$ joining $u(x_\alpha^-)$ with $u(x_\alpha^+)$. In this case, we let J_α be the arc-length of the curve. Observe that we always have $J_\alpha > 0$. Following [7, 14], we define the potential for future wave interaction as

$$\begin{aligned} Q(u) \doteq & \sum_{i \leq j} \iint_{x<y} |u_x^j(x)|\,|u_x^i(y)|\,dxdy \\ & + \sum_\alpha \left[\sum_{i \leq k_\alpha} J_{x_\alpha}^{k_\alpha} \int_{x_\alpha}^{\infty} |u_x^i(x)| dx + \sum_{i \geq k_\alpha} J_{x_\alpha}^{k_\alpha} \int_{-\infty}^{x_\alpha} |u_x^i(x)| dx \right] \\ & + \sum_{k_\alpha \leq k_\beta, x_\alpha > x_\beta} J_{x_\alpha}^{k_\alpha} J_{x_\beta}^{k_\beta}, \end{aligned} \tag{3.2}$$

and the instantaneous amount of interaction as

$$\Lambda(u) \doteq \tilde{\Lambda}(u) + \sum_\alpha \Lambda_\alpha(u), \tag{3.3}$$

where

$$\tilde{\Lambda}(u) \doteq \sum_{i<j} \int_{-\infty}^{\infty} (\lambda_j(u(x)) - \lambda_i(u(x)))|u_x^i(x)|\,|u_x^j(x)|\,dx, \tag{3.4}$$

$$\Lambda_\alpha(u) = \sum_{j \leq k_\alpha} (\dot{x}_\alpha - \lambda_j(x_\alpha^+))|u_x^j(x_\alpha^+)| J_{x_\alpha}^{k_\alpha} + \sum_{j \geq k_\alpha} (\lambda_j(x_\alpha^-) - \dot{x}_\alpha)|u_x^j(x_\alpha^-)| J_{x_\alpha}^{k_\alpha}. \tag{3.5}$$

The total amount of waves in u will be measured as

$$V(u) \doteq \sum_i \int_{-\infty}^{\infty} |u_x^i(x)|\, dx + \sum_\alpha J_\alpha. \tag{3.6}$$

In the following, $C_1, C_2, \ldots$ will denote suitable constants whose value depends only on the vector field F and on its derivatives inside Ω. For example, in the equations (2.10) we clearly have an estimate of the form

$$|G_{ijk}(u)| \leq C_1 \qquad \forall u \in \Omega, \; i,j,k = 1,\ldots,m. \tag{3.7}$$

We shall always assume that the domain Ω is sufficiently small so that (2.21) holds whenever $u^-, u^+ \in \Omega$ satisfy the Rankine-Hugoniot conditions.

LEMMA 3.1. *The functions W^j in* (2.22) *satisfy bounds of the form*

$$\left|\frac{\partial W^j}{\partial w_i^\pm}\right| \leq C_2|u^+ - u^-| \qquad i \neq j, \; i^\pm \in \mathcal{I}, \; j^\pm \in \mathcal{O}, \tag{3.8}$$

$$\left|\frac{\partial W^i}{\partial w_i^\pm} - 1\right| \leq C_2|u^+ - u^-| \qquad i \neq k_\alpha, \; i^\pm \in \mathcal{I}, \; i^\mp \in \mathcal{O}, \tag{3.9}$$

$$\left|\frac{\partial W^i}{\partial w_{k_\alpha}^\pm}\right| \leq C_2|u^+ - u^-|^2 \qquad i^\pm \in \mathcal{O}. \tag{3.10}$$

for all u^-, u^+ in Ω, connected by a shock or by a contact discontinuity of the k_α-th family.

PROOF. Since the $W^{j^\pm}$ are linear functions of $w_i^\pm$, depending smoothly on u^-, u^+, the bounds (3.8), (3.9) follow easily from (2.20), (2.21). To prove the sharper bound (3.10), by the implicit function theorem it suffices to show that

$$\left|\frac{\partial \Phi_i}{\partial w_{k_\alpha}^\pm}\right| = \left|\frac{\partial \Phi_i(u^-, u^+, w^-, w^+)}{\partial w^\pm} \cdot r_{k_\alpha}(u^\pm)\right| \leq C|u^+ - u^-|^2. \tag{3.11}$$

Set $\varepsilon = |u^+ - u^-|/|r_{k_\alpha}(u^-)|$. Writing $r_{k_\alpha}^- = r_{k_\alpha}(u^-)$ and using the Landau order symbol we now have:

$$u^+ - u^- = \varepsilon r_{k_\alpha}^- + O(\varepsilon^2), \qquad Dl_i(u^-, u^+) = Dl_i(u^-, u^-) + O(\varepsilon),$$

$$l_i(u^-, u^+) = l_i(u^-, u^-) + Dl_i(u^-, u^-) \cdot (0, \varepsilon r_{k_\alpha}^-) + O(\varepsilon^2),$$

$$
(3.12)\quad
\begin{aligned}
\frac{\partial \Phi_i}{\partial w^-} \cdot r_{k_\alpha}^- &= \langle Dl_i(u^-, u^+) \cdot (r_{k_\alpha}^-, 0),\ u^+ - u^- \rangle + \langle l_i(u^-, u^+),\ -r_{k_\alpha}^- \rangle \\
&= \langle Dl_i(u^-, u^-) \cdot (r_{k_\alpha}^-, 0),\ \varepsilon r_{k_\alpha}^- \rangle \\
&\qquad + \langle Dl_i(u^-, u^-) \cdot (0, \varepsilon r_{k_\alpha}^-),\ -r_{k_\alpha}^- \rangle + O(\varepsilon^2) \\
&= O(\varepsilon^2).
\end{aligned}
$$

The computation for $\partial \Phi_i / \partial w_{k_\alpha}^+$ is entirely similar. By the smoothness of Φ_i on Ω, (3.12) thus implies (3.10). □

Combining the estimates (3.8)-(3.10) with (2.24) one obtains

$$
(3.13)\quad \left| \frac{\partial U^j}{\partial u_x^{i\pm}} \right| \le C_3 |u^+ - u^-| \qquad i \ne j,\ i^\pm \in \mathcal{I},\ j^\pm \in \mathcal{O},
$$

$$
(3.14)\quad \left| \frac{\partial U^i}{\partial u_x^{i\pm}} - 1 \right| \le C_3 |u^+ - u^-| \qquad i \ne k_\alpha,\ i^\pm \in \mathcal{I},\ i^\mp \in \mathcal{O},
$$

$$
(3.15)\quad \left| \frac{\partial U^i}{\partial u_x^{k_\alpha^\pm}} \right| \le C_3 |u^+ - u^-|^3 \qquad i^\pm \in \mathcal{O}.
$$

If the k_α-th characteristic field is linearly degenerate, (3.10) and (3.15) can be replaced by

$$
(3.16)\quad \frac{\partial W^i}{\partial w_{k_\alpha}^\pm} = 0, \qquad \frac{\partial U^i}{\partial u_x^{k_\alpha^\pm}} = 0 \qquad i^\pm \in \mathcal{O}.
$$

From (2.15), (3.5) and (3.13)-(3.16) it now follows

$$
(3.17)\quad \sum_{i \ne k_\alpha} |u_x^{i^+} - u_x^{i^-}| \le C_4 \Lambda_\alpha(u),
$$

$$
(3.18)\quad \sum_{i \ne k_\alpha} \left| (\lambda_i^+ - \dot{x}_\alpha) |u_x^{i^+}| - (\lambda_i^- - \dot{x}_\alpha) |u_x^{i^-}| \right| \le C_4 \Lambda_\alpha(u).
$$

LEMMA 3.2. *There exists a constant C_5 such that, if $u = u(t, x)$ is a piecewise Lipschitz continuous solution of* (1.1)*, with a jump in the k_α-th characteristic family at $x_\alpha(t)$, then for almost every t the following estimate holds:*

$$
(3.19)\quad \left| \dot{J}_\alpha + \frac{\kappa}{2} J_\alpha (u_x^{k_\alpha^+} + u_x^{k_\alpha^-}) \right| \le C_5 \Lambda_\alpha(u).
$$

The constant κ is here defined as

$$
(3.20)\quad \kappa \doteq \begin{cases} 1 & \text{if the } k_\alpha\text{-th family is genuinely nonlinear,} \\ 0 & \text{if the } k_\alpha\text{-th family is linearly degenerate.} \end{cases}
$$

PROOF. We begin by introducing some notation. Given $u^-, u^+ \in \Omega$, let the states $u^- = u_0,\ u_1, \ldots, u_m = u^+$, be such that each couple (u_{i-1}, u_i) satisfies the Rankine-Hugoniot equations

$$F(u_i) - F(u_{i-1}) = \lambda(u_{i-1}, u_i)(u_i - u_{i-1}).$$

If the i-th characteristic field is genuinely nonlinear, we define the (signed) strength of the i-th shock determined by the jump (u^-, u^+) as

$$J^i(u^-, u^+) \doteq \lambda_i(u_{i-1}) - \lambda_i(u_i). \tag{3.21}$$

If the i-th field is linearly degenerate, for some ε one has $u_i = (\exp \varepsilon r_i)(u_{i-1})$. In this case we set $J^i(u^-, u^+) \doteq \varepsilon_i$. As usual $(\exp \varepsilon_i r_i)(\omega)$ denotes here the value at time $t = \varepsilon$ of the solution of the Cauchy problem

$$\dot{u} = r_i(u), \qquad u(0) = \omega.$$

To prove the lemma, observe that $J_\alpha = J^{k_\alpha}(u^-, u^+)$, with $u^- = u(t, x_\alpha^-)$ and $u^+ = u(t, x_\alpha^+)$. Its derivative w.r.t. time can be written as

$$\dot{J}_\alpha = DJ^{k_\alpha}(u^-, u^+) \cdot \left(\sum_i (\dot{x}_\alpha - \lambda_i^-) u_x^{i^-} r_i^-,\ \sum_i (\dot{x}_\alpha - \lambda_i^+) u_x^{i^+} r_i^+ \right)$$

$$\begin{aligned}
&= [DJ^{k_\alpha}(u^-, u^+) - DJ^{k_\alpha}(u^+, u^+)] \\
&\qquad \cdot \left(\sum_i (\dot{x}_\alpha - \lambda_i^-) u_x^{i^-} r_i^-,\ \sum_i (\dot{x}_\alpha - \lambda_i^+) u_x^{i^+} r_i^+ \right) \\
&\quad + DJ^{k_\alpha}(u^+, u^+) \cdot \left(\sum_{i \neq k_\alpha} (\dot{x}_\alpha - \lambda_i^-) u_x^{i^-} r_i^+,\ \sum_{i \neq k_\alpha} (\dot{x}_\alpha - \lambda_i^+) u_x^{i^+} r_i^+ \right) \\
&\quad + DJ^{k_\alpha}(u^+, u^+) \cdot \left(\sum_i (\dot{x}_\alpha - \lambda_i^-) u_x^{i^-} (r_i^- - r_i^+), 0 \right) \\
&\quad + DJ^{k_\alpha}(u^+, u^+) \cdot \left((\dot{x}_\alpha - \lambda_{k_\alpha}^-) u_x^{k_\alpha^-} r_{k_\alpha}^+,\ (\dot{x}_\alpha - \lambda_{k_\alpha}^+) u_x^{k_\alpha^+} r_{k_\alpha}^+ \right).
\end{aligned} \tag{3.22}$$

Of the four terms on the right hand side of (3.22), the first and the third one are bounded by a constant multiple of $\Lambda_\alpha(u)$, while the second vanishes. The last term is computed by

$$-[(\dot{x}_\alpha - \lambda_{k_\alpha}^+) u_x^{k_\alpha^+} + (\lambda_{k_\alpha}^- - \dot{x}_\alpha) u_x^{k_\alpha^-}]. \tag{3.23}$$

In the linearly degenerate case, $\dot{x}_\alpha = \lambda_{k_\alpha}^+ = \lambda_{k_\alpha}^-$, hence (3.19) holds. In the genuinely nonlinear case the bound (3.19) is derived from (3.23) using the estimates

$$|\dot{x}_\alpha - \lambda^+_{k_\alpha} - J_\alpha/2| \le C_6 J^2_\alpha,$$

$$|\lambda^-_{k_\alpha} - \dot{x}_\alpha - J_\alpha/2| \le C_6 J^2_\alpha.$$

□

A similar argument yields

LEMMA 3.3. *Under the same assumptions of Lemma* 3.2, *there exists a constant* C_7 *such that*

$$\left| D\lambda_{k_\alpha}(u^-, u^+)\cdot \left(\sum_i u^{i-}_x r^-_i, \sum_i u^{i+}_x r^+_i \right) - \frac{\kappa}{2}\left(u^{k^+_\alpha}_x + u^{k^-_\alpha}_x\right) \right| \le C_7 \frac{\Lambda_\alpha(u)}{J_\alpha} \tag{3.24}$$

where κ *is the constant in* (3.20).

If $u = u(t,x)$ is a piecewise Lipschitz solution of (1.1), for almost every t the functions $V(u)$ and $Q(u)$ satisfy bounds of the form:

$$\frac{d}{dt} V(u(t)) \le C_8 \Lambda(u), \tag{3.25}$$

$$\frac{d}{dt} Q(u(t)) \le -\Lambda(u) + C_9 \Lambda(u) V(u). \tag{3.26}$$

As long as shocks do not interact, these bounds can be established by a straightforward differentiation in (3.6), (3.2), using the estimates (3.13)-(3.15) and (3.17)-(3.20). Notice that, if the total variation of u is suitably small, we can assume

$$\frac{d}{dt} Q(u(t)) \le -\frac{1}{2} \Lambda(u). \tag{3.27}$$

4. - A Locally Contractive Metric

The goal of this section is to introduce a family of norms $\|\cdot\|_u$ on the tangent spaces T_u and to study how the lengths of tangent vectors change in time, along piecewise Lipschitz continuous solutions of (1.1). Let u be a function in the class $PLSD$, having a discontinuity at each one of the points $x_1 < x_2 < \cdots < x_N$, with the jump at x_α occurring along the k_α-th characteristic family. For any $(v, \xi) \in T_u \doteq \mathbf{L}^1 \times \mathbb{R}^N$, define the components

$$v_i(x) \doteq \langle l_i(u(x)),\ v(x) \rangle. \tag{4.1}$$

e now introduce the weighted norm

$$\|(v,\xi)\|_u \doteq \sum_{\alpha=1}^{N} J_{x_\alpha}^{k_\alpha}|\xi_\alpha| S_{k_\alpha}^u(x_\alpha) + \sum_{i=1}^{m} \int_{-\infty}^{\infty} |v_i(x)| S_i^u(x)\,dx, \tag{4.2}$$

where

$$S_i^u(x) \doteq R_i^u(x) + MQ(u) + \delta, \tag{4.3}$$

$$R_i^u(x) \doteq \left[\sum_{j\le i}\int_x^{\infty} + \sum_{j\ge i}\int_{-\infty}^{x}\right] |u_x^j(y)|dy + \left[\sum_{\substack{k_\alpha\le i\\ x_\alpha>x}} + \sum_{\substack{k_\alpha\ge i\\ x_\alpha<x}}\right] J_{x_\alpha}^{k_\alpha}, \tag{4.4}$$

and M,δ are suitable constants whose precise value will be specified later. Observe that $R_i^u(x)$ describes the total amount of waves which are approaching a wave of the i-th family located at x. The main result of this section shows that if the total variation of u is suitable small, then M,δ can be chosen so that the norm (4.2) is nonincreasing along solutions of the linearized system (2.27)-(2.29).

THEOREM 4.1. *There exist constants $M,\delta,\delta' > 0$ for which the following holds. Let $u = u(t,x)$ be a solution of* (1.1) *such that each $u(t,\cdot)$ is in $PLSD$ and has total variation smaller than δ'. Let $(v(t),\xi(t)) \in T_{u(t)}$ be a generalized tangent vector satisfying the linearized system* (2.27)-(2.29). *Then the norm $\|(v(t),\xi(t))\|_{u(t)}$ defined at* (4.2) *is a nonincreasing function of t, as long as the shocks in u do not interact.*

PROOF. From (2.27), one obtains a system of m scalar equations of the form

$$(v_i)_t + (\lambda_i(u)v_i)_x = \sum_{j\ne k} H_{ijk}(u)u_x^j v_k, \tag{4.5}$$

where the H_{ijk} are C^1 functions of u (see [1] for details). More precisely, for fixed i one has

$$(v_i)_t + (\lambda_i(u)v_i)_x = \sum_{k\ne i} (r_k \bullet \lambda_i)\{u_x^k v_i - u_x^i v_k\} + \sum_{j\ne k} \langle l_i,\ [r_j, r_k]\rangle(\lambda_j - \lambda_i)u_x^j v_k.$$

Along each line of discontinuity $x = x_\alpha(t)$, the jump conditions (2.28), (2.29) become

$$\begin{aligned}\Big\langle & Dl_i(u(x_\alpha^-), u(x_\alpha^+)) \\ & \cdot \Big(\sum_j (\xi_\alpha u_x^{j^-} + v_j^-) r_j^-, \ \sum_j (\xi_\alpha u_x^{j^+} + v_j^+) r_j^+ \Big), \ u(x_\alpha^+) - u(x_\alpha^-) \Big\rangle \\ & + \Big\langle l_i(u(x_\alpha^-), u(x_\alpha^+)), \ \sum_j \Big[(\xi_\alpha u_x^{j^+} + v_j^+) r_j^+ - (\xi_\alpha u_x^{j^-} + v_j^-) r_j^- \Big] \Big\rangle = 0\end{aligned} \tag{4.6}$$

for each $i \neq k_\alpha$, and

$$\dot\xi_\alpha = D\lambda_{k_\alpha}(u(x_\alpha^-), u(x_\alpha^+)) \cdot \Big(\sum_j (\xi_\alpha u_x^{j^-} + v_j^-) r_j^-, \ \sum_j (\xi_\alpha u_x^{j^+} + v_j^+) r_j^+ \Big), \tag{4.7}$$

respectively. Differentiating (4.4) and using (2.9), one obtains

$$\begin{aligned}(R_i^u(x))_t &+ \lambda_i(u(x))(R_i^u(x))_x \\ &= \Big[\sum_{j \le i} \int_x^\infty + \sum_{j \ge i} \int_{-\infty}^x \Big] \Big[(\operatorname{sgn} u_x^j(y)) \sum_{k<\ell} G_{jk\ell}(u(y)) u_x^k(y) u_x^\ell(y) \Big] dy \\ &\quad - \sum_{j \neq i} |\lambda_j(u(x)) - \lambda_i(u(x))| \, |u_x^j(x)| \\ &\quad + \Big[\sum_{\substack{j \le i, \ j \neq k_\alpha \\ x_\alpha > x}} + \sum_{\substack{j \ge i, \ j \neq k_\alpha \\ x_\alpha < x}} \Big] \big[(\lambda_j(u(x_\alpha^+)) - \dot x_\alpha) |u_x^j(x_\alpha^+)| \\ &\qquad\qquad - (\lambda_j(u(x_\alpha^-)) - \dot x_\alpha) |u_x^j(x_\alpha^-)| \big] \\ &\quad + \Big[\sum_{\substack{k_\alpha \le i \\ x_\alpha > x}} + \sum_{\substack{k_\alpha \ge i \\ x_\alpha < x}} \Big] \Big(\dot J_\alpha - (\dot x_\alpha - \lambda_{k_\alpha}(x_\alpha^+)) |u_x^{k_\alpha^+}| \\ &\qquad\qquad - (\lambda_{k_\alpha}(x_\alpha^-) - \dot x_\alpha) |u_x^{k_\alpha^-}| \Big).\end{aligned} \tag{4.8}$$

From (4.8), using (3.18) and (3.19) one obtains the estimate

$$(4.9)\qquad \begin{aligned}&(R_i^u(x))_t + \lambda_i(u(x))(R_i^u(x))_x\\ &\le -\sum_{j\ne i} |\lambda_j(u(x)) - \lambda_i(u(x))||u_x^j(x)| + C_{10}\Lambda(u).\end{aligned}$$

Recalling (3.5), the time derivative of $R_{k_\alpha}^u$ along the line of jump $x_\alpha(t)$ can be estimated by

$$\begin{aligned}\frac{d}{dt} R_{k_\alpha}^u(x_\alpha(t)) = &\left[\sum_{j\le k_\alpha}\int_x^{\infty} + \sum_{j\ge k_\alpha}\int_{-\infty}^{x}\right]\\ &\left[(\operatorname{sgn} u_x^j(y)) \sum_{k<\ell} G_{jk\ell}(u(y))u_x^k(y)u_x^\ell(y)\right] dy\\ &- \sum_{j\le k_\alpha} (\dot{x}_\alpha - \lambda_j(u(x_\alpha^+)))|u_x^j(x_\alpha^+)|\\ &- \sum_{j\ge k_\alpha} (\lambda_j(u(x_\alpha^-)) - \dot{x}_\alpha)|u_x^j(x_\alpha^-)|\\ (4.10)\qquad &+ \left[\sum_{\substack{j<k_\alpha\\ x_\beta>x_\alpha}} + \sum_{\substack{j>k_\alpha\\ x_\beta<x_\alpha}}\right] [(\lambda_j(u(x_\beta^+)) - \dot{x}_\beta)|u_x^j(x_\beta^+)|\\ &\qquad - (\lambda_j(u(x_\beta^-)) - \dot{x}_\beta)|u_x^j(x_\beta^-)|]\\ &+ \left[\sum_{\substack{k_\beta\le k_\alpha\\ x_\beta>x_\alpha}} + \sum_{\substack{k_\beta\ge x_\alpha\\ x_\beta<x_\alpha}}\right] \Big(\dot{J}_\beta - (\dot{x}_\beta - \lambda_{k_\beta}(x_\beta^+))|u_x^{k_\beta^+}|\\ &\qquad - (\lambda_{k_\beta}(x_\beta^-) - \dot{x}_\beta)|u_{k_\beta}^-|\Big)\\ &\le -\frac{\Lambda_\alpha(u)}{J_\alpha} + C_{11}\Lambda(u).\end{aligned}$$

The time derivative of the norm (4.2) can now be written as the sum of four terms:

$$\frac{d}{dt}\Big\|(v(t),\xi(t))\Big\|_{u(t)}$$

$$= \sum_{\alpha=1}^{N} \left[\dot J_\alpha|\xi_\alpha| S^u_{k_\alpha}(x_\alpha) + J_\alpha(\operatorname{sgn}\xi_\alpha)\dot\xi_\alpha S^u_{k_\alpha}(x_\alpha) + J_\alpha|\xi_\alpha|\dot S^u_{k_\alpha}(x_\alpha)\right]$$

$$+ \sum_{i=1}^{m}\int_{-\infty}^{\infty} [(S^u_i(x))_t + \lambda_i(u(x))(S^u_i(x))_x]|v_i(x)|\,dx$$

$$+ \sum_{i=1}^{m}\int_{-\infty}^{\infty} S^u_i(x)(\operatorname{sgn} v_i(x)) \sum_{k\neq \ell} H_{ik\ell}(u(x))u^k_x(x)v_\ell(x)\,dx$$

$$+ \sum_{\alpha=1}^{N}\sum_{i=1}^{m} \left[S^u_i(x^+_\alpha)|v_i(x^+_\alpha)|(\lambda_i(u(x^+_\alpha)) - \dot x_\alpha)\right.$$

$$\left. - S^u_i(x^-_\alpha)|v_i(x^-_\alpha)|(\lambda_i(u(x^-_\alpha)) - \dot x_\alpha)\right]$$

$$\doteq E_1 + E_2 + E_3 + E_4.$$

To estimate (4.11), define the instantaneous amount of interaction Ψ between u and v as

$$\Psi(u,v) \doteq \tilde\Psi(u,v) + \sum_\alpha \Psi_\alpha(u,v), \tag{4.12}$$

$$\tilde\Psi(u,v) \doteq \sum_{i\neq j}\int_{-\infty}^{\infty} |\lambda_j(u(x)) - \lambda_i(u(x))|\,|u^i_x(x)||v_j(x)|\,dx, \tag{4.13}$$

$$\Psi_\alpha(u,v) \doteq J^{k_\alpha}_{x_\alpha}\left[\sum_{i\geq k_\alpha}(\lambda_i(u(x^-_\alpha)) - \dot x_\alpha)|v_i(x^-_\alpha)| + \sum_{i\leq k_\alpha}(\dot x_\alpha - \lambda_i(u(x^+_\alpha)))|v_i(x^+_\alpha)|\right]. \tag{4.14}$$

Define also

$$M^u_S \doteq \max\{S^u_i(x);\quad x\in\mathbb{R},\quad i = 1,\dots,m\}. \tag{4.15}$$

A couple of preliminary estimates will be needed.

LEMMA 4.2. *For some constant C_{12}, at every point of jump x_α one has*

$$\sum_{j\neq k_\alpha} |v_j(x_\alpha^+) - v_j(x_\alpha^-)| \leq C_{12}(|\xi_\alpha|\Lambda_\alpha(u) + \Psi_\alpha(u,v)). \tag{4.16}$$

PROOF. From (2.28) and the estimates (3.8)-(3.10), for $j^\pm \in \mathcal{O}$, $j\neq k_\alpha$, it follows

$$\xi_\alpha u_x^{j^\pm} + v_j^\pm = \sum_{i^\pm\in I} \frac{\partial W^{j^\pm}}{\partial w_i^\pm}\cdot(\xi_\alpha u_x^{i^\pm} + v_i^\pm), \tag{4.17}$$

$$\begin{aligned} |v_j^+ - v_j^-| \leq |\xi_\alpha|\,|u_x^{j^+} - u_x^{j^-}| + CJ_\alpha \sum_{\substack{i^\pm\in I\\ i\neq j,k_\alpha}} |\xi_\alpha u_x^{i^\pm} + v_i^\pm| \\ + C\kappa J_\alpha^2\left\{|\xi_\alpha u_x^{k_\alpha^+} + v_{k_\alpha}^+| + |\xi_\alpha u_x^{k_\alpha^-} + v_{k_\alpha}^-|\right\}, \end{aligned} \tag{4.18}$$

for some constant C, with κ as in (3.20). Using (3.17) we thus obtain (4.16).

LEMMA 4.3. *For some constant C_{13} and with κ as in* (3.20), *at every point of jump one has*

$$\begin{aligned} &\left|D\lambda_{k_\alpha}(u(x_\alpha^-),u(x_\alpha^+))\cdot(v(x_\alpha^-),v(x_\alpha^+)) - \frac{\kappa}{2}(v_{k_\alpha}^- + v_{k_\alpha}^+)\right| \\ &\leq C_{13}\cdot\frac{|\xi_\alpha|\Lambda_\alpha(u) + \Psi_\alpha(u,v)}{J_\alpha}. \end{aligned} \tag{4.19}$$

PROOF. The derivative of the eigenvalue λ_{k_α} can be written as

$$\begin{aligned} D\lambda_{k_\alpha}(u^-,u^+)\cdot(v^-,v^+) &= \sum_{i\neq k_\alpha} D\lambda_{k_\alpha}(u^-,u^+)\cdot(v_i^- r_i^+; v_i^+ r_i^+) \\ &\quad + D\lambda_{k_\alpha}(u^-,u^+)\cdot(0, v_{k_\alpha}^+(r_{k_\alpha}^+ - r_{k_\alpha}^-)) \\ &\quad + [D\lambda_{k_\alpha}(u^-,u^+) - D\lambda_{k_\alpha}(u^-,u^-)] \\ &\qquad \cdot(v_{k_\alpha}^- r_{k_\alpha}^-, v_{k_\alpha}^+ r_{k_\alpha}^-) \\ &\quad + D\lambda_{k_\alpha}(u^-,u^-)\cdot(v_{k_\alpha}^- r_{k_\alpha}^-, v_{k_\alpha}^+ r_{k_\alpha}^-) \\ &\doteq B_1 + B_2 + B_3 + B_4. \end{aligned} \tag{4.20}$$

By (2.15) and (4.16), the first three terms in (4.20) satisfy an estimate of the form

$$|B_1| + |B_2| + |B_3| \leq C\cdot\frac{|\xi_\alpha|\Lambda_\alpha(u) + \Psi_\alpha(u,v)}{J_\alpha}$$

for some constant C. Concerning the last term, (2.4) implies

$$B_4 = \frac{v_{k_\alpha}^- + v_{k_\alpha}^+}{2} \quad \text{or} \quad B_4 = 0, \tag{4.21}$$

in the genuinely nonlinear and in the linearly degenerate case, respectively. This proves (4.19).

We are now ready to estimate each of the four terms in (4.11). Call $S \subseteq \{1, \ldots, N\}$ the set of indices α such that the jump at x_α is a genuinely nonlinear shock. From (2.29), using first the bounds (3.19), (3.24), (4.10), then (4.19) and (3.27) we obtain:

$$\begin{aligned}
E_1 \le & \sum_{\alpha\in S} \left[\frac{1}{2} J_\alpha(u_x^{k_\alpha^+} + u_x^{k_\alpha^-})|\xi_\alpha| S_{k_\alpha}^u(x_\alpha)\right] + C_5 \sum_\alpha \Lambda_\alpha(u)|\xi_\alpha| S_{k_\alpha}^u(x_\alpha) \\
& - \sum_{\alpha\in S} \left[\frac{1}{2} J_\alpha|\xi_\alpha|(u_x^{k_\alpha^+} + u_x^{k_\alpha^-}) S_{k_\alpha}^u(x_\alpha)\right] + C_7 \sum_\alpha |\xi_\alpha|\Lambda_\alpha(u) S_{k_\alpha}^u(x_\alpha) \\
& + \sum_\alpha J_\alpha S_{k_\alpha}^u(x_\alpha)(\operatorname{sgn}\xi_\alpha) D\lambda_{k_\alpha}(u(x_\alpha^-), u(x_\alpha^+)) \cdot (v(x_\alpha^-), v(x_\alpha^+)) \\
& + \sum_\alpha J_\alpha|\xi_\alpha| \left[M\dot{Q}(u) + C_{11}\Lambda(u) - \Lambda_\alpha(u)/J_\alpha\right] \\
\le & \sum_\alpha \Bigg[(C_5 + C_7)|\xi_\alpha|\Lambda_\alpha(u) M_S^u + C_{13}(|\xi_\alpha|\Lambda_\alpha(u) + \Psi_\alpha(u,v)) M_S^u \\
& \qquad J_\alpha|\xi_\alpha|\left(-\frac{M}{2}\Lambda(u) + C_{11}\Lambda(u)\right) - |\xi_\alpha|\Lambda_\alpha(u)\Bigg] \\
& + \sum_{\alpha\in S} S_{k_\alpha}^u(x_\alpha) J_\alpha(\operatorname{sgn}\xi_\alpha) \frac{v_{k_\alpha}^+ + v_{k_\alpha}^-}{2}.
\end{aligned} \tag{4.22}$$

Concerning E_2, (4.9) and (3.27) yield

$$E_2 \le \left(-\frac{M}{2} + C_{10}\right)\Lambda(u) \sum_{i=1}^m \int_{-\infty}^{\infty} |v_i(x)|\,dx - \tilde{\Psi}(u,v). \tag{4.23}$$

The quantity E_3 clearly satisfies a bound of the form

$$E_3 \le C_{14} M_S^u \tilde{\Psi}(u,v). \tag{4.24}$$

Finally, observe that

$$
\begin{aligned}
S_i^u(x_\alpha^+) &= S_i^u(x_\alpha^-) + J_\alpha \qquad \text{if} \qquad i < k_\alpha, \\
S_i^u(x_\alpha^+) &= S_i^u(x_\alpha^-) - J_\alpha \qquad \text{if} \qquad i > k_\alpha, \\
S_{k_\alpha}^u(x_\alpha^-) &= S_{k_\alpha}^u(x_\alpha^+) = S_{k_\alpha}^u(x_\alpha) + J_\alpha.
\end{aligned}
$$

With this in mind, recalling (4.16) and (3.24), one obtains the bound

$$
\begin{aligned}
E_4 = &\sum_\alpha \sum_{i \le k_\alpha} J_\alpha |v_i^+| (\lambda_i^+ - \dot{x}_\alpha) - \sum_\alpha \sum_{i \ge k_\alpha} J_\alpha |v_i^-| (\lambda_i^- - \dot{x}_\alpha) \\
&+ \sum_\alpha \sum_{i < k_\alpha} S_i^u(x_\alpha^-) \left[|v_i^+| (\lambda_i^+ - \dot{x}_\alpha) - |v_i^-| (\lambda_i^- - \dot{x}_\alpha) \right] \\
&+ \sum_\alpha \sum_{i > k_\alpha} S_i^u(x_\alpha^+) \left[|v_i^+| (\lambda_i^+ - \dot{x}_\alpha) - |v_i^-| (\lambda_i^- - \dot{x}_\alpha) \right] \\
&+ \sum_{\alpha \in S} S_{k_\alpha}^u(x_\alpha) \left[|v_{k_\alpha}^+| (\lambda_{k_\alpha}^+ - \dot{x}_\alpha) - |v_{k_\alpha}^-| (\lambda - \dot{x}_\alpha) \right] \\
\le &\sum_\alpha \left[-\Psi_\alpha(u,v) + C_{15} M_S^u (|\xi_\alpha| \Lambda_\alpha(u) + \Psi_\alpha(u,v)) \right] \\
&- \frac{1}{2} \sum_{\alpha \in S} S_{k_\alpha}^u(x_\alpha) J_\alpha (|v_{k_\alpha}^+| + |v_{k_\alpha}^-|)
\end{aligned}
\tag{4.25}
$$

for a suitable constant C_{15}. Summing together the right hand sides of (4.22)-(4.25), we obtain an estimate of the form

$$
\begin{aligned}
\frac{d}{dt} \left\| (v(t), \xi(t)) \right\|_{u(t)} \le &\sum_\alpha \left[C_{16} |\xi_\alpha \Lambda_\alpha(u) M_S^u + C_{17} J_\alpha |\xi_\alpha| \Lambda(u) \right. \\
&\left. + C_{18} \Psi_\alpha(u,v) M_S^u \right] \\
&- \sum_\alpha \left[\Psi_\alpha(u,v) + |\xi_\alpha| \Lambda_\alpha(u) + \frac{M}{2} J_\alpha |\xi_\alpha| \Lambda(u) \right] \\
&+ \left(C_{10} - \frac{M}{2} \right) \Lambda(u) \int_{-\infty}^{\infty} |v_i(x)| dx \\
&- \tilde{\Psi}(u,v) + C_{14} \tilde{\Psi}(u,v) M_S^u.
\end{aligned}
\tag{4.26}
$$

To satisfy the conclusion of the theorem, first choose M so large that

$$
M > \max\{2C_{10}, 2C_{17}\}. \tag{4.27}
$$

For such fixed M, there exist now two constants δ, δ_1 such that, if the total variation $T.V.(u)$ is smaller than δ_1, then the supremum M_S^u in (4.15) satisfies a bound of the form

$$M_S^u \leq \min\{C_{16}^{-1}, C_{18}^{-1}, C_{14}^{-1}\}. \tag{4.28}$$

If the inequalities (4.27), (4.28) hold, the righthand side of (4.26) is then nonpositive, for almost every t. Finally, we can choose $\delta' > 0$ so small such that, if $T.V.(u(0,\cdot)) \leq \delta'$ then $T.V.(u(t,\cdot)) \leq \delta_1$ for all $t \geq 0$. This proves the theorem. □

5. - The Case of Interacting Shocks

The analysis in §4 was concerned with the evolution of generalized tangent vectors, as long as the discontinuities in the solution u of (1.1) do not interact. In this section we show that the weighted norm $\|(v(t), \xi(t))\|_{u(t)}$ cannot increase even at times when two shocks interact. As usual, let $u(t,\cdot)$ be a solution of (1.1) with values in $PLSD$ and with suitably small total variation. Let $x_\alpha(t), x_\beta(t)$ be lines where u has a jump of the k_α-th, k_β-th characteristic family, respectively. To fix the ideas, let $k_\alpha \geq k_\beta$ and let $x_\alpha(t) < x_\beta(t)$ before the interaction time t_0. For notational simplicity, we assume $t_0 = 0$ and $x_\alpha(t_0) = x_\beta(t_0) = 0$, which is not restrictive. Define

$$u_* = \lim_{x \to 0-} u(0,x), \qquad x^* = \lim_{x \to 0+} u(0,x) \tag{5.1}$$

and let $u = \varphi(t,x)$ be the self-similar solution of the standard Riemann problem (1.1) with initial data

$$u(0,-x) = u_*, \qquad u(0,x) = u^*, \qquad \forall x > 0. \tag{5.2}$$

It is well known [8, 15] that φ consists of $m+1$ constant states $\omega_0 = u_*, \omega_1, \ldots, \omega_m = u^*$, each couple (ω_{i-1}, ω_i) being separated by a shock or a contact discontinuity of the i-th characteristic family (if $\lambda_i(\omega_i) \leq \lambda_i(\omega_{i-1})$), or by a centered rarefaction wave (if $\lambda_i(\omega_i) > \lambda_i(\omega_{i-1})$). Call $\mathcal{S} \subseteq \{1, \ldots, m\}$ the set of indices corresponding to a (nontrivial) shock or a contact discontinuity and let $\mathcal{R}$ be the set of those indices which correspond to a rarefaction wave. If, for $t < 0$, $u(t,\cdot)$ has N points of jump, the number of discontinuities in $u(t,\cdot)$ after the interaction takes place is thus $N' = N - 2 + |\mathcal{S}|$.

For $t < 0$, let a regular variation of $u(t,\cdot)$ be given, generating the tangent vector $(v(t), \xi(t)) \in T_{u(t)} = \mathbf{L}^1 \times \mathbb{R}^N$. By Theorem 3.10 in [3], under generic conditions, for $t > 0$ a tangent vector $(v(t), \xi(t)) \in T_{u(t)} = \mathbf{L}^1 \times \mathbb{R}^{N'}$ is still well defined. In order to relate the values of u_x, v, ξ before and after the time of interaction, some notation is needed. Define the limits as $t \to 0-$:

$$v_i^-(x) = v_i(0-,x), \qquad \xi_\gamma^- = \xi_\gamma(0-), \qquad \dot{x}_\gamma^- = \dot{x}_\gamma(0-) \tag{5.3}$$

For $i \in S$, $t > 0$, call $\hat{J}_i(t)$ the sizes of the new discontinuities generated at the origin by the interaction of J_α, J_β. For the definition of these sizes, recall (3.1). The components of the vector $\xi(t) \in \mathbb{R}^{N'}$ will be written

$$\{\xi_\gamma(t), \hat{\xi}_i(t); \quad \gamma \neq \alpha, \beta, \; i \in S\} \qquad (t > 0) \tag{5.4}$$

We denote by $J_\gamma^+, \hat{J}_i^+, \xi_\gamma^+, \hat{\xi}_i^+$ respectively the limits of $J_\gamma, \hat{J}_i, \xi_\gamma, \hat{\xi}_i$ as $t \to 0+$. The equations that relate the values of u_x, v, ξ across the interaction, derived in [3], are as follows. For all $i \in \mathcal{R}$, one has

$$\lim_{t \to 0+} \int_{-\infty}^{\infty} \left| u_x^i(t,x) - u_x^i(0-,x) - \frac{1}{t} \chi_{[t\lambda_i(\omega_{i-1}),\; t\lambda_i(\omega_i)]}(x) \right| dx = 0, \tag{5.5}$$

$$\lim_{t \to 0+} \int_{-\infty}^{\infty} \left| v_i(t,x) - v_i^-(x) + \frac{\xi_\beta^-(t^{-1}x - \dot{x}_\alpha) - \xi_\alpha^-(t^{-1}x - \dot{x}_\beta^-)}{t(\dot{x}_\alpha^- - \dot{x}_\beta^-)} \cdot \chi_{[t\lambda_i(\omega_{i-1}),\; t\lambda_i(\omega_i)]}(x) \right| dx = 0. \tag{5.6}$$

On the other hand, for $i \notin \mathcal{R}$ one has

$$\lim_{t \to 0+} \int_{-\infty}^{\infty} |u_x^i(t,x) - u_x^i(0-,x)| \, dx = 0, \tag{5.7}$$

$$\lim_{t \to 0+} \int_{-\infty}^{\infty} |v_i(t,x) - v_i^-(x)| \, dx = 0. \tag{5.8}$$

Moreover, the components of $\xi(t)$, $t > 0$, satisfy

$$\xi_\gamma^+ = \xi_\gamma^- \qquad \forall \gamma \neq \alpha, \beta, \tag{5.9}$$

$$\hat{\xi}_i^+ = \frac{\xi_\alpha^-(\lambda_i(\omega_{i-1}, \omega_i) - \dot{x}_\beta^-) - \xi_\beta^-(\lambda_i(\omega_{i-1}, \omega_i) - \dot{x}_\alpha^-)}{\dot{x}_\alpha^- - \dot{x}_\beta^-}. \tag{5.10}$$

For an intuitive justification of the formulas (5.5)-(5.10), observe that, calling $\tau^\varepsilon, \eta^\varepsilon$ the time and location of the interaction in the solution u^ε, an easy computation yields

$$A \doteq \lim_{\varepsilon \to 0+} \frac{\tau^\varepsilon}{\varepsilon} = \frac{\xi_\beta^- - \xi_\alpha^-}{\dot{x}_\alpha^- - \dot{x}_\beta^-}, \qquad B \doteq \lim_{\varepsilon \to 0+} \frac{\eta^\varepsilon}{\varepsilon} = \frac{\dot{x}_\alpha^- \xi_\beta^- - \dot{x}_\beta^- \xi_\alpha^-}{\dot{x}_\alpha^- - \dot{x}_\beta^-}.$$

If $\varphi(t,x)$ denotes as before the solution of the standard Riemann problem (1.1), (5.2), in a neighborhood of the origin for $t>0$ the solution u^ε can be approximated as

$$u^\varepsilon(t,x)\approx\varphi(t-\tau^\varepsilon,\ x-\eta^\varepsilon).$$

Hence

$$v(0+,x)\approx v(0-,x)-(A\varphi_t(t,x)+B\varphi_x(t,x)). \tag{5.11}$$

Taking the inner products of (5.11) with $l_i(u)$, $i=1,\dots,m$, leads to (5.6), (5.8).

Using (5.5)-(5.10), the change in the norm (4.1) of the tangent vector (v,ξ) across the interaction can now be computed by

$$\begin{aligned}
&\lim_{t\to 0+}\left[\|(v(t),\xi(t))\|_{u(t)}-\|(v(0-),\xi(0-))\|_{u(0-)}\right]\\
&=\sum_{\gamma\neq\alpha,\beta} J_\gamma|\xi_\gamma|\,[S^+_{k_\gamma}(x_\gamma)-S^-_{x_\gamma}(x_\gamma)]\\
&\quad+\left[\sum_{i\in\mathcal{S}}\hat J_i^+|\hat\xi_i^+|S_i^+(\hat x_i)-J_\alpha^-|\xi_\alpha^-|S^-_{k_\alpha}(x_\alpha)-J_\beta^-|\xi_\beta|S^-_{k_\beta}(x_\beta)\right]\\
&\quad+\sum_{i=1}^m\int_{-\infty}^{\infty}|v_i^-(x)|\,[S_i^+(x)-S_i^-(x)]\,dx\\
&\quad+\lim_{t\to0+}\sum_{i\in\mathcal{R}}\int_{t\lambda_i(\omega_{i-1})}^{t\lambda_i(\omega_i)}\left|\frac{\xi_\beta^-(t^{-1}x-\dot x_\alpha^-)-\xi_\alpha^-(t^{-1}x-\dot x_\beta^-)}{t(\dot x_\alpha^- -\dot x_\beta^-)}\right|S_i(x)dx\\
&=E_1+E_2+E_3+E_4.
\end{aligned}\tag{5.12}$$

The limits of S_i as $t\to0\pm$ are here denoted by S_i^+,S_i^- respectively. To estimate (5.12), consider first the case where $k_\alpha\neq k_\beta$, i.e., the two interacting discontinuities belong to different characteristic families. In this case, the following estimates are well known.

$$|\hat J_i^+|\le C_1J_\alpha^-J_\beta^-\qquad(i\in\mathcal{S},i\neq k_\alpha,k_\beta), \tag{5.13}$$

$$|\hat J^+_{k_\alpha}-J_\alpha^-|+|\hat J^+_{k_\beta}-J_\beta^-|\le C_1J_\alpha^-J_\beta^-, \tag{5.14}$$

$$\lambda_i(\omega_i)-\lambda_i(\omega_{i-1})\le C_1J_\alpha^-J_\beta^-\quad(i\in\mathcal{R}), \tag{5.15}$$

$$|\lambda_{k_\alpha}(\omega_{k_\alpha-1},\omega_{k_\alpha})-\dot x_\alpha^-|\le C_1J_\alpha^-J_\beta^-, \tag{5.16}$$

$$|\lambda_{k_\beta}(\omega_{k_\beta-1}, \omega_{k_\beta}) - \dot{x}_\beta^-| \le C_1 J_\alpha^- J_\beta^-, \tag{5.17}$$

for some constant C_1. From (5.10), using (5.16), (5.17) one now obtains

$$\max\left\{|\hat{\xi}_{k_\alpha}^+ - \xi_\alpha^-|,\ |\hat{\xi}_{k_\beta}^+ - \xi_\beta^-|\right\} \le C_2\left(|\xi_\alpha^-| + |\xi_\beta^-|\right) J_\alpha^- J_\beta^-, \tag{5.18}$$

$$|\hat{\xi}_i^+| \le C_2\left(|\xi_\alpha^-| + |\xi_\beta^-|\right). \tag{5.19}$$

Moreover, we have

$$\left|\frac{\xi_\beta^-(y - \dot{x}_\alpha^-) - \xi_\alpha^-(y - \dot{x}_\beta^-)}{\dot{x}_\alpha^- - \dot{x}_\beta^-}\right| \le C_2\left(|\xi_\alpha^-| + |\xi_\beta^-|\right), \tag{5.20}$$

for all $y \in [\lambda_i(\omega_{i-1}), \lambda_i(\omega_i)]$, $i \in \mathcal{R}$, and a suitable constant C_2.

Concerning the amount of approaching waves $Q(u)$, defined at (3.2), the bounds (5.13)-(5.15) imply

$$Q(u(0+)) - Q(u(0-)) \le -J_\alpha^- J_\beta^- + C_4 J_\alpha^- J_\beta^- V(u(0-)) \le -\frac{1}{2} J_\alpha^- J_\beta^-, \tag{5.21}$$

provided that the total variation of u is sufficiently small. Using (5.13)-(5.15) and (5.21) we obtain

$$\begin{aligned} S_i^+(x) - S_i^-(x) \le & \left[|\hat{J}_{k_\alpha}^+ - J_\alpha^-| + \left|\hat{J}_{k_\beta}^+ - J_\beta^-\right| + \sum_{\substack{i\in S \\ i\neq k_\alpha, k_\beta}} \hat{J}_i^+ \right] \\ & + \sum_{i\in\mathcal{R}} [\lambda_i(\omega_i) - \lambda_i(\omega_{i-1})] - \frac{M}{2} J_\alpha^- J_\beta^- \le 0 \end{aligned} \tag{5.22}$$

for all $x \neq 0$, if M is suitably large. Using (5.22) it is clear that, in (5.12), one has

$$E_1 \le 0, \qquad E_3 \le 0. \tag{5.23}$$

By (5.13)-(5.14), the second term in (5.12) satisfies the bound:

$$
\begin{aligned}
E_2 \leq\ & mC_1C_2 \left(|\xi_\alpha^-| + |\xi_\beta^-|\right) J_\alpha^- J_\beta^- M_S^{u(0+)} \\
& + \Big[\left|\hat{J}_{k_\alpha}^+ - J_\alpha^-\right| \, |\xi_{k_\alpha}^+| \, S_{k_\alpha}^+(\hat{x}_{k_\alpha}) + J_\alpha^- |\xi_{k_\alpha}^+ - \xi_\alpha^-| \, S_{k_\alpha}^+(\hat{x}_{k_\alpha}) \\
& \qquad + J_\alpha^- |\xi_\alpha^-| \, (S_{k_\alpha}^+(\hat{x}_{k_\alpha}) - S_{k_\alpha}^-(x_\alpha))\Big] \\
& + \Big[\left|\hat{J}_{k_\beta}^+ - J_\beta^-\right| \, |\xi_{k_\beta}^+| \, S_{k_\beta}^+(\hat{x}_{k_\beta}) + J_\beta^- |\xi_{k_\beta}^+ - \xi_\beta^-| \, S_{k_\beta}^+(\hat{x}_{k_\beta}) \\
& \qquad + J_\beta^- |\xi_\beta^-| \, (S_{k_\beta}^+(\hat{x}_{k_\beta}) - S_{k_\beta}^-(x_\beta))\Big] \\
\leq\ & C_5 \left(|\xi_\alpha^-| + |\xi_\beta^-|\right) J_\alpha^- J_\beta^- M_S^{u(0+)} - J_\alpha^- |\xi_\alpha^-| J_\beta^- - J_\beta^- |\xi_\beta^-| J_\alpha^- .
\end{aligned}
\tag{5.24}
$$

Concerning the fourth term, the change of variable $y = t^{-1}x$ together with (5.20), (5.15) leads to the estimate

$$
\begin{aligned}
E_4 &\leq \sum_{i\in\mathcal{R}} [\lambda_i(\omega_i) - \lambda_i(\omega_{i-1})] \cdot C_2 \left(|\xi_\alpha^-| + |\xi_\beta^-|\right) M_S \\
&\leq mC_1C_2 \left(|\xi_\alpha^-| + |\xi_\beta^-|\right) M_S^{u(0+)} J_\alpha^- J_\beta^- .
\end{aligned}
\tag{5.25}
$$

Together, (5.24) and (5.25) imply $E_2 + E_4 \leq 0$, as soon as

$$
(C_5 + mC_1C_2) M_S^{u(0+)} < 1. \tag{5.26}
$$

This concludes the analysis in the case $k_\alpha \neq k_\beta$.

Next, consider the case where $k_\alpha = k_\beta$, i.e., the two interacting shocks belong to the same characteristic family. In this case, the speeds of the shocks before and after the interaction can be estimated as follows.

$$
\left|\dot{x}_\alpha^- - \dot{x}_\beta^- - \frac{1}{2}\left(J_\alpha^- + J_\beta^-\right)\right| \leq C_6 \left(J_\alpha^- + J_\beta^-\right)^2, \tag{5.27}
$$

$$
\left|\lambda_{k_\alpha}(\omega_{k_\alpha - 1}, \omega_{k_\alpha}) - \left(J_\alpha^- \dot{x}_\alpha + J_\beta^- \dot{x}_\beta^-\right)\left(J_\alpha^- + J_\beta^-\right)^{-1}\right| \leq C_6 J_\alpha^- J_\beta^- . \tag{5.28}
$$

If the size of the interacting shocks is small enough, (5.27) implies

$$
\dot{x}_\alpha^- - \dot{x}_\beta^- \geq \frac{1}{4}\left(J_\alpha^- + J_\beta^-\right). \tag{5.29}
$$

Concerning the sizes of the new shocks and their displacements, using (5.28), (5.29) in (5.10) we obtain the bounds

$$\left|\hat{J}^+_{k_\alpha} - J^-_\alpha - J^-_\beta\right| \le C_7 J^-_\alpha J^-_\beta \left(J^-_\alpha + J^-_\beta\right), \tag{5.30}$$

$$\begin{aligned} |\hat{\xi}^+_{k_\alpha}| \le & \left[|\xi^-_\alpha| \left|\frac{J^-_\alpha(\dot{x}^-_\alpha - \dot{x}^-_\beta)}{J^-_\alpha + J^-_\beta}\right| + |\xi^-_\beta| \left|\frac{J^-_\beta(\dot{x}^-_\alpha - \dot{x}^-_\beta)}{J^-_\alpha + J^-_\beta}\right| \right. \\ & \left. + \left(|\xi^-_\alpha| + |\xi^-_\beta|\right) C_6 J^-_\alpha J^-_\beta \right] \cdot \frac{1}{\dot{x}^-_\alpha - \dot{x}^-_\beta} \\ \le & \left[|\xi^-_\alpha| J^-_\alpha + |\xi^-_\beta| J^-_\beta + \left(|\xi^-_\alpha| + |\xi^-_\beta|\right) \cdot 4C_6 J^-_\alpha J^-_\beta \right] \cdot \frac{1}{J^-_\alpha + J^-_\beta}. \end{aligned} \tag{5.31}$$

Moreover, the estimates (5.13), (5.15), (5.19) can now be replaced by

$$|\hat{J}^+_i| \le C_7 J^-_\alpha J^-_\beta \left(J^-_\alpha + J^-_\beta\right) \qquad i \in \mathcal{S},\ i \ne k_\alpha, \tag{5.32}$$

$$\lambda_i(\omega_i) - \lambda_i(\omega_{i-1}) \le C_7 J^-_\alpha J^-_\beta \left(J^-_\alpha + J^-_\beta\right) \qquad i \in \mathcal{R}, \tag{5.33}$$

$$|\hat{\xi}^+_i| \le C_8 \frac{|\xi^-_\alpha| + |\xi^-_\beta|}{\dot{x}^-_\alpha - \dot{x}^-_\beta} \le 4C_8 \frac{|\xi^-_\alpha| + |\xi^-_\beta|}{J_\alpha + J_\beta}. \tag{5.34}$$

Observe that in the present case (5.21) remains valid, while (5.22) is replaced by

$$\begin{aligned} S^+_i(x) - S^-_i(x) \le & \left|\hat{J}^+_{k_\alpha} - J^-_\alpha J^-_\beta\right| \sum_{\substack{i \in \mathcal{S} \\ i \ne k_\alpha}} \hat{J}^+_i \\ & + \sum_{i \in \mathcal{R}} [\lambda_i(\omega_i) - \lambda_i(\omega_{i-1})] - \frac{M}{2} J^-_\alpha J^-_\beta \le 0 \end{aligned} \tag{5.35}$$

for all $x \ne 0$, if M is suitably large. This again implies the two inequalities in (5.23). The second term in (5.12) can now be bounded by

$$\begin{aligned} E_2 \le & \, m \cdot C_7 J^-_\alpha J^-_\beta \left(J^-_\alpha + J^-_\beta\right) \cdot 4C_8 \left(|\xi^-_\alpha| + |\xi^-_\beta|\right) \left(J^-_\alpha + J^-_\beta\right)^{-1} M_S^{u(0+)} \\ & + \left[J^-_\alpha |\xi^-_\alpha| (S^+_{k_\alpha}(\hat{x}_{k_\alpha}) - S^-_{k_\alpha}(x_\alpha)) + J^-_\beta |\xi^-_\beta| (S^+_{k_\alpha}(\hat{x}_{k_\alpha}) - S^-_{k_\alpha}(x_\beta)) \right] \\ & + \left| \hat{J}^+_{k_\alpha} |\hat{\xi}^+_{k_\alpha}| - J^-_\alpha |\xi^-_\alpha| - J^-_\beta |\xi^-_\beta| \right| M_S^{u(0+)} \\ \le & \, C_9 \left(|\xi^-_\alpha| + |\xi^-_\beta|\right) J^-_\alpha J^-_\beta M_S^{u(0+)} - J^-_\alpha |\xi^-_\alpha| J^-_\beta - J^-_\beta |\xi^-_\beta| J^-_\alpha. \end{aligned} \tag{5.36}$$

Concerning the fourth term, the change of variable $y = t^{-1}x$, together with (5.29), (5.33) leads to the estimate

$$(5.37)\qquad \begin{aligned} E_4 &\le \sum_{i\in\mathcal{R}} [\lambda_i(\omega_i) - \lambda_i(\omega_{i-1})]\, C_{10} \left(|\xi_\alpha^-| + |\xi_\beta^-|\right) \left(J_\alpha^- + J_\beta^-\right)^{-1} \cdot M_S^{u(0+)} \\ &\le C_{11} \left(|\xi_\alpha^-| + |\xi_\beta^-|\right) M_S^{u(0+)} J_\alpha^- J_\beta^- . \end{aligned}$$

Together, (5.36) and (5.37) imply $E_2 + E_4 \le 0$, as soon as

$$(5.38)\qquad (C_9 + C_{11})\, M_S^{u(0+)} \le 1.$$

At this stage, we can first choose M large enough so that (5.21), (5.35) hold; then choose $\delta, \delta' > 0$ small enough so that, if the total variation of u remains smaller than δ', (5.26), (5.38) hold. The previous analysis thus proves that the conclusion of Theorem 4.1 remains valid also when shock interaction occurs, i.e.:

THEOREM 5.1. *There exist constants* $M, \delta, \delta' > 0$ *for which the following holds. Let* $u = u(t,x)$ *be a piecewise Lipschitz solution of* (1.1), *with total variation* $< \delta'$ *and with finitely many lines of discontinuity which interact only two at a time. Let* $(v(t), \xi(t)) \in T_{u(t)}$ *be any generalized tangent vector satisfying* (2.27)-(2.29) *and assume that* (5.5)-(5.10) *hold at each time* τ *when two discontinuities of* u *interact. Then the norm* $\|(v(t), \xi(t))\|_{u(t)}$, *defined at* (4.2)-(4.4) *is a nonincreasing function of* t.

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