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Large Time Behaviour of a Diffusion Equation with Strong Convection

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1. - Introduction

In this paper we study the large time behaviour of solutions of the equation

$$u_t - (u^q)_x = u_{xx} \quad \text{in } (0,\infty)\times(0,\infty); \tag{1.1}$$

we always assume $q > 1$. Equation (1.1) is complemented with initial data

$$u(x,0) = \varphi(x) \quad \text{in } (0,\infty)\times\{0\}, \tag{1.2}$$

and homogeneous Neumann boundary conditions

$$u_x(0,t) = 0 \quad \text{in } \{0\}\times(0,\infty). \tag{1.3}$$

A unique classical solution of problem (1.1)-(1.3) is known to exist, if the initial data are non-negative, bounded and sufficiently smooth (see Section 2).

In spite of its simplicity, equation (1.1) allows one to investigate the mutual effects of convection and diffusion. It can also be thought of as a particular case of a more general model equation, which encompasses diffusion, convection and source terms (see [RK]). The interest of such models, also from the applicative point of view, can hardly be overemphasized (e.g. see [BE]).

Reaction-diffusion equations with absorption, yet without convective terms, have been widely investigated. The asymptotic behaviour of their solutions for large times reveals to be markedly different, depending on the initial data and on the mutual size of diffusion and absorption. In fact, depending on this quantities, the limiting behaviour is described in a suitable sense by some problems, where either diffusion or absorption terms have disappeared (see in particular [KP1], [KP2] and references therein).

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Much in the same way, it can be expected that any solution of problem (1.1)-(1.3) approaches in a suitable sense some solution either of the hyperbolic conservation law

$$v_t - (v^q)_x = 0 \quad \text{in } (0,\infty)\times(0,\infty), \tag{1.4}$$

or of the heat equation, as $t \to \infty$. We also expect that the prevalence of either situations will depend on q and on the initial data.

Let us make the following assumptions:

$$(A_0) \quad \begin{cases} \text{(i)} & \varphi \in L^\infty(0,\infty), \quad \varphi \geq 0 \text{ in } (0,\infty); \\ \text{(ii)} & \text{there exists } \sigma \in (0,1) \text{ such that } \varphi \in C^{2+\sigma}([0,r]) \\ & \text{for any } r > 0; \end{cases}$$

$$(A_1) \quad \text{there exist } \alpha > 0 \text{ and } A > 0 \text{ such that } \begin{cases} \text{(i)} & \alpha < \dfrac{1}{q-1}; \\ \text{(ii)} & \lim\limits_{x\to\infty} x^\alpha \varphi(x) = A; \end{cases}$$

$$(A_2) \quad \begin{cases} \text{(i)} & \varphi' \in L^\infty(0,\infty), \ \varphi'(0) = 0, \ \varphi' \leq 0 \text{ in } [0,\infty); \\ \text{(ii)} & \text{there exist } \tilde{x} > 0, \ C \in (0,A) \text{ such that } x^{\alpha+1}\varphi'(x) \geq -C \\ & \text{for any } x > \tilde{x}. \end{cases}$$

Let us consider equation (1.4) with Cauchy data

$$v(x,0) = Ax^{-\alpha} \quad \text{for } x > 0. \tag{1.5}$$

The method of characteristics easily gives a classical solution of problem (1.4)-(1.5), which is the unique entropy solution of the problem (see Section 2). Observe that, due to the sign of the convection term in equation (1.1), characteristics point towards the time axis. In particular, no boundary condition at $x = 0$ is needed for problem (1.4)-(1.5) to be well posed. Define

$$\beta := \alpha(q-1) + 1, \tag{1.6}$$

$$P_{a,b}(t) := \{x > 0 | at^{\frac{1}{\beta}} \leq x \leq bt^{\frac{1}{\beta}}\} \quad (t \geq 0). \tag{1.7}$$

THEOREM 1.1. *Let the assumptions* (A_0)-(A_2) *be satisfied. Let* u *be the unique solution of problem* (1.1)-(1.3) *and let* v *be the unique entropy solution of problem* (1.4)-(1.5). *Then for any* $0 < a < b < \infty$ *we have*

$$t^{\frac{\alpha}{\beta}} \sup_{x\in P_{a,b}(t)} |u(x,t) - v(x,t)| \to 0 \quad \text{as } t \to \infty.$$

A similar, yet weaker convergence result holds if only assumptions (A_0)-(A_1) are made (see Theorem 6.1).

The above results are easily interpreted. Due to the boundary condition (1.3), the solution of problem (1.1)-(1.3) is confined to the first quadrant. On the other hand, by assumptions (A_0)-(A_1) its initial data are large at $x = \infty$. This enhances convection over diffusion (observe that the characteristics of problem (1.4)-(1.5) emanate from $x = \infty$ towards the axis $x = 0$); hence the large time behaviour of solutions is "hyperbolic".

Let us mention that qualitative results in the same spirit have been proved in [EVZ] for the Cauchy problem

$$\begin{cases} u_t - (u^q)_x = u_{xx} & \text{in } \mathbb{R} \times (0, \infty) \\ u = u_0 & \text{in } \mathbb{R} \times \{0\}, \end{cases}$$

if $u_0 \in L^1(\mathbb{R})$ and $q \in (1, 2)$ (the case $q > 2$ had been previously investigated in [EZ]). In this case the limiting behaviour of solutions is described by a suitable source-type solution of the hyperbolic conservation law

$$v_t - (v^q)_x = 0 \quad \text{in } \mathbb{R} \times (0, \infty)$$

(for the existence and uniqueness of such a solution see [LP]). In our case, due to assumption (A_1), the initial data of problem (1.1)-(1.3) need not belong to $L^1(0, \infty)$. Hence the large time behaviour is described by a solution of equation (1.4), which is markedly different from the source type solutions considered in [EVZ]. Let us mention that related results are proved in [PS].

Theorem 1.1 will be proved introducing the following family of functions:

$$u_k(x, t) := k^\alpha u(kx, \; k^\beta t) \quad (x > 0, \; t > 0). \tag{1.8}$$

Upon substitution in (1.1)-(1.3), it is easily seen that for any $k > 0$ the function u_k solves the problem

$$\begin{cases} u_{kt} - (u_k^q)_x = k^{-(2-\beta)} u_{kxx} & \text{in } (0, \infty) \times (0, \infty) \\ u_{kx}(0, t) = 0 & \text{in } (0, \infty) \\ u_k(x, 0) = \varphi_k(x) & \text{in } (0, \infty) \end{cases} \tag{1.9}$$

where

$$\varphi_k(x) := k^\alpha \varphi(kx).$$

Since $\beta < 2$ (see assumption (A_1)-(i)) and assumption (A_1)-(ii) holds, problem (1.9) formally reduces to problem (1.4)-(1.5) as $k \to \infty$.

To make the above remarks rigorous, we need uniform estimates for both for u_k and u_{kx}. Preliminary estimates of the solution of problem (1.1)-(1.3) and of its gradient are proved in Sections 4 and 5, respectively. In doing so, a crucial step is constructing a nontrivial subsolution of problem (1.1)-(1.3); this is made using a suitable solution of equation (1.4) (see Section 3). Then uniform estimates of u_k and u_{kx} easily follow by assumptions (A_1)-(ii) and (A_2)-(ii) respectively; relying on them Theorem 1.1 is proved by well known arguments (see Section 6).

2. - Background

Let $Q_T := (0,\infty)\times(0,T]$ for any $T>0$; set also $Q := (0,\infty)\times(0,\infty)$. Let us state the following result, concerning existence and uniqueness of classical solutions of problem (1.1)-(1.3).

THEOREM 2.1. *(a) Let assumption* (A_0) *be satisfied; suppose also* $\varphi'(0)=0$. *Then there exists a unique solution* $u\in L^\infty(Q)\cap C^{2,1}(\overline{Q})\cap C^{2+\gamma,1+\gamma/2}([0,r]\times[0,\infty))$ *for any* $r>0$ *(where* $\gamma=\gamma(q,\sigma)\in(0,\sigma]$*) of problem* (1.1)-(1.3). *Moreover,* $u\geq 0$ *in* Q.
(b) Assume further that $\varphi'\in L^\infty(0,\infty)$. *Then* $u\in W^{1,\infty}(Q)$.

The proof makes use of classical approximation arguments (e.g., see [LSU, p. 495]), thus it will be omitted.

Concerning problem (1.4)-(1.5) we have the following definition (see [Kr]).

DEFINITION 2.1. *A function* $v: Q_T\to(0,\infty)$ *is an entropy solution of problem* (1.4)-(1.5) *in* Q_T *if:*

(i) *there exists a constant* $M>0$ *such that*

$$0\leq v(x,t)\leq Mx^{-\alpha}\quad in\quad Q_T;$$

(ii) *for any* $L\in\mathbb{R}$ *and any* $\varsigma\in C_0^\infty(Q_T)$, $\varsigma\geq 0$

$$\int\int_{Q_T}\{|v-L|\varsigma_t-\operatorname{sgn}(v-L)(v^q-L^q)\varsigma_x\}\mathrm{d}x\mathrm{d}t\geq 0;$$

(iii) *for any* $0<r_1<r_2<\infty$

$$\operatorname*{ess\,lim}_{t\to 0}\int_{r_1}^{r_2}|v(x,t)-Ax^{-\alpha}|\mathrm{d}x=0.$$

Entropy super- and subsolutions are similarly defined. Comparison results can be found in [NT]; hence we obtain the following result.

THEOREM 2.2. *There exists at most one entropy solution of problem* (1.4)-(1.5) *in* Q_T $(T>0)$.

Existence of an entropy solution of problem (1.4)-(1.5) is easily proved. Since the initial data are decreasing, a classical solution is found by the method of characteristics. This gives the equality

$$x=A^{\frac{1}{\alpha}}v^{-\frac{1}{\alpha}}-qv^{q-1}t\quad\text{in } Q. \tag{2.1}$$

Hence we obtain

$$0\leq v(x,t)\leq Ax^{-\alpha}\quad\text{in } Q. \tag{2.2}$$

By Theorem 2.2 the function v defined implicitly in (2.1) is the unique entropy solution of problem (1.4)-(1.5) in Q. Observe that by scaling invariance we have

$$v(x,t) = t^{-\frac{\alpha}{\beta}} f(\xi), \quad \xi := xt^{-\frac{1}{\beta}}, \tag{2.3}$$

where f is the unique solution of the problem

$$\begin{cases} \left(\frac{\xi}{\beta} + qf^{q-1}\right) f' + \frac{\alpha}{\beta} f = 0 & \text{in } (0,\infty) \\ f > 0 & \text{in } (0,\infty) \\ \xi^\alpha f(\xi) \to A & \text{as } \xi \to \infty. \end{cases} \tag{2.4}$$

In the sequel we shall encounter functions which solve problem (1.4)-(1.5) in a sense slightly different from that of Definition 2.1 (see Theorem 6.1). This is made precise in the following:

DEFINITION 2.2. *A function $v : Q_T \to (0,\infty)$ is a mild entropy solution of problem* (1.4)-(1.5) *in Q_T if:*

(i) *there exists a constant $M > 0$ such that*

$$0 \leq v(x,t) \leq Mx^{-\alpha} \quad \textit{in} \quad Q_T;$$

(ii) *for any $L \in \mathbb{R}$ and any $\varsigma \in C_0^\infty(\overline{Q}_T)$, $\operatorname{supp}\varsigma \subseteq \overline{Q}_T \backslash \{x = 0\}$, $\varsigma \geq 0$*

$$\iint_{Q_T} \{|v - L|\varsigma_t - \operatorname{sgn}(v - L)(v^q - L^q)\varsigma_x\} dx dt + \int_0^\infty |Ax^{-\alpha} - L|\varsigma(x,0) dx \geq 0.$$

It can be checked that any mild entropy solution of bounded variation in Q_T is an entropy solution (in the sense of Definition 2.1). Hence there exists at most one mild entropy solution of bounded variation in Q_T to problem (1.4)-(1.5).

3. - The associated conservation law

We aim at proving estimates of the solution of problem (1.1)-(1.3) by using suitable solutions of the associated conservation law (1.4). For this purpose we complement equation (1.4) with initial data ψ satisfying the following assumption:

$$\begin{cases} \text{(i)} & \psi \in L^\infty(0,\infty) \cap C^2([0,\infty]), \ \psi' \leq 0 \text{ in } [0,\infty); \\ \text{(ii)} & \text{there exist } \overline{x} > 0, \ B \in (0,A) \text{ such that } \\ & \psi(x) = Bx^{-\alpha} \quad \text{for } x > \overline{x}. \end{cases} \tag{3.1}$$

Existence, uniqueness and comparison results for entropy solutions of problem (1.4)-(3.1) are well known (e.g. see [Kr]). Clearly, problems (1.4)-(1.5) and (3.1) differ in that the initial data for the latter is bounded in any right neighbourhood of the origin.

The domain of influence of the interval $(0, \overline{x})$ for such a solution is contained in the region

$$\Omega_1 := \{(x,t) |\ t \geq 0,\ \ 0 \leq x \leq \overline{x}(1 - t/\overline{t})\}, \tag{3.2}$$

where

$$\overline{t} := \frac{B^{-(q-1)}}{q} \overline{x}^{\beta}. \tag{3.3}$$

(see Fig. 3.1). We also set

$$\Omega_2 := \{(x,t) | t \geq 0,\ \ x \geq \max\{\overline{x}(1 - t/\overline{t}),\ 0\}\}, \tag{3.4}$$

$$\sigma := \{(x,t) | t \in [0, \overline{t}],\ \ x = \overline{x}(1 - t/\overline{t})\}.$$

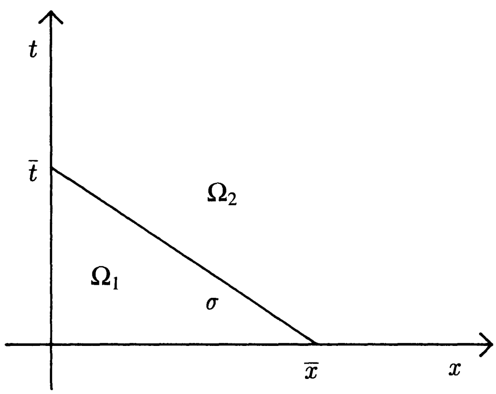

Fig. 3.1

We shall denote by w the unique entropy solution of problem (1.4)-(3.1). In the region Ω_2 it is given implicitly by equality (2.1), which now reads

$$x = B^{\frac{1}{\alpha}} w^{-\frac{1}{\alpha}} - q w^{q-1} t. \tag{3.5}$$

In the following we shall need several inequalities concerning the solution w and its derivatives. This is the content of the following lemmas.

LEMMA 3.1. *In Ω_2 we have:*

(i) $0 \le w \le B \min\{\overline{x}^{-\alpha},\ x^{-\alpha}\}$;

(ii) $w \le (Bq^{-\alpha})^{\frac{1}{\beta}} t^{-\frac{\alpha}{\beta}}$.

PROOF. It follows by (3.5) and comparison results. □

LEMMA 3.2. *In* Ω_2 *we have*

$$-\alpha B^{-\frac{1}{\alpha}} w^{\frac{1}{\alpha}+1} \le w_x \le -\frac{\alpha}{\beta} B^{-\frac{1}{\alpha}} w^{\frac{1}{\alpha}+1}.$$

PROOF. From (3.5) we obtain

$$w_x = -\frac{1}{H} \quad \text{in } \Omega_2, \tag{3.6}$$

where

$$H(w,t) := \alpha^{-1} B^{\frac{1}{\alpha}} w^{-\frac{1}{\alpha}-1} + q(q-1) w^{q-2} t. \tag{3.7}$$

By (3.5) and (3.7) we have

$$H = \frac{\beta}{\alpha} B^{\frac{1}{\alpha}} w^{-\frac{1}{\alpha}-1} - (q-1)\frac{x}{w},$$

hence

$$\frac{\alpha}{\beta} B^{-\frac{1}{\alpha}} w^{\frac{1}{\alpha}+1} \le \frac{1}{H} \le \alpha B^{-\frac{1}{\alpha}} w^{\frac{1}{\alpha}+1}. \tag{3.8}$$

Then the conclusion follows. □

COROLLARY 3.1. *In* Ω_2 *we have*

(i) $w_x \ge -\alpha B \min\{\overline{x}^{-\alpha-1}, x^{-\alpha-1}\}$;

(ii) $w_x \ge -\alpha q^{-\frac{\alpha+1}{\beta}} B^{\frac{2-q}{\beta}} t^{-\frac{\alpha+1}{\beta}}$.

LEMMA 3.3. *There exist constants* $c_1,\ c_2 > 0$ *such that in* Ω_2

$$c_1 w^{\frac{2}{\alpha}+1} \le w_{xx} \le c_2 w^{\frac{2}{\alpha}+1}.$$

PROOF. We deduce from (3.6) that

$$w_{xx} = \frac{H_w}{H^2} w_x = -\frac{H_w}{H^3}, \tag{3.9}$$

where by (3.7)

$$\begin{aligned} H_w &= -(\alpha+1)\alpha^{-2} B^{\frac{1}{\alpha}} w^{-\frac{1}{\alpha}-2} + q(q-1)(q-2) w^{q-3} t \\ &= -(\alpha+1)\alpha^{-2} B^{\frac{1}{\alpha}} w^{-\frac{1}{\alpha}-2} + (q-1)(q-2) w^{-2}\{B^{\frac{1}{\alpha}} w^{-\frac{1}{\alpha}} - x\} \\ &= -\gamma\alpha^{-2} B^{\frac{1}{\alpha}} w^{-\frac{1}{\alpha}-2} - (q-1)(q-2) x w^{-2}. \end{aligned}$$

Here we set

$$\gamma := -(q-1)(q-2)\alpha^2 + \alpha + 1.$$

It is easily seen that $\gamma > 0$ for any $q > 1$ and $0 < \alpha < 1/(q-1)$. From the above expressions we obtain

$$-\gamma\alpha^{-2}B^{\frac{1}{\alpha}}w^{-\frac{1}{\alpha}-2} \leq H_w \leq -(\alpha+1)\alpha^{-2}B^{\frac{1}{\alpha}}w^{-\frac{1}{\alpha}-2} \quad \text{if } 1 < q \leq 2, \tag{3.10}$$

or

$$-(\alpha+1)\alpha^{-2}B^{\frac{1}{\alpha}}w^{-\frac{1}{\alpha}-2} \leq H_w \leq -\gamma\alpha^{-2}B^{\frac{1}{\alpha}}w^{-\frac{1}{\alpha}-2} \quad \text{if } q > 2. \tag{3.11}$$

The conclusion follows immediately. □

LEMMA 3.4. *There exists $c_3 > 0$ such that in Ω_2*

$$w_{xxx} \geq -c_3 w^{\frac{3}{\alpha}+1}.$$

PROOF. From (3.6), (3.9) we obtain

$$w_{xxx} = -3\,\frac{H_w^2}{H^5} + \frac{H_{ww}}{H^4},$$

where

$$\begin{aligned} H_{ww} &= (\alpha+1)(2\alpha+1)\alpha^{-3}B^{\frac{1}{\alpha}}w^{-\frac{1}{\alpha}-3} + q(q-1)(q-2)(q-3)w^{q-4}t \\ &= \delta\alpha^{-3}B^{\frac{1}{\alpha}}w^{-\frac{1}{\alpha}-3} - (q-1)(q-2)(q-3)xw^{-3}. \end{aligned}$$

Here

$$\delta := (q-1)(q-2)(q-3)\alpha^3 + (\alpha+1)(2\alpha+1)$$

and equality (3.7) has been used. If $1 < q \leq 2$ or $q \geq 3$, then

$$H_{ww} \geq 0.$$

Hence

$$w_{xxx} \geq -3\,\frac{H_w^2}{H^5},$$

thus the conclusion follows by (3.8), (3.10), (3.11).

If $2 < q < 3$, from the above expression of H_{ww} we get the inequalities

$$\delta\alpha^{-3}B^{\frac{1}{\alpha}}w^{-\frac{1}{\alpha}-3} \leq H_{ww} \leq (\alpha+1)(2\alpha+1)\alpha^{-3}B^{\frac{1}{\alpha}}w^{-\frac{1}{\alpha}-3}.$$

The conclusion follows again by (3.8), (3.11). □

LEMMA 3.5. *There exists $c_4 > 0$ such that in Ω_2*

$$w_{xt} \leq c_4 w^{\frac{2}{\alpha}+q}.$$

PROOF. From (3.6) we obtain

$$w_{xt} = -qw^{q-1}\frac{H_w}{H^3}.$$

Then the conclusion follows by (3.8), (3.10), (3.11). □

Let us mention for completeness another estimate from below of w, which shows that the bounds in Lemma 3.1 are sharp.

LEMMA 3.6. *In* Ω_2

$$w(x,t) \geq B(x^\beta + q\beta B^{q-1}t)^{-\alpha/\beta}. \tag{3.12}$$

PROOF. By Lemma 3.1-(i) and Lemma 3.2 w is a supersolution of the linear problem

$$\begin{cases} y_t - qB^{q-1}x^{-\alpha(q-1)}y_x = 0 & \text{in } \Omega_2 \\ y(x,0) = Bx^{-\alpha} & (x > \overline{x}). \end{cases}$$

The expression in the right-hand side of (3.12) is the unique solution of this problem. Hence the claim follows. □

4. - Estimates of u

A first estimate for the solution of problem (1.1)-(1.3) is given in the following proposition.

PROPOSITION 4.1. *Let assumptions* (A_0)-(A_1) *be satisfied. Then there exists a positive constant* M_0 *such that*

$$0 \leq u(x,t) \leq \min\{\|\varphi\|_\infty,\ M_0x^{-\alpha}\} \quad \textit{in } Q. \tag{4.1}$$

PROOF. By the maximum principle we have

$$0 \leq u \leq \|\varphi\|_\infty \quad \text{in } Q.$$

By assumption (A_1)-(ii) there exists $x_0 > 0$ such that

$$\varphi(x) < 2Ax^{-\alpha} \quad \text{for any} \quad x > x_0.$$

Set

$$M_0 := \max\left\{2A,\ x_0^\alpha\|\varphi\|_\infty,\ \left[\left(\frac{\alpha+1}{q}\right)x_0^{\beta-2}\right]^{\frac{1}{q-1}}\right\};$$

define also

$$\overline{u}(x,t) := M_0x^{-\alpha} \quad (x > 0).$$

Due to the definition of M_0, for any $x \in (x_0, \infty) \times (0, \infty)$ we have

$$\overline{u}_t - (\overline{u}^q)_x - \overline{u}_{xx} > \alpha q M_0^q x^{-\alpha q - 1} \left(1 - \frac{\alpha + 1}{q M_0^{q-1}} x_0^{\beta - 2} \right) > 0.$$

Similarly,

$$u(x_0, t) \le \|\varphi\|_\infty \le M_0 x_0^{-\alpha} = \overline{u}(x_0, t) \quad \text{for any } t > 0,$$

$$u(x, 0) \le 2Ax^{-\alpha} \le M_0 x^{-\alpha} = \overline{u}(x, 0) \quad \text{for any } x > x_0.$$

Hence

$$u(x,t) \le M_0 x^{-\alpha} \quad \text{in } (x_0, \infty) \times (0, \infty).$$

Since

$$\|\varphi\|_\infty \le M_0 x_0^{-\alpha} \le M_0 x^{-\alpha} \quad \text{for any } x \in (0, x_0]$$

the conclusion follows. □

As we shall see in Section 6, gradient estimates of u are also needed to prove Theorem 1.1. In the case $q > 2$ the proof of such estimates requires a more refined estimate from below of u. This is the content of the following Proposition.

Let $B \in (0, A)$. Due to assumption (A_1) there exists $x_0 > 0$ such that

$$\varphi(x) > Bx^{-\alpha} \quad \text{for any } x > x_0. \tag{4.2}$$

Set also

$$x_B := (\beta \alpha^{-1} B^{q-1})^{-\frac{1}{2-\beta}}, \tag{4.3}$$

$$\overline{x} > \max\{x_0, \ x_B\}; \tag{4.4}$$

then consider the time $\overline{t}$ and the region Ω_2 associated with the above choice of $\overline{x}$ and B (see (3.3), (3.4)). We have the following result.

PROPOSITION 4.2. *Let $q > 2$ and assumptions (A_0)-(A_1) be satisfied. For any fixed $B \in (0, A)$ and $\overline{x}$ as in* (4.4) *consider the corresponding region Ω_2; let w denote the unique entropy solution of problem* (1.4)-(3.1) *in Q. Then there exists $M_1 \in (0, 1]$ such that*

$$u \ge M_1 w \quad \text{in } \Omega_2. \tag{4.5}$$

For the proof we need the following preliminary estimate.

PROPOSITION 4.3. *Let the assumptions of Proposition* 4.2 *be satisfied. For any fixed $B \in (0, A)$ and $\overline{x}$ as in* (4.4) *consider the corresponding time $\overline{t}$. Then there exists a positive constant M_2 such that*

$$u(0, t) \ge M_2 t^{-\frac{\alpha}{\beta}} \quad \text{for any } t \in [\overline{t}, \infty). \tag{4.6}$$

The proof of Proposition 4.3 will be given in the sequel. Assuming its validity we proceed to prove Proposition 4.2. Observe that the half-line $\sigma_0 := \{(0,t) | t \geq \bar{t}\}$ is contained in Ω_2 by definition.

PROOF OF PROPOSITION 4.2. Observe that

$$m := \min_{(x,t)\in\sigma} u(x,t) > 0$$

by the strong maximum principle. Set

$$M_1 := \min\left\{ M_2 \left(\frac{q^\alpha}{B}\right)^{\frac{1}{\beta}},\ \frac{m}{B}\bar{x}^{-\alpha},\ 1 \right\};$$

define also

$$\underline{u} := M_1 w \quad \text{in } Q.$$

Due to the choice of M_1 we have in Ω_2

$$\underline{u}_t - (\underline{u}^q)_x - \underline{u}_{xx} = M_1[w_t - M_1^{q-1}(w^q)_x - w_{xx}] \leq -M_1 w_{xx} \leq 0;$$

here we have made use of Lemmas 3.2, 3.3. Similarly,

(i) $\underline{u}(0,t) = M_1 w(0,t) \leq M_2 t^{-\frac{\alpha}{\beta}} \leq u(0,t)$ for any $t \geq \bar{t}$, due to Lemma 3.1 and Proposition 4.3;

(ii) $\underline{u}|_\sigma = M_1 B \bar{x}^{-\alpha} \leq m \leq u|_\sigma$, due to equality (3.5);

(iii) $\underline{u}(x,0) = M_1 B x^{-\alpha} \leq u(x,0)$ for any $x \geq \bar{x}$, due to definitions (4.2), (4.4).

Then the conclusion follows. □

Let us now turn to the proof of Proposition 4.3. For this purpose let us consider the following problems:

$$(4.7)\qquad \begin{cases} \epsilon f_\epsilon'' = -\left(\dfrac{\xi}{\beta} + q f_\epsilon^{q-1}\right) f_\epsilon' - \dfrac{\alpha}{\beta} f_\epsilon & \text{in } (0,\infty) \\ f_\epsilon > 0 & \text{in } (0,\infty) \\ f_\epsilon(0) = c, \quad f_\epsilon'(0) = 0, \end{cases}$$

where ϵ and c are positive constants;

$$(4.8)\qquad \begin{cases} \left(\dfrac{\xi}{\beta} + q f_h^{q-1}\right) f_h' + \frac{\alpha}{\beta} f_h = 0 & \text{in } (0,\infty) \\ f_h > 0 & \text{in } (0,\infty) \\ \xi^\alpha f_h(\xi) \to B \quad \text{if } \xi \to +\infty. \end{cases}$$

Observe that problems (4.8) and (2.4) coincide if $B = A$. Hence the unique global solution of problem (4.8) is implicitly given by the following equality (see (2.1)):

$$\xi = B^{\frac{1}{\alpha}} f_h^{-\frac{1}{\alpha}} - q f_h^{q-1} \quad (\xi \in [0, \infty)). \tag{4.9}$$

In particular we have

$$f_h(0) = q^{-\frac{\alpha}{\beta}} B^{\frac{1}{\beta}} =: c_B. \tag{4.10}$$

It is immediately seen that

$$f_h' < 0, \;\; f_h'' \geq 0 \text{ in } [0, \infty), \tag{4.11}$$

which in turn implies

$$f_h' \geq -\frac{\alpha}{\beta q} c_B^{2-q}. \tag{4.12}$$

We shall write $f_\epsilon(\cdot, c)$, $f_h(\cdot, c_B)$ to stress the dependence on the initial data. The existence of a unique solution of problem (4.7) in some maximal interval $(0, \xi_\epsilon)$ follows by classical results. The following properties of this solution will be proved in the sequel.

LEMMA 4.1. *Let $q > 2$ and $\alpha < \frac{1}{q-1}$. Then for any $\epsilon > 0$ and $c \geq c_B$*

$$f_\epsilon > f_h \quad \text{in } (0, \xi_\epsilon). \tag{4.13}$$

COROLLARY 4.1. *Let $q > 2$ and $\alpha < \frac{1}{q-1}$. Then for any $\epsilon > 0$ and $c > 0$*

$$f_\epsilon' < 0 \quad \text{in } (0, \xi_\epsilon). \tag{4.14}$$

Moreover, for any $\epsilon > 0$, $c > 0$ and $\xi_1 \in (0, \xi_\epsilon)$

$$f_\epsilon' \geq -\frac{\alpha}{\beta} c \xi_1 \quad \text{in } (0, \xi_1). \tag{4.15}$$

COROLLARY 4.2. *Let $q > 2$ and $\alpha < \frac{1}{q-1}$. Then for any $\epsilon > 0$ and $c > 0$ there exists a unique global solution of problem* (4.7).

COROLLARY 4.3. *Let $q > 2$ and $\alpha < \frac{1}{q-1}$. Then for any $\epsilon > 0$ and $c > 0$ there exists a unique $\overline{\xi} \in (0, \infty)$ such that $f_\epsilon''(\overline{\xi}) = 0$. Moreover,*

(i) $f_\epsilon'' < 0$ in $(0, \overline{\xi})$, (ii) $f_\epsilon'' > 0$ in $(\overline{\xi}, \infty)$.

LEMMA 4.2. *Let* $q > 2$ *and* $\alpha < \frac{1}{q-1}$. *Then for any* $\epsilon \in \left(0, \frac{\beta q}{\alpha} c_B^{2(q-1)}\right)$ *there exists* $c \in (0, c_B)$, *such that* $f_\epsilon(\cdot, c) < f_h(\cdot, c_B)$ *in* $(0, \infty)$.

COROLLARY 4.4. *Let* $q > 2$ *and* $\alpha < \frac{1}{q-1}$. *Then for any* $\epsilon \in \left(0, \frac{\beta q}{\alpha} c_B^{2(q-1)}\right)$ *there exist* $\overline{c} \in (0, c_B)$, $\overline{\xi} \in (0, \infty)$ *such that:*

(i) $f_\epsilon(\overline{\xi}, \overline{c}) = f_h(\overline{\xi}, c_B)$;

(ii) $f'_\epsilon(\overline{\xi}, \overline{c}) = f'_h(\overline{\xi}, c_B)$;

(iii) $f''_\epsilon(\overline{\xi}, \overline{c}) = 0$.

Using the above result we can now prove Proposition 4.3.

PROOF OF PROPOSITION 4.3. (i) From definitions (3.3), (4.3), (4.10) and (4.4) we have

$$\overline{t} = \frac{B^{-(q-1)}}{q} \overline{x}^\beta > \frac{B^{-(q-1)}}{q} x_B^\beta = \left[\frac{\beta q}{\alpha} c_B^{2(q-1)}\right]^{-\frac{\beta}{2-\beta}}.$$

Fix

$$\epsilon = \overline{t}^{-\frac{2-\beta}{\beta}} < \frac{\beta q}{\alpha} c_B^{2(q-1)}; \tag{4.16}$$

let $\overline{c} \in (0, c_B)$, $\overline{\xi} \in (0, \infty)$ satisfy equalities (i)-(iii) in Corollary 4.4. It is easily seen that there exists a unique $x_1 > 0$ such that

$$\overline{\xi}\, \overline{t}^{\frac{1}{\beta}} = x_1 - qB^{q-1} x_1^{-\alpha(q-1)} \overline{t}. \tag{4.17}$$

Due to (4.17) the point $(\overline{\xi}\, \overline{t}^{\frac{1}{\beta}}, \overline{t})$ lies on the line of equation

$$x = x_1 - qB^{q-1} x_1^{-\alpha(q-1)} t;$$

namely, on the characteristic of problem (1.4)-(3.1) issued at $x = x_1$ (see 2.1).

Let us consider the following regions (see Fig 4.1):

$$D_1 := \{(x,t) | t > \overline{t},\ 0 < x < \overline{\xi}\, \overline{t}^{\frac{1}{\beta}}\},$$

$$D_2 := \{(x,t) | t > 0,\ x > \max\{x_1 - qB^{q-1} x_1^{-\alpha(q-1)} t,\ \overline{\xi}\, \overline{t}^{\frac{1}{\beta}}\}\},$$

$$D := D_1 \cup D_2.$$

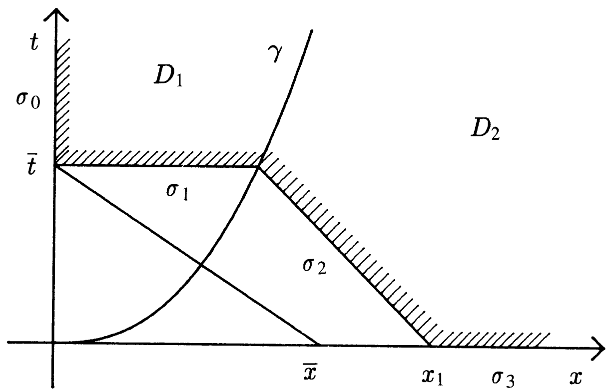

Fig. 4.1

Set also

$$\gamma := \{(x,t) \in Q | x = \bar{\xi} t^{\frac{1}{\beta}}\},$$
$$\sigma_0 := \{(0,t) | t \geq \bar{t}\},$$
$$\sigma_1 := \{(x,\bar{t}) | 0 \leq x \leq \bar{\xi}\bar{t}^{\frac{1}{\beta}}\},$$
$$\sigma_2 := \{(x,t) | 0 \leq t \leq \bar{t},\ x = x_1 - qB^{q-1} x_1^{-\alpha(q-1)} t\},$$
$$\sigma_3 := \{(x,0) | x \geq x_1\}.$$

(Observe that $\sigma_0 \subseteq \overline{D}$ by definition.)

Define

$$v_1 := t^{-\frac{\alpha}{\beta}} f_\epsilon(xt^{-\frac{1}{\beta}}) \quad \text{for } (x,t) \in \overline{D}_1,$$
$$v_2 := t^{-\frac{\alpha}{\beta}} f_h(xt^{-\frac{1}{\beta}}) \quad \text{for } (x,t) \in \overline{D}_2,$$

where f_ϵ is the unique solution of problem (4.7) such that $f_\epsilon(0) = \bar{c}$. Clearly, v_1 and v_2 are of class C^2 in $\overline{D}_1$, respectively $\overline{D}_2$. Define also

$$v := \begin{cases} v_1 & \text{in } \overline{D}_1 \\ v_2 & \text{in } \overline{D}_2. \end{cases}$$

Due to the choice of $\bar{\xi}$, v is well defined and of class C^1 in $\overline{D}$ (see Corollary 4.4).

(ii) Let us prove that for any $M_3 \in (0,1]$ the function

$$\underline{u} := M_3 v$$

satisfies

$$L[\underline{u}] = \underline{u}_{xx} + (\underline{u}^q)_x - \underline{u}_t \geq 0 \quad \text{in the open region } D = D_1 \cup D_2. \tag{4.18}$$

In the region D_1 we have

$$\begin{aligned} L[\underline{u}] &= M_3 t^{-\frac{\alpha}{\beta}-1}\left[t^{-\frac{2-\beta}{\beta}} f_\epsilon'' + \left(\frac{\xi}{\beta} + qM_3^{q-1} f_\epsilon^{q-1}\right) f_\epsilon' + \frac{\alpha}{\beta} f_\epsilon\right] \\ &\geq M_3 t^{-\frac{\alpha}{\beta}-1}\left[\epsilon f_\epsilon'' + \left(\frac{\xi}{\beta} + qM_3^{q-1} f_\epsilon^{q-1}\right) f_\epsilon' + \frac{\alpha}{\beta} f_\epsilon\right] \\ &= M_3\, q(M_3^{q-1} - 1) t^{-\frac{\alpha}{\beta}-1} f_\epsilon^{q-1} f_\epsilon' \geq 0; \end{aligned}$$

here the definition of the region D_1, the choice (4.16) of ϵ and inequality (i) in Corollary 4.3 have been used.

In the region D_2 we have similarly

$$\begin{aligned} L[\underline{u}] &= M_3 t^{-\frac{\alpha}{\beta}-1}\left[t^{-\frac{2-\beta}{\beta}} f_h'' + \left(\frac{\xi}{\beta} + qM_3^{q-1} f_h^{q-1}\right) f_h' + \frac{\alpha}{\beta} f_h\right] \\ &\geq M_3 t^{-\frac{\alpha}{\beta}-1}\left[t^{-\frac{2-\beta}{\beta}} f_h'' + q(M_3^{q-1} - 1) f_h^{q-1} f_h'\right] \geq 0, \end{aligned}$$

due to inequalities (4.11). This proves inequality (4.18).

(iii) Now we can prove that for any $M_3 \in (0,1]$ small enough:

$$u \geq M_3 v \quad \text{in } \overline{D}. \tag{4.19}$$

Since by definition

$$v(0,t) = v_1(0,t) = \bar{c} t^{-\frac{\alpha}{\beta}},$$

inequality (4.6) follows with $M_2 := \bar{c} M_3$, thus proving the result. To prove inequality (4.19) we make use of a slight generalization of classical comparison results. Since $v_1 \in C^2(\overline{D}_1)$ and $v_2 \in C^2(\overline{D}_2)$, from (4.18) we have

$$D_{xx}^{+}\underline{u} + (\underline{u}^q)_x - \underline{u}_t \geq 0 \quad \text{in } \overline{D}_1 \backslash (\sigma_0 \cup \sigma_1), \tag{4.20}$$

$$D_{xx}^{-}\underline{u} + (\underline{u}^q)_x - \underline{u}_t \geq 0 \quad \text{in } \overline{D}_2 \backslash (\sigma_2 \cup \sigma_3). \tag{4.21}$$

Set

$$\begin{aligned} a &:= q\, u^{q-1}, \\ b &:= q\, \frac{\underline{u}^{q-1} - u^{q-1}}{\underline{u} - u}\, \underline{u}_x, \\ z &:= \underline{u} - u \quad \text{in } \overline{D}. \end{aligned}$$

Observe that a, b are bounded in $\overline{D}$ since $q > 2$, $\underline{u} \in C^1(\overline{D})$ and inequality (4.1) holds. Inequalities (4.20)-(4.21) imply:

$$D^-_{xx} z + a z_x + b z - z_t \geq 0 \quad \text{in } D_1 \cup \gamma, \tag{4.22}$$

$$D^+_{xx} z + a z_x + b z - z_t \geq 0 \quad \text{in } D_2 \cup \gamma. \tag{4.23}$$

As for the boundary conditions, observe that

$$z_x(0,t) = M_3 v_{1x}(0,t) = M_3 t^{-\frac{\alpha+1}{\beta}} f'_\epsilon(0) = 0 \quad \text{for any } t > \overline{t}. \tag{4.24}$$

Moreover it is easily seen that:

(a) $\underline{u}|_{\sigma_1} = M_3 v_1(\cdot, \overline{t})$ is non-increasing,

(b) $\underline{u}|_{\sigma_2} = M_3 B x_1^{-\alpha} =$ constant,

(c) $m := \min_{\sigma_1 \cup \sigma_2} u > 0$.

Since $v \in C(\overline{D})$, by (a) and (b) above we have

$$\underline{u}|_{\sigma_1 \cup \sigma_2} \leq \underline{u}(0, \overline{t}) = M_3 \overline{c}\,\overline{t}^{\,-\frac{\alpha}{\beta}}.$$

Hence by (c)

$$\underline{u}|_{\sigma_1 \cup \sigma_2} \leq u|_{\sigma_1 \cup \sigma_2}, \tag{4.25}$$

provided that

$$M_3 \leq m \overline{t}^{\frac{\alpha}{\beta}} / \overline{c}.$$

Since $x_1 \geq \overline{x}$, for any $x \geq x_1$ and $M_3 \in (0,1]$ we have

$$\underline{u}(x,0) = M_3 v_2(x,0) = M_3 B x^{-\alpha} \leq \varphi(x) = u(x,0).$$

Then if $M_3 \in (0,1]$ we have

$$\underline{u}|_{\sigma_3} \leq u|_{\sigma_3}. \tag{4.26}$$

By (4.25)-(4.26) we have $\underline{u}|_{\sigma_2 \cup \sigma_3} \leq u|_{\sigma_2 \cup \sigma_3}$. Hence by the strong maximum principle

$$\underline{u}|_{\sigma'_1} \leq u|_{\sigma'_1}, \tag{4.27}$$

where

$$\sigma'_1 := \{(x, \overline{t}) \in D_2\}.$$

This in turn implies

$$\underline{u}|_{\sigma_1 \cup \sigma'_1} \leq u|_{\sigma_1 \cup \sigma'_1}. \tag{4.28}$$

Due to inequalities (4.22)-(4.24) and (4.28), the conclusion follows using the classical maximum principle (which still holds in the present situation) in the set $\{(x,t)\in D | t>\bar{t}\}$. □

We conclude this section proving Lemmas 4.1-4.2 and Corollaries 4.1-4.4.

PROOF OF LEMMA 4.1. Define

$$\psi_\epsilon(\xi) := f_\epsilon(\xi) - f_h(\xi) \quad \text{in } (0,\xi_\epsilon).$$

From (4.7)-(4.8) we easily obtain

$$\begin{aligned}\epsilon\psi_\epsilon'(\xi) \geq {}& -\frac{\xi}{\beta}\psi_\epsilon(\xi) - [f_\epsilon^q(\xi) - f_h^q(\xi)] \\ & + \frac{1-\alpha}{\beta}\int_0^\xi \psi_\epsilon(x)dx - \epsilon f_h'(\xi) \qquad (\xi\in(0,\xi_\epsilon)).\end{aligned} \tag{4.29}$$

Set

$$\bar{\xi} := \sup\{\xi\in[0,\xi_\epsilon) | \psi_\epsilon(\xi) > 0\}$$

(observe that $\bar{\xi}>0$, since $\psi_\epsilon(0)\geq 0$, $\psi_\epsilon'(0)>0$). Suppose $\bar{\xi}<\xi_\epsilon$; then $\psi_\epsilon(\bar{\xi})=0$, $\psi_\epsilon'(\bar{\xi})\leq 0$. Since

$$q>2 \Longrightarrow \alpha < \frac{1}{q-1} < 1,$$

from (4.11) and (4.29) we obtain $\psi_\epsilon'(\bar{\xi})>0$. The contradiction proves the result. □

PROOF OF COROLLARY 4.1. For any $c>0$ choose $B>0$ so small that $c\geq c_B$ (see (4.10)). Then inequality (4.13) implies

$$f_\epsilon > 0 \quad \text{in } (0,\xi_\epsilon)$$

for any $\epsilon>0$. Set

$$\tilde{\xi} := \sup\{\xi\in[0,\xi_\epsilon) | f_\epsilon'(\xi)<0\}$$

(observe that $\tilde{\xi}>0$ since $f_\epsilon'(0)=0$, $f_\epsilon''(0)<0$). Suppose $\tilde{\xi}<\xi_\epsilon$; then $f_\epsilon'(\tilde{\xi})=0$, $f_\epsilon''(\tilde{\xi})\geq 0$. On the other hand, for any $\epsilon>0$

$$\epsilon f_\epsilon''(\tilde{\xi}) = -\frac{\alpha}{\beta}f_\epsilon(\tilde{\xi}) < 0$$

by the above inequality; the contradiction proves the first claim.

As for the second, it follows from the inequalities

$$\begin{aligned}\epsilon f_\epsilon'(\xi) &= -\frac{\xi}{\beta}f_\epsilon(\xi) + [c^q - f_\epsilon^q(\xi)] + \frac{1-\alpha}{\beta}\int_0^\xi f_\epsilon(x)dx \\ &> -\frac{\xi}{\beta}f_\epsilon(\xi) + \frac{1-\alpha}{\beta}\xi f_\epsilon(\xi) > -\frac{\alpha}{\beta}c\xi_1 \quad (0<\xi<\xi_1<\xi_\epsilon);\end{aligned}$$

here we have made use of the assumption $q > 2$ and of inequality (4.14). This completes the proof. □

PROOF OF COROLLARY 4.2. This follows immediately from inequalities (4.13)-(4.15). □

PROOF OF COROLLARY 4.3. According to inequalities (4.13)-(4.14) and Corollary 4.2 we have

$$f_\epsilon > 0, \quad f'_\epsilon < 0 \quad \text{in } (0, \infty),$$

which implies that f'_ϵ has at least a minimum point $\bar{\xi} \in (0, \infty)$. In fact, from the obvious equality

$$f_\epsilon(\xi) = c + \int_0^\xi f'_\epsilon(x) dx,$$

we deduce that f'_ϵ cannot be non-increasing all over $\mathbb{R}^+$. Observe that at any stationary point $\tilde{\xi}$ of f'_ϵ we have

$$\begin{aligned} f'''_\epsilon(\tilde{\xi}) &= -f'_\epsilon(\tilde{\xi}) \left[\frac{1+\alpha}{\beta} + q(q-1) f_\epsilon^{q-2}(\tilde{\xi}) f'_\epsilon(\tilde{\xi}) \right] \\ &= -f'_\epsilon(\tilde{\xi}) \left[\frac{1+\alpha}{\beta} \cdot \frac{\tilde{\xi}}{\beta} + q \, \frac{2-\beta+\alpha}{\beta} \, f_\epsilon^{q-1}(\tilde{\xi}) \right] \frac{1}{\frac{\tilde{\xi}}{\beta} + q f_\epsilon^{q-1}(\tilde{\xi})} > 0 \end{aligned}$$

by inequality (4.14). Hence $\bar{\xi}$ is the unique minimum point of f'_ϵ in $(0, \infty)$. Since $f''_\epsilon(\tilde{\xi}) = 0$ the conclusion follows. □

PROOF OF LEMMA 4.2. Set

$$\psi_\epsilon(\xi) := f_\epsilon - f_h \quad \text{in } [0, \infty).$$

As in the proof of Lemma 4.1, from (4.7)-(4.8) we obtain

$$\begin{aligned} \epsilon \psi'_\epsilon(\xi) &= -\frac{\xi}{\beta} \psi_\epsilon(\xi) - [f_\epsilon^q - f_h^q](\xi) + c^q - c_B^q \\ &\quad + \frac{1-\alpha}{\beta} \int_0^\xi \psi_\epsilon - \epsilon f'_h(\xi), \quad (\xi \in (0, \infty)). \end{aligned} \tag{4.30}$$

For any $\epsilon \in \left(0, \frac{\beta q}{\alpha} c_B^{2(q-1)}\right)$ choose $c \in (0, c_B)$ such that

$$c_B^q - c^q > \epsilon \frac{\alpha}{\beta q} c_B^{2-q}. \tag{4.31}$$

Set

$$\xi^* := \sup\{\xi \in [0,\infty) | \psi_\epsilon(\xi) < 0\}$$

(observe that $\xi^* > 0$ since $\psi_\epsilon(0) = c - c_B < 0$). Suppose $\xi^* < \infty$; then $\psi_\epsilon(\xi^*) = 0$, $\psi'_\epsilon(\xi^*) \geq 0$. However, by (4.30), (4.12) we have

$$\epsilon\psi'_\epsilon(\xi^*) = c^q - c_B^q + \frac{1-\alpha}{\beta}\int_0^{\xi^*} \psi_\epsilon(x)dx - \epsilon f'_h(\xi^*) < c^q - c_B^q + \epsilon\frac{\alpha}{\beta q} c_B^{2-q} < 0,$$

due to inequality (4.31). The contradiction proves the result. □

PROOF OF COROLLARY 4.4. Fix $\epsilon \in \left(0, \frac{\beta q}{\alpha} c_B^{2(q-1)}\right)$. According to Corollary 4.3, there exists in $(0,\infty)$ a unique $\overline{\xi} = \overline{\xi}(c)$ such that

$$f''_\epsilon(\overline{\xi}(c), \overline{c}) = 0 \quad (c > 0).$$

It follows by classical results that $\overline{\xi}$ depends continuously on c in $(0,\infty)$.

By Lemma 4.1 the point $(\overline{\xi}(c_B), f_\epsilon(\overline{\xi}(c_B), c_B))$ lies above the graph of f_h. By Lemma 4.2 there exists $\tilde{c} \in (0, c_B)$ such that the point $(\overline{\xi}(\tilde{c}), f_\epsilon(\overline{\xi}(\tilde{c}), \tilde{c}))$ lies below the same graph. Let c increase between $\tilde{c}$ and c_B. Since the functions $\overline{\xi}(\cdot)$ and $f_\epsilon(\cdot,\cdot)$ are continuous, there exists at least a number $\overline{c} \in (\tilde{c}, c_B)$ such that

$$f_\epsilon(\overline{\xi}(\overline{c}), \overline{c}) = f_h(\overline{\xi}(\overline{c}), c_B),$$

$$f''_\epsilon(\overline{\xi}(\overline{c}), \overline{c}) = 0.$$

The above equalities together with (4.7)-(4.8) imply

$$f'_\epsilon(\overline{\xi}(\overline{c}), \overline{c}) = f'_h(\overline{\xi}(\overline{c}), c_B).$$

Setting $\overline{\xi} = \overline{\xi}(\overline{c})$ the conclusion follows. □

5. - Estimates of u_x

Observe that $z := u_x$ satisfies the problem:

$$(5.1) \qquad \begin{cases} z_t - qu^{q-1}z_x - q(q-1)u^{q-2}z^2 = z_{xx} & \text{in } (0,\infty)\times(0,\infty) \\ z = 0 & \text{in } \{0\}\times(0,\infty) \\ z = \varphi'(x) & \text{in } (0,\infty)\times\{0\}. \end{cases}$$

The following estimate will be of use in the sequel.

PROPOSITION 5.1. *Let assumptions* (A_0)-(A_2) *be satisfied. Then there exists a positive constant* $N_0 > 0$ *such that*

$$0 \geq u_x \geq -\min\{\|\varphi'\|_\infty,\ N_0 x^{-\alpha-1}\} \quad in\ Q. \tag{5.2}$$

In the case $q > 2$ the proof of Proposition 5.1 makes use of an additional comparison argument.

Let $B \in (0, A)$. Due to assumptions (A_1), (A_2)-(ii) there exists $\hat{x}_0 > 0$ such that

$$\varphi(x) > Bx^{-\alpha}, \quad \varphi'(x) > -Cx^{-\alpha-1} \quad \text{for any } x > \hat{x}_0. \tag{5.3}$$

Now define

$$\overline{x} > \max\{\hat{x}_0,\ x_B\} \tag{5.4}$$

(where x_B is defined in (4.3)). We have the following result.

PROPOSITION 5.2. *Let the assumptions of Proposition* 4.2 *be satisfied, where* $\overline{x}$ *is defined in* (5.4). *Moreover let assumption* (A_2) *hold. Then there exists a positive constant* N_1 *such that*

$$0 \geq u_x \geq N_1 w_x \quad in\ \Omega_2. \tag{5.5}$$

PROOF. Since $u \in W^{1,\infty}$ under the present assumptions (see Theorem 2.1 - (b)), from problem (5.1) we obtain by the maximum principle

$$0 \geq u_x \geq -\|\varphi'\|_\infty \quad \text{in } Q. \tag{5.6}$$

Set

$$N_1 := \max\left\{ \frac{c_4 - qM_1^{q-1}c_1 + B^{\frac{2-\beta}{\alpha}} c_3 \overline{x}^{\beta-2}}{q(q-1)M_1^{q-2}\alpha^2 B^{-\frac{2}{\alpha}}\beta^{-2}},\ \frac{\beta}{\alpha B}\overline{x}^{\alpha+1}\|\varphi'\|_\infty,\ \frac{C}{\alpha B} \right\}, \tag{5.7}$$

where the constants c_k, M_1 in the right-hand side were introduced in Lemmas 3.2-3.5 and in Proposition 4.2. Define also

$$\underline{z} := N_1 w_x \quad \text{in } Q.$$

Due to the definition of N_1, in Ω_2 we have

$$\begin{aligned}
&\underline{z}_t - qu^{q-1}\underline{z}_x - q(q-1)u^{q-2}\underline{z}^2 - \underline{z}_{xx} \\
&\leq N_1\{w_{xt} - qM_1^{q-1}w^{q-1}w_{xx} - q(q-1)M_1^{q-2}N_1 w^{q-2}w_x^2 - w_{xxx}\} \\
&\leq N_1 w^{\frac{2}{\alpha}+q}\{c_4 - qM_1^{q-1}c_1 - q(q-1)M_1^{q-2}N_1\alpha^2 B^{-\frac{2}{\alpha}}\beta^{-2} \\
&+ B^{\frac{2-\beta}{\alpha}}\overline{x}^{\beta-2}c_3\} \leq 0;
\end{aligned}$$

here we have made use of Lemmas 3.2 - 3.5 and of Proposition 4.2. Similarly,

(i) $\underline{z}(0,t) \leq 0$ for any $t \geq \bar{t}$,
due to Lemma 3.2;

(ii) $\underline{z}|_\sigma \leq -\frac{\alpha}{\beta} B^{-\frac{1}{\alpha}} (w|_\sigma)^{\frac{1}{\alpha}+1} N_1 = -\frac{\alpha}{\beta} B^{-\frac{1}{\alpha}} (B\overline{x}^{-\alpha})^{\frac{1}{\alpha}+1} N_1 \leq -\|\varphi'\|_\infty \leq u_x|_\sigma$,
due to Lemma 3.2 and inequality (5.6);

(iii) $\underline{z}(x,0) = -\alpha B N_1 x^{-\alpha-1} \leq -Cx^{-\alpha-1} < u_x(x,0)$ for any $x > \overline{x}$.

Then the conclusion follows. □

Now we can prove Proposition 5.1.

PROOF OF PROPOSITION 5.1. (i) Let us first consider the case $1 < q \leq 2$. Set

$$\tilde{x} := \max\left\{\hat{x}_0, \left(\frac{\|\varphi'\|_\infty}{M_0}\right)^{\frac{1}{\alpha}}\right\} \tag{5.8}$$

where M_0 is the positive constant in inequality (4.1), and

$$N_0 := \max\left\{\frac{(\alpha+1)(\alpha+2)}{q(q-1)} M_0^{2-q} \tilde{x}^{\beta-2},\ \|\varphi'\|_\infty \tilde{x}^{\alpha+1},\ C\right\}. \tag{5.9}$$

Observe that the definition of $\tilde{x}$ and inequality (4.1) imply

$$u(x,t) \leq M_0 x^{-\alpha} \quad \text{in } (\tilde{x},\infty)\times(0,\infty). \tag{5.10}$$

Define also

$$\underline{z}(x) := -N_0 x^{-\alpha-1} \quad (x > 0).$$

Due to the definition of N_0, in $(\tilde{x},\infty)\times(0,\infty)$ we have

$$\begin{aligned}&\underline{z}_t - qu^{q-1}\underline{z}_x - q(q-1)u^{q-2}\underline{z}^2 - \underline{z}_{xx}\\ &\quad\leq (\alpha+1)(\alpha+2)N_0 x^{-\alpha-3}\left[1 - \frac{q(q-1)}{(\alpha+1)(\alpha+2)} M_0^{q-2}\tilde{x}^{2-\beta} N_0\right] \leq 0.\end{aligned}$$

Similarly,

(i′) $\underline{z}(\tilde{x}) = -N_0\tilde{x}^{-\alpha-1} \leq -\|\varphi'\|_\infty \leq u_x(\tilde{x},t)$ for any $t > 0$,
due to inequality (5.6);

(ii′) $\underline{z}(x,0) = -N_0 x^{-\alpha-1} \leq -Cx^{-\alpha-1} \leq u_x(x,0)$ for any $x > \tilde{x}$,
due to (5.3), (5.8).

By the maximum principle and classical approximation arguments (e.g., see [LSU, p. 495]) we obtain

$$u_x(x,t) \geq -N_0 x^{-\alpha-1} \quad \text{in } (\tilde{x},\infty)\times(0,\infty).$$

We also have, due to definition (5.9),

$$\|\varphi'\|_\infty \leq N_0 x^{-\alpha-1} \quad \text{for any } x \in (0, \tilde{x}].$$

The inequality

$$0 \geq u_x \quad \text{in } Q$$

is proved similarly. Then the conclusion follows in the present case.

(ii) Let $q > 2$. By Lemma 3.1-(i) and Proposition 5.2 we have

$$u_x \geq -\alpha B N_1 x^{-\alpha-1} \quad \text{for } x \geq \overline{x}.$$

Due to definition (5.7) we also have

$$\|\varphi'\|_\infty \leq \frac{\alpha B}{\beta} N_1 x^{-\alpha-1} \quad \text{for any } x \in (0, \overline{x}].$$

Setting

$$N_0 := \frac{\alpha B}{\beta} N_1$$

the conclusion follows. □

6. - Convergence results

In order to prove Theorem 1.1 we have to investigate the convergence of the family $\{u_k\}_{k>0}$ defined in (1.8) as $k \to \infty$. Let us first establish some a priori bounds.

LEMMA 6.1. *Let assumptions* (A_0)-(A_1) *be satisfied. Then for any* $k > 0$

$$0 \leq u_k(x,t) \leq M_0 x^{-\alpha} \quad \textit{in } Q; \tag{6.1}$$

here $M_0 > 0$ *is the constant of inequality* (4.1).

PROOF. By inequality (4.1) and definition (1.8) we have

$$0 \leq u_k(x,t) \leq k^\alpha M_0 (kx)^{-\alpha} = M_0 x^{-\alpha} \quad \text{in } Q.$$ □

LEMMA 6.2. *Let assumptions* (A_0)-(A_2) *be satisfied. Then for any* $k > 0$

$$0 \geq u_{kx}(x,t) \geq -N_0 x^{-\alpha-1} \quad \textit{in } Q. \tag{6.2}$$

PROOF. By inequality (5.2) and definition (1.8) we have

$$0 \geq u_{kx}(x,t) \geq -k^{\alpha+1} N_0 (kx)^{-\alpha-1} = -N_0 x^{-\alpha-1} \quad \text{in } Q.$$ □

LEMMA 6.3. *Let assumptions* (A_0)-(A_2) *be satisfied. Define*

$$Q_\rho := \{(x,t) | x \geq \rho,\ t \geq \rho\} \qquad (\rho > 0).$$

Then for any $\rho > 0$ *there exist* $H > 0$, $\delta > 0$ *such that for any* $k > 0$

$$|u_k(x,t_2) - u_k(x,t_1)| \leq H|t_2 - t_1|^{1/2} \quad \text{in } Q_\rho \tag{6.3}$$

whenever $|t_2 - t_1| < \delta$.

PROOF. It follows from inequalities (6.2) by the results in [Gi]. □

Now we can prove the following convergence result.

PROPOSITION 6.1. *Let assumptions* (A_0)-(A_2) *be satisfied. Let* v *be the unique entropy solution of problem* (1.4)-(1.5). *Then* $u_k \to v$ *as* $k \to \infty$, *uniformly on compact subsets of* Q.

PROOF. Consider the family of sets $Q_{1/n} (n \in \mathbb{N})$. By Lemmas 6.1-6.3 and Ascoli's Theorem, for any $n \in \mathbb{N}$ there exist a subsequence $\{u_k^{(n)}\}$ and a function $v^{(n)} \in C(Q_{1/n})$ such that

$$u_k^{(n)} \to v^{(n)} \quad \text{as } k \to \infty$$

uniformly on compact subsets of $Q_{1/n}$. By a classical diagonal argument we find a subsequence $\{u_{k_l}\}$ and a function $\overline{v} \in C(Q)$ such that

$$u_{k_l} \to \overline{v} \quad \text{as } l \to \infty \tag{6.4}$$

uniformly on compact subsets of Q.

Let us prove that $\overline{v}$ is an entropy solution of problem (1.4)-(1.5) in Q_T for any $T > 0$; then $\overline{v} = v$ in Q by uniqueness (see Theorem 2.2) and the conclusion follows. Condition (i) of Definition 2.1 is obviously satisfied (see (6.1), (6.4)). For (ii) we can use a standard argument (see [Kr], [La]). In fact, let $\eta \in C^2(\mathbb{R})$, $p \in C^1(\mathbb{R})$ such that $\eta'' \geq 0$ and

$$p'(u) = -qu^{q-1}\eta'(u) \qquad (u \in \mathbb{R}).$$

From problem (1.9) we obtain easily

$$[\eta(u_k)]_t + [p(u_k)]_x - k^{-(2-\beta)}[\eta(u_k)]_{xx} \leq 0 \quad \text{in } Q. \tag{6.5}$$

Set

$$\eta_m(u) := |u - L| * \rho_m(u), \tag{6.6}$$

$$p_m(u) := \{-\text{sgn}(u - L)(u^q - L^q)\} * \rho_m(u) \quad (u, L \in \mathbb{R};\ m \in \mathbb{N}), \tag{6.7}$$

where ρ_m is a suitable mollifier. Since

$$\eta_m(u) \to |u - L|,$$
$$p_m(u) \to -\mathrm{sgn}(u - L)(u^q - L^q)$$

in $L^1(Q)$ as $m \to \infty$, substituting (6.6)-(6.7) in (6.5) and taking the limit as $m \to \infty$ we obtain

$$\int\int_{Q_T} \{|u_k - L|\varsigma_t - \mathrm{sgn}(u_k - L)(u_k^q - L^q)\varsigma_x + k^{-(2-\beta)}|u_k - L|\varsigma_{xx}\}\mathrm{d}x\mathrm{d}t \geq 0 \tag{6.8}$$

for any $L \in \mathbb{R}$, $T > 0$ and any $\varsigma \in C_0^\infty(Q_T)$, $\varsigma \geq 0$. Since convergence (6.4) is uniform in the compact subsets of Q and inequality (6.1) holds, from (6.8) we easily obtain

$$\int\int_{Q_T} \{|\overline{v} - L|\varsigma_t - \mathrm{sgn}(\overline{v} - L)(\overline{v}^q - L^q)\varsigma_x\}\mathrm{d}x\mathrm{d}t \geq 0 \quad (T > 0)$$

for any $L \in \mathbb{R}$ and any $\varsigma \in C_0^\infty(Q_T)$, $\varsigma \geq 0$. This proves the claim.

Let us finally prove that property (iii) in Definition 2.1 is also satisfied. For this purpose the following claim is expedient:

(C) Let $T > 0$, $h_0 > 0$. There exists a constant $c_0 = c_0(r_1, r_2) > 0$ such that

$$\int_{r_1}^{r_2} ||u_k(x, t + \tau) - u_k(x, t)|\mathrm{d}x \leq c_0 \min_{|h| \leq h_0} \left\{|h| + \frac{\tau}{h^2}\right\} \tag{6.9}$$

for any $0 \leq t \leq t + \tau < T$.

The proof of claim (C) will be given in the sequel. Set

$$\begin{aligned}\int_{r_1}^{r_2} |\overline{v}(x,t) - Ax^{-\alpha}|\mathrm{d}x = &\int_{r_1}^{r_2} |\overline{v}(x,t) - u_{k_l}(x,t)|\mathrm{d}x \\ &+ \int_{r_1}^{r_2} |u_{k_l}(x,t) - \varphi_{k_l}(x)|\mathrm{d}x \\ &+ \int_{r_1}^{r_2} |\varphi_{k_l}(x) - Ax^{-\alpha}|\mathrm{d}x.\end{aligned} \tag{6.10}$$

Let $\epsilon > 0$ be fixed. Due to inequality (6.9) there exists $\tau_\epsilon > 0$ such that

$$\int_{r_1}^{r_2} |u_{k_l}(x,t) - \varphi_{k_l}(x)|\mathrm{d}x < \frac{\epsilon}{3} \tag{6.11}$$

for any $t \in [0, \tau_\epsilon)$. Since convergence (6.4) is uniform in compact subsets of Q and inequality (6.1) holds, for any ϵ, τ_ϵ as above there exists $k_1 \in \mathbb{N}$ such that

$$\int_{r_1}^{r_2} |\overline{v}(x,t) - u_{k_l}(x,t)| \mathrm{d}x < \frac{\epsilon}{3} \tag{6.12}$$

for any $k_l > k_1$. Due to assumption (A_1)-(ii), for any $\epsilon > 0$ there exists $k_2 \in \mathbb{N}$ such that

$$\int_{r_1}^{r_2} |\varphi_{k_l}(x) - Ax^{-\alpha}| \mathrm{d}x < \frac{\epsilon}{3} \tag{6.13}$$

for any $k_l > k_2$. Now fix $k_l > \max\{k_1,\ k_2\}$ in equality (6.10). Due to inequalities (6.11)-(6.13), for any $\epsilon > 0$ there exists $\tau_\epsilon > 0$ such that

$$\int_{r_1}^{r_2} |\overline{v}(x,t) - Ax^{-\alpha}| \mathrm{d}x < \epsilon$$

for any $t \in [0, \tau_\epsilon)$. This proves (iii) assuming (C).

Now let us turn to the proof of claim (C). This follows from [Kr, Lemma 5, p. 233] as soon as the following inequalities are proved:

$$\int_{r_1}^{r_2} |u_k(x+h,t) - u_k(x,t)| \mathrm{d}x \le \overline{c}_0 |h| \tag{6.14}$$

for any $|h| \le h_0$, $\overline{c}_0 > 0$ being a suitable constant;

$$\left| \int_{r_1}^{r_2} [u_k(x,t+\tau) - u_k(x,t)] \varsigma(x) \mathrm{d}x \right| \le \tilde{c}_0 \|\varsigma\|_{C^2}\, \tau \tag{6.15}$$

for any t, $t+\tau \in (0,T]$ $(\tau > 0)$ and any $\varsigma \in C_0^2([r_1, r_2])$, $\tilde{c}_0 > 0$ being a suitable constant.

Inequality (6.14) is easily proved using the uniform estimate (6.2). Inequality (6.15) follows similarly by problem (1.9) and estimate (6.1). This completes the proof. □

Now we can prove Theorem 1.1.

PROOF OF THEOREM 1.1. By Proposition 6.1 we have in particular

$$k^\alpha u(kx, k^\beta) \to v(x,1) \quad \text{as } k \to \infty$$

uniformly on compact subsets $K \subset (0,\infty)$. Choosing

$$t' = k^{\beta}, \quad x' = kx = x(t')^{\frac{1}{\beta}}$$

and dropping the primes again, we obtain by (2.3)

$$\sup_{x\in K} |t^{\frac{\alpha}{\beta}} u(xt^{\frac{1}{\beta}}, t) - f(x)| \to 0 \quad \text{as } t \to \infty.$$

Hence the conclusion follows. □

Let us state the following result, analogous to Theorem 1.1.

THEOREM 6.1. *Let assumptions* (A_0)-(A_1) *be satisfied. Let* u *be the unique solution of problem* (1.1)-(1.3). *Then there exists a mild entropy solution* v *of problem* (1.4)-(1.5) *and a diverging sequence* $\{t_k\} \subseteq (0,\infty)$ *such that*

$$\lim_{k\to\infty} t_k^{\frac{\alpha}{\beta}} |u(xt_k^{\frac{1}{\beta}}, t_k) - v(xt_k^{\frac{1}{\beta}}, t_k)| = 0$$

for any $x \in (0,\infty)$.

The proof of Theorem 6.1 makes use of the following local energy estimate.

LEMMA 6.4. *Let assumptions* (A_0)-(A_1) *be satisfied. Then for any* $\varsigma \in C_0^{\infty}(\overline{Q}_T)$, $\operatorname{supp}\varsigma \subseteq \overline{Q}_T \backslash \{x = 0\}$, $\varsigma \geq 0$ *there exists a positive constant* H_0 *such that*

$$k^{\frac{2-\beta}{2}} \|u_{kx}\varsigma\|_{L^2(Q_T)} \leq H_0 \tag{6.16}$$

for any $k > 0$.

PROOF. Let $\chi \in C_0^1(\overline{Q}_T)$, $\operatorname{supp}\chi \subseteq \overline{Q}_T\backslash\{x = 0\}$, $\chi \geq 0$. From problem (1.9) we have

$$\iint_{Q_T} \{-u_{kt} - (u_k^q)_x]\chi + k^{-(2-\beta)} u_{kx}\chi_x\} dx dt = 0. \tag{6.17}$$

Choose

$$\chi \equiv \chi_k := e^{u_k}\varsigma^2;$$

observe that $\chi_k \in C_0^1(\overline{Q}_T)$ by regularity results. We easily obtain

$$\begin{aligned} \iint_{Q_T} u_{kt} e^{u_k}\varsigma^2 dx dt &= \iint_{Q_T} \{(e^{u_k}\varsigma^2)_t - 2e^{u_k}\varsigma\varsigma_t\} dx dt \\ &= -2\iint_{Q_T} e^{u_k}\varsigma\varsigma_t dx dt - \int_0^{\infty} e^{\varphi_k}\varsigma^2(x,0) dx. \end{aligned} \tag{6.18}$$

Define

$$\rho(u) := q \int_0^u s^{q-1} e^s \mathrm{d}s;$$

then we have

$$\iint_{Q_T} (u_k^q)_x e^{u_k} \varsigma^2 \mathrm{d}x\mathrm{d}t = -2 \iint_{Q_T} \rho(u_k) \varsigma \varsigma_x \mathrm{d}x\mathrm{d}t. \tag{6.19}$$

From (6.17)-(6.19) we obtain

$$k^{-(2-\beta)} \iint_{Q_T} e^{u_k} u_{kx}^2 \varsigma^2 \mathrm{d}x\mathrm{d}t = +2 \int_0^\infty e^{\varphi_k} \varsigma^2(x,0)\mathrm{d}x$$
$$+ 2 \iint_{Q_T} e^{u_k} \{\varsigma\varsigma_t - [k^{-(2-\beta)} u_{kx} + \rho(u_k)]\varsigma\varsigma_x\}\mathrm{d}x\mathrm{d}t.$$

Observe that for any $\lambda \in (0,1)$

$$\iint_{Q_T} e^{u_k} |u_{kx}\varsigma\varsigma_x| \mathrm{d}x\mathrm{d}t \le \frac{1}{2} \iint_{Q_T} e^{u_k} \left[\lambda u_{kx}^2 \varsigma^2 + \frac{\varsigma_x^2}{\lambda}\right] \mathrm{d}x\mathrm{d}t.$$

Moreover, since $\operatorname{supp} \varsigma \subseteq \overline{Q}_T \backslash \{x = 0\}$, by Lemma 6.1 there exists $C > 0$ such that for any $k > 0$

$$\sup_{(x,t) \in \operatorname{supp} \varsigma} u_k(x,t) \le C.$$

Then from (6.20) we obtain

$$k^{-(2-\beta)} \iint_{Q_T} u_{kx}^2 \varsigma^2 \mathrm{d}x\mathrm{d}t$$
$$\le \frac{2}{1-\lambda} e^C \left\{ \iint_{Q_T} \left\{ |\varsigma\varsigma_t| + \frac{\varsigma_x^2}{2\lambda} + \rho(C)|\varsigma\varsigma_x| \right\} \mathrm{d}x\mathrm{d}t + \int_0^\infty e^{\varphi_k}\varsigma^2(x,0)\mathrm{d}x \right\} =: H_0.$$

This completes the proof. □

Observe that the constant H_0 in (6.16) depends only on $\operatorname{supp} \varsigma$ and on the constants M_0, N_0.

PROOF OF THEOREM 6.1. Consider the family of sets $\{Q_{1/n}\} (n \in \mathbb{N})$. By Lemma 6.1 the family $\{u_k\}$ is uniformly bounded in $Q_{1/n}$ for any n. Hence there exist a subsequence $\{u_k^{(n)}\}$ and a function $v^{(n)}$ such that

$$u_k^{(n)} \to v^{(n)} \quad \text{as } k \to \infty$$

in the $L^\infty(Q_{1/n})$ – weak* topology. By Lemma 6.4 and compensated compactness results (see [Ta]) we also have

$$u_k^{(n)} \to v^{(n)} \quad \text{as } k \to \infty$$

in $L^r_{\text{loc}}(Q_{1/n})$ for any $r \in [1, \infty)$. Therefore by a diagonal argument there exist a subsequence $\{u_{k_m}\}$ and a function v such that

$$u_{k_m} \to v \quad \text{as } m \to \infty$$

in $L^r_{\text{loc}}(Q)$ for any $r \in [1, \infty)$. This in turn implies existence of a subsequence $\{u_{k_l}\}$ such that

$$u_{k_l} \to v \quad \text{as } l \to \infty$$

almost everywhere in Q.

Let us prove that v is a mild entropy solution of problem (1.4)-(1.5) in Q_T. Condition (i) of Definition 2.2 is clearly satisfied. To check condition (ii) let $\eta \in C^2(\mathbb{R})$, $\eta'' \geq 0$; set

$$p(u) := q \int_0^u s^{q-1} \eta'(s) \mathrm{d}s \quad (u \in \mathbb{R}).$$

For any $\varsigma \in C_0^\infty(\overline{Q}_T)$ $\operatorname{supp} \varsigma \subseteq \overline{Q}_T \backslash \{x = 0\}$, $\varsigma \geq 0$ we have

$$\begin{aligned}
\int\int_{Q_T} \{\eta(u_k)\varsigma_t + p(u_k)\varsigma_x\} &+ \int_0^\infty \eta(\varphi_k)\varsigma(x,0)\mathrm{d}x \\
&= k^{-(2-\beta)} \int\int_{Q_T} \{\eta'(u_k)u_{kx}\varsigma_x + \eta''(u_k)u_{kx}^2 \\
&\geq k^{-(2-\beta)} \int\int_{Q_T} \eta'(u_k)u_{kx}\varsigma_x \\
&\geq -k^{-(2-\beta)} \|\eta'(u_k)\|_{L^2(\operatorname{supp} \varsigma_x)} \|u_{kx}\varsigma_x\|_{L^2(Q_T)} \\
&\geq -k^{-\frac{2-\beta}{2}} H_0 \|\eta'(u_k)\|_{L^2(\operatorname{supp} \varsigma_x)},
\end{aligned}$$

due to inequality (6.16). By Lemma 6.1 the right-hand side of the above inequality is infinitesimal as $k \to \infty$. Moreover, it follows plainly by assumption (A_1)-(ii) that

$$\lim_{k \to \infty} \int_0^\infty \eta(\varphi_k)\varsigma(x,0)\mathrm{d}x = \int_0^\infty \eta(Ax^{-\alpha})\varsigma(x,0)\mathrm{d}x.$$

Proceeding as in the proof of Lemma 6.4 the claim follows. □

Since $u_{k_l} \to v$ a.e. in Q as $l \to \infty$, we also have

$$u_{k_l}(\cdot, t) \to v(\cdot, t) \quad \text{as } l \to \infty$$

a.e. in $(0, \infty)$, for any $t > 0$ not belonging to some set $E \subset (0, \infty)$ of zero measure. Fix $\tilde{t} \in (0, \infty) \backslash E$. Define

$$t_l := \tilde{t} k_l^{\beta}, \quad x' := x(\tilde{t})^{-\frac{1}{\beta}};$$

then the conclusion follows as in the proof of Theorem 1.1. □

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