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On Global Real Analytic Solutions of the Degenerate Kirchhoff Equation

KUNIHIKO KAJITANI - KAORU YAMAGUTI

1. - Introduction

We shall consider the problem of existence and uniqueness of real analytic solutions of the Cauchy problem for the degenerate Kirchhoff equation

$$\text{(1.1)} \qquad \begin{cases} \partial_t^2 u + M((Au,u))Au = f(t,x), & (t,x) \in (0,\infty)\times\mathbb{R}^n \\ u(0,x) = u_0(x), \ u_t(0,x) = u_1(x), & x \in \mathbb{R}^n, \end{cases}$$

where $Au(t,x) = \sum_{i,j=1}^{n} D_j(a_{ij}(x)D_iu(t,x))$, $D_j = \frac{1}{\sqrt{-1}}\frac{\partial}{\partial x_j}$, $(Au(t,\cdot),u(t,\cdot))$ is an inner product of $Au(t,x)$ and $u(t,x)$ in $L^2(\mathbb{R}^n_x)$ and $M(\eta)$ is a non-negative function in $C^1([0,\infty))$.

When $A = \sum_{j=1}^{n} D_j^2$ the equation (1.1) is called the Kirchhoff equation, which has been studied by many authors (cf. [1], [2], [3], [8], [9] and [10]). In this paper, we shall treat the case where A is degenerate elliptic, that is, $[a_{ij}(x); i,j = 1,\ldots,n]$ is a real symmetric matrix and

$$\text{(1.2)} \qquad a(x,\xi) = \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \ge 0$$

for $x \in \mathbb{R}^n$ and $\xi = (\xi_1,\cdots,\xi_n) \in \mathbb{R}^n$. Moreover we assume that there are $c_0 > 0$ and $\rho_0 > 0$ such that

$$\text{(1.3)} \qquad |D_x^\alpha a_{ij}(x)| \le c_0\rho_0^{-|\alpha|}|\alpha|!$$

for $x \in \mathbb{R}^n_x$, $\alpha = (\alpha_1,\ldots,\alpha_n) \in \mathbb{N}^n$ and i, $j = 1,\ldots,n$, and that $M(\eta) \in C^1([0,\infty)]$ and

$$\text{(1.4)} \qquad M(\eta) \ge 0$$

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for $\eta \in [0, \infty)$. We introduce some functional spaces. For a topological space X and an interval $I \subset \mathbb{R}^1$, we denote by $C^k(I; X)$ the set of functions from I to X which are k times continuously differentiable with respect to $t \in I$ in X. For $s \in \mathbb{R}$ and $\rho > 0$ we define a Hilbert space $H^s_\rho = \{u(x) \in L^2(R^n_x); \langle\xi\rangle^s e^{\rho\langle\xi\rangle}\hat{u}(\xi) \in L^2(\mathbb{R}^n_\xi)\}$, where $\hat{u}(\xi)$ stands for Fourier transform of u and $\langle\xi\rangle = (1+\xi_1^2+\cdots\xi_n^2)^{1/2}$. For $\rho < 0$ we define H^s_ρ as the dual space of $H^{-s}_{-\rho}$. For $\rho = 0$ we denote by $H^s = H^s_0$ the usual Sobolev space. Then note that the dual space of H^s_ρ becomes $H^{-s}_{-\rho}$ for any s, $\rho \in \mathbb{R}$.

For $\rho \in \mathbb{R}$ define an operator $e^{\rho\langle D\rangle}$ from H^s_ρ to H^s as follows:

$$e^{\rho\langle D\rangle}u(x) = \int_{\mathbb{R}^n_\xi} e^{ix\cdot\xi+\rho\langle\xi\rangle}\hat{u}(\xi)\tilde{d}\xi$$

for $u \in H^s_\rho$, where $\tilde{d}\xi = (2\pi)^{-n}d\xi$. Note that $(e^{\rho\langle D\rangle})^{-1} = e^{-\rho\langle D\rangle}$ is a mapping from H^s to H^s_ρ.

We prove the following result:

MAIN THEOREM. *Assume that* (1.2) *through* (1.4) *are valid. Let* $0 < \rho_1 < \rho_0/\sqrt{n}$. *Put* $\rho(t) = \rho_1 e^{-\gamma t}$ *for* $\gamma > 0$. *Then there exists* $\gamma > 0$ *such that for any* $u_0 \in H^2_{\rho_1}$, $u_1 \in H^1_{\rho_1}$ *and for any* $f(t,x)$ *satisfying* $e^{\rho(t)\langle D\rangle}f \in C^0([0,\infty); H^1)$, *the Cauchy Problem* (1.1) *has the unique solution* $u(t,x)$ *satisfying* $e^{\rho(t)\langle D\rangle}u \in \bigcap_{j=0}^{2} C^{2-j}([0,\infty); H^j)$.

The idea in the proof of our main theorem is based on the method introduced in [5] in order to find the global real analytic solution of the Cauchy problem for a Kowalevskian system. Roughly speaking, we transform an unknown function u such as $v = e^{\rho(t)\langle D\rangle}u$ and then change the hyperbolic equation (1.1) of the unknown function u into the parabolic equation of v. Thanks to parabolicity, we can prove local existence of a solution v of the modified problem in the usual Sobolev spaces by the use of the principle of a contraction mapping. Finally we can show the existence of a time global solution of the original equation (1.1) modifying the energy estimate which was introduced in [3] in the case of $A = -\Delta$.

2. - Preliminaries

Let S^m be the *class of symbols of pseudo-differential operators of order* m whose element $a(x,\xi)$ in $C^\infty(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$ satisfies

$$|a^{(\alpha)}_{(\beta)}(x,\xi)| \leq C_{\alpha,\beta}\langle\xi\rangle^{m-|\alpha|}$$

for x, $\xi \in \mathbb{R}^n$ and for all multi-indeces α, $\beta \in \mathbb{N}^n$, where $a^{(\alpha)}_{(\beta)}(x,\xi) =$

$\left(\frac{\partial}{\partial \xi}\right)^{\alpha}\left(\frac{1}{\sqrt{-1}}\frac{\partial}{\partial x}\right)^{\beta} a(x,\xi)$. We define a pseudo-differential operator $a(x,D)$ as usual

$$a(x,D)u(x) = \int_{\mathbb{R}^n_\xi} e^{ix\cdot\xi} a(x,\xi)\hat{u}(\xi)\tilde{d}\xi$$

for $u \in S$ where S denotes the *Schwartz space of rapidly decreasing functions* in $\mathbb{R}^n$. Then we have the following well-known fact:

PROPOSITION 2.1. (i) *For $a(x,\xi) \in S^m$ and $s \in \mathbb{R}$, there is $C_s > 0$ such that*

$$\|a(x,D)u\|_s \leq C_s\|u\|_{s+m} \tag{2.1}$$

for $u \in H^{s+m}$.

(ii) *Assume $a(x,\xi) \in S^2$ is non-negative. Then there are positive numbers C_1 and C_2 such that*

$$\Re(a(x,D)u,u)_s \geq -C_1\|u\|_s \tag{2.2}$$

and

$$\begin{aligned}\sum_{|\alpha|=1}&\{\|a_{(\alpha)}(x,D)u\|_{s-1}^2 + \|a^{(\alpha)}(x,D)u\|_s^2\}\\ &\leq C_2(2C_1\|u\|_s^2 + \Re(a(x,D)u,u)_s)\end{aligned} \tag{2.3}$$

for $u \in H^{s+2}$.

For a proof refer to [6] and [4] for (i) and (ii) respectively.

Now let us state some properties of the Hilbert space H^s_ρ.

LEMMA 2.2. (i) *Let $\rho > 0$. Then it holds that*

$$\|D_x^\alpha w\|_{H^s} \leq \|w\|_{H^s_\rho}\rho^{-|\alpha|}|\alpha|! \tag{2.5}$$

and

$$|D_x^\alpha w(x)| \leq C_n\|w\|_{H^s_\rho}\rho^{-(|\alpha|+n+|s|)}(|\alpha|+n+|s|)! \tag{2.6}$$

for $x \in \mathbb{R}^n$, $\alpha \in \mathbb{N}^n$ and $w \in H^s_\rho$.

(ii) *Let $u(x)$ be a function in H^∞ and $s \in \mathbb{R}$. If u satisfies*

$$\|D_x^\alpha u\|_{H^s} \leq c_0\rho_1^{-|\alpha|}|\alpha|! \tag{2.7}$$

for every multi-index $\alpha \in \mathbb{N}^n$, then $u(x)$ belongs to H^s_ρ for $\rho < \rho_1/\sqrt{n}$.

PROOF. (i) It is easy to verify (2.5) using the fact that $|\xi^\alpha| \le \langle\xi\rangle^{|\alpha|}$. Representing $D_x^\alpha w$ by Fourier transformation, we get

$$\begin{aligned} |D_x^\alpha w(x)| &= \left| \int e^{ix\cdot\xi}\xi^\alpha \hat{w}(\xi) \tilde{d}\xi \right| \\ &\le \int \langle\xi\rangle^{|\alpha|} |\hat{w}(\xi)| \tilde{d}\xi \\ &\le \left\{ \int (\rho^{-\rho\langle\xi\rangle}\langle\xi\rangle^{|\alpha|+|s|})^2 \tilde{d}\xi \right\}^{\frac{1}{2}} \|w\|_{H_\rho^2} \end{aligned}$$

which implies (2.6).

(ii) Since

$$\xi^\alpha \hat{u}(\xi) = \int_{\mathbb{R}_x^n} e^{-ix\cdot\xi} D_x^\alpha u(x)dx,$$

we have the estimate by virtue of (2.7) that

$$\|\langle\xi\rangle^j \hat{u}\|_{H^s}^2 \le (c_n c_0 (\sqrt{n}\rho_1^{-1})^j j!)^2$$

for any j. Hence we obtain

$$\|e^{\rho\langle\xi\rangle}\hat{u}(\xi)\|_{H^s}^2 \le \sum_{j=0}^{\infty} \left\| \frac{\rho^j \langle\xi\rangle^j}{j!} \hat{u} \right\|_{H^s}^2 \le \frac{c_n^2 c_0^2}{1-\sqrt{n}\rho\rho_1^{-1}}$$

if $\sqrt{n}\rho\rho_1^{-1} < 1$. □

Let $a(x)$ be a real analytic function in $\mathbb{R}^n$ satisfying

$$|D_x^\alpha a(x)| \le c_0 \rho_0^{-|\alpha|} |\alpha|! \tag{2.8}$$

for all $x \in \mathbb{R}^n$ and for all multi-indices $\alpha \in \mathbb{N}^n$. Define a multiplier $a\cdot$ as $(a\cdot u)(x) = a(x)u(x)$. Let us define $a(\rho; x, D)u(x) = e^{\rho\langle D\rangle} a \cdot e^{-\rho\langle D\rangle} u(x)$ for $u \in L^2(\mathbb{R}^n)$ and denote its symbol by $a(\rho; x, \xi)$.

PROPOSITION 2.3. *Suppose that* $a(x)$ *satisfies* (2.8).

(i) *If a function* u *belongs to the class* $H_{\rho_1}^s$ *and* $0 < \rho_1 < \rho_0$, *then* $a \cdot u$ *belongs to the class* H_ρ^s *for* $0 < \rho < \rho_1/\sqrt{n}$.

(ii) $a(\rho; x, D)$ *is a pseudodifferential operator of order* 0 *and its symbol has the representation*

$$a(\rho; x, \xi) = a(x) + \rho a_1(x, \xi) + \rho^2 a_2(\rho; x, \xi) + r(\rho; x, \xi), \tag{2.10}$$

where

$$a_1(x, \xi) = -\sum_{j=1}^{n} D_{x_j} a(x) \xi_j \langle\xi\rangle^{-1}, \tag{2.11}$$

and a_2 *and* r *respectively satisfy*

$$|a_{2(\beta)}^{(\alpha)}(\rho; x, \xi)| \le C_{\alpha\beta\rho_0}\langle\xi\rangle^{-|\alpha|}, \tag{2.12}$$

$$|r_{(\beta)}^{(\alpha)}(\rho; x, \xi)| \le C_{\alpha\beta\rho_0}\langle\xi\rangle^{-1-|\alpha|} \tag{2.13}$$

for $x,\ \xi \in \mathbb{R}^n$, $|\rho| < n^{-1}\rho_0$ *and* $\alpha,\ \beta \in \mathbb{N}^n$.

PROOF. (i) Assume $\rho > 0$. Taking into account the fact that

$$\sum_{\alpha'\le\alpha} \binom{\alpha}{\alpha'} |\alpha-\alpha'|!|\alpha'|!\rho_0^{-|\alpha'|}\rho_1^{-|\alpha-\alpha'|} \le \frac{\rho_0}{\rho_0-\rho_1}\,\rho_1^{-|\alpha|}|\alpha|!$$

if $\rho_1 < \rho_0$, we have the estimate that

$$\begin{aligned}\|D_x^\alpha(a\cdot u)\|_{H^s} &= \left\|\sum_{\alpha'<\alpha}\binom{\alpha}{\alpha'} D_x^{\alpha'}a\cdot D_x^{\alpha-\alpha'}u(\,\cdot\,)\right\|_{H^s}\\ &\le c_0\|u\|_{H^s_{\rho_1}}\sum\binom{\alpha}{\alpha'}\rho_0^{-|\alpha'|}|\alpha'|!\rho_1^{-|\alpha-\alpha'|}|\alpha-\alpha'|!\\ &\le c_0\,\frac{\rho}{\rho_0-\rho_1}\,\|u\|_{H^s_{\rho_1}}\rho^{-|\alpha|}|\alpha|!\end{aligned}$$

from (2.5) and (2.7). Therefore it follows from (ii) of Lemma 2.2 that $a\cdot u \in H^s_\rho$ for $\rho < \rho_1/\sqrt{n}$.

(ii) For $u \in S$ and $\varepsilon > 0$ we put $\hat{u}_\varepsilon(\xi) = e^{-\varepsilon|\xi|^2}\hat{u}(\xi)$. $u_\varepsilon(x)$ denotes the inverse Fourier transformation of $\hat{u}_\varepsilon(\xi)$. Then $u_\varepsilon(x)$ is in H^s_τ for every $\tau > 0$ and $e^{-\rho\langle D\rangle}u_\varepsilon(x)$ is also in H^s_τ for all $\tau > 0$ and $\rho \in \mathbb{R}^1$. Therefore it follows from (i) that $a\cdot e^{-\rho\langle D\rangle}u_\varepsilon$ is in H^s_τ if $\tau < \rho_0/\sqrt{n}$. Note that $a\cdot e^{-\rho\langle D\rangle}u_\varepsilon \in L^1(\mathbb{R}^n_x)$ and $e^{\rho\langle\xi\rangle}\mathcal{F}[a\cdot e^{-\rho\langle D\rangle}u_\varepsilon](\xi) \in L^1(\mathbb{R}^n_\xi)$ for $|\rho| < \rho_0/\sqrt{n}$ and $\varepsilon > 0$. So we can write

$$\begin{aligned}&e^{\rho\langle D\rangle}(a\cdot e^{-\rho\langle D\rangle}u_\varepsilon)(x)\\ &= \int e^{ix\cdot\eta+\rho\langle\eta\rangle}\,\bar{d}\eta\int e^{-iy\cdot\eta}(a\cdot e^{-\rho\langle D\rangle}u_\varepsilon)(y)dy\\ &= \lim_{\delta\to+0}\int e^{ix\cdot\eta+\rho\langle\eta\rangle-\delta|\eta|^2}\,\bar{d}\eta\int e^{-iy\cdot\eta-\delta|x-y|^2}(a\cdot e^{-\rho\langle D\rangle}u_\varepsilon)(y)dy\\ &= \lim_{\delta\to+0}\int\int\int e^{i(x-y)\cdot\eta+\rho\langle\eta\rangle-\delta|x-y|^2-\delta|\eta|^2}a(y)e^{iy\cdot\xi-\rho\langle\xi\rangle}\hat{u}_\varepsilon(\xi)\,\bar{d}\eta\, dy\,\bar{d}\xi\\ &= \lim_{\delta\to+0}\int e^{ix\cdot\xi}a_\delta(x,\xi)\hat{u}_\varepsilon(\xi)\,\bar{d}\xi,\end{aligned}$$

where $a_\delta(x,\xi)$ is given by

$$a_\delta(x,\xi) = \int\int e^{-iy\cdot\eta-\delta|y|^2-\delta|\xi+\eta|^2+\rho(\langle\xi+\eta\rangle-\langle\xi\rangle)}a(x+y)\,dy\,\bar{d}\eta.$$

Putting

$$\langle \xi+\eta\rangle - \langle \xi\rangle = \sum_{j=1}^{n} \eta_j \int_0^1 (\xi_j+\theta\eta_j)\langle \xi+\theta\eta\rangle^{-1} d\theta$$
$$= \eta\cdot w(\xi,\eta),$$

we can re-write $a_\delta(x,\xi)$ using Stokes formula:

$$a_\delta(x,\eta) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i(y-i\rho w(\xi,\eta))\cdot\eta-\delta|y|^2-\delta|\xi+\eta|^2} a(x+y)dy\tilde{d}\eta$$
$$= \int_{\mathbb{R}^n} \tilde{d}\eta \int_{\mathbb{R}^n - iw(\xi,\eta)} e^{-iz\cdot\eta-\delta(z+i\rho w(\xi,\eta))^2\delta|\xi+\eta|^2} a(x+z+i\rho w(\xi,\eta))dz$$
$$= \int_{\mathbb{R}^n} \tilde{d}\eta \int_{\mathbb{R}^n} e^{-iy\cdot\eta-\delta(y+i\rho w(\xi,\eta))^2-\delta|\xi+\eta|^2} a(x+y+i\rho w(\xi,\eta))dy$$

for $\rho < \rho_0/n$, where we write $z^2 = \sum_{j=1}^{n} z_j^2$ for $z\in C^n$. Thus, by Taylor's expansion, we obtain

$$\lim_{\delta\to+0} a_\delta(x,\xi) = \mathrm{Os} - \int\int e^{-y\cdot\eta} a(x+y+i\rho w(\xi,\eta))dy\tilde{d}\eta$$
$$= a(x+i\rho w(\xi,0)) + r(\rho;x,\xi),$$

where

$$r(\rho;x,\xi) = \lim_{\delta\to 0}\int\int \Bigg(e^{-iy\cdot\eta-\delta(y+i\rho w(\xi,\eta)^2-\delta|\xi+\eta|^2}$$
$$\sum_{|\alpha|=1} \partial_\eta^\alpha\{D_y^\alpha a(x+y+i\rho w(\xi,\eta))\}\Bigg)dy\tilde{d}\eta$$

satisfies (2.13) (See Lemma 2.4 in [6]). Another application of Taylor's expansion yields

$$a(x+i\rho\omega(\xi,0))$$
$$= a(x+i\rho\xi\langle\xi\rangle^{-1})$$
$$= a(x)+\rho a_1(x,\xi)+\rho^2 a_2(\rho;x,\xi),$$

where $a_1(x,\xi)$ and $a_2(\rho;x,\xi)$ satisfy (2.11) and (2.12) respectively. Since

$$u_\varepsilon(x)\to u(x) \text{ in } S \text{ as } \varepsilon\to+0,$$

we have

$$(e^{\rho\langle D\rangle}a \cdot e^{-\rho\langle D\rangle}u)(x) = \lim_{\varepsilon\to+0} a(\rho; x, D)u_\varepsilon(x),$$

if $u \in \mathcal{S}$. □

Let $P(t) = [p_{ij}(t,x,D)]_{i,j=1,\dots,d}$ be a matrix consisting of pseudo-differential operators whose symbols $p_{ij}(t,x,\xi)$ belong to the class $C([0,T]; S^1)$. Let us consider the following Cauchy problem

$$\begin{cases} \dfrac{d}{dt}U(t) = P(t)U(t) + F(t), \ \ t \in (0,T), \\ U(0) = U_0, \end{cases} \tag{2.14}$$

where $U(t) = {}^t(U_1(t,x),\dots,U_d(t,x))$ is an unknown vector-valued function and $F(t) = {}^t(F_d(t,x),\dots,F_d(t,x))$, $U_0 = {}^t(U_{01},\dots,U_{0d})$ are known vector-valued functions. Then we have:

PROPOSITION 2.4. *Suppose that* $\det(\lambda - p(t,x,\xi)) \neq 0$ *for* $\lambda \in C^1$ *with* $\Re\lambda > -c_0\langle\xi\rangle$, $t \in [0,T]$ *and* $|\xi| \gg 1$. *Take an arbitrary real number* s. *Then for any* $U_0 \in (H^{s+1}(\mathbb{R}^n))^d$ *and for any* $F(t) \in C^0([0,T]; (H^{s+1})^d))$, *there exists a unique solution* $U(t) \in C^1([0,T]; (H^s)^d \cap C^0([0,T]; (H^{s+1})^d)$ *of* (2.14).

This proposition will be used in Section 4 to prove existence of local solutions of the Cauchy problem (1.6). The proof of this proposition is given in Proposition 4.5 in [7].

3. - A priori estimates of solutions for the linear problem

Let $0 < T < \infty$ and $m(t)$ be a non-negative function in $C^0([0,T])$ and $\rho(t)$ a positive function in $C^1([0,T)) \cap C^0([0,T])$ such that $\rho_t(t) < 0$ for $t \in [0,T)$. Consider the following Cauchy Problem,

$$\begin{cases} (\partial_t - \Lambda_t)^2 v(t) + m(t)A_\Lambda v(t) = g(t), \ \ t \in (0,T) \\ v(0) = v_0, \\ \partial_t v(0) = v_1, \end{cases} \tag{3.1}$$

where $\Lambda(t) = \rho(t)\langle D\rangle$, $\Lambda_t(t) = \rho_t(t)\langle D\rangle$ and $A_\Lambda = e^{\Lambda(t)}Ae^{-\Lambda(t)}$. Then by (ii) of Proposition 2.3 we have

$$A_\Lambda = A + \rho(t)a_1(x,D) + \rho(t)^2 a_2(\rho(t); x, D) + r(\rho(t); x, D), \tag{3.2}$$

where

$$a(x,\xi)=\sum_{i,j} a_{ij}(x)\xi_i\xi_j,$$

$$a_1(x,\xi)=-\sum_{|\alpha|=1} a_{(\alpha)}(x,\xi)\xi^\alpha\langle\xi\rangle^{-1}\in C^0([0,T];S^2),$$

$$a_2(t;x,\xi)\in C^0([0,T];S^2),$$

and

$$r(\rho(t);x,\xi)\in C^0([0,T];S^1).$$

Let $\tilde{m}(t)$ and $\lambda(t)$ be positive functions in $C^1([0,T])$ and assume $\lambda'(t)\le 0$ for $t\ge 0$. Define

$$E(t)^2=\frac{1}{2}\{\|(\partial_t-\Lambda_t)v(t)\|_s^2+\lambda(t)\|v(t)\|_{s+1}^2+\tilde{m}(t)(A\langle D\rangle^s v(t),\langle D\rangle^s v(t))_{L^2}\} \tag{3.3}$$

for $t\in[0,T)$, where $(\cdot,\cdot)_s$ and $\|\cdot\|_s$ stand for an inner product and a norm of H^s respectively.

Assume that $v(t)\in\bigcap_{j=0}^{2} C^{2-j}([0,T);H^{j+s})$ is a solution of (3.1). Differentiating (3.3) we have

$$\begin{aligned}
2E'(t)E(t) = {} & \Re(-m(t)A_\Lambda v+g,(\partial_t-\Lambda_t)v)_s \\
& +\rho_t(t)\|(\partial_t-\Lambda_t)v\|_{s+\frac{1}{2}}^2 \\
& +\tilde{m}_t(t)(A\langle D\rangle^s v,\langle D\rangle^s v)_{L^2} \\
& +\Re((\partial_t-\Lambda_t)v,v)_{s+1}\lambda(t)+\lambda'(t)\|v(t)\|_{s+1}^2 \\
& +\tilde{m}(t)\{\Re(\langle D\rangle^{-s}A\langle D\rangle^s v,(\partial_t-\Lambda_t)v)_s+\Re(\Lambda_t\langle D\rangle^{-s}Av,v)_s\} \\
& +\rho_t(t)\|v\|_{s+\frac{3}{2}}^2\lambda(t) \\
\le {} & \Re(g,(\partial_t-\Lambda_t)v)_s+\tilde{m}(t)\Re(\Lambda_t\langle D\rangle^{-s}A\langle D\rangle^s v,v)_s \\
& +|\tilde{m}_t(t)|((A\langle D\rangle^s v,\langle D\rangle^s v)_s \\
& +\Re((\tilde{m}(t)\langle D\rangle^{-s}A\langle D\rangle^s-m(t)A_\Lambda)v,(\partial_t-\Lambda_t)v)_s \\
& +\frac{1}{2}\rho_t(t)\|(\partial_t-\Lambda_t)v\|_{s+\frac{1}{2}}^2+\lambda(t)\left\{\rho_t+\frac{\lambda}{|\rho_t|}\right\}\|v\|_{s+\frac{3}{2}}^2 \\
\le {} & \|g(t)\|_s E(t)+\tilde{m}(t)\Re(\Lambda_t)\langle D\rangle^{-s}A\langle D\rangle^s v,v)_s \\
& +\frac{|\tilde{m}(t)|}{\tilde{m}(t)}E(t)^2 \\
& +\||\Lambda_t|^{-\frac{1}{2}}(\tilde{m}(t)\langle D\rangle^{-s}A\langle D\rangle^s-m(t)A_\Lambda)v\|_s^2 \\
& +\frac{1}{4}\rho_t(t)\|(\partial_t-\Lambda_t)v\|_{s+\frac{1}{2}}^2+\lambda\left\{\rho_t+\frac{\lambda}{|\rho_t|}\right\}\|v\|_{s+\frac{3}{2}}^2
\end{aligned} \tag{3.4}$$

for $t \in [0,T)$. Since A is a positive operator, by taking into account (2.2) we have

$$\begin{aligned}
&\Re(\Lambda_t\langle D\rangle^{-s}A\langle D\rangle^s v, v)_s \\
&= \rho_t(t)\Re(\langle D\rangle^{1-s}A\langle D\rangle^s v, v)_s \\
&\le \rho_t(t)(A\langle D\rangle^{s+\frac{1}{2}}v, \langle D\rangle^{s+\frac{1}{2}}v)_{L^2} + c|\rho_t(t)|\|v\|_{s+1}^2 \\
&\le c\,\frac{|\rho_t|}{\lambda(t)}\,E(t)^2
\end{aligned} \tag{3.5}$$

where c is a positive constant depending only on s and A.

The equality

$$\begin{aligned}
&\tilde{m}(t)\langle D\rangle^{-s}A\langle D\rangle^s - m(t)A_\Lambda \\
&= (\tilde{m}(t) - m(t))\langle D\rangle^{-s}A\langle D\rangle^s + m(t)(A - A_\Lambda) + m(t)(\langle D\rangle^{-s}A\langle D\rangle^s - A)
\end{aligned} \tag{3.6}$$

and (3.2) lead us to the estimate

$$\begin{aligned}
&\||\Lambda_t(t)|^{-\frac{1}{2}}(\tilde{m}(t)\langle D\rangle^{-s}A\langle D\rangle^s - m(t)A_\Lambda)v\|_s \\
&\le |\tilde{m}(t) - m(t)|\||\Lambda_t|^{-\frac{1}{2}}\langle D\rangle^{-s}A\langle D\rangle^s v\|_s \\
&\quad + m(t)\{\rho(t)\||\Lambda_t|^{-\frac{1}{2}}a_1 v\|_s + \rho(t)^2\||\Lambda_t|^{-\frac{1}{2}}a_2 v\|_s \\
&\qquad + \||\Lambda_t|^{-\frac{1}{2}}rv\|_s + cm(t)\|v\|_{s+1}\} \\
&\le |\rho_t(t)|^{-\frac{1}{2}}\Big\{c|\tilde{m}(t) - m(t)|\|v\|_{s+\frac{3}{2}} + m(t)\rho(t)\|a_1 v\|_{s-\frac{1}{2}} \\
&\qquad + cm(t)\rho(t)^2\|v\|_{s+\frac{3}{2}} + \frac{cm(t)}{\sqrt{\lambda(t)}}\,E(t)\Big\}
\end{aligned} \tag{3.7}$$

for $t \in [0,T)$. Besides, by virtue of (2.3) we have

$$\begin{aligned}
\|a_1 v\|_{s-\frac{1}{2}}^2 &\le c\{2c\|v\|_{s+\frac{1}{2}}^2 + \Re(Av, v)_{s+\frac{1}{2}}\} \\
&\le c\{3c\|v\|_{s+\frac{1}{2}}^2 + (A\langle D\rangle^{s+\frac{1}{2}}v, \langle D\rangle^{s+\frac{1}{2}}v)\},
\end{aligned} \tag{3.8}$$

where c is a positive constant depending only on s and the coefficients of A. Therefore, from (3.4) through (3.8), we come to the conclusion that

$$\begin{aligned}
&2E'(t)E(t) \le \|g(t)\|_s E(t) \\
&\quad + \left\{\frac{c|\rho_t(t)|}{\lambda(t)} + \frac{m(t)^2\rho^2}{|\rho_t(t)|\lambda(t)} + \frac{|\tilde{m}_t(t)|}{\tilde{m}(t)} + \frac{m(t)^2}{\lambda(t)|\rho_t(t)|}\right\}E(t)^2 \\
&\quad + \frac{\rho_t(t)}{4}\|(\partial_t - \Lambda_t)v\|_{s+\frac{1}{2}}^2
\end{aligned} \tag{3.9}$$

$$+\left\{\lambda(t)\left(\rho_t(t)+\frac{\lambda(t)}{|\rho_t(t)|}\right)+c\,\frac{|\tilde{m}(t)-m(t)|^2}{|\rho_t(t)|}+cm(t)^2\,\frac{\rho(t)^4}{|\rho_t(t)|}\right\}\|v\|^2_{s+\frac{3}{2}}$$
$$+\left\{\tilde{m}(t)\rho_t(t)+cm(t)^2\,\frac{\rho(t)^2}{|\rho_t(t)|}\right\}(A\langle D\rangle^{s+\frac{1}{2}},\langle D\rangle^{s+\frac{1}{2}}v)_{L^2}$$

for $t\in[0,T)$.

PROPOSITION 3.1. *Assume that $m(t)$ is a non-negative function in $C^1([0,T])$. Let $\tilde{m}(t)=m(t)+\varepsilon e^{-\gamma t}$, $\lambda(t)=e^{-2\gamma t}$, $\rho(t)=\rho_1 e^{-\gamma t}$, and $v(t)\in\bigcap_{j=0}^{2} C^{2-j}([0,T);H^{j+s})$. Then there are $\varepsilon>0$ and $\gamma>0$ such that if $v(t)$ satisfies (3.1) we have*

$$E(t)\le e^{\int_0^t p(\tau)d\tau}E(0)+\int_0^t e^{\int_\tau^t p(\sigma)d\sigma}\|g(\tau)\|_s d\tau \tag{3.10}$$

for $t\in[0,T)$, where

$$p(t)=c\gamma e^{\gamma t}+m(t)^2\,\frac{\rho_1^2}{\gamma}\,e^{\gamma t}+\frac{|m_t(t)|}{\tilde{m}(t)}+\frac{m(t)^2e^{3\gamma t}}{\gamma}. \tag{3.11}$$

PROOF. It suffices to prove that the terms in the right-hand side of (3.9) except for the first one and the second one are negative, if $\varepsilon>0$ and $\gamma>0$ are suitably chosen. In fact the third term is negative because of $\rho_t(t)<0$. The fourth term is

$$\begin{aligned}&\lambda(t)\left\{\rho_t(t)+\frac{\lambda(t)}{|\rho_t(t)|}\right\}+c\,\frac{|\tilde{m}(t)-m(t)|^2}{|\rho_t(t)|}+cm(t)^2\,\frac{\rho(t)^4}{|\rho_t(t)|}\\&=-\frac{\rho_1\gamma}{2}\,e^{-3\gamma t}+c\varepsilon^2\,\frac{e^{-\gamma t}}{\rho_1\gamma}+c\,\frac{\rho_1^3}{\gamma}\,m(t)^2e^{-3\gamma t}<0\end{aligned} \tag{3.13}$$

if we take

$$\gamma^2>\frac{3}{2\rho_1^2}+c\rho_1^2m(t)^2,\quad \varepsilon=\rho^{-\gamma T}. \tag{3.14}$$

Moreover we have the fifth term

$$\begin{aligned}&\tilde{m}(t)\rho_t(t)+cm(t)^2\,\frac{\rho(t)^2}{|\rho_t(t)|}\\&\le-\rho_1\gamma m(t)e^{-\gamma t}+c\,\frac{\rho_1}{\gamma}\,m(t)^2e^{-\gamma t}<0\end{aligned} \tag{3.15}$$

if we take $\gamma>0$ such that

$$\gamma^2>c\max_{0\le t\le T}m(t). \tag{3.16}$$

Therefore, choosing $\varepsilon > 0$ and $\gamma > 0$ such that (3.14) and (3.16) are valid, we obtain (3.11) from (3.9). □

For $m(t) \in L^1([0, T])$ and $\varepsilon > 0$, we define

$$\tilde{m}(t) = \int_0^T \chi_\varepsilon(t-\tau)m(\tau)d\tau + \varepsilon \tag{3.17}$$

where $\chi_\varepsilon(t) = \dfrac{1}{\varepsilon}\chi\left(\dfrac{t}{\varepsilon}\right)$ and $\chi(t) \in C_0^\infty((0,1))$ satisfying that $\chi(t) \geq 0$ and $\displaystyle\int_0^1 \chi(t)dt = 1$.

PROPOSITION 3.2. *Assume that $m(t)$ is a non-negative function in $C^1([0, T)) \cap L^1([0, T])$. Let $\tilde{m}(t)$ be a function defined by (3.17) and $v(t) \in \bigcap_{j=0}^{2} C^{2-j}(0, T); H^{s+j})$. Then there are $\rho(t)$ and $\lambda(t)$ in $C^1([0, T))$ with $\rho_t(t) \in L^1([0, T])$ and $\varepsilon > 0$ such that if $v(t)$ satisfies (3.1) we have*

$$E(t) \leq e^{\int_0^t p(\tau)d\tau} E(0) + \int_0^t e^{\int_\tau^t p(\sigma)d\sigma} \|g(\tau)\|_s d\tau \tag{3.18}$$

for $t \in [0, T)$, where $E(t) = E(t, s, \tilde{m}(t), \rho(t))$ is defined by (3.3) and

$$p(t) = \frac{c|\rho_t(t)|}{\lambda(t)} + \frac{m(t)^2}{|\rho_t(t)|}(\rho(t)^2 + 1) + \frac{|\tilde{m}_t(t)|}{\tilde{m}(t)}, \tag{3.19}$$

where c depends only on s and A.

PROOF. If we choose $\rho(t)$ and $\varepsilon > 0$ suitably, we can prove that the terms in the right-hand side of (3.9) except for the first one and the second one are negative. We can take $\rho(t)$ with $\rho_t(t) < 0$ such that the first terms of (3.13) and (3.15) are negative respectively. In fact, it suffices to find a function $\rho(t)$ satisfying

$$\begin{cases} \rho_t(t) \leq -c\left\{\dfrac{|\tilde{m}(t) - m(t)|}{\sqrt{\lambda(t)}} + \dfrac{m(t)\rho(t)^2}{\sqrt{\lambda(t)}} + \dfrac{m(t)\rho(t)}{\sqrt{\tilde{m}(t)}} + \sqrt{\lambda(t)}\right\} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (t \in (0, T)) \\ \rho(0) = \rho_1. \end{cases} \tag{3.20}$$

Put

$$(3.21)\qquad \begin{aligned} \rho(t) &= \rho_1 e^{-ct} - \int_0^t \frac{|\tilde{m}(\tau) - m(\tau)|}{\sqrt{\rho_1}}\, d\tau, \\ \lambda(t) &= \rho_1^2 e^{-2c\left\{t+\int_0^t m(\tau)(1+1/\tilde{m}(\tau))d\tau\right\}}. \end{aligned}$$

Here we take $\varepsilon > 0$ sufficiently small such that $\rho(t) > 0$ for $t \in [0,T)$. Since $\rho(t) \leq \sqrt{\lambda(t)}$ and $\lambda(t) \leq \lambda(0)$, we can see easily that $\rho(t)$ defined by (3.21) satisfies (3.20). Hence, we obtain (3.18) from (3.9) defining $p(t)$ by (3.19). □

4. - Existence of solutions for the linear problem

In this section, we consider the following linear Cauchy problem:

$$(4.1)\qquad \begin{cases} \partial_t^2 u(t) + m(t)Au(t) = f(t), & t \in (0,T) \\ u(0) = u_0, \\ \partial_t u(0) = u_1. \end{cases}$$

Following the idea of the proof of the theorem in [5], we shall prove that the Cauchy problem (4.1) has a unique solution.

THEOREM 4.1. *Assume that* (1.2) *and* (1.3) *are valid. Let* $0 < \rho_1 < \rho_0/\sqrt{n}$, $s \in \mathbb{R}$ *and* $m(t) \in C^0([0,T])$. *Then there is* $\gamma > 0$ *such that for any* $u_0 \in H_{\rho_1}^{s+2}$, $u_1 \in H_{\rho_1}^{s+2}$ *and* $e^{\Lambda(t)} f(t) \in C^0([0,t]; H^{s+1})$, (4.1) *has a unique solution* $u(t)$ *satisfying* $e^{\Lambda(t)} u(t) \in \bigcap_{j=0}^{2} C^{2-j}([0,T]; H^{s+j})$, *where* $\Lambda(t) = \rho_1 e^{-\gamma t}\langle D\rangle$. *Moreover if* $m(t) \in C^1([0,T])$, *the solution* $u(t)$ *satisfies*

$$(4.2)\qquad \begin{aligned} &\{\|e^{\Lambda(t)}\partial_t u(t)\|_s^2 + e^{-2\gamma t}\|e^{\Lambda(t)}u(t)\|_{s+1}^2\}^{1/2} \\ &\leq e^{\int_0^t p(\sigma)d\sigma}\Big[\{\|e^{\rho_1\langle D\rangle}u_1\|_s^2 + (m(0)+\varepsilon)(Ae^{\rho_1\langle D\rangle}\langle D\rangle^s u_0, e^{\rho_1\langle D\rangle}\langle D\rangle^s u_0)_{L^2} \\ &\qquad + \|e^{\rho_1\langle D\rangle}u_0\|_{s+1}^2\}^{\frac{1}{2}} + \int_0^t \|e^{\Lambda(\sigma)}f(\sigma)\|_s d\sigma\Big], \end{aligned}$$

for $t \in [0,T]$, *where* $p(t)$, γ *and* ε *are given by Proposition* 3.1.

PROOF. Put $v(t) = e^{\Lambda(t)}u(t)$. If $v(t)$ is a solution of (3.1), it is evident that $u(t)$ satisfies (4.1). So it suffices to prove that problem (3.1) has a solution.

Now we put

$$V_1(t) = \langle D\rangle v(t),$$
$$V_2(t) = (\partial_t - \Lambda_t)v(t),$$
$$V(t) = {}^t(V_1(t), V_2(t)).$$

Then if $v(t)$ is a solution of (3.1), $V(t)$ satisfies

$$\begin{cases} \dfrac{d}{dt}V(t) = P(t)V(t) + F(t), \ t > 0 \\ V(t) = V_0, \end{cases} \tag{4.3}$$

where $F(t) = {}^t(0, g(t))$, $V_0 = {}^t(v_0, v_1)$ and

$$P(t) = \begin{pmatrix} \Lambda_t & \langle D\rangle \\ m(t)A_\Lambda\langle D\rangle^{-1} & \Lambda_t \end{pmatrix}. \tag{4.4}$$

Conversely, it is evident that if $V(t)$ is a solution of (4.3), then $v(t) = \langle D\rangle^{-1}V_1(t)$ becomes a solution of (3.1). It follows from (4.4) and (ii) of Proposition 2.3 that $P(t)$ is a pseudo-differential operator of order 1 with symbol satisfying

$$\begin{aligned}\det(\lambda I - p(t; x, \xi)) = (\lambda + \gamma\rho(t)\langle\xi\rangle)^2 - \gamma^2\rho(t)\langle\xi\rangle \\ +m(t)\{a(x,\xi) + \rho(t)a_1(x,\xi) + \rho(t)^2a_2(a,\xi) + r(x,\xi)\}.\end{aligned}$$

Since $m(t) \geq 0$, $a(x,\xi) \geq 0$ and $r \in S^1$ there are $\gamma_0 > 0$ and $R_0 > 0$ such that $\det(\lambda - p(t,x,\xi)) \neq 0$ for $\Re\lambda \geq -2^{-1}\gamma e^{-\gamma T}\langle\xi\rangle$, $\gamma \geq \gamma_0 \sup\limits_{0\leq t\leq T} m(t)$ and $|\xi| \geq R_0\gamma^{-1}e^{2\gamma T}$. Therefore it follows from Proposition 2.4 that there exists a unique solution $V(t)$ of (4.3) and consequently $v(t) = \langle D\rangle^{-1}V_1(t)$ satisfies (3.1) and belongs to $\bigcap\limits_{j=0}^{2} C^{2-j}([0,T]; H^{s+j})$. Put $u(t) = e^{-\Lambda(t)}v(t)$. Then $u(t)$ satisfies (4.1) and $e^{\Lambda(t)}u(t)$ is in $\bigcap\limits_{j=0}^{2} C^{2-j}([0,T]; H^{s+j})$. If $m(t)$ is in $C^1([0,T])$, it follows from Proposition 3.1 that $v(t)$ satisfies (3.10) so $u(t)$ satisfies (4.2). In particular, if $u_0 = u_1 \equiv 0$, $f(t) \equiv 0$ and $e^{\Lambda(t;\gamma)}u(t) \subset \bigcap\limits_{j=0}^{2} C^{2-j}([0,T]; H^{s+j})$ for some $\gamma > 0$, $u(t)$ identically vanishes. This implies the uniqueness of the the solution of (4.1). Note that $v(t)$ may depend on γ but $u(t) = e^{-\Lambda(t)}v(t)$ does not depend on γ. In fact, $\tilde{u}(t) = u(t;\gamma) - u(t;\gamma')$ satisfies (4.1) with $u_0 = u_1 = f(t) \equiv 0$ and $e^{\Lambda(t;\overline{\gamma})}\tilde{u}(t) \in \bigcap\limits_{j=0}^{2} C^{2-j}([0,T]; H^{s+j})$, where $\overline{\gamma} = \max(\gamma, \overline{\gamma})$. Therefore we have $\tilde{u}(t) \equiv 0$ from the uniqueness of solution of (4.1), and consequently $u(t;\gamma) = u(t;\gamma')$. □

Finally we remark that it follows from (4.2) that $u(t)$, the solution of (4.1), satisfies

$$
(4.5)\quad
\begin{aligned}
&\|\partial_t u(t)\|_s + e^{-\gamma t}\|u(t)\|_{s+1} \\
&\leq \|e^{\Lambda(t)}\partial_t u(t)\|_s + e^{-\gamma t}\|e^{\Lambda(t)}u(t)\|_{s+1} \\
&\leq c e^{\int_0^t p(\sigma)d\sigma} \left\{ \|e^{\rho_1\langle D\rangle}u_1\|_s + \|e^{\rho_1\langle D\rangle}u_0\|_{s+1} + \int_0^t \|e^{\Lambda(\sigma)}f(\sigma)\|_s d\sigma \right\}
\end{aligned}
$$

for $t \in [0, T]$, where the positive constant c is independent of γ.

5. - Local existence of solutions of the nonlinear problem

Let $0 \leq \tau < T_1 < \infty$. For $T \in (\tau, T_1]$ we consider the Cauchy problem

$$
(5.1)\quad
\begin{cases}
\partial_t^1 u(t) + M((Au(t), u(t)))Au(t) = f(t), & \tau < t < T \\
u(\tau) = u_0, \\
u(\tau) = u_1.
\end{cases}
$$

THEOREM 5.1. *Assume that the conditions* (1.2), (1.3) *and* (1.4) *are valid. Let* $s \in \mathbb{R}$ *and* $0 < \rho_2 < \rho_0/\sqrt{n}$. *Then for any* $u_0 \in H^2_{\rho_2}$, $u_1 \in H^1_{\rho_2}$ *and* $e^{\Lambda(t)}f(t) \in C^0([\tau, T_1]; H^1)$ *where* $\Lambda(t) = \rho_2 e^{-\gamma(t-\tau)}\langle D\rangle$, *there are* $T \in (\tau, T_1]$ *and* $\gamma_0 > 0$ *such that the Cauchy problem* (4.1) *has a unique solution* $u(t)$ *satisfying* $e^{\Lambda(t)}u(t) \in \bigcap_{j=0}^{2} C^{2-j}([\tau, T]; H^j)$ *for any* $\gamma \geq \gamma_0$.

PROOF. We may assume $\tau = 0$ without loss of generality. We shall prove the existence of solutions of (5.1) by the principle of *contraction mapping*. For $T > 0$ and $s \in \mathbb{R}$, we introduce a space of functions

$$X_T^s = C^0([0, T]; H^{s+1}) \cap C^1([0, T]; H^s)$$

equipped with its *norm* $\|\cdot\|_{X_T^s}$ as

$$
(5.2)\quad \|w\|_{X_T^s} = \sup_{0 \leq t \leq T} \left\{ \frac{1}{2}(\|\partial_t u(t)\|_s^2 + \|u(t)\|_{s+1}^2) \right\}^{1/2}
$$

for every $w \in X_T^s$. We now define two functions

$$
(5.3)\quad
\begin{aligned}
m(t) &= m(t; w) = M(\eta(t; w)), \\
\eta(t; w) &= \sum_{i,j=1}^{n} (a_{ij} D_j w(t), D_i w(t))_{L_2},
\end{aligned}
$$

for each $w \in X_T^1$. Note that $m(t) \in C^1([0,T])$ if $w \in X_T^1$, and it satisfies

$$\sup_{0\le t\le T}\{m(t) + |m'(t)|\} \le K(\|w\|_{X_T^1}), \tag{5.4}$$

where K is a positive and continuous function defined in $[0,\infty)$.

Let us consider the Cauchy problem (4.1) with $m(t) = m(t;w)$. Then it follows from Theorem 4.1 that there exists a unique solution $u(t)$ of (4.1) satisfying that $e^{\Lambda(t)}u(t) \in \bigcap_{j=0}^{2} C^{2-j}([0,T];H^j)$, where $\Lambda(t) = \rho_1 e^{-\gamma t}\langle D\rangle$. So the correspondence with each $w \in X_T^1$ to $u \in X_T^1$ defines a map

$$\Psi : X_T^1 \ni w \mapsto u \in X_T^1$$

such that

$$\Psi(w) = u;\ \ \partial_t^2 u + m(t;w)Au = f,\ \ u(0) = u_0,\ \ \partial_t u(0) = u_1.$$

We shall prove that Ψ is a contraction mapping if T is sufficiently small. For $k > 0$, let us define a set

$$B_T(k) = \left\{ e^{\Lambda(t)}u(t) \in \bigcap_{j=0}^{2} C^{2-j}([0,T];H^j);\ \|u\|_{X_T^1} \le k \right\}.$$

Then we can prove that for every $k \gg 1$ there is a real number $T = T(k) > 0$ such that $\Psi(w) \in B_T(k)$ as long as $w \in B_T(k)$. Actually, we can gain an estimate

$$\|\Psi(w)\|_{X_T^1} \le c e^{\int_0^T (p(t)+\gamma)dt} \quad \text{for } w \in B_T(k), \tag{5.5}$$

which is deduced from the estimate (4.5) with $s = 1$ and the fact that

$$\Lambda(t,\xi) = \rho_2 e^{-\gamma t}\langle\xi\rangle \le \rho_2\langle\xi\rangle.$$

Note that the constant c appearing in (5.5) is independent of T, k and w. Since $p(t)$ is determined by (3.1) and (5.3), we can find a function $\overline{p}(t,k) \in C^0([0,T]\times[0,\infty))$, by virtue of (5.4), such that

$$p(t) + \gamma \le \overline{p}(t,k) \quad t \in (0,T)$$

if $w \in B_T(k)$. Since the constant c in (5.5) is independent of k and the function $\overline{p}(t,k)$ is continuous in (t,k), we can find $T = T(k) > 0$ such that

$$c e^{\int_0^T \overline{p}(t,k)dt} = k$$

for every $k > c$. Hence (5.5) implies that $\Psi(w)$ belongs to $B_T(k)$ provided $w \in B_T(k)$.

Next we shall prove Ψ is Lipschitz continuous in X_T^0, that is, with sufficiently small $T > 0$ we have the inequality

$$\|\Psi(w) - \Psi(w')\|_{X_T^0} \leq \frac{1}{2}\|w - w'\|_{X_T^0} \tag{5.6}$$

for any w, $w' \in B_T(k)$. Since the difference $\Psi(w) - \Psi(w')$ satisfies

$$\begin{aligned}
&(\partial_t^2 + m(t;w)A)(\Psi(w) - \Psi(w')) = (m(t;w') - m(t;w))A\Psi(w'),\ t > 0;\\
&(\Psi(w) - \Psi(w'))(0) = 0,\\
&\partial_t(\Psi(w) - \Psi(w'))(0) = 0,
\end{aligned}$$

we obtain, by virtue of (4.5) with $s = 0$

$$\begin{aligned}
\|\Psi(w) - \Psi(w')\|_{X_T^0} &\leq c e^{\int_0^T p(\sigma)d\sigma}\\
&\times \int_0^T |m(\sigma;w) - m(\sigma;w')| \|e^{\Lambda(\sigma)}A\Psi(w')\|_{L^2} d\sigma.
\end{aligned} \tag{5.7}$$

On the other hand, an application of Proposition 2.3 to A and the estimate (4.5) with $s = 1$ yield

$$\begin{aligned}
\|e^{\Lambda(\sigma)}A\Psi(w')(\sigma)\|_{L^2} &\leq c\|e^{\Lambda(\sigma)}\Psi(w')(\sigma)\|_2\\
&\leq c_1 e^{\int_0^\sigma \overline{p}(\tau,k)d\tau} \leq C_1(k)
\end{aligned}$$

for $w' \in B_T(k)$. Moreover, taking into account (5.4) we gain

$$\begin{aligned}
|m(\sigma;w) - m(\sigma;w')| &\leq \|M((Aw,w)) - M((Aw',w'))|\\
&\leq C_2(k)\|w - w'\|_{X_T^0}.
\end{aligned}$$

Hence, from (5.7) we have $C_3(k) > 0$ satisfying

$$\|\Psi(w) - \Psi(w')\|_{X_T^0} \leq C_3(k)T\|w - w'\|_{X_T^0}$$

for w, $w' \in B_T(k)$, which proves assertion (4.6) if $T \leq (2C_3(k))^{-1}$.

Thus once we choose $T = \min\{T(k), (2C_3(k))^{-1}\}$, we can find the solution u of (5.1) with the initial plane $\tau = 0$ which belongs to $B_T(k)$. $\square$

6. - Existence of time global solutions for the nonlinear problem

In this section we shall prove our *main theorem*. According to D'Ancona and Spagnolo [3], we introduce the following energy,

$$e(t)^2 = \frac{1}{2}\{\|\partial_t u(t) + u(t)\|^2 + \|u(t)\|^2 + F(\eta(t))\} \tag{6.1}$$

where $F(\eta) = \int_0^\eta M(\lambda)d\lambda$, $\eta(t) = ((Au(t), u(t))_{L^2}$ and $\|\cdot\|$ stands for a norm of $L^2(\mathbb{R}^n)$.

PROPOSITION 6.1 ([3]). *Assume that* $M(\eta)$ *is a non-negative continuous function in* $[0,\infty)$ *and* $f(t) \in C^0([0,T]; L^2)$. *If* $u(t)$ *is a solution of the Cauchy problem of* (1.1) *in* $(0,T)$ *such that* $u \in \bigcap_{j=0}^{2} C^{2-j}([0,T); H^j)$, *then we have the energy inequality*

$$e(t)^2 + \int_0^t e^{\frac{5}{2}(t-\tau)} M(\eta(\tau))\eta(\tau)d\tau \le e^{\frac{5}{2}t}e(0)^2 + \int_0^t e^{\frac{5}{2}(t-\tau)}\|f(\tau)\|^2 d\tau \tag{6.2}$$

for $t \in [0,T)$.

PROOF. Differentiating (6.1), we get from (1.1)

$$\begin{aligned}\frac{d}{dt}(e(t)^2) &= \Re(f(t) + \partial_t u(t), \partial_t u(t) + u(t)) - M(\eta(t))\eta(t)\\ &\le \frac{1}{2}\|f(t)\|^2 + \frac{5}{2}e(t)^2 - M(\eta(t))\eta(t)\end{aligned}$$

for $t \in [0,T)$, which yields (6.2). □

PROPOSITION 6.2 ([3]). *If* (6.2) *holds and* $T < \infty$, *then* $M(\eta(t)) \in L^1([0,T])$.

PROOF. From (6.2), it is evident that $M(\eta(t))\eta(t) \in L^1([0,T])$. On the other hand

$$\begin{aligned}\int_0^t M(\eta(\tau))d\tau &= \int_{[0,t]\cap\{\tau;\eta(\tau)>1\}} M(\eta(\tau))d\tau + \int_{[0,t]\cap\{\tau;\eta(\tau)\le 1\}} M(\eta(\tau))d\tau\\ &\le \int_0^t M(\eta(\tau))\eta(t)d\tau + t\sup_{0\le\eta\le 1} M(\eta)\end{aligned}$$

for all $t \in [0,T)$, which implies that $M(\eta(t)) \in L^1([0,T])$, □

Now we can prove our main theorem. Let $\Lambda(t,\gamma) = \rho_1 e^{-\gamma t \langle D \rangle}$, and let T^* be a real number defined by

$$T^* = \max \left\{ T > 0;\ \text{there exist } \gamma > 0 \text{ and a solution } u(t) \text{ satisfying (1.1)} \right.$$
$$\left. \text{in } (0,T) \text{ such that } e^{\Lambda(t,\gamma)} u(t) \in \bigcap_{j=0}^{2} C^{2-j}([0,T); H^j) \right\}.$$

Theorem 4.1 ensures that $T^* > 0$. We claim that $T^* = \infty$. Suppose that $T^* < \infty$. Then it follows from Proposition 6.2 that $m(t) = M((Au(t), u(t))$ is in $L^1([0,T^*])$. Hence, Proposition 3.2 and the fact that $m(t) \in C^1([0,T^*)) \cap L^1([0,T^*])$ yield that $v(t) = e^{\Lambda(t)} u(t)$ which satisfies (3.18) with $s = 0, 1$ and $T = T^*$, where $\Lambda(t) = \rho(t)\langle D \rangle$ and $\rho(t)$ is what is introduced in (3.21). Let us take $\gamma > 0$ such that $\rho_1 e^{-\gamma t} \leq \rho(t)$ for $t \in [0, T^*)$. Then the definition of T^* and (3.18) imply $e^{\Lambda(t,\gamma)} u(t) \in \bigcap_{j=0}^{2} C^{2-j}([0,T^*]; H^j)$, where $\Lambda(t,\gamma) = \rho_1 e^{-\gamma t} \langle D \rangle$. Hence we have the limits $u(T^* - 0)$ and $\partial_t u(T^* - 0)$ which satisfy $e^{\Lambda(T^*,\gamma)} u(T^* - 0) \in H^2$ and $e^{\Lambda(T^*,\gamma)} \partial_t u(T^* 0) \in H^1$. Therefore, applying Theorem 5.1 with $\rho_2 = \rho_1 e^{-\gamma T^*}$, we have a solution $\tilde{u}(t)$ of the Cauchy problem (5.1) in $(T^*, T)(T > T^*)$ with initial data $\tilde{u}(T^*) = u(T^* - 0)$ and $\partial_t \tilde{u}(T^*) = \partial_t u(T^* - 0)$, which satisfies

$$\exp(\rho_2 e^{-\gamma(t-T^*)} \langle D \rangle) \tilde{u}(t) \in \bigcap_{j=0}^{2} C^{2-j}([T^*, T]; H^j).$$

Then $\Lambda(t,\gamma) = \rho_2 e^{-\gamma(T-T^*)} \langle D \rangle$ implies that $e^{\Lambda(t,\gamma)} \tilde{u}(t) \in \bigcap_{j=0}^{2} C^{2-j}([T^*, T]; H^j)$. Now let us define

$$w(t) = \begin{cases} u(t), & t \in (0, T^*) \\ \tilde{u}(t), & t \in [T^*, T). \end{cases}$$

Then $w(t)$ has to satisfy (1.1) in $(0,T)$ and $e^{\Lambda(t,\gamma)} w(t) \in \bigcap_{j=0}^{2} C^{2-j}([0,T); H^j)$. This contradicts the definition of T^*. Thus, we have proved that $T^* = \infty$. Since $M(\eta)$ is of class C^1, we can prove easily the uniqueness of the solution of (1.1). □

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