ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

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Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 21, nº 2 (1994), p. 235-266

<http://www.numdam.org/item?id=ASNSP_1994_4_21_2_235_0>

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The Dirichlet-Neumann Operator on Continuous Functions

JOACHIM ESCHER(*)

1. - Introduction and main results

Concern of this paper is to investigate a semigroup which is a closely related to elliptic problems with so-called dynamical boundary conditions of the following type:

$$(E)\qquad \begin{aligned} \mathcal{A}u &= 0 && \text{in } \Omega\times(0,\infty),\\ \partial_t(\gamma u)+\mathcal{B}u &= 0 && \text{on } \Gamma\times(0,\infty),\\ \gamma u(\,\cdot\,,0) &= z_0 && \text{on } \Gamma. \end{aligned}$$

Here, $(\mathcal{A},\mathcal{B})$ denotes a normally elliptic boundary value problem of second order on a bounded smooth domain $\Omega\subset\mathbb{R}^n$, $n\geq 2$. This means that $(\mathcal{A},\mathcal{B})$ is given by

$$(1.1)\qquad \mathcal{A}u := -\partial_j(a_{jk}\partial_k u)+a_j\partial_j u+a_0 u \text{ and } \mathcal{B}u := a_{jk}\nu^j\gamma\partial_k u+b_0\gamma u,$$

where we use obvious summation conventions throughout. We assume that the coefficients of these operators are smooth, i.e.,

$$(1.2)\qquad a_{jk}=a_{kj},\ a_j,\ a_0,\ b_0\in C^\infty(\overline{\Omega}),\quad 1\leq j,\ k\leq n.$$

Besides, γ denotes the trace operator with respect to the boundary Γ of Ω. Finally, we suppose that the operator $\mathcal{A}$ is uniformly strongly elliptic, i.e.,

$$(1.3)\qquad a_{jk}(x)\xi^j\xi^k>0 \text{ for } x\in\overline{\Omega} \text{ and } \xi\in\mathbb{R}^n\backslash\{0\}.$$

The natural spaces to treat problem (E) are the Besov spaces $B^s_{pp}(\Gamma)$ over the boundary Γ, where $s\in\mathbb{R}$ and $p\in(1,\infty)$. More precisely, it is shown in [10]

(*) Supported by Schweizerischer Nationalfonds.

Pervenuto alla Redazione il 19 Marzo 1993 e in forma definitiva il 14 Gennaio 1994.

(cf. also [12] in the case $s = 1 - 1/p$) that, given any initial value z_0 in $B^s_{pp}(\Gamma)$ with $s > 0$, there exists a unique solution $u(\cdot, z_0) \in C([0,\infty), H_p^{s+1/p}(\Omega))$ of problem (E). The trace of this solution, i.e.,

$$\gamma u(t, z_0) =: T(t)z_0 \text{ for } t \geq 0,\ \ z_0 \in B^s_{pp}(\Gamma), \tag{1.4}$$

defines a strongly continuous analytic semigroup $(T(t))_{t\geq 0}$ on the Besov space $B^s_{pp}(\Gamma)$. Note that due to our assumption $s > 0$ and due to the trace theorem the definition in (1.4) is meaningful. However, it is possible to extend these semigroups to any Besov space $B^s_{pp}(\Gamma)$ with $s \in \mathbb{R}$, cf. [10, Theorem 1.5]. Let $\boldsymbol{B}_{s,p}$ denote the corresponding generator of the semigroup $(T(t))_{t\geq 0}$ on $B^s_{pp}(\Gamma)$, $s \in \mathbb{R}$. Following A.P. Calderón [7], J. Sylvester and G. Uhlmann [29-31], and A. Nachmann [20] (cf. also J.L. Lions [15]), $\boldsymbol{B}_{s,p}$ is called generalized *Dirichlet-Neumann operator* (see Remark 4.2).

The aim of this paper is to extend the results in [10] by establishing well-posedness of the above problem in the space $C(\Gamma)$. To be more precise, observe that

$$C(\Gamma) \hookrightarrow B^s_{pp}(\Gamma) \text{ if } s < 0. \tag{1.5}$$

Thus, given any $s < 0$, we may define the $C(\Gamma)$-realization $\boldsymbol{B}$ of $\boldsymbol{B}_{s,p}$, i.e.,

$$\begin{gathered} \operatorname{dom}(\boldsymbol{B}) := \{z \in \operatorname{dom}(\boldsymbol{B}_{s,p}) \cap C(\Gamma); \boldsymbol{B}_{s,p}z \in C(\Gamma)\} \\ \text{and } \boldsymbol{B}z := \boldsymbol{B}_{s,p}z \text{ for } z \in \operatorname{dom}(\boldsymbol{B}). \end{gathered} \tag{1.6}$$

Using these notations, our main result reads as follows:

THEOREM. *The operator* $\boldsymbol{B}$ *is well-defined, i.e., it is independent of* $p \in (1,\infty)$ *and* $s < 0$, *and* $-\boldsymbol{B}$ *generates a strongly continuous, positive, compact and analytic semigroup on* $C(\Gamma)$.

The proof of the Theorem is given in the main part of this paper. It is based on the same ideas as introduced in the articles of Stewart [27, 28]. On one hand one uses sharp a priori estimates in the case of constant coefficients and with Ω being the halfspace $\mathbb{H}^n := \mathbb{R}^{n-1} \times (0,\infty)$ with boundary $\Gamma = \mathbb{R}^{n-1}$ and for functions with small support. These estimates can be established by introducing appropriate pseudo-differential operators and by applying the Mihlin-Hörmander multiplier theorem.

On the other hand the "local structure" of the norm in $C(\Gamma)$ is heavily used. This means that, given $\varepsilon > 0$, we have that

$$\|z\|_{C(\Gamma)} = \sup_{x\in\Gamma} \|z\|_{C(\Gamma\cap\mathbb{B}_n(x,\varepsilon))},$$

where $\mathbb{B}_n(x,\varepsilon)$ denotes the open ball in $\mathbb{R}^n$ with center x and radius ε. This local structure makes it possible to overcome the obstacle that the number of covering balls $\mathbb{B}_n(x,\varepsilon)$, $x \in \Gamma$, tends to infinity as ε goes to zero, and therefore it enables us to work with parameter-dependent radii $\varepsilon = \varepsilon(\lambda)$ of these covering balls $\mathbb{B}_n(x,\varepsilon)$.

The material in this paper is organized as follows. In Section 2 we collect some known facts about Besov and Bessel potential spaces, which are needed in our treatment. Section 3 contains sharp a priori estimates for a class of pseudo-differential operators with constant symbols. In Section 4 we introduce a scale of analytic semigroups on the Besov spaces over the surface Γ. Some (more or less) technical estimates for two commutators are derived in Section 5. Finally, in Section 6 we prove the main result of this paper.

Acknowledgement. I would like to express my gratitude to the Equipe de Mathémathiques de Besançon and in particular to messieurs Prof. W. Arendt and Dr. V. Keyantuo for many interesting and helpful discussions.

2. - Preliminaries

In this section we collect some basic facts about Besov spaces. We refer to [6, 32, 33, 34] for proofs of the statements below. Furthermore we estimate the behaviour of the norm of some of these spaces under dilation (cf. Lemma 2.1).

Let $\mathcal{S}(\mathbb{R}^n)$ be the Fréchet space of all rapidly decreasing C^∞-functions and let $\mathcal{S}'(\mathbb{R}^n)$ denote its dual space, i.e., the space of all tempered distributions over $\mathbb{R}^n$. The Fourier transform on $\mathbb{R}^n$ is denoted by $\mathcal{F}$. It is well-known that $\mathcal{F} \in \mathrm{Isom}(\mathcal{S}(\mathbb{R}^n)) \cap \mathrm{Isom}(\mathcal{S}'(\mathbb{R}^n))$. Besides, let $L_p(\mathbb{R}^n) := (L_p(\mathbb{R}^n); |\cdot|_p)$, $1 \le p \le \infty$, denote the Lebesgue spaces over $\mathbb{R}^n$.

Next, we introduce the following open covering $\{O_j\}_{j\in\mathbb{N}}$ of $\mathbb{R}^n$: let $O_0 := \{x \in \mathbb{R}^n; |x| < 2\}$ and $O_j := \{x \in \mathbb{R}^n; 2^{j-1} < |x| < 2^{j+1}\}$ for $j = 1,2,3,\ldots$. Furthermore, pick a smooth partition of unity $\{\pi_j\}_{j\in\mathbb{N}}$ on $\mathbb{R}^n$ subordinate to the covering $\{O_j\}_{j\in\mathbb{N}}$. Given p, $q \in [1,\infty]$ and $s \in \mathbb{R}$, we set

$$|u|_{s,p,q} := \|2^{sj}|\mathcal{F}^{-1}\pi_j\mathcal{F}u|_p\|_{l_q(\mathbb{N})} \text{ for } u \in \mathcal{S}'(\mathbb{R}^n), \tag{2.1}$$

and we define the *Besov spaces over* $\mathbb{R}^n$ to be

$$B^s_{pq}(\mathbb{R}^n) := (\{u \in \mathcal{S}'(\mathbb{R}^n), |u|_{s,p,q} < \infty\}, |\cdot|_{s,p,q}). \tag{2.2}$$

It is well-known that these spaces are well-defined Banach spaces satisfying

$$\mathcal{S}(\mathbb{R}^n) \hookrightarrow B^s_{p1}(\mathbb{R}^n) \hookrightarrow B^s_{pq}(\mathbb{R}^n) \hookrightarrow B^s_{p\infty}(\mathbb{R}^n) \hookrightarrow B^t_{p1}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n) \text{ for } t < s, \tag{2.3}$$

$$B^0_{p1}(\mathbb{R}^n) \hookrightarrow L_p(\mathbb{R}^n) \hookrightarrow B^0_{p\infty}(\mathbb{R}^n) \text{ if } p < \infty, \tag{2.4}$$

and

$$S(\mathbb{R}^n) \overset{d}{\hookrightarrow} B^s_{pq}(\mathbb{R}^n) \text{ if } p \vee q < \infty. \tag{2.5}$$

Moreover, the following generalized Sobolev embedding theorem holds

$$B^s_{pq}(\mathbb{R}^n) \hookrightarrow BUC^t(\mathbb{R}^n) \text{ if } s > t + \frac{n}{p}. \tag{2.6}$$

If $p \in [1, \infty)$, we identify the dual space $[L_p(\mathbb{R}^n)]'$ of $L_p(\mathbb{R}^n)$ with $L_{p'}(\mathbb{R}^n)$, where $p' := p/(p-1)$, according to the duality pairing

$$\langle \varphi, u \rangle := \int_{\mathbb{R}^n} \varphi(x) u(x) dx, \quad (\varphi, u) \in L_{p'}(\mathbb{R}^n) \times L_p(\mathbb{R}^n). \tag{2.7}$$

Let us further note the following duality properties of the Besov spaces

$$[B^s_{pq}(\mathbb{R}^n)]' = B^{-s}_{p'q'}(\mathbb{R}^n) \text{ for } s \in \mathbb{R},\ p \vee q < \infty, \tag{2.8}$$

with respect to the duality pairing induced by (2.7).

It is also worthwhile to mention that the Besov spaces are stable under interpolation. Here, we restrict ourselves to the complex interpolation method and refer again to [6, 32, 33, 34] for further interpolation properties. More precisely, let $[\cdot\,;\cdot]_\theta$, $\theta \in (0,1)$, denote the standard complex interpolation functor and let p_0, p_1, q_0, $q_1 \in (1, \infty)$ and s_0, $s_1 \in \mathbb{R}$ be given. Then

$$[B^{s_0}_{p_0 q_0}(\mathbb{R}^n); B^{s_1}_{p_1 q_1}(\mathbb{R}^n)]_\theta = B^2_{pq}(\mathbb{R}^n) \text{ for } \theta \in (0,1), \tag{2.9}$$

where $s := (1-\theta)s_0 + s_1$, $\dfrac{1}{p} := \dfrac{1-\theta}{p_0} + \dfrac{\theta}{p_1}$ and $\dfrac{1}{q} := \dfrac{1-\theta}{q_0} + \dfrac{\theta}{q_1}$.

In the following, we will use the Besov spaces only in the case where $p = q$. Thus, we set

$$B^s_p(\mathbb{R}^n) := B^s_{pp}(\mathbb{R}^n) \text{ and } |\cdot|_{s,p} := |\cdot|_{s,p,p},$$

to shorten our notation.

Finally, let $\varphi \in S(\mathbb{R}^n)$, $u \in S'(\mathbb{R}^n)$ and $\rho > 0$ be given. We define the *dilation* σ_ρ by $\sigma_\rho \varphi(x) := \varphi(\rho x)$, $x \in \mathbb{R}^n$, and $\langle \sigma_\rho u, \varphi \rangle := \rho^{-n} \langle u, \sigma_{1/\rho} \varphi \rangle$.

LEMMA 2.1. *Assume that* $p \in (1, \infty)$ *and that* $\rho \in (0, 1]$. *Then there exist positive constants* α_0, α_1, β_0, *and* β_1 *(independent of* ρ*) such that*

a) $\alpha_1 \rho^{\frac{n}{p}} |\sigma_\rho u|_{1-1/p,p} \leq |u|_{1-1/p,p} \leq \beta_1 \rho^{\frac{n+1}{p}-1} |\sigma_\rho u|_{1-1/p,p}$ *for* $u \in B^{1-1/p}_p(\mathbb{R}^n)$,

b) $\alpha_0 \rho^{\frac{n+1}{p}} |\sigma_\rho u|_{-1/p,p} \leq |u|_{-1/p,p} \leq \beta_0 \rho^{\frac{n}{p}} |\sigma_\rho u|_{-1/p,p}$ *for* $u \in B^{-1/p}_p(\mathbb{R}^n)$.

PROOF. a) First, we introduce an equivalent norm on the space $B_p^{1-1/p}(\mathbb{R}^n)$. Set

$$I_{1-1/p,p}(u) := \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x) - u(y)|^p}{|x-y|^{n+p-1}}\, d(x,y) \text{ and}$$

$$|||u|||_{1-1/p,p}^p := |u|_p^p + I_{1-1/p,p}(u) \text{ for } u \in B_p^{1-1/p}(\mathbb{R}^n).$$

It is well known (cf. [1, Theorem 7.48]) that $|||\cdot|||_{1-1/p,p}$ defines an equivalent norm on $B_p^{1-1/p}(\mathbb{R}^n)$. Moreover, the transformation theorem for the Lebesgue integral yields

$$|\sigma_\rho u|_p^p = \rho^{-n}|u|_p^p \text{ and } I_{1-1/p,p}(\sigma_\rho u) = \rho^{-n+p-1} I_{1-1/p,p}(u), \tag{2.10}$$

for $u \in B_p^{1-1/p}(\mathbb{R}^n)$. Hence, we find that

$$|||u|||_{1-1/p,p}^p = \rho^n[|\sigma_\rho u|_p^p + \rho^{1-p} I_{1-1/p,p}(\sigma_\rho u)] \text{ for } u \in B_p^{1-1/p}(\mathbb{R}^n). \tag{2.11}$$

Observe now that $\rho^{1-p} \geq 1$ for $\rho \in (0,1]$ and $p \in (1,\infty)$. Thus, we conclude from (2.11) that

$$\rho^{\frac{n}{p}} |||\sigma_\rho u|||_{1-1/p,p} \leq |||u|||_{1-1/p,p} \leq \rho^{\frac{n+1}{p}-1} |||\sigma_\rho u|||_{1-1/p,p}. \tag{2.12}$$

Since $|||\cdot|||_{1-1/p,p}$ defines an equivalent norm on $B_p^{1-1/p}(\mathbb{R}^n)$, the first assertion follows from (2.12).

b) Recall that

$$[B_p^{-1/p}(\mathbb{R}^n)]' = B_{p'}^{1/p}(\mathbb{R}^n) = B_{p'}^{1-1/p'}(\mathbb{R}^n),$$

cf. (2.8). Consequently, we have

$$|u|_{-1/p,p} = \sup\{|\langle u,\varphi\rangle|\,|\varphi|_{1/p,p'}^{-1};\ \varphi \in B_{p'}^{1/p}(\mathbb{R}^n)\backslash\{0\}\} \text{ for } u \in B_p^{-1/p}(\mathbb{R}^n). \tag{2.13}$$

Further, we mention that it suffices to prove assertion b) for $u \in S(\mathbb{R}^n)$ since $S(\mathbb{R}^n) \overset{d}{\hookrightarrow} B_p^{-1/p}(\mathbb{R}^n)$, by (2.5). Recall also that $B_{p'}^{1/p}(\mathbb{R}^n) \hookrightarrow L_{p'}(\mathbb{R}^n)$, cf. (2.3) and (2.4). Hence, the duality pairing in (2.13) is given by

$$\langle u,\varphi\rangle = \int_{\mathbb{R}^n} u\varphi\, dx \text{ for } u \in S(\mathbb{R}^n).$$

Consequently, a change of variable shows that

$$|u|_{-1/p,p} = \sup\{\rho^n|\langle\sigma_\rho u, \sigma_\rho\varphi\rangle|\,|\varphi|_{1/p,p'}^{-1};\ \varphi \in B_{p'}^{1/p}(\mathbb{R}^n)\backslash\{0\}\}. \tag{2.14}$$

Finally, replacing p by p', we conclude from a) that

$$\beta_1^{-1}\rho^{\frac{n+1}{p}}|\sigma_\rho\varphi|_{1/p,p'}^{-1} \leq \rho^n|\varphi|_{1/p,p'}^{-1} \leq \alpha_1^{-1}\rho^{\frac{n}{p}}|\sigma_\rho\varphi|_{1/p,p'}^{-1} \text{ for } \varphi \in B_{p'}^{1-1/p'}(\mathbb{R}^n).$$

These estimates together with (2.14) imply now the assertion. □

We also introduce the so-called *Bessel potential spaces* $H_p^s(\mathbb{R}^n)$ for $s \in \mathbb{R}$ and $p \in (1, \infty)$:

$$H_p^s(\mathbb{R}^n) := (\{u \in \mathcal{S}'(\mathbb{R}^n); \mathcal{F}^{-1}\Lambda_1^s \mathcal{F} u \in L_p(\mathbb{R}^n)\}; \|\cdot\|_{H_p^s(\mathbb{R}^n)}),$$

where $\Lambda_1(\xi) := (1 + |\xi|^2)^{1/2}$ for $\xi \in \mathbb{R}^n$ and $\|\cdot\|_{H_p^s(\mathbb{R}^n)} := |\mathcal{F}^{-1}\Lambda_1^s \mathcal{F} \cdot|_p$.

Observe that, $H_p^0(\mathbb{R}^n) = L_p(\mathbb{R}^n)$. In general, it is well-known that the Bessel potential spaces with integer exponents coincide with the Sobolev spaces, i.e.,

$$H_p^k(\mathbb{R}^n) = W_p^k(\mathbb{R}^n) \text{ for } k \in \mathbb{Z}. \tag{2.15}$$

However, let us mention that $B_p^0(\mathbb{R}^n) = L_p(\mathbb{R}^n)$ iff $p = 2$, cf. [32, Theorem 2.1.2].

Also, the Bessel potential spaces are stable under complex interpolation:

$$[H_p^{s_0}(\mathbb{R}^n); H_p^{s_1}(\mathbb{R}^n)]_\theta = H_p^{(1-\theta)s_0+\theta s_1}(\mathbb{R}^n) \text{ for } \theta \in (0,1), \ s_0, \ s_1 \in \mathbb{R}. \tag{2.16}$$

In contrast to this, we obtain the Besov spaces as real interpolation spaces of Bessel potential spaces. More precisely, let p, $q \in (1, \infty)$ and s_0, $s_1 \in \mathbb{R}$ be given and let $(\cdot\,;\,\cdot)_{\theta,q}$ denote the real interpolation functor. Then, given $\theta \in (0,1)$, we have

$$(H_p^{s_0}(\mathbb{R}^n); H_p^{s_1}(\mathbb{R}^n))_{\theta,q} = B_{pq}^{(1-\theta)s_0+\theta s_1}(\mathbb{R}^n) \text{ if } s_0 \neq s_1. \tag{2.17}$$

Our next step is to introduce the so-called "local" spaces. To this end, let U be an open subset of $\mathbb{R}^n$ and let r_U denote the restriction map with respect to U, i.e., $r_U u := u|U$ for $u \in L_1(\mathbb{R}^n)$. Given $s \geq 0$, we define $H_p^s(U)$ and $B_p^s(U)$ to be the images under r_U of $H_p^s(\mathbb{R}^n)$ and $B_p^s(\mathbb{R}^n)$ in $L_1(U)$, respectively. Observe that these spaces are well-defined, since $H_p^s(\mathbb{R}^n)$ and $B_p^s(\mathbb{R}^n)$ are subspaces of $L_1(\mathbb{R}^n)$ for $s \geq 0$, cf. (2.4) and (2.15). We equip these spaces with the corresponding quotient topologies. Moreover, let $\mathring{H}_p^s(U)$ and $\mathring{B}_p^s(U)$ denote the closure of the test-functions $\mathcal{D}(U)$ in $H_p^s(U)$ and $B_p^s(U)$, respectively. Finally, we set

$$H_p^{-s}(U) := [\mathring{H}_{p'}^s(U)]' \text{ and } B_p^{-s}(U) := [\mathring{B}_{p'}^s(U)]' \text{ for } p \in (1, \infty), \ s \geq 0. \tag{2.18}$$

Recall that we have assumed that Ω is a bounded domain in $\mathbb{R}^n$ having a smooth boundary Γ.

Thus, there exists a smooth atlas for $\overline{\Omega}$. That is, given $r > 0$, there are an integer m_r, open subsets U_j of $\mathbb{R}^n$, and smooth diffeomorphisms $\varphi_j \in \mathrm{Diff}^\infty(\mathbb{B}_n(0,r), U_j)$, $1 \leq j \leq m_r$, such that

$$\begin{aligned} &\overline{\Omega} \subset \bigcup_{j=1}^{m_r} U_j, \ \varphi_j(\mathbb{B}_n(0,r) \cap \mathbb{H}^n) = U_j \cap \Omega \text{ if } U_j \cap \Gamma = \emptyset, \\ &\text{and } \varphi_j(\mathbb{B}_n(0,r) \cap \partial\mathbb{H}^n) = U_j \cap \Gamma \text{ if } U_j \cap \Gamma \neq \emptyset, \ 1 \leq j \leq m_r. \end{aligned} \tag{2.19}$$

In particular, it follows from [33, Theorem 3.3.4] that the operators $r_{U_j} \in \mathcal{L}(B_p^s(\mathbb{R}^n), B_p^s(U_j))$ and $r_{\mathbb{B}_n(0,r)} \in \mathcal{L}(B_p^s(\mathbb{R}^n), B_p^s(\mathbb{B}_n(0,r))$ are retractions. Therefore, Theorem 3.3.6 in [33] implies that

(2.20) the interpolation properties (2.9), (2.16) and (2.17) remain true if we replace $\mathbb{R}^n$ by U_j or $\mathbb{B}_n(0,r)$.

Given $u \in \mathcal{D}'(U_j)$ and $v \in \mathcal{D}'(\mathbb{B}_n(0,r))$ we define the following pullback and pushforward operators

$$\varphi_j^* u := u \circ \varphi_j \text{ and } \varphi_{j*} v := v \circ \varphi_j^{-1}. \tag{2.21}$$

Then we have

$$\varphi_j^* \in \operatorname{Isom}(H_p^k(U_j), H_p^k(\mathbb{B}_n(0,r)) \text{ and } [\varphi_j^*]^{-1} = \varphi_{j*}, \tag{2.22}$$

for $k \in \mathbb{Z}$ and $1 \leq j \leq m_r$. In fact, this follows immediately from the transformation theorem for the Lebesgue integral, the chain rule, and (2.15) if $k \in \mathbb{N}$. If $-k \in \mathbb{N}$ we use again a duality argument.

By interpolation, cf. (2.16) and (2.20), it follows from (2.22) that

$$\varphi_j^* \in \operatorname{Isom}(H_p^s(U_j), H_p^s(\mathbb{B}_n(0,r)) \text{ and } [\varphi_j^*]^{-1} = \varphi_{j*}, \tag{2.23}$$

for $s \in \mathbb{R}$ and $1 \leq j \leq m_r$.

Finally, let $(\tilde{U}_j, \tilde{\varphi}_j^{-1})_{1 \leq j \leq m_r}$ denote the smooth atlas for the submanifold Γ induced by the atlas $(U_j, \varphi_j^{-1})_{1 \leq j \leq m_r}$ of Ω, i.e., $\tilde{U}_j := U_j \cap \Gamma$ and $\tilde{\varphi}_j(x') := \varphi_j(x', 0)$ with $x = (x', x_n) \in \mathbb{H}^n$. Then we define

$$B_p^s(\tilde{U}_j) := (\{z \in \mathcal{D}'(\Gamma); \tilde{\varphi}_j^* z \in B_p^s(\mathbb{B}_{n-1}(0,r))\}, \|\cdot\|_{B_p^s(\tilde{U}_j)}),$$

where $\|z\|_{B_p^s(\tilde{U}_j)} := \|\tilde{\varphi}_j^* z\|_{B_p^s(\mathbb{B}_{n-1}(0,r))}$. As an immediate consequence of this definition we note that

$$\tilde{\varphi}_j^* \in \operatorname{Isom}(B_p^s(\tilde{U}_j), B_p^s(\mathbb{B}_{n-1}(0,r)) \text{ and } [\tilde{\varphi}_j^*]^{-1} = \tilde{\varphi}_{j*} \text{ for } 1 \leq j \leq m_r. \tag{2.24}$$

Let $\{\theta_j\}_{1 \leq j \leq m_r}$ be a smooth partition of unity on $\overline{\Omega}$ subordinate to the open covering $\{U_j\}_{1 \leq j \leq m_r}$.

Given $s \in \mathbb{R}$ and $p \in (1, \infty)$, we set

$$B_p^s(\Gamma) := (\{z \in \mathcal{D}'(\Gamma); \tilde{\varphi}_j^*(\theta_j z) \in B_p^s(\tilde{U}_j)\}, \|\cdot\|_{s,p}),$$

where

$$\|z\|_{s,p} = \sum_{j=1}^{m_r} |\tilde{\varphi}_j^*(\theta_j z)|_{s,p} \text{ for } z \in B_p^s(\Gamma).$$

The following Lemma collects some of the basic properties of these spaces. A proof follows from the corresponding results in $\mathbb{R}^{n-1}$ by a well known localization procedure, cf. [17, 33].

LEMMA 2.2. *Let $p \in (1, \infty)$ and $s,\ t,\ r \in \mathbb{R}$ with $s \leq t,\ r > 0$ be given. Then the following assertions hold:*

a) *$B_p^s(\Gamma)$ is a well-defined (i.e., independent of the choice of the atlas), reflexive, and separable Banach space.*

b) $\mathcal{D}(\Gamma) \overset{d}{\hookrightarrow} B_p^t(\Gamma) \overset{d}{\hookrightarrow} B_p^s(\Gamma) \overset{d}{\hookrightarrow} \mathcal{D}'(\Gamma)$.

c) $B_p^r(\Gamma) \overset{d}{\hookrightarrow} L_p(\Gamma)$.

d) $B_p^s(\Gamma) \overset{d}{\hookrightarrow} C^r(\Gamma)$, *if* $s - \dfrac{n-1}{p} > r$.

e) $[B_p^s(\Gamma)]' = B_{p'}^{-s}(\Gamma)$, *according to the duality pairing induced by the identification of $[L_p(\Gamma)]'$ with $L_{p'}(\Gamma)$.*

f) $[B_p^s(\Gamma), B_p^t(\Gamma)]_\theta = B_p^{s(1-\theta)+t\theta}(\Gamma)$ *for* $\theta \in (0,1)$.

g) $\gamma \in \mathcal{L}(H_p^s(\Omega), B_p^{s-1/p}(\Gamma))$ *if* $s > 1/p$.

Moreover, all the preceding embeddings are compact.

3. - Pseudo-differential operators with constant symbols

It is a powerful tool to localize and to transform differential operators with variable coefficients on a bounded domain to a problem with constant coefficients on a half space. This procedure leads in a natural way to pseudo-differential operators with constant symbols. The main concern of this section is to study these pseudo-differential operators according to problem (E) by means of the Mihlin-Hörmander multiplier theorem.

Throughout this section we assume that

$$a_{jk}^\pi \in \mathbb{R},\ 1 \leq j,\ k \leq n \text{ such that } a_{jk}^\pi \xi^j \xi^k > 0 \text{ for } \xi \in \mathbb{R}^n \backslash \{0\}. \tag{3.1}$$

Given $u \in H_p^2(\mathbb{H}^n)$, where $\mathbb{H}^n := \{x = (x', x_n) \in \mathbb{R}^n; x_n > 0\}$, we set

$$\mathcal{A}'_\pi u := -a_{jk}^\pi \partial_j \partial_k u, \qquad \mathcal{B}'_\pi u := a_{nk}^\pi \gamma \partial_k. \tag{3.2}$$

It is well known that

$$(\lambda + \mathcal{A}'_\pi, \gamma) \in \operatorname{Isom}(H_p^2(\mathbb{H}^n), L_p(\mathbb{H}^n) \times B_p^{2-1/p}(\mathbb{R}^{n-1})) \tag{3.3}$$

for $\lambda \in [\operatorname{Re} z \geq 0] \backslash \{0\}$.

Without restriction we may assume that $a_{nn}^{\pi} = 1$. Moreover, given $\xi \in \mathbb{R}^{n-1}$ and $\alpha \in \mathbb{C}$, we define

$$b(\alpha, \xi) := [\alpha^2 + a_1(\xi) - a_2^2(\xi)]^{1/2}, \tag{3.4}$$

where

$$a_1(\xi) := \sum_{j,k=1}^{n-1} a_{jk}^{\pi} \xi^j \xi^k \text{ and } a_2(\xi) := \sum_{j=1}^{n-1} a_{jn}^{\pi} \xi^j.$$

The function $b(\alpha, \cdot)$ serves as the symbol of the following pseudo-differential operator

$$\boldsymbol{B}_{\pi}(\alpha)z := \mathcal{F}^{-1} b(\alpha, \cdot)\mathcal{F} z \text{ for } z \in \mathcal{S}'(\mathbb{R}^{n-1}).$$

Roughly speaking, the symbol $b(\alpha, \cdot)$ arises by applying (partial) Fourier transform to solve certain boundary value problems on the half space (cf. the proof of Lemma 3.6, where also a characterization of $\boldsymbol{B}_{\pi}(\alpha)$ will be given).

Observe that, due to (3.1), the function $b(\alpha, \cdot)$ is a multiplier on $\mathcal{S}(\mathbb{R}^{n-1})$ for each $\alpha \in \mathbb{C}$ with $\alpha^2 \in [\operatorname{Re} \lambda \geq 0]\backslash\{0\}$, i.e., $b(\alpha, \cdot)$ belongs to $\mathcal{O}_M$, (cf. [23]). Hence, we know that

$$\boldsymbol{B}_{\pi}(\alpha) \in \mathcal{L}(\mathcal{S}(\mathbb{R}^{n-1})) \cap \mathcal{L}(\mathcal{S}'(\mathbb{R}^{n-1})).$$

However, we need more detailed information about the multiplier properties of $b(\alpha, \cdot)$ for our purposes. To this end, we introduce the following Banach space

$$\begin{aligned} \mathcal{M} :=(\{a \in C^{[n/2]+1}(\mathbb{R}^{n-1});\, &|\xi|^{|\beta|}|\partial^{\beta} a(\xi)| < \infty, \\ &\xi \in \mathbb{R}^{n-1}, |\beta| \leq [n/2]+1\}; \|\cdot\|_{\mathcal{M}}), \end{aligned}$$

where $\|a\|_{\mathcal{M}} := \sup\{|\xi|^{|\beta|}|\partial^{\beta} a(\xi)|; \xi \in \mathbb{R}^{n-1}, |\beta| \leq [n/2]+1\}$. It follows Leibniz' rule that $\mathcal{M}$ is a continuous multiplication algebra. However, the fundamental property of $\mathcal{M}$ is the fact that it is a subspace of the space of all multipliers on $L_p(\mathbb{R}^{n-1})$. Indeed, it follows from Mihlin-Hörmanders multiplier theorem (cf. [26, Theorem 4.3.2]) that

$$\mathcal{F}^{-1} a \mathcal{F} \in \mathcal{L}(L_p(\mathbb{R}^{n-1})) \text{ for } a \in \mathcal{M} \text{ and } 1 < p < \infty. \tag{3.6}$$

Moreover, the elements of $\mathcal{M}$ are also multipliers for the Besov and the Bessel potential spaces. In fact, by the definition of $H_p^s(\mathbb{R}^{n-1})$ we have

$$\mathcal{F}^{-1} \Lambda_1^s \mathcal{F} \in \operatorname{Isom}(H_p^{s+t}(\mathbb{R}^{n-1}), H_p^t(\mathbb{R}^{n-1})) \text{ for } 1 < p < \infty. \tag{3.7}$$

Thus by interpolation it follows from (3.6), (3.7) (with $t = 0$), (2.16), and (2.17) that

$$\mathcal{F}^{-1} a \mathcal{F} \in \mathcal{L}(H_p^s(\mathbb{R}^{n-1})) \cap \mathcal{L}(B_p^s(\mathbb{R}^{n-1})) \text{ for } a \in \mathcal{M},\ s \in \mathbb{R} \text{ and } 1 < p < \infty. \tag{3.8}$$

LEMMA 3.1. *Given $\alpha^2 \in [\operatorname{Re}\lambda \geq 0]\backslash\{0\}$, we have*

$$\Lambda_1^{-1} b(\alpha, \cdot), \ \Lambda_1[\mu + b(\alpha, \cdot)]^{-1} \in \mathcal{M},$$

and there is a positive constant c_α such that

$$\|\Lambda_1^{-1} b(\alpha, \cdot)\|_{\mathcal{M}} + \|\Lambda_1[\mu + b(\alpha, \cdot)]^{-1}\|_{\mathcal{M}} \leq c_\alpha \text{ for } \mu \in [\operatorname{Re} z > 0].$$

PROOF. a) We first introduce some notation. Let $\Lambda(\alpha, \xi) := (|\alpha|^2 + |\xi|^2)^{1/2}$ and $(\eta^*, \xi^*) := (|\eta|^2 + |\xi|^2)^{-1/2}(\eta, \xi)$ for $\alpha \in \mathbb{C}$, $\eta \in \mathbb{C}^k$ and $\xi \in \mathbb{R}^{n-1}$. Observe that $\Lambda_1 = \Lambda(1, \cdot)$ and that, given $\alpha^2 \in [\operatorname{Re}\lambda \geq 0]\backslash\{0\}$, there is a constant c_α such that

$$\begin{aligned} &\Lambda_1^{-1}\Lambda(\alpha, \cdot), \ \Lambda_1\Lambda^{-1}(\alpha, \cdot) \in \mathcal{M} \\ &\text{and } \|\Lambda_1^{-1}\Lambda(\alpha, \cdot)\|_{\mathcal{M}} + \|\Lambda_1\Lambda^{-1}(\alpha, \cdot)\|_{\mathcal{M}} \leq c_\alpha. \end{aligned} \tag{3.9}$$

Furthermore, we note that

$$\Lambda^{-1}(\alpha, \xi) b(\alpha, \xi) = b(\alpha^*, \xi^*) \text{ for } \alpha^2 \in [\operatorname{Re}\lambda \geq 0]\backslash\{0\}, \ \xi \in \mathbb{R}^{n-1}.$$

Differentiating this identity with respect to ξ we find that $\Lambda^{-1}(\alpha, \cdot) b(\alpha, \cdot) \in \mathcal{M}$ and consequently, (3.9) yields

$$\Lambda_1^{-1} b(\alpha, \cdot) \in \mathcal{M} \text{ with } \|\Lambda_1^{-1} b(\alpha, \cdot)\|_{\mathcal{M}}, \ \alpha^2 \in [\operatorname{Re}\lambda \geq 0]\backslash\{0\}.$$

b) Again, by homogeneity we conclude that

$$\partial_\xi^\beta \Lambda(\alpha, \xi) = \Lambda^{1-|\beta|}(\alpha, \xi) \partial_\xi^\beta \Lambda(\alpha^*, \xi^*).$$

Thus, there is a constant $c_\alpha > 0$ such that

$$|\partial_\xi^\beta \Lambda(\alpha, \xi)| \leq c_\alpha |\xi|^{1-|\beta|} \text{ for } \xi \in \mathbb{R}^{n-1}, \ \beta \in \mathbb{N}^{n-1}.$$

On the other hand, we have

$$\lambda[\mu + b(\alpha, \xi)] = [\lambda\mu + b(\lambda\alpha, \lambda\xi)] \text{ for } \lambda > 0.$$

Therefore, given $\gamma \in \mathbb{N}^{n-1}$, it follows that

$$\partial_\xi^\gamma [\mu + b(\alpha, \xi)]^{-1} = \lambda^{1+|\gamma|} \partial_z^\gamma [\lambda\mu + b(\lambda\alpha, z)]^{-1}|_{z=\lambda\xi}$$

and consequently, setting $\lambda := (|\alpha|^2 + |\mu|^2 + |\xi|^2)^{-1/2}$, we find a positive constant c such that

$$\begin{aligned} &|\partial_\xi^\gamma [\mu + b(\alpha, \xi)]^{-1}| \leq c(|\mu| + |\xi|)^{-1-|\gamma|} \\ &\text{for } \xi \in \mathbb{R}^{n-1}, \ \gamma \in \mathbb{N}^{n-1}, \ \mu \in [\operatorname{Re}\lambda > 0]. \end{aligned} \tag{3.10}$$

Leibniz' rule now implies that

$$|\xi^\beta \partial_\xi^\beta(\Lambda(\alpha,\xi)[\mu+b(\alpha,\xi)]^{-1})| = \left|\xi^\beta \sum_{\gamma\leq\beta}\binom{\beta}{\gamma}\partial_\xi^\gamma\Lambda(\alpha,\xi)\partial_\xi^{\beta-\gamma}[\mu+b(\alpha,\xi)]^{-1}\right|$$
$$\leq c|\xi|^{|\beta|}\sum_{\gamma\leq\beta}|\xi|^{1-|\gamma|}|\xi|^{-1-|\beta|+|\gamma|}\leq c,$$

for $\mu \in [\operatorname{Re} z > 0]$, $\xi \in \mathbb{R}^{n-1}$ and $\beta \in \mathbb{N}^{n-1}$. This means that $\Lambda(\alpha,\cdot)[\mu + b(\alpha,\cdot)]^{-1} \in \mathcal{M}$ and that

$$\|\Lambda(\alpha,\cdot)[\mu+b((\alpha,\cdot)]^{-1}\|_{\mathcal{M}} \leq c_\alpha \text{ for } \mu\in[\operatorname{Re}\lambda>0].$$

Finally, we apply again (3.9) to complete the proof. □

COROLLARY 3.2. *Given $s \in \mathbb{R}$ and $\alpha^2 \in [\operatorname{Re} z \geq 0]\backslash\{0\}$, there exists a positive constant c such that for all $\mu \in [\operatorname{Re} z > 0]$ and $z \in B_p^s(\mathbb{R}^{n-1})$:*

a) $\mu + \boldsymbol{B}_\pi(\alpha) \in \operatorname{Isom}(B_p^s(\mathbb{R}^{n-1}), B_p^{s-1}(\mathbb{R}^{n-1}))$;

b) $|z|_{s,p} + |\mu|\cdot|z|_{s-1,p} \leq c\|(\mu+\boldsymbol{B}_\pi(\alpha))z\|_{s-1,p}$.

PROOF. a) Observe that

$$\mathcal{F}^{-1}\Lambda_1^s\mathcal{F} \in \operatorname{Isom}(B_p^{s+t}(\mathbb{R}^{n-1}), B_p^t(\mathbb{R}^{n-1})) \text{ for } s,\ t\in\mathbb{R},\ 1<p<\infty. \tag{3.11}$$

This follows from (3.7) and (2.17) by interpolation. The first assertion is now an immediate consequence of (3.11), (3.8) and Lemma 3.1.

b) Since $\mathcal{M}$ is a multiplication algebra, Lemma 3.1 yields

$$b(\alpha,\cdot)[\mu+b(\alpha,\cdot)]^{-1}\in\mathcal{M} \text{ and } \|b(\alpha,\cdot)[\mu+b(\alpha,\cdot)]^{-1}\|_{\mathcal{M}}\leq c_\alpha$$

for all $\mu \in [\operatorname{Re} z > 0]$. Now, observe that

$$\mu[\mu+b(\alpha,\cdot)]^{-1} = 1 - b(\alpha,\cdot)[\mu+b(\alpha,\cdot)]^{-1}.$$

Thus the assertion follows from (3.8). □

Finally, let $\alpha^2 \in [\operatorname{Re} z \geq 0]\backslash\{0\}$ be fixed. We introduce the following symbol:

$$b'(\mu,\xi) := \frac{b(\alpha|\mu|,\xi)+\mu}{b(\alpha,\xi)+\mu} \text{ for } \xi\in\mathbb{R}^{n-1} \text{ and } \mu\in[\operatorname{Re} z\geq 0].$$

Then we have:

LEMMA 3.3. *There exists a constant $c > 0$ such that*

$$b'(\mu,\cdot)\in\mathcal{M} \textit{ and } \|b'(\mu,\cdot)\|_{\mathcal{M}}\leq c$$

for all $\mu \in [\operatorname{Re}\lambda > 0]$.

PROOF. Note that $\lambda\mu + b(\lambda|\mu|\alpha, \lambda\xi) = \lambda[\mu + b(\alpha|\mu|, \xi)]$ for $\lambda > 0$. Thus, letting $\lambda := (|\mu|^2 + |\xi|^2)^{1/2}$, we find that

$$\begin{aligned} &|\partial_\xi^\gamma[b(\alpha|\mu|, \xi) + \mu]| \leq c(|\xi|^2 + |\mu|^2)^{\frac{1-|\gamma|}{2}} \\ &\text{for } \mu \in [\operatorname{Re} z \geq 0],\ \xi \in \mathbb{R}^{n-1},\ \gamma \in \mathbb{N}^{n-1}. \end{aligned} \tag{3.12}$$

Now, by an argument similar to that given in the proof of Lemma 3.1, the assertion follows from (3.12), (3.10) and Leibniz' rule again. □

COROLLARY 3.4. *Given* $\alpha^2 \in [\operatorname{Re}\lambda \geq 0]\backslash\{0\}$, *there exists a positive constant* c *such that*

$$\left|\left(\frac{\mu}{|\mu|} + \boldsymbol{B}_\pi(\alpha)\right)\sigma_{1/|\mu|}z\right|_{-1/p,p} \leq c|\mu|^{\frac{n}{p}-1}|(\mu + \boldsymbol{B}_\pi(\alpha))z|_{-1/p,p}$$

for all $\mu \in [\operatorname{Re}\lambda \geq 1]$ *and* $z \in B_p^{1-1/p}(\mathbb{R}^{n-1})$.

PROOF. Pick $\mu \in [\operatorname{Re} z \geq 1]$. Then $\rho := 1/|\mu| \in (0, 1]$. Furthermore let $\mathcal{F}$ denote the Fourier transform on $\mathbb{R}^{n-1}$. Note that

$$\mathcal{F}\sigma_\rho = \rho^{1-n}\sigma_{\rho^{-1}}\mathcal{F},$$

which follows again from basic transformation properties of the Lebesgue integral. Thus we find

$$\begin{aligned} (\rho\mu + \boldsymbol{B}_\pi(\alpha))\sigma_\rho z &= \rho\mu\sigma_\rho z + \mathcal{F}^{-1}b(\alpha, \cdot)\mathcal{F}(\sigma_\rho z) \\ &= \rho\mu\sigma_\rho z + \rho^{1-n}\mathcal{F}^{-1}b(\alpha, \cdot)\sigma_{\rho^{-1}}\mathcal{F}z \\ &= \rho\mu\sigma_\rho z + \rho^{1-n}\mathcal{F}^{-1}\rho b\left(\frac{\alpha}{\rho}, \frac{\cdot}{\rho}\right)\sigma_{\rho^{-1}}\mathcal{F}z \\ &= \rho\mu\sigma_\rho z + \rho^{2-n}\mathcal{F}^{-1}\sigma_{\rho^{-1}}\left(b\left(\frac{\alpha}{\rho}, \cdot\right)\mathcal{F}z\right) \\ &= \rho\left\{\mu\sigma_\rho z + \sigma_\rho\left(\mathcal{F}^{-1}b\left(\frac{\alpha}{\rho}, \cdot\right)\mathcal{F}z\right)\right\} \\ &= \rho\sigma_\rho\left[\left(\mu + \boldsymbol{B}_\pi\left(\frac{\alpha}{\rho}\right)\right)z\right] \\ &= \rho\sigma_\rho\left[\left(\mu + \boldsymbol{B}_\pi\left(\frac{\alpha}{\rho}\right)\right)(\mu + \boldsymbol{B}_\pi(\alpha))^{-1}(\mu + \boldsymbol{B}_\pi(\alpha))\right]z. \end{aligned}$$

Applying Lemma 2.1, we obtain

$$
\begin{aligned}
&|(\rho\mu + \boldsymbol{B}_\pi(\alpha))\sigma_\rho z|_{-1/p,p} \\
(3.13) \qquad &\leq c\rho^{1-\frac{n}{p}} \left| \left(\mu + \boldsymbol{B}_\pi\left(\frac{\alpha}{\rho}\right)\right)(\mu + \boldsymbol{B}_\pi(\alpha))^{-1}(\mu + \boldsymbol{B}_\pi(\alpha))z \right|_{-1/p,p}.
\end{aligned}
$$

On the other hand, observe that $\left(\mu + \boldsymbol{B}_\pi\left(\frac{\alpha}{\rho}\right)\right)(\mu + \boldsymbol{B}_\pi(\alpha))^{-1} = \mathcal{F} b'(\mu, \cdot)\mathcal{F}$. Hence, Lemma 3.3 yields

$$
\left\| \left(\mu + \boldsymbol{B}_\pi\left(\frac{\alpha}{\rho}\right)\right)(\mu + \boldsymbol{B}_\pi(\alpha))^{-1} \right\|_{\mathcal{L}(B_p^{-1/p}(\mathbb{R}^{n-1}))} \leq c
$$

for all $\mu \in [\operatorname{Re}\lambda \geq 1]$. So (3.13) implies the assertion. □

COROLLARY 3.5. *Assume that $p > n$ and that $\alpha^2 \in [\operatorname{Re}\lambda \geq 0]\backslash\{0\}$. Then there is a positive constant c such that*

$$
|\mu|^{\frac{n}{p}}|z|_{1-1/p,p} + |\mu| \cdot \|z\|_{BC(\mathbb{R}^{n-1})} \leq c|\mu|^{\frac{n}{p}}|(\mu + \boldsymbol{B}_\pi(\alpha))z|_{-1/p,p}
$$

for all $\mu \in [\operatorname{Re}\lambda \geq 1]$ and $z \in B_p^{1-1/p}(\mathbb{R}^{n-1})$.

PROOF. Pick $\mu \in [\operatorname{Re}\lambda \geq 1]$ and set again $\rho := |\mu|^{-1} \in (0,1]$. From Corollary 3.2b) we know that there is a $c > 0$ such that

$$
|\sigma_\rho z|_{1-1/p,p} \leq c|(\mu\rho + \boldsymbol{B}_\pi(\alpha))\sigma_\rho z|_{-1/p,p}.
$$

On the other hand, we have assumed that $p > n$. Thus Sobolev's embedding theorem implies that

$$
B_p^{1-1/p}(\mathbb{R}^{n-1}) \hookrightarrow C_0(\mathbb{R}^{n-1}).
$$

Hence

$$
\|z\|_{BC(\mathbb{R}^{n-1})} = \|\sigma_\rho z\|_{BC(\mathbb{R}^{n-1})} \leq c|(\mu\rho + \boldsymbol{B}_\pi(\alpha))\sigma_\rho z|_{-1/p,p}.
$$

From Corollary 3.4 we now obtain

$$
|\mu| \cdot \|z\|_{BC(\mathbb{R}^{n-1})} \leq c|\mu|^{\frac{n}{p}}|(\mu + \boldsymbol{B}_\pi(\alpha))z|_{-1/p,p}.
$$

Finally, Corollary 3.2b) implies that

$$
|\mu|^{\frac{n}{p}}|z|_{1-1/p,p} \leq c|\mu|^{\frac{n}{p}}|(\mu + \boldsymbol{B}_\pi(\alpha))z|_{-1/p,p},
$$

which completes the proof. □

In the following we collect some of the basic properties of the so-called Lions-Magenes extension of $(\mathcal{A}'_\pi, \gamma)$. We refer to [4, 16] for a proof of the assertions below. First of all, the operator $\mathcal{A}'_\pi$ is closable in $L_p(\mathbb{H}^n)$. Let $\overline{\mathcal{A}'_\pi}$

denote its closure and let $\mathcal{D}_{\pi,0} := \operatorname{dom}(\overline{\mathcal{A}}'_\pi)$ be the domain of this closure. Then there is a unique extension $(\overline{\mathcal{B}}'_\pi, \overline{\gamma}'_\pi) \in \mathcal{L}(\mathcal{D}_{\pi,0}, B_p^{-1-1/p}(\mathbb{R}^{n-1}) \times B_p^{-1/p}(\mathbb{R}^{n-1}))$ of $(\mathcal{B}'_\pi, \gamma)$ to $\mathcal{D}_{\pi,0}$ such that Green's formula holds. In addition, it can be shown that

$$(\lambda + \overline{\mathcal{A}}'_\pi, \overline{\gamma}'_\pi) \in \operatorname{Isom}(\mathcal{D}_{\pi,0}, L_p(\mathbb{H}^n) \times B_p^{-1/p}(\mathbb{R}^{n-1})) \text{ for } \lambda \in [\operatorname{Re} z \geq 0] \backslash \{0\}.$$

Now let $\mathcal{D}_{\pi,\theta} := [\mathcal{D}_{\pi,0}, H_p^2(\mathbb{H}^n)]_\theta$, $\theta \in (0,1)$. Then, using (3.3), it follows by interpolation that

$$(\lambda + \overline{\mathcal{A}}'_\pi, \overline{\gamma}'_\pi) \in \operatorname{Isom}(\mathcal{D}_{\pi,\theta}, L_p(\mathbb{H}^n) \times B_p^{2\theta-1/p}(\mathbb{R}^{n-1})), \quad \theta \in [0,1), \tag{3.14}$$

for $\lambda \in [\operatorname{Re} z \geq 0] \backslash \{0\}$. Thus we may define

$$\overline{\mathcal{T}}'_\pi(\alpha) := (\alpha^2 + \overline{\mathcal{A}}'_\pi, \overline{\gamma}'_\pi)^{-1} | \{0\} \times B_p^{2\theta-1/p}(\mathbb{R}^{n-1}) \tag{3.15}$$

for $\alpha \in \mathbb{C}$ such that $\alpha^2 \in [\operatorname{Re} \lambda \geq 0] \backslash \{0\}$. Observe that $\overline{\mathcal{T}}'_\pi(\alpha) \in \mathcal{L}(B_p^{2\theta-1/p}(\mathbb{R}^{n-1}), \mathcal{D}_{\pi,2\theta})$ and $\mathcal{D}_{\pi,2\theta} \hookrightarrow H_p^{2\theta}(\mathbb{H}^n)$ imply that

$$\overline{\mathcal{T}}'_\pi(\alpha) \in \mathcal{L}(B_p^{2\theta-1/p}(\mathbb{R}^{n-1}), H_p^{2\theta}(\mathbb{H}^n)), \quad \theta \in [0,1), \tag{3.16}$$

for $\alpha^2 \in [\operatorname{Re} \lambda \geq 0] \backslash \{0\}$. Moreover, since $(\alpha^2 + \overline{\mathcal{A}}'_\pi, \overline{\gamma}'_\pi)$ is an extension of $(\alpha^2 + \mathcal{A}'_\pi, \gamma)$, we find that

$$\overline{\mathcal{T}}'_\pi(\alpha) z = \mathcal{T}'_\pi(\alpha) z \in H_p^2(\mathbb{H}^n) \text{ for } z \in B_p^{2-1/p}(\mathbb{R}^{n-1}), \tag{3.17}$$

where $\mathcal{T}'_\pi(\alpha) := (\alpha^2 + \mathcal{A}'_\pi, \gamma'_\pi)^{-1} | \{0\} \times B_p^{2-1/p}(\mathbb{R}^{n-1})$. The following Lemma gives a characterization of the pseudo-differential operators $\boldsymbol{B}_\pi(\alpha)$ using these Lions-Magenes extensions.

LEMMA 3.6. *Given $\alpha^2 \in [\operatorname{Re} \lambda \geq 0] \backslash \{0\}$ and $z \in B_p^{1-1/p}(\mathbb{R}^{n-1})$, we have*

$$\boldsymbol{B}_\pi(\alpha) z = \overline{\mathcal{B}}'_\pi \overline{\mathcal{T}}'_\pi(\alpha) z.$$

PROOF. Recall that

$$a_1(\xi) = \sum_{j,k=1}^{n-1} a_{jk}^\pi \xi^j \xi^k, \quad a_2(\xi) = \sum_{j=1}^{n-1} a_{jn}^\pi \xi^j \text{ and } a_{nn}^\pi = 1.$$

We now define

$$p_{\xi,\alpha}(z) := \alpha^2 + a_1(\xi) - 2i a_2(\xi) z - z^2 \text{ for } z \in \mathbb{C}.$$

Observe that the polynomial $p_{\xi,\alpha}$ has no zeros on $i\mathbb{R}$, since $\sum_{j,k=1}^{n} a_{jk}^\pi \eta^j \eta^k \geq 0$ for $\eta \in \mathbb{R}^n$ and by the choice of α. Moreover, we have $p_{\xi,\alpha}(-z) = p_{-\xi,\alpha}(z)$.

Consequently, given $\xi \in \mathbb{R}^{n-1}$, the polynomial $p_{\xi,\alpha}$ admits a unique root in the left half-plane, which is given by

$$\lambda(\xi,\alpha) = -ia_2(\xi) - [\alpha^2 + a_1(\xi) - a_2^2(\xi)]^{1/2}, \quad \xi \in \mathbb{R}^{n-1}.$$

Denoting by $\mathcal{F}$ the Fourier tranform in $\mathbb{R}^{n-1}$, we now set

$$u(x', x_n) := [\mathcal{F}^{-1} e^{\lambda(\cdot,\alpha)x_n} \mathcal{F} z](x') \text{ for } x' \in \mathbb{R}^{n-1}, \ x_n \geq 0.$$

Since $\operatorname{Re}\lambda(\xi,\alpha) < 0$ for $\xi \in \mathbb{R}^{n-1}$, it follows from the Mihlin-Hörmander multiplier theorem that $u \in H_p^2(\mathbb{H}^n)$. Moreover, recalling that $p_{\xi,\alpha}(\lambda(\xi,\alpha)) = 0$, we find

$$(\alpha^2 + \mathcal{A}'_\pi)u = (\alpha^2 - a^\pi_{jk}\partial_j\partial_k)u = 0 \text{ in } \mathbb{H}^n, \quad \gamma u = z \text{ on } \mathbb{R}^{n-1}.$$

In other words we have

$$[\mathcal{T}'_\pi(\alpha)z](x', x_n) = [\mathcal{F}^{-1} e^{\lambda(\xi,\alpha)x_n} \mathcal{F} z](x').$$

Now using the definition of $\mathcal{B}'_\pi$ and $b(\alpha, \cdot)$, it follows that

$$\mathcal{B}'_\pi \mathcal{T}'_\pi(\alpha) z = \mathcal{F}^{-1} b(\alpha, \cdot) \mathcal{F} z \text{ for } z \in B_p^{2-1/p}(\mathbb{R}^{n-1}).$$

Finally, observe that $\overline{\mathcal{B}}'_\pi \overline{\mathcal{T}}'_\pi(\alpha) z = \mathcal{B}'_\pi \mathcal{T}'_\pi(\alpha) z$ for $z \in B_p^{2-1/p}(\mathbb{R}^{n-1})$. Thus the density of $B_p^{1-1/p}(\mathbb{R}^{n-1})$ in $B_p^{2-1/p}(\mathbb{R}^{n-1})$ and Lemma 3.2a) complete the proof. □

It should be mentioned that the operator $\mathcal{B}'_\pi \mathcal{T}'_\pi(\alpha)$ was introduced by Hintermann in [12] and that Corollary 3.2 is a special case of [12, Lemma 1.8]. However, due to our simpler situation (a single second order equation), we have been able to give a much more transparent proof of Corollary 3.2 than the one in [12, Lemma 1.8].

4. - Scales of analytic semigroups

The main concern of this section is to construct two families of generators of strongly continuous analytic semigroups on the Besov spaces $B_q^s(\Gamma)$. Moreover, we establish some duality properties of these scales.

Given $u \in H_p^2(\Omega)$, we set

$$\mathcal{A}u := -\partial_j(a_{jk}\partial_k u) + a_j\partial_j u + a_0 u \text{ and } \mathcal{B}u := a_{jk}\nu^j \gamma \partial_k u + b_0 \gamma u.$$

We assume that the coefficients of $(\mathcal{A}, \mathcal{B})$ satisfy the regularity assumption (1.2) and the ellipticity condition (1.3). It is well known (cf. [3, 4]) that there exists a constant $\lambda_0 \in \mathbb{R}$ such that

$$
\begin{aligned}
&(\lambda + \mathcal{A}, \gamma) \in \operatorname{Isom}(H_p^2(\Omega), L_p(\Omega) \times B_p^{2-1/p}(\Gamma)),\\
&(\lambda + \mathcal{A}^{\#}, \gamma) \in \operatorname{Isom}(H_{p'}^2(\Omega), L_{p'}(\Omega) \times B_{p'}^{2-1/p'}(\Gamma)),
\end{aligned} \tag{4.1}
$$

for $\lambda \in [\operatorname{Re} z \geq \lambda_0]$. Here, $\gamma \in \mathcal{L}(H_q^s(\Omega), B_q^{s-1/q}(\Gamma))$, for $q \in (1, \infty)$ and $s > 1/q$, denotes the trace operator with respect to Γ. Moreover, $\mathcal{A}^{\#}$ stands for the formally adjoint operator, i.e.,

$$\mathcal{A}^{\#} v := -\partial_j(a_{jk}\partial_k v) - \partial_j(a_j v) + a_0 v$$

for $v \in H_{p'}^2(\Omega)$. As in Section 3 we define the generalized Dirichlet operators by

$$
\begin{aligned}
\mathcal{T} &:= \mathcal{T}(\lambda) := (\lambda + \mathcal{A}, \gamma)^{-1} | \{0\} \times B_p^{2-1/p}(\Gamma),\\
\mathcal{T}^{\#} &:= \mathcal{T}^{\#}(\lambda) := (\lambda + \mathcal{A}^{\#}, \gamma)^{-1} | \{0\} \times B_{p'}^{2-1/p'}(\Gamma)
\end{aligned}
$$

for $\lambda \in [\operatorname{Re} z \geq \lambda_0]$. Observe that

$$\mathcal{T} \in \mathcal{L}(B_p^{2-1/p}(\Gamma), H_p^2(\Omega)), \quad \mathcal{T}^{\#} \in \mathcal{L}(B_{p'}^{2-1/p'}(\Gamma), H_{p'}^2(\Omega)). \tag{4.2}$$

Moreover, given $z \in B_p^{2-1/p}(\Gamma)$, let $u := \mathcal{T} z$. Then $u \in H_p^2(\Omega)$ is the unique solution of the inhomogeneous elliptic boundary value problem

$$(\lambda + \mathcal{A})u = 0 \text{ in } \Omega, \quad \gamma u = z \text{ on } \Gamma.$$

We use these solution operators to construct the following linear operators

$$\boldsymbol{B}_{1-1/p,p} := \mathcal{B}\mathcal{T} \text{ and } \boldsymbol{B}_{1-1/p',p'} := \mathcal{B}\mathcal{T}^{\#}. \tag{4.3}$$

Due to the trace theorem (Lemma 2.2g) and the regularity assumption (1.2), we have

$$\mathcal{B} \in \mathcal{L}(H_q^2(\Omega), B_q^{1-1/q}(\Gamma)), \quad q \in (1, \infty).$$

Consequently, (4.2) implies that

$$\boldsymbol{B}_{1-1/q,q} \in \mathcal{L}(B_q^{2-1/q}(\Gamma), B_q^{1-1/q}(\Gamma)), \quad q \in (1, \infty). \tag{4.4}$$

In virtue of $B_q^{2-1/q}(\Gamma) \overset{d}{\hookrightarrow} B_q^{1-1/q}(\Gamma)$ we can (and will) also consider $\boldsymbol{B}_{1-1/q,q}$ as an unbounded, densely defined linear operator in $B_q^{1-1/q}(\Gamma)$.

Furthermore, we have the embeddings

$$B_q^s(\Gamma) \overset{d}{\hookrightarrow} B_q^{1-1/q}(\Gamma) \text{ if } s > 1 - 1/q. \tag{4.5}$$

Thus the B_q^s-realization $\boldsymbol{B}_{s,q}$ of $\boldsymbol{B}_{1-1/q,q}$, i.e.,

$$\begin{aligned} &\mathrm{dom}(\boldsymbol{B}_{s,q}) := \{z \in B_q^{2-1/q}(\Gamma); \boldsymbol{B}_{1-1/q,q} z \in B_q^s(\Gamma)\}, \\ &\boldsymbol{B}_{s,q} z := \boldsymbol{B}_{1-1/q,q} z \text{ for } z \in \mathrm{dom}(\boldsymbol{B}_{s,q}), \end{aligned} \tag{4.6}$$

is well defined for $s > 1 - 1/q$. On the other hand, we also known that

$$B_q^{1-1/q}(\Gamma) \overset{d}{\hookrightarrow} B_q^s(\Gamma) \text{ if } s < 1 - 1/q. \tag{4.7}$$

So, we can consider $\boldsymbol{B}_{1-1/q,q}$ as an operator in $B_q^s(\Gamma)$ too. It is shown in [10, Theorem 1.5] that $\boldsymbol{B}_{1-1/q,q}$ is closable in $B_q^s(\Gamma)$. Let $\boldsymbol{B}_{s,q}$ denote its closure. Summarizing, we have constructed scales of linear operators on the Besov spaces $B_q^s(\Gamma)$, $s \in \mathbb{R}$, $q \in (1,\infty)$. In the following theorem we will describe some of the basic properties of these scales. In particular, it turns out that these operators are negative generators of analytic semigroups on $B_q^s(\Gamma)$ and that it is possible to describe presisely their domains. Moreover, we will characterize the dual operators of $\boldsymbol{B}_{s,q}$.

Let E_0 and E_1 be Banach spaces such that $E_1 \overset{d}{\hookrightarrow} E_0$. We define

$$\mathcal{H}(E_1, E_0) := \{A \in \mathcal{L}(E_1, E_0); -A \text{ generates a strongly continuous analytic semigroup on } E_0\}.$$

Then Theorem 1.5 in [10] implies the following result:

THEOREM 4.1. *Suppose that* $s \in \mathbb{R}$ *and that* $q \in (1,\infty)$. *Then*

$$\boldsymbol{B}_{s,q} \in \mathcal{H}(B_q^{s+1}(\Gamma), B_q^s(\Gamma)) \text{ and } [\boldsymbol{B}_{s,q}]' = \boldsymbol{B}_{-s,q'}$$

(as unbounded operators in $[B_q^s(\Gamma)]' = B_{q'}^{-s}(\Gamma)$*).*

REMARK 4.2. a) Suppose that $a_{jk} = \delta_{jk}$ (Kronecker symbol), that $a_j = 0$, $b_0 = 0$ and that $a_0 > 0$ on $\overline{\Omega}$. Then the operator $\mathcal{T}$ is the solution operator of the Dirichlet problem

$$-\Delta u + a_0 u = 0 \text{ in } \Omega, \quad \gamma u = z \text{ on } \Gamma,$$

and $\mathcal{B}$ reduces to the Neumann boundary operator ∂_ν. Consequently, in this situation the operator $\mathcal{B}\mathcal{T}$ becomes the so-called Dirichlet-Neumann operator Λ introduced by A.P. Calderón [7], J. Sylvester and G. Uhlmann [29] and J.L. Lions [15] in the Hilbert space setting.

As exhibited in the introduced, the generalized Dirichlet-Neumann operator plays an important role in the study of dynamic boundary conditions. These boundary conditions appear in various models in theoretical physics (heat conduction, water waves, acoustic waves), see [2, 8, 9, 11, 13, 19], as well as in colloid chemistry and in chemical reactor theory, see [5, 14, 24, 35].

In addition, the Dirichlet-Neumann operator Λ is used in inverse scattering problems and in electrical prospection. In the study of these problems the operator Λ has been recently investigated in a series of papers by J. Sylvester, G. Uhlmann and A. Nachmann, [20, 21, 29, 30, 31], see also [25].

b) Let us remark that the operators constructed above coincide with those of the introduction. This follows from the well-known fact that generators of strongly continuous semigroups are unique and the observation that the elliptic problem (E) is well posed on the Besov spaces $B_p^s(\Gamma)$ (see [10], Section 3).

c) The regularity assumption (1.2) for the coefficients as well as the assumption that Γ is of class C^∞ are not really necessary. We have introduced this strong regularity only to be able to construct the scale $\boldsymbol{B}_{s,p}$ for all $s \in \mathbb{R}$. Indeed, one can weaken these regularity hypotheses in the following sense:

Suppose that for some $\sigma \geq 0$ we have

$$a_{jk} = a_{kj} \in C^{1+\sigma}(\overline{\Omega}), \quad 1 \leq j,\ k \leq n, \quad a_j,\ a_0,\ b_0 \in C^\sigma(\overline{\Omega}),$$
$$1 \leq j \leq n, \text{ and } \Gamma \text{ is of class } C^{1+\sigma}.$$

Then the conclusions of Theorem 4.1 remain true, provided $s \in [-\sigma - 1, \sigma + 1]$.

5. - Estimates for two commutators

Let E_0 and E_1 be Banach spaces such that $E_1 \hookrightarrow E_0$. We introduce the following linear subspace of $\mathcal{L}(E_1, E_0)$:

$$\mathcal{L}_e(E_1, E_0) := \{A \in \mathcal{L}(E_1, E_0); \exists c > 0 \text{ with } \|Ax\|_{E_0} \leq c\|x\|_{E_0}, x \in E_1\}.$$

If we suppose in addition that E_1 is dense in E_0 then, given $A \in \mathcal{L}_e(E_1, E_0)$, there exists a unique extension $A^e \in \mathcal{L}(E_0)$ of A, where A is considered as a (generally) unbounded operator in E_0.

Next assume that $A \in \mathcal{L}(E_1, E_0)$ and that $B \in \mathcal{L}(E_0)$ such that likewise $B \in \mathcal{L}(E_1)$. Obviously, in this situation we have

$$[A, B] := AB - BA \in \mathcal{L}(E_1, E_0).$$

However, the main results in this section show that there are nontrivial examples such that

$$[A, B] \in \mathcal{L}_e(E_1, E_0).$$

Let $\psi \in \mathcal{D}(\mathbb{R}^m)$ and $z \in B_p^s(\mathbb{R}^m)$ be given. Then it is well known [34, Theorem 4.2.2] that $\psi z \in B_p^s(\mathbb{R}^m)$ and that there is a positive constant c such that

$$|\psi z|_{s,p} \leq c|z|_{s,p}, \quad z \in B_p^s(\mathbb{R}^m). \tag{5.1}$$

We often write ψ for the multiplication operator induced by $\psi \in \mathcal{D}(\mathbb{R}^m)$. Moreover, if $\psi \in \mathcal{D}(V)$, where $V \subset \mathbb{R}^m$ is open, we frequently consider ψ as an element of $\mathcal{D}(\mathbb{R}^m)$.

Finally, we need some notation from the theory of pseudo-differential operators. Let $K \subset \mathbb{R}^m$ be compact and let α, $\beta \in \mathbb{N}^m$ be given. If k, δ, ρ are real numbers, $0 < \rho \leq 1$, $0 \leq \delta < 1$, we set

$$p^k_{K,\alpha,\beta}(a) := \sup\{(1 + |\xi|)^{-k+\rho|\alpha|-\delta|\beta|}|\partial^\alpha_\xi \partial^\beta_x a(x, \xi)|; x \in K, \xi \in \mathbb{R}^m\}$$

for $a \in C^\infty(\mathbb{R}^{2m})$. The space of all symbols of order k and type (ρ, δ) is given as

$$S^k_{\rho,\delta}(\mathbb{R}^m) := \{a \in C^\infty(\mathbb{R}^{2m}); p^k_{K,\alpha,\beta}(a) < \infty \text{ for } \alpha, \beta \in \mathbb{N}^m \text{ and } K \subset \mathbb{R}^m \text{ compact}\}.$$

Given $p \in S^k_{\rho,\delta}(\mathbb{R}^m)$, we define the following formal operators $p(X, D)$ by

$$p(X, D)u(x) := \int_{\mathbb{R}^m} e^{i\langle x,\xi\rangle} p(x, \xi)(\mathcal{F}u(\xi))d\xi, \quad u \in \mathcal{S}(\mathbb{R}^m).$$

Then $\Psi^k_{\rho,\delta}$ denotes the space of all pseudo-differential operators of order k and type (ρ, δ), i.e., $\Psi^k_{\rho,\delta}$ consists of all operators which are locally of the form $p(X, D)$.

Let us now return to the pseudo-differential operators of Section 3. We choose $\alpha \in \mathbb{C}\backslash\{0\}$ such that $\alpha^2 \in [\operatorname{Re}\lambda \geq \lambda_0^+]$, where $\lambda_0 \in \mathbb{R}$ denotes the constant of (4.1). To economize our notation, we suppress in the following the dependence on α. We also fix $1 \leq i \leq m_r$, where m_r is given by (2.19).

LEMMA 5.1. *Given $\psi \in \mathcal{D}(U_i)$ we have*

$$[\tilde{\varphi}_i\psi, \boldsymbol{B}_\pi] \in \Psi^0_{1,0}.$$

PROOF. Obviously, the multiplication operator $\tilde{\varphi}_i^*\psi$ induced by $\tilde{\varphi}_i^*\psi$ belongs to $\Psi^0_{1,0}$. Besides, $\tilde{\varphi}_i^*\psi$ is properly supported (see [23, p. 180] for a definition).

On the other hand, it follows from

$$b(\lambda\alpha, \lambda\xi) = \lambda b(\alpha, \xi) \text{ for } \lambda \geq 0, \ \xi \in \mathbb{R}^{n-1}$$

that

$$\partial^\beta_\xi b(\alpha, \xi) = (\alpha^2 + |\xi|^2)^{\frac{1-|\beta|}{2}} \partial^\beta_\xi b(\alpha^*, \xi^*) \text{ for } \xi \in \mathbb{R}^{n-1}, \ \beta \in \mathbb{N}^{n-1}.$$

Consequently, we have $b(\alpha, \cdot) \in S^1_{1,0}(\mathbb{R}^{n-1})$. By definition, this means that $\boldsymbol{B}_\pi = \mathcal{F}^{-1}b(\alpha, \cdot)\mathcal{F} \in \Psi^1_{1,0}$. Now, Corollary 3.5.10 in [23] can be applied and yields the result. □

COROLLARY 5.2. *Assume that* $\psi \in \mathcal{D}(U_i)$. *Then, given* $s \in \mathbb{R}$ *and* $p \in (1,\infty)$, *we have*

$$[\tilde{\varphi}_i^* \psi, \boldsymbol{B}_\pi] \in \mathcal{L}_e(B_p^{s+1}(\mathbb{R}^{n-1}), B_p^s(\mathbb{R}^{n-1})).$$

PROOF. Theorem 6.2.2 in [34] ensures that $\Psi_{1,0}^0 \subset \mathcal{L}(B_p^s(\mathbb{R}^{n-1}))$ for $s \in \mathbb{R}$ and $p \in (1,\infty)$. □

Our next goal is to establish a result similar to Corollary 5.2 for the operators $\boldsymbol{B}_{s,p}$. To this end we note that each $\theta \in \mathcal{D}(U_i)$ induces a (pointwise) multiplication operator m_θ on $B_p^s(\Gamma)$, given by

$$(m_\theta z)(x) := \theta(x) z(x) \text{ for } z \in B_p^s(\Gamma),\ x \in \Gamma.$$

If $s \leq 0$ this definition has to be understood in the sense of distributions, of course. Observe that we have

$$m_\theta \in \mathcal{L}(B_p^s(\Gamma), B_p^s(\tilde{U}_i)) \text{ for } s \in \mathbb{R},\ p \in (1,\infty). \tag{5.2}$$

This follows from [34, Theorem 4.2.2] and the localization procedure based on (2.19). Again, we write θ for the operator m_θ if there is no risk of confusion.

LEMMA 5.3. *Let* $\theta \in \mathcal{D}(U_i)$ *and* $p \in (1,\infty)$ *be given. Then we have*

$$[\boldsymbol{B}_{-1/p,p}, \theta] \in \mathcal{L}_e(B_p^{1-1/p}(\Gamma), B_p^{-1/p}(\Gamma)).$$

PROOF. a) We denote by

$$\overline{\mathcal{T}}^{\#} \in \mathcal{L}(B_{p'}^{1-1/p'}(\Gamma), H_{p'}^1(\Omega)) \tag{5.3}$$

the so-called Lions-Magenes extension of $\mathcal{T}^{\#}$ to $B_{p'}^{1-1/p'}(\Gamma)$, cf. [10, (2.4)]. Then we have

$$\overline{\mathcal{T}}^{\#}\varphi = \mathcal{T}^{\#}\varphi \in H_{p'}^2(\Omega) \text{ for } \varphi \in B_{p'}^{2-1/p'}(\Gamma).$$

Given $\varphi \in B_{p'}^{1-1/p'}(\Gamma)$, we set

$$S_\theta^{\#}\varphi := 2a_{jk}\partial_k\theta \cdot \partial_j(\overline{\mathcal{T}}^{\#}\varphi) + [\partial_j(a_{jk}\partial_k\theta) + a_j\partial_j\theta]\overline{\mathcal{T}}^{\#}\varphi.$$

Since $\partial_j \in \mathcal{L}(H_{p'}^1(\Omega), \mathcal{L}_{p'}(\Omega))$, it follows from (5.3) that

$$S_\theta^{\#} \in \mathcal{L}(B_{p'}^{1-1/p'}(\Gamma), L_{p'}(\Omega)). \tag{5.4}$$

Next pick $\varphi \in B_{p'}^{1-1/p'}(\Gamma)$. Then by Lemma 2.2b), there is a sequence $(\varphi_n)_{n\in\mathbb{N}} \subset B_{p'}^{2-1/p'}(\Gamma)$ such that $\varphi_n \to \varphi$ in $B_{p'}^{1-1/p'}(\Gamma)$ as $n \to \infty$. Since $\theta\varphi_n \in B_{p'}^{2-1/p'}(\Gamma)$ as $n \to \infty$. Since $\theta\varphi_n \in B_{p'}^{2-1/p'}(\Gamma)$, $n \in \mathbb{N}$, it follows that

$$\overline{\mathcal{T}}^{\#}\theta\varphi_n - \theta\overline{\mathcal{T}}^{\#}\varphi_n = \mathcal{T}^{\#}\theta\varphi_n - \theta\mathcal{T}^{\#}\varphi_n \in H_{p'}^2(\Omega) \cap \overset{\circ}{H}{}_{p'}^1(\Omega). \tag{5.5}$$

Letting $n \to \infty$, we find that

$$\overline{\mathcal{T}}^{\#}\theta\varphi - \theta\overline{\mathcal{T}}^{\#}\varphi \in \overset{\circ}{H}{}^1_{p'}(\Omega) \text{ for } \varphi \in B_{p'}^{1-1/p'}(\Gamma). \tag{5.6}$$

b) Next, we consider $(\alpha^2 + \mathcal{A}^{\#})|H^2_{p'}(\Omega) \cap \overset{\circ}{H}{}^1_{p'}(\Omega)$ as an unbounded operator in $L_{p'}(\Omega)$. Then it is known that this operator is closable in $H^{-1}_{p'}(\Omega)$. Let $A^{\#}_D$ denote its closure. It can be shown that

$$A^{\#}_D \in \text{Isom}(\overset{\circ}{H}{}^1_{p'}(\Omega, H^{-1}_{p'}(\Omega)) \text{ and } [A^{\#}_D]^{-1}|L_{p'}(\Omega) \in \mathcal{L}(L_{p'}(\Omega), H^2_{p'}(\Omega)). \tag{5.7}$$

For a proof of these facts we refer to [4, Section 8].

Moreover, given $v \in \overset{\circ}{H}{}^1_p(\Omega)$, (5.5) implies that

$$\begin{aligned}
\langle v, A^{\#}_D(\overline{\mathcal{T}}^{\#}\theta\varphi_n - \theta\overline{\mathcal{T}}^{\#}\varphi_n)\rangle &= \int_\Omega v(\alpha^2 + \mathcal{A}^{\#})\mathcal{T}^{\#}\theta\varphi_n dx \\
&\quad - \int_\Omega v(\alpha^2 + \mathcal{A}^{\#})[\theta\mathcal{T}^{\#}\varphi_n]dx \\
&= \int_\Omega v(\alpha^2 + \mathcal{A}^{\#})\mathcal{T}^{\#}\theta\varphi_n dx \\
&\quad - \int_\Omega v\theta(\alpha^2 + \mathcal{A}^{\#})\mathcal{T}^{\#}\varphi_n dx \\
&\quad + \int_\Omega v[\partial_j(a_{jk}\partial_k\theta)\cdot\mathcal{T}^{\#}\varphi_n \\
&\quad + 2a_{jk}\partial_k\theta\cdot\partial_j(\mathcal{T}^{\#}\varphi_n) + a_j\partial_j\theta\cdot\mathcal{T}^{\#}\varphi_n]dx.
\end{aligned}$$

By definition of $\mathcal{T}^{\#}$ the first two integrals vanish. Thus

$$\langle v, A^{\#}_D(\overline{\mathcal{T}}^{\#}\theta\varphi_n - \theta\overline{\mathcal{T}}^{\#}\varphi_n)\rangle = \langle v, S^{\#}_\theta\varphi_n\rangle, \quad v \in \overset{\circ}{H}{}^1_p(\Omega).$$

Now letting $n \to \infty$, it follows from (5.4) and (5.7) that

$$\langle v, A^{\#}_D(\overline{\mathcal{T}}^{\#}\theta\varphi - \theta\overline{\mathcal{T}}^{\#}\varphi) - S^{\#}_\theta\varphi\rangle = 0, \quad v \in \overset{\circ}{H}{}^1_p(\Omega).$$

Since $H^{-1}_{p'}(\Omega) = [\overset{\circ}{H}{}^1_p(\Omega)]'$ this implies that

$$\overline{\mathcal{T}}^{\#}\theta\varphi - \theta\overline{\mathcal{T}}^{\#}\varphi = [A^{\#}_D]^{-1}S^{\#}_\theta\varphi, \quad \varphi \in B_{p'}^{1-1/p'}(\Gamma).$$

Consequently, (5.4) and (5.7) yield

$$[\overline{\mathcal{T}}^{\#},\theta]\in\mathcal{L}(B_{p'}^{1-1/p'}(\Gamma),H_{p'}^{2}(\Omega)). \tag{5.8}$$

c) Let $\varphi\in B_{p'}^{2-1/p'}(\Gamma)$ be given. Then we have

$$\begin{aligned}\mathcal{B}[\overline{\mathcal{T}}^{\#},\theta]\varphi+\varphi\mathcal{B}\theta&=\mathcal{B}\mathcal{T}^{\#}\theta\varphi-\mathcal{B}(\theta\mathcal{T}^{\#}\varphi)+\varphi\mathcal{B}\theta\\&=\mathcal{B}\mathcal{T}^{\#}\theta\varphi-\theta\mathcal{B}\mathcal{T}^{\#}\varphi=[\boldsymbol{B}_{1-1/p',p'},\theta]\varphi,\end{aligned}$$

where we have used (4.3) to obtain the last equality. Thus the fact that $\mathcal{B}\in\mathcal{L}(H_{p'}^{2}(\Omega),B_{p'}^{1-1/p'}(\Gamma))$, (5.8), and equality $\operatorname{dom}(\boldsymbol{B}_{1-1/p',p'})=B_{p'}^{2-1/p'}(\Gamma))$ imply that

$$[\boldsymbol{B}_{1-1/p',p'},\theta]\in\mathcal{L}_{e}(B_{p'}^{2-1/p'}(\Gamma),B_{p'}^{1-1/p'}(\Gamma)). \tag{5.9}$$

d) Finally, let $z\in B_{p}^{1-1/p}(\Gamma)$ and $\varphi\in B_{p'}^{2-1/p'}(\Gamma)$ be given. Again, choose a sequence $(\varphi_n)_{n\in\mathbb{N}}$ such that $\varphi_n\to\varphi$ in $B_{p'}^{1-1/p'}(\Gamma)$ as $n\to\infty$.

Note that $\boldsymbol{B}_{-1/p,p}=[\boldsymbol{B}_{1-1/p',p'}]'$ by Theorem 4.1. Thus we have

$$\begin{aligned}\langle[\boldsymbol{B}_{-1/p,p},\theta]z,\varphi_n\rangle&=\langle\boldsymbol{B}_{-1/p,p}\theta z,\varphi_n\rangle-\langle\theta\boldsymbol{B}_{-1/p,p}z,\varphi_n\rangle\\&=\langle z,\theta\boldsymbol{B}_{1-1/p',p'}\varphi_n\rangle-\langle z,\boldsymbol{B}_{1-1/p',p'}\theta\varphi_n\rangle\\&=-\langle z,[\boldsymbol{B}_{1-1/p',p'},\theta]\varphi_n\rangle.\end{aligned} \tag{5.10}$$

Since $[\boldsymbol{B}_{1-1/p',p'},\theta]\in\mathcal{L}_{e}(B_{p'}^{2-1/p'}(\Gamma),B_{p'}^{1-1/p'}(\Gamma))$, we can find a positive constant c such that

$$\|[\boldsymbol{B}_{1-1/p',p'},\theta]\varphi_n\|_{1-1/p',p'}\le c\|\varphi_n\|_{1-1/p',p'}.$$

Consequently, letting $n\to\infty$, we obtain together with (5.10) and $B_p^{-1/p}(\Gamma)=[B_{p'}^{1-1/p'}(\Gamma)]'$ that

$$\|\varphi\|_{1-1/p',p'}^{-1}|\langle[\boldsymbol{B}_{-1/p,p},\theta]z,\varphi\rangle|\le c\|z\|_{-1/p,p}. \tag{5.11}$$

Taking the supremum of all $\varphi \in B_{p'}^{1-1/p'}(\Gamma)\backslash\{0\}$ in (5.11), we find that

$$\|[\boldsymbol{B}_{-1/p,p}, \theta]z\|_{-1/p,p} \leq c\|z\|_{-1/p,p}.$$

This proves the Lemma. □

6. - Proof of the main result

In the preceding section we have derived estimates for the commutators $[\boldsymbol{B}_\pi, \theta]$ and $[\boldsymbol{B}_{-1/p,p}, \theta]$. To apply the estimates of Section 3 it remains to establish appropriate bounds for $\tilde{\varphi}_i^* \theta \boldsymbol{B}_{-1/p,p} - \boldsymbol{B}_\pi \tilde{\varphi}_i^* \theta$ for each $1 \leq i \leq m_r$. This will be done by means of local coordinates. After these technical preliminaries we estimate the resolvent of the $C(\Gamma)$-realization of $\boldsymbol{B}_{-1/p,p}$. From this estimate the main result follows then easily.

In the following we fix $i \in \{1, \ldots, m_r\}$. Let us first introduce some notation to work with the local coordinates $(\mathbb{B}_n(0,r)\varphi_i)$. We first note that we can assume without restriction that $D\varphi_i(0) = \mathrm{id}_{\mathbb{R}^n}$.

Furthermore, we set $x_0 := \varphi_i(0) \in \tilde{U}_i$ and $a_{jk}^\pi := a_{jk}(x_0)$. Besides, we define $g_{jk} := (\partial_j \varphi_i | \partial_k \varphi_i)$, $g := \det[g_{jk}]_{1 \leq j,k \leq n}$, and $\hat{g} := \varphi_*^i \det[g_{jk}]_{1 \leq j,k \leq n}^{-1}$. Let us also remark that, given $k \in \mathbb{N}$, we may assume that there exists a positive constant c such that $\|\varphi_i\|_{BC^k(\mathbb{B}_n(0,r))} \leq c$ for $r \in (0,1]$ and $1 \leq i \leq m_r$.

Recall that $\{\theta_i\}_{1 \leq i \leq m_r}$ is a partition of unity subordinate to the open covering $\{U_i\}_{1 \leq i \leq m_r}$ of $\overline{\Omega}$. Moreover, letting $V_i := \varphi_i(\mathbb{B}_n(0, r/2))$, we may assume that $\{V_i\}_{1 \leq i \leq m_r}$ is an open covering of $\overline{\Omega}$ and that $\theta_i | V_i = 1$. Additionally, in the following let ψ_i, $\overline{\psi}_i \in \mathcal{D}(U_i)$ such that $\psi_i | \operatorname{supp} \theta_i = 1$ and $\overline{\psi}_i | \operatorname{supp} \psi_i = 1$.

If there is no risk of confusion, we suppress the index i in our notation, i.e., we write $\varphi = \varphi_i$, $U = U_i$, $\theta = \theta_i$, and so on.

Finally, let $\overline{\mathcal{T}}$ denote the so-called Lions-Magenes extension of the solution operator $\mathcal{T}$ defined in (4.2). Let us note the following basic properties of this extension (cf. [10, Section 2]).

(6.1) $\overline{\mathcal{T}} z = \mathcal{T} z$ for $z \in B_p^{2-1/p}(\Gamma)$ and $\overline{\mathcal{T}} \in \mathcal{L}(B_p^{s-1/p}(\Gamma), H_p^s(\Omega))$, $s \in [0,2]$.

LEMMA 6.1. *Let $z \in B_p^{1-1/p}(\Gamma)$ and $h' \in B_{p'}^{1-1/p'}(\mathbb{R}^{n-1})$ be given. Then we have*

(a)
$$\langle (\tilde{\varphi}^* \psi) \boldsymbol{B}_\pi \tilde{\varphi}^* (\theta z), h' \rangle = \int_{\mathbb{H}^n} a_{jk}^\pi \partial_j \varphi^* (\psi \overline{\mathcal{T}}(\theta z) \cdot \partial_k (\varphi^* \overline{\psi} \cdot \overline{\mathcal{T}}_\pi' h') + \alpha^2 \varphi^* (\psi \overline{\mathcal{T}}(\theta z)) \cdot \varphi^* \overline{\psi} \cdot \overline{\mathcal{T}}_\pi' h' dx.$$

b) $$\langle \tilde\varphi^*(\psi \boldsymbol{B}_{-1/p,p}(\theta z)), h'\rangle = \int_{\mathbb{H}^n} a'_{jk}\partial_j \varphi^*(\psi \overline{\mathcal{T}}(\theta z)) \cdot \partial_k[\varphi^*\overline{\psi} \cdot \overline{\mathcal{T}}'_\pi h']dx$$
$$+ \int_{\mathbb{H}^n} [a'_j \partial_j \varphi^*(\psi \overline{\mathcal{T}}(\theta z))$$
$$+ (a'_0 + \alpha^2)\varphi^*(\psi \overline{\mathcal{T}}(\theta z))][\varphi^*\overline{\psi} \cdot \overline{\mathcal{T}}'_\pi h']dx$$
$$+ \int_{\mathbb{R}^{n-1}} b'_0 \tilde\varphi^*(\theta z) \cdot h' dx',$$

where $a'_{jk} := \varphi^*[\psi a_{lm}\partial_l(\varphi^{-1})^j \partial_m(\varphi^{-1})^k]$, $a'_j := \varphi^*[\psi a_l \partial_l(\varphi^{-1})^j]$, $a_0 := \varphi^*[\psi a_0]$ *and* $b'_0 := \varphi^*(\psi b_0)$.

PROOF. a) Observe that $\varphi^*(\psi\overline{\mathcal{T}}(\theta z)) = \varphi^*(\psi \mathcal{T}(\theta z)) \in H^2_p(\mathbb{H}^n)$, cf. (6.1). Besides, the integrand in a) has compact support. Thus Gauss' theorem yields

$$I := \int_{\mathbb{H}^n} a^\pi_{jk}\partial_j \varphi^*(\psi \mathcal{T}(\theta z)) \cdot \partial_k(\varphi^*\overline{\psi} \cdot \overline{\mathcal{T}}'_\pi h') + \alpha^2 \varphi^*(\psi \mathcal{T}(\theta z)) \cdot \varphi^*\overline{\psi} \cdot \overline{\mathcal{T}}'_\pi h' dx$$
$$= \int_{\mathbb{H}^n} a^\pi_{jk}\partial_j[\varphi^*(\psi \mathcal{T}(\theta z)) - \mathcal{T}'_\pi \varphi^*(\theta z)]\partial_k \overline{\mathcal{T}}'_\pi h' dx$$
$$+ \int_{\mathbb{H}^n} \alpha^2 \overline{\mathcal{T}}'_\pi h'[\varphi^*(\psi(\mathcal{T}(\theta z)) - \mathcal{T}'_\pi \varphi^*(\theta z)]dx$$
$$\int_{\mathbb{H}^n} a^\pi_{jk}\partial_j \mathcal{T}'_\pi \varphi^*(\theta z)\partial_k \overline{\mathcal{T}}'_\pi h' + \alpha^2 \mathcal{T}'_\pi \varphi^*(\theta z)\overline{\mathcal{T}}'_\pi h dx$$
$$= -\int_{\mathbb{R}^{n-1}} a^\pi_{nk}\gamma\partial_k \overline{\mathcal{T}}'_\pi h' \cdot \gamma[\varphi^*(\psi \mathcal{T}(\theta z)) - \mathcal{T}'_\pi \varphi^*(\theta z)]dx'$$
$$+ \int_{\mathbb{H}^n} [\varphi^*(\psi \mathcal{T}(\theta z)) - \mathcal{T}'_\pi \varphi^*(\theta z)][-a^\pi_{jk}\partial_j\partial_k + \alpha^2]\overline{\mathcal{T}}'_\pi h' dx$$
$$- \int_{\mathbb{R}^{n-1}} a^\pi_{jk}\gamma\partial_j \mathcal{T}'_\pi \varphi^*(\theta z) \cdot \gamma(\varphi^*\psi \overline{\mathcal{T}}'_\pi h')dx'$$
$$+ \int_{\mathbb{H}^n} [-a^\pi_{jk}\partial_j\partial_k(\mathcal{T}'_\pi \varphi^*(\theta z)) + \alpha^2 \mathcal{T}'_\pi \varphi^*(\theta z)]\varphi^*\psi \overline{\mathcal{T}}'_\pi h' dx$$

$$= \int_{\mathbb{R}^{n-1}} \overline{\mathcal{B}}_\pi' \overline{\mathcal{T}}_\pi' h' \cdot \gamma[\varphi^*(\psi \mathcal{T}(\theta z)) - \mathcal{T}_\pi' \varphi^*(\theta z)]$$

$$+ \int_{\mathbb{H}^n} [(\alpha^2 + \overline{\mathcal{A}}_\pi') \overline{\mathcal{T}}_\pi' h'][\varphi^*(\psi \mathcal{T}(\theta z)) - \mathcal{T}_\pi' \varphi^*(\theta z)] dx$$

$$+ \int_{\mathbb{R}^{n-1}} \mathcal{B}_\pi' \mathcal{T}_\pi' \varphi^*(\theta z) \cdot \gamma(\varphi^* \psi \overline{\mathcal{T}}_\pi' h') dx'$$

$$+ \int_{\mathbb{H}^n} [(\alpha^2 + \mathcal{A}_\pi') \mathcal{T}_\pi' \varphi^*(\theta z)] \varphi^* \psi \overline{\mathcal{T}}_\pi' h' dx.$$

Recalling the definition of $\mathcal{T}$, $\mathcal{T}_\pi'$ and $\overline{\mathcal{T}}_\pi'$, we find that $\gamma[(\varphi^*(\psi \mathcal{T}(\theta z)) - \mathcal{T}_\pi' \varphi^*(\theta z)] = 0$, $(\alpha^2 + \overline{\mathcal{A}}_\pi')\overline{\mathcal{T}}_\pi' = 0$ and $\gamma(\varphi^* \psi \cdot \overline{\mathcal{T}}_\pi' h') = \tilde{\varphi}^* \psi \cdot h'$. Thus the first two and the last integrand vanish. Now it follows from Lemma 3.6 that

$$I = \langle \tilde{\varphi}^* \psi \overline{\mathcal{B}}_\pi' \mathcal{T}_\pi' \varphi^*(\theta z), h' \rangle = \langle \tilde{\varphi}^* \psi \boldsymbol{B}_\pi' \varphi^*(\theta z), h' \rangle.$$

Furthermore, observe that $\boldsymbol{B}_\pi' \varphi^*(\theta \cdot) \in \mathcal{L}(B_p^{1-1/p}(\Gamma), B_p^{-1/p}(\mathbb{H}^n))$, as Corollary 3.2, (2.24) and (5.2) show. Since $B_p^{2-1/p}(\Gamma)$ is dense in $B_p^{1-1/p}(\Gamma)$, the assertion follows.

b) The transformation theorem for the Lebesgue integral implies that

$$I := \int_{\mathbb{H}^n} \{a_{jk}' \partial_j \varphi^*(\psi \overline{\mathcal{T}}(\theta z)) \cdot \partial_k (\varphi^* \overline{\psi} \cdot \overline{\mathcal{T}}_\pi' h'] + [a_j' \partial_j \varphi^*(\psi \overline{\mathcal{T}}(\theta z))$$

$$+ (a_0' + \alpha^2) \varphi^*(\psi \overline{\mathcal{T}}(\theta z))][\varphi^* \overline{\psi} \cdot \overline{\mathcal{T}}_\pi' h']\} dx$$

$$= \int_{\mathbb{H}^n} \varphi^*[\psi \sqrt{g} \{a_{lm} \partial_l (\psi \overline{\mathcal{T}}(\theta z)) \cdot \partial_m \varphi_*(\varphi^* \psi \cdot \overline{\mathcal{T}}_\pi' h')$$

$$+ (a_l \partial_l (\psi \overline{\mathcal{T}}(\theta z)) + (a_0 + \alpha^2)(\psi \overline{\mathcal{T}}(\theta z))) \varphi_*(\varphi^* \psi \cdot \overline{\mathcal{T}}_\pi' h')\}] \sqrt{\hat{g}} dx$$

$$= \int_\Omega \{a_{jk} \partial_j (\overline{\mathcal{T}}(\theta z)) \cdot \partial_k \varphi_*(\varphi^* \psi \cdot \overline{\mathcal{T}}_\pi' h') + a_j \partial_j (\overline{\mathcal{T}}(\theta z))$$

$$+ (a_0 + \alpha^2) \overline{\mathcal{T}}(\theta z) \varphi_*(\varphi^* \psi \cdot \overline{\mathcal{T}}_\pi' h')\} \sqrt{\hat{g}} dx$$

$$= \int_\Omega \partial_j [a_{jk} \partial_k (\overline{\mathcal{T}}(\theta z)) \cdot \varphi_*(\varphi^* \psi \cdot \overline{\mathcal{T}}_\pi' h') \cdot \sqrt{\hat{g}}] dx$$

$$+ \int_\Omega (\alpha^2 + \mathcal{A}) \overline{\mathcal{T}}(\theta z) \cdot \varphi_*(\varphi^* \psi \cdot \overline{\mathcal{T}}_\pi' h') \sqrt{\hat{g}} dx.$$

Since $z \in B_p^{2-1/p}(\Gamma)$, we have that $\overline{T}(\theta z) = \mathcal{T}(\theta z)$, cf. (6.1). Consequently, the definition of $\mathcal{T}$ shows that the last integral vanishes. Moreover, Gauss' theorem yields

$$I = \int_{\partial\Omega} a_{jk}\nu^k \partial_j(\mathcal{T}(\theta z)) \cdot \gamma(\varphi_*(\varphi^*\psi \cdot \overline{\mathcal{T}}'_\pi h'))\gamma\sqrt{\hat{g}}d\sigma$$

$$= \int_{\partial\Omega} [\mathcal{B}\mathcal{T}(\theta z) \cdot \psi \cdot \tilde{\varphi}_* h' - b_0\theta z \cdot \psi \cdot \tilde{\varphi}_* h']\gamma\sqrt{\hat{g}}d\sigma$$

$$= \int_{\mathbb{R}^{n-1}} \tilde{\varphi}^*(\psi \boldsymbol{B}_{-1/p,p}(\theta z))h' dx' - \int_{\mathbb{R}^{n-1}} \tilde{\varphi}^*(\psi b_0 \theta z) h' dx'$$

$$= \langle \tilde{\varphi}^*(\psi \boldsymbol{B}_{-1/p,p}(\theta z)), h' \rangle - \int_{\mathbb{R}^{n-1}} b_0' \tilde{\varphi}^*(\theta z) h' dx'.$$

Again by density, this implies the assertion. □

LEMMA 6.2. *Given $\varepsilon > 0$, there exists a positive constant $c(\varepsilon)$ such that*

$$\left| \int_{\mathbb{H}^n} [a_j' \partial_j \varphi^*(\psi \overline{\mathcal{T}}(\theta z)) + a_0' \varphi^*(\psi \overline{\mathcal{T}}(\theta z))][\varphi^* \overline{\psi} \cdot \overline{\mathcal{T}}'_\pi h'] dx \right| + \left| \int_{\mathbb{R}^{n-1}} b_0' \tilde{\varphi}^*(\theta z) \cdot h' dx' \right|$$

$$\leq (\varepsilon \|\theta z\|_{1-1/p,p} + c(\varepsilon)\|\theta z\|_{-1/p,p}) |h'|_{1-1/p',p'}$$

for all $z \in B_p^{1-1/p}(\Gamma)$ and $h' \in B_{p'}^{1-1/p'}(\mathbb{R}^{n-1})$.

PROOF. Using (2.23), (6.1) and (3.16), it follows that

$$\|a_0' \varphi^*(\psi \overline{\mathcal{T}}(\theta z))\|_{L_p(\mathbb{H}^n)} \leq \|\theta z\|_{-1/p,p} \text{ and } \|\varphi^* \overline{\psi} \cdot \overline{\mathcal{T}}'_\pi h'\|_{L_{p'}(\mathbb{H}^n)} \leq |h'|_{1-1/p',p'}.$$

Moreover, since $B_p^{-1/p}(\mathbb{R}^{n-1}) = [B_{p'}^{1-1/p'}(\mathbb{R}^{n-1})]'$, we have (observe that $\tilde{\varphi}^*(\theta z) \in B_p^{1-1/p}(\mathbb{R}^{n-1}) \hookrightarrow L_p(\mathbb{R}^{n-1}) \hookrightarrow B_p^{-1/p}(\mathbb{R}^{n-1})$)

$$\left| \int_{\mathbb{R}^{n-1}} b_0' \tilde{\varphi}^*(\theta z) \cdot h' dx' \right| = \left| \langle b_0' \tilde{\varphi}^*(\theta z), h' \rangle_{B_{p'}^{1-1/p'}(\mathbb{R}^{n-1})} \right|$$

$$\leq c\|\theta z\|_{-1/p,p} \cdot |h'|_{1-1/p',p'}.$$

Thus we find that

$$\left| \int_{\mathbb{H}^n} [a_0' \varphi^*(\psi \overline{\mathcal{T}}(\theta z))][\varphi^* \overline{\psi} \cdot \overline{\mathcal{T}}'_\pi h'] dx \right| + \left| \int_{\mathbb{R}^{n-1}} b_0' \tilde{\varphi}^*(\theta z) \cdot h' dx' \right| \tag{6.2}$$

$$\leq c\|\theta z\|_{-1/p,p} \cdot |h'|_{1-1/p',p'}.$$

Next fix $\delta \in (0, 1/p)$. Then it is well known that $H_{p'}^\delta(\mathbb{H}^n) = \overset{\circ}{H}{}_{p'}^\delta(\mathbb{H}^n)$, thus $[H_{p'}^\delta(\mathbb{H}^n)]' = H_p^{-\delta}(\mathbb{H}^n)$, cf. [32, Theorem 4.5.2]. Hence it follows that

$\partial_j \in \mathcal{L}(H_p^{1-\delta}(\mathbb{H}^n), H_p^{-\delta}(\mathbb{H}^n))$. Now we conclude from (2.23) and (6.1) that there is a $c > 0$ such that

$$\|a'_j \partial_j \varphi^*(\psi \mathcal{T}(\theta z))\|_{H_p^{-\delta}(\mathbb{H}^n)} \leq c\|\theta z\|_{1-\delta-1/p,p} \text{ for } z \in B_p^{1-1/p}(\Gamma). \tag{6.3}$$

But we also have $B_p^{1-\delta-1/p}(\Gamma) = [B_p^{-1/p}(\Gamma), B_p^{1-1/p}(\Gamma)]_{1-\delta}$ by Lemma 2.2f). Hence, the interpolation inequality

$$\|z\|_{1-\delta-1/p,p} \leq c\|z\|_{-1/p,p}^{\delta} \cdot \|z\|_{1-1/p,p}^{1-\delta}, \quad z \in B_p^{1-1/p}(\Gamma),$$

holds. Using Young's inequality in the form

$$xy \leq (1-\delta)x^{\frac{1}{1-\delta}} + \delta y^{\frac{1}{\delta}}, \quad x,\ y \geq 0,$$

we find for each $\varepsilon > 0$ a constant $c(\varepsilon) > 0$ such that

$$\|z\|_{1-\delta-1/p,p} \leq \varepsilon\|z\|_{1-1/p,p} + c(\varepsilon)\|z\|_{-1/p,p}, \quad z \in B_p^{1-1/p}(\Gamma). \tag{6.4}$$

On the other hand it follows from (3.16) that

$$\begin{aligned} \|\varphi^*\overline{\psi} \cdot \mathcal{T}'_\pi h'\|_{H_{p'}^{\delta}(\mathbb{H}^n)} &\leq c|h'|_{\delta-1/p',p'} \\ &\leq c'|h'|_{1-1/p',p'}, \quad h' \in B_{p'}^{1-1/p'}(\mathbb{R}^{n-1}). \end{aligned} \tag{6.5}$$

Combining (6.2)-(6.5) the Lemma follows. □

LEMMA 6.3. *Given $\varepsilon > 0$ there exist positive constants $c(\varepsilon)$ and c (independent of r) such that*

$$\begin{aligned} \|\tilde{\varphi}^*(\theta \boldsymbol{B}_{-1/p,p} z) - \boldsymbol{B}_\pi \tilde{\varphi}^*(\theta z)\|_{-1/p,p} &\leq (\varepsilon + c \cdot r)\|\theta z\|_{1-1/p,p} \\ &+ c(\varepsilon)\|z\|_{-1/p,p}, \quad z \in B_p^{1-1/p,p}(\Gamma). \end{aligned}$$

PROOF. Pick $h' \in B_{p'}^{1-1/p'}(\mathbb{R}^{n-1})$ with $|h'|_{1-1/p',p'} = 1$.
Furthermore, let

$$T_1 := \langle \tilde{\varphi}^*(\psi[\theta, \boldsymbol{B}_{-1/p,p}]z), h' \rangle,$$

$$T_2 := \langle [\tilde{\varphi}^*\psi, \boldsymbol{B}_\pi]\tilde{\varphi}^*(\theta z), h' \rangle,$$

$$T_3 := \int_{\mathbb{H}^n} [a'_j \partial_j \varphi^*(\psi \mathcal{T}(\theta z)) + a'_0 \varphi^*(\psi \mathcal{T}(\theta z))][\varphi^*\overline{\psi} \cdot \mathcal{T}'_\pi h']dx$$

$$+ \int_{\mathbb{R}^{n-1}} b'_0 \tilde{\varphi}^*(\theta z) \cdot h' dx',$$

$$T_4 := \int_{\mathbb{H}^n} (a'_{jk} - a^\pi_{jk})\partial_j \varphi^*(\psi \mathcal{T}(\theta z)) \cdot \partial_k[\varphi^*\overline{\psi} \cdot \mathcal{T}'_\pi h']dx.$$

Applying Lemma 6.1, we find that

$$\langle \tilde{\varphi}^*(\theta \boldsymbol{B}_{-1/p,p} z) - \boldsymbol{B}'_\pi \tilde{\varphi}^*(\theta z), h' \rangle = T_1 + T_2 + T_3 + T_4.$$

To estimate T_1 and T_2 we use (2.24), Lemma 5.3, Corollary 5.2, (5.1), the fact that $[B_p^{-1/p}(\Gamma)]' = B_{p'}^{1-1/p'}(\Gamma)$ and the hypothesis $|h'|_{1-1/p',p'} = 1$, and we find that

$$|T_1| + |T_2| \leq c\|z\|_{-1/p,p} \text{ for } z \in B_p^{1-1/p}(\Gamma), \tag{6.6}$$

where c is a positive constant.

Given $\varepsilon > 0$, there is a $c(\varepsilon) > 0$ such that

$$|T_3| \leq \varepsilon\|\theta z\|_{1-1/p,p} + c(\varepsilon)\|\theta z\|_{-1/p,p} \text{ for } z \in B_p^{1+1/p}(\Gamma), \tag{6.7}$$

due to Lemma 6.2.

Moreover, using the same arguments which lead to (6.6) as well as (6.1) and (3.16), we conclude that

$$|T_4| \leq c \sup_{y \in \mathbb{B}_n(0,r)} |a'_{jk}(y) - a^\pi_{jk}| \cdot \|\theta z\|_{1-1/p,p}.$$

Now, observe that due to $\varphi_i(0) = x_0$ and $D\varphi_i(0) = \mathrm{id}_{\mathbb{R}^n}$, we have $a'_{jk}(0) = a_{jk}(x_0) = a^\pi_{jk}$. Consequently, the mean value theorem implies that

$$\sup_{y \in \mathbb{B}_n(0,r)} |a'_{jk}(y) - a^\pi_{jk}| \leq c \cdot r,$$

with a constant c independent of r. This proves the Lemma. □

After those (more or less) technical preliminaries we are now going to prove the main estimate.

Given $s < 0$ and $p \in (1, \infty)$, let $\boldsymbol{B}$ denote that $C(\Gamma)$-realization of $\boldsymbol{B}_{s,p}$, i.e.,

$$\mathrm{dom}(\boldsymbol{B}) := \{z \in B_p^{1+s}(\Gamma) \cap C(\Gamma); \boldsymbol{B}_{s,p} z \in C(\Gamma)\}$$

$$\text{and } \boldsymbol{B}z := \boldsymbol{B}_{s,p} z \text{ for } z \in \mathrm{dom}(\boldsymbol{B}).$$

Then we have

LEMMA 6.4. *$\boldsymbol{B}$ is a well-defined closed linear operator in $C(\Gamma)$ having dense domain. Moreover, $\boldsymbol{B}$ is the closure of $\boldsymbol{B}_{s,p}$ in $C(\Gamma)$ if $s > \dfrac{n-1}{p}$.*

PROOF. Observe that by Lemma 2.2c) and e) we know that

$$C(\Gamma) \hookrightarrow B_p^s(\Gamma) \text{ for } s < 0.$$

Consequently, the assertions follow from the definition of $\boldsymbol{B}_{s,p}$, from the first part of Theorem 4.1 (which in particular ensures that $\mu + \boldsymbol{B}_{s,p} \in \mathrm{Isom}(B_p^{s+1}(\Gamma), B_p^s(\Gamma))$ for some $\mu \in \mathbb{R}$) and from Lemma 2.2b) and d). □

THEOREM 6.5. *Suppose that $p > n$. Then there exist constants $c > 0$ and $\mu_* \geq 1$ such that*

$$|\mu|^{n/p} \sup_{1\leq i\leq m_r} \|z\|_{B_p^{1-1/p}(\tilde{U}_i)} + |\mu| \cdot \|z\|_{C(\Gamma)} \leq c\|(\mu + \boldsymbol{B})z\|_{C(\Gamma)}$$

for all $\mu \in [\mathrm{Re}\,\lambda \geq \mu_*]$ and all $z \in \mathrm{dom}(\boldsymbol{B})$, where $r := |\mu|^{\frac{n}{1-n}}$ and m_r is as in (2.19).

PROOF. Combining Corollary 3.5 and Lemma 6.3, we find that for each $\varepsilon > 0$ and each $i \in \{1, \ldots, m_r\}$:

$$\begin{aligned} &|\mu|^{n/p}|\tilde{\varphi}_i^*(\theta_i z)|_{1-1/p,p} + |\mu|\|\tilde{\varphi}_i^*(\theta_i z)\|_{BC(\mathbb{R}^{n-1})} \\ &\qquad\leq c|\mu|^{n/p}|(\mu + \boldsymbol{B}_\pi(\alpha))\tilde{\varphi}_i^*(\theta_i z)|_{-1/p,p} \\ &\qquad\leq c|\mu|^{n/p}\{|\tilde{\varphi}_i^*(\theta_i(\mu + \boldsymbol{B})z)|_{-1/p,p} \\ &\qquad+ |\boldsymbol{B}_\pi(\alpha)\tilde{\varphi}_i^*(\theta_i z) - \tilde{\varphi}_i^*(\theta_i \boldsymbol{B}z)|_{-1/p,p}\} \\ &\qquad\leq c|\mu|^{n/p}\{|\tilde{\varphi}_i^*(\theta_i(\mu + \boldsymbol{B})z)|_{-1/p,p} \\ &\qquad+ (\varepsilon + c \cdot r)\|\theta_i z\|_{1-1/p} + c(\varepsilon)\|z\|_{-1/p,p}\}. \end{aligned} \tag{6.8}$$

Now observe that $L_p(\mathbb{R}^{n-1}) \hookrightarrow B_p^{-1/p}(\mathbb{R}^{n-1})$, as Lemma 2.2 shows. Hence we find the estimate

$$\begin{aligned} |\tilde{\varphi}_i^*(\theta_i(\mu + \boldsymbol{B})z)|_{-1/p,p} &\leq c\|\tilde{\varphi}_i^*(\theta_i(\mu + \boldsymbol{B})z)\|_{L_p(\mathbb{R}^{n-1})} \\ &= c\left(\int_{\mathbb{B}_{n-1}(0,r)} |\tilde{\varphi}_i^*(\theta_i(\mu + \boldsymbol{B})z)|^p dx\right)^{1/p} \\ &\leq cr^{\frac{n-1}{p}}\|\tilde{\varphi}_i^*(\theta_i(\mu + \boldsymbol{B})z)\|_{BC(\mathbb{B}_{n-1}(0,r))} \\ &\leq cr^{\frac{n-1}{p}}\|(\mu + \boldsymbol{B})z)\|_{C(\Gamma)}. \end{aligned} \tag{6.9}$$

On the other hand, using 2.24 we find a constant $c' > 0$ such that

$$\begin{aligned} c'(|\mu|^{n/p}\|\theta_i z\|_{B_p^{1-1/p}(\tilde{U}_i)} + |\mu| \cdot \|\theta_i z\|_{BC(\tilde{U}_i)}) &\leq |\mu|^{n/p}|\tilde{\varphi}_i^*(\theta_i z)|_{1-1/p,p} \\ &+ |\mu| \cdot \|\tilde{\varphi}_i^*(\theta_i z)\|_{BC(\mathbb{R}^{n-1})}. \end{aligned}$$

Hence it follow from (6.8) and (6.9) that

$$\begin{aligned} &|\mu|^{n/p}\|\theta_i z\|_{B_p^{1-1/p}(\tilde{U}_i)} + |\mu| \cdot \|\theta_i z\|_{BC(\tilde{U}_i)} \\ &\qquad\leq c|\mu|^{\frac{n}{p}} r^{\frac{n-1}{p}}\|(\mu + \boldsymbol{B})z\|_{C(\Gamma)} + |\mu|^{\frac{n}{p}}\{c(\varepsilon)\|z\|_{-1/p,p} \\ &\qquad+ c(\varepsilon + c \cdot r)\|\theta_i z\|_{1-1/p,p}\}. \end{aligned} \tag{6.10}$$

Next observe that $\|z\|_{C(\Gamma)} = \sup_{1\le i\le m_r} \|z\|_{BC(\Gamma\cap V_i)} = \sup_{1\le i\le m_r} \|\theta_i z\|_{BC(\tilde{U}_i)}$ since $\{V_i\}_{1\le i\le m_r}$ is an open covering of $\overline{\Omega}$ and since $\theta_i|V_i = 1$. We also note that $C(\Gamma) \hookrightarrow B_p^{-1/p}(\Gamma)$. Thus (6.10) implies that

$$\begin{aligned}
&|\mu|^{n/p} \sup_{1\le i\le m_r} \|z\|_{B_p^{1-1/p}(\tilde{U}_i)} + |\mu|\cdot\|z\|_{C(\Gamma)} \\
&\le c|\mu|^{\frac{n}{p}} r^{\frac{n-1}{p}} \|(\mu+\boldsymbol{B})z\|_{C(\Gamma)} + |\mu|^{\frac{n}{p}}\{c(\varepsilon)\|z\|_{C(\Gamma)} \\
&+ c(\varepsilon + c\cdot r) \sup_{1\le i\le m_r} \|z\|_{B_p^{1-1/p}(\tilde{U}_i)}\}.
\end{aligned} \tag{6.11}$$

Finally, we choose $\varepsilon := \dfrac{1}{4c}$, $r := |\mu|^{\frac{n}{1-n}}$ and $\mu_* := [2c(\varepsilon)]^{\frac{p}{p-n}} \vee [4c^2]^{\frac{n-1}{n}} \vee 1$. Then we have $|\mu|^{\frac{n}{p}} r^{\frac{n-1}{p}} = 1$, $c(\varepsilon)|\mu|^{\frac{n}{p}} \le \dfrac{|\mu|}{2}$ and $c(\varepsilon + c\cdot r) \le \dfrac{1}{2}$ for $\mu \in [\operatorname{Re}\lambda \ge \mu_*]$. It remains to subtract the last two terms on the right-hand side in (6.11) to complete the proof. □

REMARK. Observe that m_r (the "size" of the covering $\{U_j\}_{1\le j\le m_r}$ of $\overline{\Omega}$) tends to ∞ as $r \to 0$ or, equivalently, as $|\mu| \to \infty$.

PROOF OF THE THEOREM. Theorem 6.5 and Lemma 6.4 imply immediately that $-\boldsymbol{B}$ generates a strongly continuous analytic semigroup on $C(\Gamma)$.

Since the embedding $B_p^{1-1/p}(\Gamma) \hookrightarrow C(\Gamma)$ is compact for $p > n$ (see Lemma 2.2), Theorem 2.3.3 in [22] yields the compactness of the semigroup generated by $-\boldsymbol{B}$.

Finally, observe that $B_p^{1-1/p}(\Gamma)$, $p > n$, is an invariant subspace for the resolvent of $-\boldsymbol{B}$ and that

$$(\mu+\boldsymbol{B})^{-1}|B_p^{1-1/p}(\Gamma) = (\mu+\boldsymbol{B}_{1-1/p,p})^{-1} \text{ for } \mu\in\rho(-\boldsymbol{B}).$$

On the other hand the positive cone $[B_p^{1-1/p}(\Gamma)]_+$ of $B_p^{1-1/p}(\Gamma)$ is dense in $[C(\Gamma)]_+$. But $-\boldsymbol{B}_{1-1/p,p}$ is a resolvent positive operator, as [10, Appendix] shows. Consequently, $-\boldsymbol{B}$ is a resolvent positive operator too. This completes the proof. □

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