ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

SHIGERU SAKAGUCHI

Critical points of solutions to the obstacle problem in the plane

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 21, nº 2 (1994), p. 157-173

<http://www.numdam.org/item?id=ASNSP_1994_4_21_2_157_0>

Critical Points of Solutions to the Obstacle Problem in the Plane

SHIGERU SAKAGUCHI

1. - Introduction

In [1] Alessandrini considered solutions of the Dirichlet problem for an elliptic equation without zero-order terms over a bounded simply connected domain in $\mathbb{R}^2$, and showed that if the set of local maximum points of the boundary datum consists of N connected components, then the interior critical points of the solution are finite in number and the following inequality holds

$$\sum_{j=1}^{k} m_j + 1 \leqq N, \tag{1.1}$$

where $m_1, m_2, \ldots, m_k$ denote the respective multiplicities of the interior critical points of the solution. It was shown by Hartman and Wintner in [3] that the zeros of the gradient of a non-constant solution (critical points) are isolated and each zero has finite integral multiplicity, if the coefficients of the equation are sufficiently smooth (see [1, p. 231]). Similar arguments and results were used in the theory of minimal surfaces by Schneider in [6].

In this paper we consider solutions of the obstacle problem over a bounded simply connected domain in $\mathbb{R}^2$. Our main purpose is to show that if the number of critical points of the obtacle is finite and the obstacle has only N local maximum points, then the same inequality (1.1) holds for the critical points of the solution in the noncoincidence set. We note that the multiplicity of a critical point in the noncoincidence set is well-defined if the solution is not constant near the critical point, since the solution satisfies an elliptic equation without zero-order terms in the noncoincidence set. Precisely, let Ω be a bounded simply connected domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$. Consider a function $\psi \in C^2(\overline{\Omega})$ which is negative on $\partial\Omega$ and has positive maximum in Ω. Let

Pervenuto alla Redazione il 26 Luglio 1991 e in forma definitiva il 20 Giugno 1993.

$a = (a_1, a_2)$ be a C^∞ vector field on $\mathbb{R}^2$ satisfying

$$\lambda|\xi|^2 \leqq \sum_{i,j} \frac{\partial a_i}{\partial p_j}(p)\xi_i\xi_j \leqq \Lambda|\xi|^2 \qquad \text{for all } p, \xi \in \mathbb{R}^2 \tag{1.2}$$

for some positive constants λ, Λ. Consider the following variational inequality:

Find $u \in \mathbb{K}$ satisfying

$$\int_\Omega a(\nabla u) \cdot \nabla(v - u)dx \geqq 0 \qquad \text{for all } v \in \mathbb{K}, \tag{1.3}$$

$$\mathbb{K} := \{v \in H_0^1(\Omega) : v \geqq \psi \text{ in } \Omega\}.$$

It is known that there exists a unique solution u to (1.3) and u belongs to $C^{1,1}(\overline{\Omega})$ (see the book of Kinderlehrer and Stampacchia [5]). Let I be the coincidence set

$$I = \{x \in \Omega : u(x) = \psi(x)\}. \tag{1.4}$$

Since ψ is negative on $\partial\Omega$, I is a compact set contained in Ω. Note that u satisfies the following:

$$\operatorname{div}(a(\nabla u)) = 0 \qquad \text{in } \Omega \backslash I, \tag{1.5}$$

$$\operatorname{div}(a(\nabla u)) \leqq 0 \qquad \text{in } \Omega, \tag{1.6}$$

and

$$u(x) = \inf_{g \in G} g(x) \qquad \text{for any } x \in \Omega, \tag{1.7}$$

where G is the set of Lipschitz continuous functions g over $\overline{\Omega}$ satisfying

$$\operatorname{div}(a(\nabla g)) \leqq 0 \text{ in } \Omega, \qquad g \geqq \psi \text{ in } \Omega \qquad \text{and } g \geqq 0 \text{ on } \partial\Omega \tag{1.8}$$

(see [5]).

Now our results are the following:

THEOREM 1. *Suppose that the number of critical points of ψ in $\{x \in \Omega : \psi(x) > 0\}$ is finite. If ψ has only N local maximum points, then the number of critical points of u is finite. Furthermore, denote by $m_1, \dots m_k$ the multiplicities of the critical points of u in $\Omega \backslash I$. Then the following inequality holds*

$$\sum_{j=1}^k m_j + 1 \leqq N. \tag{1.9}$$

THEOREM 2. *If ψ has only N global maximum points and has no other critical point in $\{x \in \Omega : \psi(x) > 0\}$ then equality holds in* (1.9).

Letting N be equal to 1, we have:

COROLLARY 3. *If ψ has only one critical point then u has only one critical point.*

REMARK 4. Kawohl showed in [4] that in the case $\Omega \subset \mathbb{R}^n$ ($n \geqq 2$) and $a(p) = p$, if Ω is starshaped with respect to the origin and $x \cdot \nabla\psi(x) < 0$ for $x \in \overline{\Omega}\backslash\{0\}$, then $x \cdot \nabla u(x) < 0$ in $\overline{\Omega}\backslash\{0\}$ and u has only one critical point. However, for general $a(p)$, or for non-starshaped domains Ω, similar results are not known. The typical case is $a(p) = \dfrac{p}{\sqrt{1+|p|^2}}$ (minimal surface case) with convex domain Ω. We note that in this case we can obtain a bound for the gradient of the solution and so we can modify $a(p)$ to have the condition (1.2) (see [5]).

Since a critical point in the noncoincidence set is a saddle point if the solution is not constant on some neighborhood of the point, we get a generalization of Theorem 1 as follows:

THEOREM 5. *Suppose that the number of connected components of local maximum points of ψ is equal to N. Then the number of saddle points of u in $\Omega\backslash I$ is finite and the same inequality* (1.9) *holds for these saddle points.*

In Section 2 we prove Theorem 1 and in Section 3 we prove Theorem 2. The proof of Theorem 5 is similar to that of Theorem 1. Section 4 provides some examples of Theorem 2. In Section 5 we generalize the results of the obstacle problem for the general non-constant boundary data.

2. - Proof of Theorem 1

We begin with the following five basic lemmas.

LEMMA 2.1. *u is not constant over any open subset of $\Omega\backslash I$.*

PROOF. Since $u = 0$ on $\partial\Omega$, (1.6) and the strong maximum principle imply that u is positive in Ω. Suppose that there exists a connected open set ω contained in $\Omega\backslash I$ where u is a positive constant. Since the theorem of Hartman and Wintner implies the unique continuation of solutions, it follows from (1.5) that u equals the same constant over the connected component $\tilde{\omega}$ of $\Omega\backslash I$ containing ω. Since $u = 0$ on $\partial\Omega$, then $\partial\tilde{\omega} \subset I$. Hence $\nabla\psi = \nabla u = 0$ on $\partial\tilde{\omega}$, since $u - \psi$ attains a minimum on I. This contradicts the assumption that the number of critical points of ψ in $\{x \in \Omega : \psi(x) > 0\}$ is finite. □

LEMMA 2.2. *For any* $t \in (0, \max_{\Omega} \psi)$ *we have the following:*

(1) *The set* $\{x \in \Omega;\ u(x) < t\}$ *is connected.*

(2) *Any connected component of* $\{x \in \Omega : u(x) > t\}$ *is simply connected.*

PROOF. Since $u = 0$ on $\partial\Omega$ and $\partial\Omega$ is connected, only one component of $\{x \in \Omega : u(x) < t\}$ reaches $\partial\Omega$. Suppose that there exists another component, say ω. Then $\partial\omega \subset \Omega$. Since $u = t$ on $\partial\omega$, it follows from (1.6) and the maximum principle that $u > t$ in ω. This is a contradiction and we get (1). Let A be a connected component of $\{x \in \Omega : u(x) > t\}$ and let γ be a simple closed curve in A. By the Jordan curve theorem there exists a bounded domain B with $\partial B = \gamma$. Since Ω is simply connected, B is contained in Ω. Then, since $u > t$ on γ, the maximum principle and (1.6) yield $u > t$ in B. This shows that B is contained in A, namely A is simply connected and we get (2). □

LEMMA 2.3. *We have the following:*

(1) *The interior critical points of* u *in* $\Omega\backslash I$ *are isolated.*

(2) u *has no local maximum point in* $\Omega\backslash I$.

(3) u *has no local minimum point in* Ω.

PROOF. In view of Lemma 2.1, we obtain (1) and (2) from (1.5) and the results of Hartman and Wintner [3] (see [1, p. 231]). (3) is a direct consequence of (1.6) and the maximum principle. □

LEMMA 2.4. *Any local maximum point of* u *in* Ω *is also a local maximum point of* ψ*, and the number of the local maximum points of* u *in* Ω *is at most* N.

PROOF. Since $u = 0$ on $\partial\Omega$, by Lemma 2.3(2) any local maximum point of u belongs to I. Then, since $u \geq \psi$ in Ω and $u = \psi$ on I, any local maximum point of u is also a local maximum point of ψ. Since ψ has only N local maximum points, we get this lemma. □

LEMMA 2.5. *Let* $x_0 \in \Omega\backslash I$ *be an interior critical point of* u *in* $\Omega\backslash I$*, and let* m *be its multiplicity. Then* $m+1$ *distinct connected components of the level set* $\{x \in \Omega : u(x) > u(x_0)\}$ *cluster round the point* x_0.

PROOF. By the results of Hartman and Wintner [3], in a neighborhood of x_0 the level line $\{x \in \Omega : u(x) = u(x_0)\}$ consists of $m+1$ simple arcs intersecting at x_0 (see [1, p. 231]). Since any component of the level set $\{x \in \Omega : u(x) > u(x_0)\}$ cannot surround a component of $\{x \in \Omega : u(x) < u(x_0)\}$ by Lemma 2.2(1), all the components of $\{x \in \Omega : u(x) > u(x_0)\}$ clustering round x_0 have to be distinct. This completes the proof. □

Since any connected component of a level set $\{x \in \Omega : u(x) > t\}$ with $t \in \mathbb{R}$ contains at least one local maximum point of u, Lemma 2.4 and Lemma 2.5 suggest to count the number of disjoint components of $\{x \in \Omega : u(x) > t\}$ by using multiplicities. The first step is the following:

LEMMA 2.6. *Let* $x_1, \ldots, x_n \in \Omega \backslash I$ *be the interior critical points of* u *in* $\Omega \backslash I$ *and let* $m_1, \ldots m_n$ *be their respective multiplicities. Suppose that* $u(x_1) = \cdots = u(x_n) = t$ *for some* $t \in \mathbb{R}$, *and suppose that all the points* $x_1, \ldots, x_n$ *together with the components of* $\{x \in \Omega : u(x) > t\}$ *clustering round these points form a connected set. Then this connected set contains exactly* $\sum_{j=1}^{n} m_j + 1$ *connected components of the level set* $\{x \in \Omega : u(x) > t\}$.

PROOF. We prove this by induction on the number n of critical points. When $n = 1$, the result holds by Lemma 2.5. Assume that $k \geq 1$ and that if $n \leq k$ then the connected set, which consists of n critical points and the components clustering round these points, contains exactly $\sum_{j=1}^{n} m_j + 1$ components of the level set $\{x \in \Omega : u(x) > t\}$. Let $n = k + 1$. Let A be the set which consists of the points $x_1, \ldots, x_{k+1}$ together with the respective components clustering round these points. Since A is connected, up to a renumbering we may assume that the points $x_1, \ldots, x_k$ together with the respective components clustering round these points form a connected set, say B. By Lemma 2.2(1) A cannot surround a component of $\{x \in \Omega : u(x) < t\}$. Therefore there is only one component of $\{x \in \Omega : u(x) > t\}$ whose boundary contains both x_{k+1} and x_j for some $1 \leq j \leq k$, since both A and B are connected. Hence, using Lemma 2.5 and applying the inductive assumption to B, we see that A contains exactly

$$\left(\sum_{j=1}^{k} m_j + 1 \right) + (m_{k+1} + 1) - 1$$

connected components of the level set $\{x \in \Omega : u(x) > t\}$. This completes the proof. □

Using this we obtain:

LEMMA 2.7. *Let* $x_1, \ldots, x_k \in \Omega \backslash I$ *be the interior critical points of* u *in* $\Omega \backslash I$ *and let* $m_1, \ldots, m_k$ *be their respective multiplicities. Then* u *has at least* $\sum_{j=1}^{k} m_j + 1$ *local maximum points in* Ω.

PROOF. In case $u(x_1) = \cdots = u(x_k) = t$, if the points $x_1, \ldots, x_k$ together with the components of $\{x \in \Omega : u(x) > t\}$ clustering round these points form q connected sets, by applying Lemma 2.6 to each set we see that these sets contain exactly $\sum_{j=1}^{k} m_j + q$ connected components of the level set $\{x \in \Omega : u(x) > t\}$. Therefore, in this case the level set always has at least $\sum_{j=1}^{k} m_j + 1$ connected components, and u has at least $\sum_{j=1}^{k} m_j + 1$ local maximum points in Ω.

Hence, without loss of generality, we may assume that

$$\begin{aligned}(2.1)\qquad & u(x_1) = \cdots = u(x_{j_1}) < u(x_{j_1+1}) = \cdots = u(x_{j_2}) < \cdots < \\ & < u(x_{j_s+1}) = \cdots = u(x_{j_{s+1}})\end{aligned}$$

where $j_{s+1} = k$ and $s \geq 1$. For $1 \leq n \leq s+1$ let I_{j_n} be the set of all components of the open sets $\{x \in \Omega : u(x) > u(x_j)\}$ $(j = 1, \ldots, j_n)$, and let J_{j_n} be the subset of those members ω of I_{j_n} such that either ω is a component of $\{x \in \Omega : u(x) > u(x_{j_n})\}$ or ω is a component of $\{x \in \Omega : u(x) > u(x_{j_i})\}$ for some $1 \leq i \leq n-1$ satisfying

$$\max_{\omega} u \leq u(x_{j_{i+1}}).$$

Note that $J_{j_1} = I_{j_1}$. By the definition, J_{j_n} consists of disjoint components. Denote by $|J_{j_n}|$ the number of elements of J_{j_n}. Let us show that $|J_{j_\ell}| \geq \sum_{j=1}^{j_\ell} m_j + 1$ by induction on the number ℓ. When $\ell = 1$, we have already shown this by Lemma 2.6. Suppose that $|J_{j_p}| \geq \sum_{j=1}^{j_p} m_j + 1$ for $p \geq 1$. Let $\ell = p+1$. Then, by (2.1) $\{x_{j_p+1}, \ldots, x_{j_{p+1}}\} \subset \cup_{\omega \in J_{j_p}} \omega$, and each x_j $(j = j_p+1, \ldots, j_{p+1})$ belongs to some $\omega \in J_{j_p}$ which is a component of $\{x \in \Omega : u(x) > u(x_{j_p})\}$. Let $\{x_{j_p+1}, \ldots, x_{j_{p+1}}\}$ be contained in exactly q components $\omega_1, \ldots, \omega_q$. Then, counting the number of components of $\{x \in \Omega : u(x) > u(x_{j_{p+1}})\}$ in each $\omega_j (j = 1, \ldots, q)$ with the help of Lemma 2.6, in view of the definition of $J_{j_{p+1}}$ we obtain

$$\begin{aligned}|J_{j_{p+1}}| &\geq |J_{j_p}| + \left(\sum_{j=j_p+1}^{j_{p+1}} m_j + q\right) - q \\ &= |J_{j_p}| + \sum_{j=j_p+1}^{j_{p+1}} m_j.\end{aligned}$$

Therefore, by the inductive assumption we get

$$|J_{j_{p+1}}| \geq \sum_{j=1}^{j_{p+1}} m_j + 1.$$

This completes the proof. □

By Lemma 2.7 and Lemma 2.4 we get

$$\sum_{j=1}^{k} m_j + 1 \leq N.$$

This shows that the number of interior critical points of u in $\Omega\backslash I$ is finite and the proof of Theorem 1 is completed, since u has no critical point on $\partial\Omega$ by virtue of Hopf's boundary point lemma (see the book of Gilbarg and Trudinger [2, Lemma 3.4, p. 34]) and $\nabla u = \nabla\psi$ on I.

3. - Proof of Theorem 2

By Theorem 1 equality in (1.9) is trivial in case $N = 1$. Therefore we consider the case $N \geq 2$. Let $p_1, \dots, p_N$ be the global maximum points of ψ. By considering $g(x) \equiv \max_\Omega \psi$ in (1.7) we get $(0 <) u \leq \max_\Omega \psi$ in Ω. Then, all the points $p_1, \dots, p_N$ belong to I and are all local maximum points of u by Lemma 2.4. Since by Theorem 1 the critical points of u are finite in number and ψ has no critical point in $\{x \in \Omega : \psi(x) > 0\}$ other than $p_1, \dots, p_N$, the set of all critical points of u consists of a finite number of saddle points in $\Omega\backslash I$ and $p_1, \dots, p_N$. Therefore, by using the implicit function theorem and the theorem of Hartman and Wintner we obtain the following about properties of the level curves of u.

LEMMA 3.1. *Let $0 < t < \max_\Omega \psi$ and let γ be a connected component of the level curve $\{x \in \Omega : u(x) = t\}$. Then we get the following:*

(1) *If γ does not contain any critical point of u, then γ is a simple C^1 regular closed curve which surrounds at least one point of $\{p_1, \dots, p_N\}$.*

(2) *If γ contains at least one critical point of u, then γ is a finite collection of simple piecewise C^1 regular closed curves, each of which meets the others exclusively at critical points of u and surrounds at least one point of $\{p_1, \dots, p_N\}$.*

(3) *Each γ does not surround any other component.*

(4) *The number of connected components of the level curve $\{x \in \Omega : u(x) = t\}$ is finite.*

PROOF. Proof of (1): Since the level curve in question is compact, the implicit function theorem guarantees that γ is a simple C^1 regular closed curve. In view of the simply-connectedness of Ω, (1.6) and the maximum principle imply that γ is the boundary of a component of $\{x \in \Omega : u(x) > t\}$. Hence, by Lemma 2.4, γ has to surround at least one point of $\{p_1, \dots, p_N\}$. Proof of (2): With the help of the theorem of Hartman and Wintner, we can prove this along the line of the proof of (1). Proof of (3): This is a consequence of the maximum principle as in the proof of (1). Proof of (4): This is a direct consequence of (1), (2) and (3), since the number of elements of the set $\{p_1, \dots, p_N\}$ is finite. $\square$

Since the critical points of u are finite in number and ψ has no critical point in $\{x \in \Omega : \psi(x) > 0\}$ other than $p_1, \dots, p_N$, it follows from Lemma 2.3(2) that there exists a small number $r > 0$ which satisfies the following:

$$\max_{\partial B_r(p_j)} u < \max_{\Omega} \psi \qquad \text{for any } j, \tag{3.1}$$

$$\nabla u(x) \neq 0 \qquad \text{for any } x \in \bigcup_{j=1}^{N} \overline{B_r(p_j)} \setminus \{p_1, \dots, p_N\}, \tag{3.2}$$

$$\overline{B_r(p_i)} \cap \overline{B_r(p_j)} = \emptyset \text{ if } i \neq j, \tag{3.3}$$

where each $B_r(p_j)$ denotes an open ball with radius r centered at p_j. Hence there exists a sufficiently small number $\delta > 0$ such that $\nabla u \neq 0$ in $\{x \in \Omega : \max_{\Omega} \psi - \delta \leq u(x) < \max_{\Omega} \psi\}$. Therefore, it follows from Lemma 3.1 that the set $\{x \in \Omega : u(x) = \max_{\Omega} \psi - \eta\}$ consists of N simple C^1 regular closed curves for any $0 < \eta \leq \delta$.

Let us show that there exists at least one critical point of u in $\Omega \setminus I$. Suppose that $\nabla u \neq 0$ in $\Omega \setminus I$. Then $\nabla u \neq 0$ in $\left\{x \in \overline{\Omega} : u(x) \leq \max_{\Omega} \psi - \frac{1}{2}\delta\right\}$, since $\nabla u \neq 0$ on $\partial\Omega$ by virtue of Hopf's boundary point lemma. Therefore, with the help of the implicit function theorem, we see that $\{x \in \Omega : u(x) = \max_{\Omega} \psi - \delta\}$ is C^1-diffeomorphic to $\partial\Omega (= \{x \in \overline{\Omega} : u(x) = 0\})$. Indeed, let $\Gamma_s = \{x \in \overline{\Omega} : u(x) = s\}$. By Lemma 3.1, for $0 < s \leq \max_{\Omega} \psi - \frac{1}{2}\delta$, Γ_s consists of a finite number of simple C^1 regular closed curves, each of which does not surround any other one. Furthermore the interior domain surrounded by Γ_s is the level set $\{x \in \Omega : u(x) > s\}$. Since the interior normal derivative of u with respect to this domain is positive on Γ_s, with the help of the implicit function theorem we see that for any $0 < s \leq \max_{\Omega} \psi - \frac{1}{2}\delta$ there exists a small number $\varepsilon > 0$ such that Γ_t is C^1-diffeomorphic to Γ_s for all $s - \varepsilon < t < s + \varepsilon$. Also, for Γ_0 there exists a small number $\varepsilon_0 > 0$ such that Γ_t is C^1-diffeomorphic to $\Gamma_0 (= \partial\Omega)$ for all $0 < t < \varepsilon_0$. Since the interval $[0, \max_{\Omega} \psi - \delta]$ is compact, we conclude that $\Gamma_{\max_{\Omega} \psi - \delta}$ is C^1- diffeomorphic to $\partial\Omega$. This is a contradiction, since $N \geq 2$. Then, there exists at least one critical point of u in $\Omega \setminus I$.

Let $x_1, \dots, x_k \in \Omega \setminus I$ be the critical points of u and let $m_1, \dots, m_k$ be the respective multiplicities. We may assume that there is no critical point of u in Ω except the points $x_1, \dots, x_k, p_1, \dots, p_N$.

As in the proof of Lemma 2.7, we first consider the case $u(x_1) = \cdots = u(x_k) = t$ for some $t \in \mathbb{R}$. Since $\nabla u \neq 0$ in $\{x \in \overline{\Omega} : u(x) < t\}$, then by the same arguments as above $\{x \in \overline{\Omega} : u(x) = s\}$ is C^1-diffeomorphic to $\partial\Omega$ for any $0 < s < t$. Therefore, by the continuity, the level curve $\{x \in \Omega : u(x) = t\}$ is connected. Indeed, in view of Lemma 3.1(4), suppose that this level curve has exactly h components ($h \geq 2$), say $\gamma_1, \dots, \gamma_h$. Then the set $\{x \in \Omega : u(x) \geq t\}$

has exactly h components $A_1, \dots, A_h$ with boundaries $\gamma_1, \dots, \gamma_h$ respectively. Put $B_j = \{x \in \Omega;\ \mathrm{dist}(x, A_j) < \eta\}$ for a small number $\eta > 0$ and for each $1 \le j \le h$. Since each A_j is compact, by choosing $\eta > 0$ sufficiently small we may assume that $B_i \cap B_j = \emptyset$ if $i \neq j$. Since $u(x) < t$ for any $x \in \overline{\Omega} \setminus \bigcup_{j=1}^{h} A_j$, there exists a number $\varepsilon > 0$ such that $u(x) \le t - \varepsilon$ for all $x \in \overline{\Omega} \setminus \bigcup_{j=1}^{h} B_j$. Consider the level curve $\Gamma_s (= \{x \in \Omega : u(x) = s\})$ for $t - \frac{1}{2}\varepsilon < s < t$. Then, in view of Lemma 3.1 we see that this level curve Γ_s consists of h simple C^1 regular closed curves, each of which is contained in $B_j \setminus A_j (j = 1, \dots, h)$. This contradicts the fact that Γ_s is C^1-diffeomorphic to $\partial\Omega$.

Hence, by Lemma 2.6 and Lemma 3.1 the number of connected components of $\{x \in \Omega : u(x) > t\}$ is exactly $\sum_{j=1}^{k} m_j + 1$ and each component contains at least one point of $\{p_1, \dots, p_N\}$. Of course all the points $p_1, \dots, p_N$ are contained in these components. Furthermore, each component contains exactly one point of $\{p_1, \dots, p_N\}$. Indeed, suppose that there exists a component containing at least two points of $\{p_1, \dots p_N\}$, say ω. By Lemma 2.2(2), we note that ω is simply connected. Furthermore, using the theorem of Hartman and Wintner, we see that there exists a small number $\varepsilon > 0$ which satisfies

$$(3.4) \qquad \{x \in \omega : u(x) = t + \varepsilon\} \qquad \text{is a simple } C^1 \text{ regular closed curve.}$$

Of course we already know that

$$(3.5) \qquad \begin{array}{c}\{x \in \omega : u(x) = \max_{\Omega} \psi - \delta\} \text{ consists of at least two simple-} \\ C^1 \text{ regular closed curves.}\end{array}$$

On the other hand, since $\nabla u \neq 0$ in $\{x \in \omega : t < u(x) < \max_{\Omega} \psi\}$, by using the implicit function theorem as above, we see that $\{x \in \omega : u(x) = t + \varepsilon\}$ is C^1-diffeomorphic to $\{x \in \omega : u(x) = \max_{\Omega} \psi - \delta\}$. This contradicts (3.4) and (3.5). Consequently, we get $\sum_{j=1}^{k} m_j + 1 = N$.

Consider the general case as in the proof of Lemma 2.7. We use the same notation as in the proof of Lemma 2.7 (see (2.1)). We want to prove that $|J_k| = \sum_{j=1}^{k} m_j + 1$. Let us prove that $|J_{j_\ell}| = \sum_{j=1}^{j_\ell} m_j + 1$ for $1 \le \ell \le s + 1$ by induction on the number ℓ. We remark that since any local maximum point of u is a global maximum point of u, by definition J_{j_n} consists of components of $\{x \in \Omega : u(x) > u(x_{j_n})\}$ only. When $\ell = 1$, we have already shown this as in the case $u(x_1) = \cdots = u(x_k) = t$ for some $t \in \mathbb{R}$. Suppose that $|J_{j_p}| = \sum_{j=1}^{j_p} m_j + 1$ for $p \geq 1$. Let $\ell = p+1$. Then $\{x_{j_p+1}, \dots, x_{j_{p+1}}\} \subset \cup_{\omega \in J_{j_p}} \omega$ and each x_j $(j = j_p + 1, \dots, j_{p+1})$ belongs to some $\omega \in J_{j_p}$ which is a component of $\{x \in \Omega : u(x) > u(x_{j_p})\}$. Let $\{x_{j_p+1}, \dots, x_{j_{p+1}}\}$ be contained in exactly q components $\omega_1, \dots, \omega_q$. In each ω_i $(1 \leq i \leq q)$ the level curve $\{x \in \omega_i : u(x) = u(x_{j_{p+1}})\}$ is connected. Indeed, since each ω_i is simply connected by Lemma 2.2(2), using the theorem of Hartman and Wintner we see that $\{x \in \omega_i : u(x) > u(x_{j_p}) + \varepsilon\}$ is a simple C^1 regular closed curve for small $\varepsilon > 0$. Furthermore, since $\nabla u \neq 0$ for $\{x \in \omega_i : u(x_{j_p}) + \varepsilon \leq u(x) < u(x_{j_{p+1}})\}$, by the same argument as in the case $u(x_1) = \dots = u(x_k) = t$ for some $t \in \mathbb{R}$ we get the above conclusion.

Therefore, in view of this and Lemma 2.6 counting the number of components of $\{x \in \Omega : u(x) > u(x_{j_{p+1}})\}$ in each ω_i $(i = 1, \dots q)$, we get

$$
\begin{aligned}
|J_{j_{p+1}}| &= |J_{j_p}| + \left(\sum_{j=j_p+1}^{j_{p+1}} m_j + q \right) - q \\
&= |J_{j_p}| + \sum_{j=j_p+1}^{j_{p+1}} m_j.
\end{aligned}
$$

This shows that $|J_k| = \sum_{j=1}^{k} m_j + 1$. Finally, since $\nabla u \neq 0$ in $\{x \in \Omega : u(x_k) < u(x) < \max_{\Omega} \psi\}$, as in the case $u(x_1) = \dots = u(x_k) = t$ for some $t \in \mathbb{R}$, we obtain a one-to-one correspondence between J_k and $\{p_1, \dots, p_N\}$. Therefore we get $|J_k| = N$, which completes the proof. □

4. - Some examples

We give a few examples in the situation of Theorem 2. The first example shows that there exists a critical point with an arbitrarily large multiplicity. Precisely, let Ω be a unit open ball in $\mathbb{R}^2$ centered at the origin. Consider $a(p)$ defined by $a(p) = b(|p|)p$ for some real valued positive function $b(\cdot)$. We introduce polar coordinates (r, θ). Fix an integer $m \geq 1$. Put $\alpha = 2\pi/(m+1)$. Consider $m+1$ balls B_k $(k = 0, 1, \dots, m)$ centered at $P_k = (1/2, k\alpha)$ with radius $r > 0$. We choose r sufficiently small to make these balls pairwise disjoint. Let φ be a radially symmetric smooth function on $B = \{x \in \mathbb{R}^2 : |x| \leq r\}$ which satisfies the following conditions:

$$
\max_B \varphi > 0 \quad \text{and} \quad \varphi < 0 \quad \text{on } \partial B, \tag{4.1}
$$

$$
\varphi(0) = \max_B \varphi \quad \text{and} \quad \nabla\varphi \neq 0 \quad \text{in } B - \{0\}. \tag{4.2}
$$

EXAMPLE 1. Consider an obstacle $\psi \in C^2(\overline{\Omega})$ which satisfies the following conditions:

$$\psi(x) = \varphi(x - P_k) \qquad \text{for } x \in B_k \ (k = 0, 1, \dots, m), \tag{4.3}$$

$$\psi(x) < 0 \qquad \text{in } \Omega \setminus \bigcup_{k=0}^{m} B_k. \tag{4.4}$$

Then, by symmetry, the origin is a critical point of the solution u. Furthermore, by Theorem 2 and the symmetry the origin is the only critical point of u in $\Omega \setminus I$ and the multiplicity of the origin is exactly m. Here $N = m+1$ in Theorem 2.

EXAMPLE 2. Consider an obstacle $\psi \in C^2(\overline{\Omega})$ satisfying (4.3) and the following conditions:

$$\psi(x) = \varphi(x) \qquad \text{for } x \in B, \tag{4.5}$$

$$\psi(x) < 0 \qquad \text{in } \Omega \setminus \left\{ \bigcup_{k=0}^{m} B_k \cup B \right\}. \tag{4.6}$$

Then, by symmetry, there exist $m+1$ critical points $(r_1, k\alpha)$ $(k = 0, \dots, m)$ for some $0 < r_1 < 1/2$, and the multiplicity of each point equals 1. Here $N = m+2$ in Theorem 2.

EXAMPLE 3. Let $Q_{j,k} = (j/3, k\alpha)$ in polar coordinates for $j = 1, 2$ and $k = 0, 1, \dots, m$. Let $B_{j,k}$ be a ball in $\mathbb{R}^2$ centered at $Q_{j,k}$ with radius r for each j and k. Of course we choose r sufficiently small. Consider an obstacle $\psi \in C^2(\overline{\Omega})$ which satisfies the following conditions:

$$\psi(x) = \varphi(x - Q_{j,k}) \qquad \text{for } x \in B_{j,k} \text{ and for any } j, k, \tag{4.7}$$

$$\psi(x) < 0 \qquad \text{in } \Omega \setminus \bigcup_{j=1}^{2} \bigcup_{k=0}^{m} B_{j,k}. \tag{4.8}$$

Then, by symmetry, the set of all critical points of the solution in $\Omega \setminus I$ consists of the origin with multiplicity m and $m+1$ points $(r_2, k\alpha)$ with multiplicity 1 $(k = 0, 1, \dots, m)$ for some $1/3 < r_2 < 2/3$. Here $N = 2m+2$ in Theorem 2.

5. - Obstacle problems for general non-constant boundary data

Let $f \in C^2(\overline{\Omega})$ be a non-constant function on $\partial\Omega$. Fix a function $\psi \in C^2(\overline{\Omega})$ satisfying $\psi < f$ on $\partial\Omega$. Consider the variational inequality:

Find $u \in \tilde{\mathbb{K}}$ satisfying

$$\int_\Omega a(\nabla u)\cdot\nabla(v-u)dx \geq 0 \qquad \text{for all } v\in\tilde{\mathbb{K}}, \tag{5.1}$$

$$\tilde{\mathbb{K}} := \{v\in H^1(\Omega): v\geq\psi \text{ in } \Omega \text{ and } v=f \text{ on } \partial\Omega\}.$$

It is known that there exists a unique solution u to (5.1) and u belongs to $C^{1,1}(\Omega)\cap C^1(\overline{\Omega})$. Let I be the coincidence set as in (1.4). Since $\psi < f$ on $\partial\Omega$, I is a compact set contained in Ω or empty. Then, only replacing "$g\geq 0$ on $\partial\Omega$" by "$g\geq f$ on $\partial\Omega$" in (1.8), we get (1.5), (1.6) and (1.7). By (1.6), (1.7) and the maximum principle we see that $\min_{\partial\Omega} f < u < \max\{\max_{\Omega}\psi, \max_{\partial\Omega} f\}$ in Ω. Our results for general non-constant boundary data are the following:

THEOREM 6. *Suppose that the number of critical points of ψ in $\{x\in\Omega: \psi(x)>w(x)\}$ is finite, where $w\in C^2(\Omega)\cap C^1(\overline{\Omega})$ is the unique solution of the Dirichlet problem*

$$\operatorname{div}(a(\nabla w)) = 0 \ \textit{in}\ \Omega, \qquad w=f\ \textit{on}\ \partial\Omega.$$

If ψ has only N local maximum points and $f|_{\partial\Omega}$ has only M local maximum points, then the number of critical points of u in Ω is finite and the following inequality holds

$$\sum_{j=1}^k m_j + 1 \leq N+M, \tag{5.2}$$

where $m_1,\dots,m_k$ are the multiplicities of the critical points of u in $\Omega\backslash I$.

THEOREM 7. *Assume the hypotheses of Theorem 6 and the following:*

(1) *ψ has only N global maximum points and has no other critical point in $\{x\in\Omega:\psi(x)>w(x)\}$.*

(2) *$f|_{\partial\Omega}$ has only M global maximum points and M global minimum points, and the tangential derivative of f on $\partial\Omega$ is not zero out of these points.*

(3) $\max_{\partial\Omega} f = \max_{\Omega}\psi$.

Then equality holds in (5.2).

REMARK 8. The assumption (2) of Theorem 7 is the same as that for the boundary data in Theorem 3.1 of Alessandrini (see [1, p. 243]). Although we can prove these theorems almost along the same lines as in the proofs of Theorem 1 and Theorem 2, what we additionally have to consider is that the solution may have a local maximum point on the boundary $\partial\Omega$ and that the level curves of the solution may meet the boundary $\partial\Omega$.

PROOF OF THEOREM 6. If $\{x\in\Omega:\psi(x)>w(x)\}$ is empty, then $u=w$ in Ω. Therefore the theorem follows from the result of Alessandrini [1]. Assume that $\{x \in \Omega : \psi(x) > w(x)\}$ is not empty. Then it follows from (1.6) and the comparison principle that $u(x) > w(x)$ in Ω. Of course I is not empty and I is contained in $\{x \in \Omega : \psi(x) > w(x)\}$.

We first prove Lemma 2.1. Suppose that there exists a connected component ω of $\Omega \backslash I$ such that u is constant in ω. Then $\partial\omega \cap \Omega$ is not empty, since f is a non-constant function on $\partial\Omega$. Hence $\partial\omega \cap I$ contains an infinite number of points. This contradicts the assumption that the number of critical points of ψ in $\{x \in \Omega;\ \psi(x) > w(x)\}$ is finite, since $I \subset \{x \in \Omega;\ \psi(x) > w(x)\}$. This shows Lemma 2.1.

Instead of Lemma 2.2 we have the following:

LEMMA 2.2*. *For any* $t \in (\min_{\partial\Omega} f, \max\{\max_{\Omega} \psi, \max_{\partial\Omega} f\})$ *we have the following:*

(1) *The set* $\{x \in \Omega;\ u(x) < t\} \cup \partial\Omega$ *is connected.*

(2) *Any connected component of* $\{x \in \Omega : u(x) > t\}$ *is simply connected.*

PROOF. It follows from the maximum principle and (1.6) that any component of $\{x \in \Omega : u(x) < t\}$ has to meet the boundary $\partial\Omega$. Therefore we get (1). The proof of (2) is the same as that of Lemma 2.2. □

We get Lemma 2.3 by the same proof. Instead of Lemma 2.4 we have the following:

LEMMA 2.4*. *Any local maximum point of* u *in* $\overline{\Omega}$ *is also a local maximum point of either* $f|_{\partial\Omega}$ *or* ψ, *and the number of local maximum points of* u *in* $\overline{\Omega}$ *is at most* $N + M$.

PROOF. By Lemma 2.3(2) any local maximum point of u in $\overline{\Omega}$ belongs to $\partial\Omega \cup I$. Let p be a local maximum point of u. If p belongs to I, then p is a local maximum point of ψ. If p belongs to $\partial\Omega$, then p is a local maximum point of $f|_{\partial\Omega}$. By the assumptions of ψ and f we get this lemma. □

Using Lemma 2.2*(1) instead of Lemma 2.2(1) we get Lemma 2.5 and Lemma 2.6 by the same proofs. Instead of Lemma 2.7 we have the following:

LEMMA 2.7*. *Let* $x_1, \dots, x_k \in \Omega \backslash I$ *be the interior critical points of* u *in* $\Omega \backslash I$ *and let* $m_1, \dots, m_k$ *be their respective multiplicities. Then* u *has at least* $\sum_{j=1}^{k} m_j + 1$ *local maximum points in* $\overline{\Omega}$.

PROOF. Since in the situation of Theorem 6 any connected component of a level set $\{x \in \overline{\Omega} : u(x) > t\}$ has at least one local maximum point of u in $\overline{\Omega}$ (not in Ω), the difference between Lemma 2.7 and Lemma 2.7* is only concerned with the location of local maximum points of u. Therefore the proof is the same as that of Lemma 2.7. □

By Lemma 2.7* and Lemma 2.4* we get inequality (5.2). This shows that the number of the interior critical points of u in $\Omega \backslash I$ is finite and the proof of Theorem 6 is complete, since $\nabla u = \nabla \psi$ on I. □

PROOF OF THEOREM 7. Let $p_1, \dots, p_N$ be the global maximum points of ψ and let $q_1, \dots, q_M$ and $z_1, \dots, z_M$ be respectively the global maximum points of $f|_{\partial\Omega}$ and the global minimum points of $f|_{\partial\Omega}$. By considering $g(x) \equiv \max_{\Omega} \psi (= \max_{\partial\Omega} f)$ in (1.7), we get $u \leq \max_{\Omega} \psi$ in Ω. Then all the points $p_1, \dots, p_N$ belong to I. Therefore by Lemma 2.4* the set of all local maximum points of u in $\overline{\Omega}$ consists of the points $p_1, \dots, p_N, q_1, \dots, q_M$. On the other hand, it follows from (1.6) and the maximum principle that $\min_{\partial\Omega} f < u$ in Ω. Hence we see that $\min_{\partial\Omega} f < u < \max_{\partial\Omega} f$ in $\Omega \backslash I$. Therefore it follows from (1.5) and Hopf's boundary point lemma that the interior normal derivative of u on $\partial\Omega$ is positive at z_j $(j = 1, \dots, M)$ and it is negative at $q_j (j = 1, \dots, M)$. Consequently, by the assumption (2) of Theorem 7 we get $\nabla u \neq 0$ on $\partial\Omega$. Hence, by Theorem 6 and assumption (1) the set of all critical points of u consists of a finite number of saddle points in $\Omega \backslash I$ and $p_1, \dots, p_N$. Therefore, by using the implicit function theorem, the theorem of Hartman and Wintner and the maximum principle, instead of Lemma 3.1 we can obtain the following about properties of the level curves of u:

LEMMA 3.1*. *Let* $\min_{\partial\Omega} f < t < \max_{\Omega} \psi (= \max_{\partial\Omega} f)$ *and let* γ *be a connected component of the level curve* $\{x \in \overline{\Omega} : u(x) = t\}$. *Then we get the following:*

(1) *If* γ *does not contain any critical point of* u, *then either* γ *is a simple* C^1 *regular closed curve in* Ω *surrounding at least one point of* $\{p_1, \dots, p_N\}$ *or* γ *is a simple* C^1 *regular arc having endpoints on* $\partial\Omega$.

(2) *If* γ *contains at least one critical point of* u, *then* γ *is a finite collection of the following two kinds of curves* (a) *or* (b), *each of which meets the others exclusively at critical points of* u: (a) *simple piecewise* C^1 *regular closed curves surrounding at least one point of* $\{p_1, \dots, p_N\}$, (b) *simple piecewise* C^1 *regular arcs having endpoints on* $\partial\Omega$.

(3) *Each* γ *does not surround any other component.*

(4) *The number of connected components of the level curve* $\{x \in \overline{\Omega} : u(x) = t\}$ *is finite.*

(5) *Every endpoint of arcs as in* (1) *and* (2) *on* $\partial\Omega$ *belongs to only one arc which has non-zero angle against the boundary* $\partial\Omega$, *and the number of endpoints is exactly* $2M$.

Since the critical points of u in $\overline{\Omega}$ are finite in number, it follows from (2) and (3) of Lemma 2.3 that there exists a small number $r > 0$ which satisfies the following:

$$\max_{\partial B_r(y) \cap \overline{\Omega}} u < \max_{\Omega} \psi, \tag{5.3}$$

$$\nabla u(x) \neq 0 \qquad \text{for any } x \in \left(\overline{B_r(y)} \cap \overline{\Omega}\right) \backslash \{y\} \tag{5.4}$$

for any $y \in \{p_1, \dots, p_N, q_1, \dots, q_M\}$, and

$$\min_{\partial B_r(z) \cap \overline{\Omega}} u > \min_{\partial\Omega} f, \tag{5.5}$$

$$\nabla u(x) \neq 0 \qquad \text{for any } x \in \overline{B_r(z)} \cap \overline{\Omega} \tag{5.6}$$

for any $z \in \{z_1, \dots, z_M\}$. Of course we choose $r > 0$ sufficiently small to make these balls pairwise disjoint. Hence there exists a sufficiently small number $\delta > 0$ such that $\nabla u \neq 0$ in $\{x \in \overline{\Omega} : u(x) \leq \min_{\partial\Omega} f + \delta \text{ or } \max_{\Omega} \psi - \delta \leq u(x) < \max_{\Omega} \psi\}$. Therefore by Lemma 3.1* we get:

LEMMA 5.1. *For any $0 < \eta \leq \delta$ we have the following:*

(1) $\{x \in \overline{\Omega} : u(x) = \max_{\Omega} \psi - \eta\}$ *consists either of* (a) *N simple C^1 regular closed curves in Ω surrounding only one point of $\{p_1, \dots, p_N\}$ or of* (b) *M simple C^1 regular arcs, each of which has endpoints on $\partial\Omega$ and is a part of the boundary of a component of the level set $\{x \in \overline{\Omega} : u(x) > \max_{\Omega} \psi - \eta\}$, where this component contains only one point of $\{q_1, \dots, q_M\}$.*

(2) $\{x \in \overline{\Omega} : u(x) = \min_{\partial\Omega} f + \eta\}$ *consists of M simple C^1 regular arcs, each of which has endpoints on $\partial\Omega$ and is a part of the boundary of a component of $\{x \in \overline{\Omega} : u(x) < \min_{\partial\Omega} f + \eta\}$, where this component contains only one point of $\{z_1, \dots, z_M\}$.*

As in the proof of Theorem 2, let us show that there exists at least one critical point of u in $\Omega \backslash I$. Suppose that $\nabla u \neq 0$ in $\Omega \backslash I$. Then $\nabla u \neq 0$ in $\left\{x \in \overline{\Omega};\ u(x) \leq \max_{\Omega} \psi - \frac{1}{2}\delta\right\}$. For simplicity we put

$$\Gamma_s = \{x \in \overline{\Omega};\ u(x) = s\}, \tag{5.7}$$

$$\overline{\Omega}_s = \{x \in \overline{\Omega};\ u(x) \geq s\}. \tag{5.8}$$

By using Lemma 3.1* instead of Lemma 3.1, with the help of the implicit function theorem, after arguments similar to those used in the proof of Theorem 2, we see that $\Gamma_{\max_{\Omega} \psi - \delta}$ is C^1-diffeomorphic to $\Gamma_{\min_{\partial\Omega} f + \delta}$. This contradicts Lemma 5.1. Therefore there exists at least one critical point of u in $\Omega \backslash I$.

Let $x_1, \dots, x_k \in \Omega \backslash I$ be the critical points of u and let $m_1, \dots, m_k$ be the respective multiplicities. We may assume that there is no critical point of u in $\overline{\Omega}$ except the points $x_1, \dots, x_k, p_1, \dots, p_N$.

We first consider the case $u(x_1) = \cdots = u(x_k) = t$ for some $t \in \mathbb{R}$. Note that $\min_{\partial\Omega} f + \delta < t < \max_{\Omega} \psi - \delta$. Since $\nabla u \neq 0$ in $\{x \in \overline{\Omega} : u(x) < t\}$, then Γ_s is C^1-diffeomorphic to $\Gamma_{\min_{\partial\Omega} f + \delta}$ for any $\min_{\partial\Omega} f < s < t$. Hence in view of Lemma 5.1(2) we see that $\overline{\Omega}_s$ is connected for any $s < t$. Therefore $\overline{\Omega}_t$ is connected. Indeed, since by virtue of Lemma 3.1* the boundary of $\overline{\Omega}_t$ consists of arcs of $\partial\Omega$ and arcs of Γ_t as in Lemma 3.1*, if $\overline{\Omega}_t$ is not connected, then by arguments similar to those used in the proof of Theorem 2 we see that $\overline{\Omega}_{t-\varepsilon}$ is not connected for small $\varepsilon > 0$. This is a contradiction. Therefore, by Lemma 2.6 and Lemma 3.1* the number of connected components of $\{x \in \overline{\Omega} : u(x) > t\}$ is exactly $\sum_{j=1}^{k} m_j + 1$ and each component contains at least one point of $\{p_1, \ldots, p_N, q_1, \ldots, q_M\}$. Of course all the points $p_1, \ldots, p_N, q_1, \ldots, q_M$ are contained in these components. Furthermore we obtain a one-to-one correspondence between these components and these points. Indeed, as in the proof of Theorem 2, suppose that there exists a component containing at least two points of $\{p_1, \ldots, p_N, q_1, \ldots, q_M\}$, say ω. By Lemma 2.2*(2) ω is simply connected. Furthermore, by using the theorem of Hartman and Wintner we see that there exists a small number $\varepsilon > 0$ satisfying

$$\{x \in \omega : u(x) \geq t + \varepsilon\} \text{ is connected.} \tag{5.9}$$

On the other hand, since $\nabla u \neq 0$ in $\{x \in \omega : t < u(x) < \max_\Omega \psi\}$, so $\{x \in \omega : u(x) = s\}$ is C^1-diffeomorphic to $\{x \in \omega : u(x) = \max_\Omega \psi - \delta\}$ for any $t < s < \max_\Omega \psi$. Therefore by Lemma 5.1(1) we get

$$\begin{aligned} &\{x \in \omega;\ u(x) \geq s\} \text{ has at least two components} \\ &\qquad\qquad \text{for any } t < s < \max_\Omega \psi. \end{aligned} \tag{5.10}$$

This contradicts (5.9). Consequently equality holds in (5.2).

Consider the general case as in the proof of Lemma 2.7 (see (2,1)). By modifying the arguments in the proof of Theorem 2 as in the case $u(x_1) = \cdots = u(x_k) = t$ for some $t \in \mathbb{R}$, we can prove the equality in (5.2) along the line of the proof of Theorem 2. Therefore we omit this proof. The proof of Theorem 7 is complete. □

Acknowledgments

I would like to thank Professor Johannes Nitsche for informing me of the paper Schneider [6]. Also I am grateful to Professor Takashi Suzuki for informing me of the paper Alessandrini [1]. The present paper was completed during my long stay at the Institute for Mathematics and its Applications in the University of Minnesota. I would gratefully acknowledge it for its support and hospitality. I would also like to thank the referee for his many valuable suggestions to improve clarity in several points.

REFERENCES

[1] G. ALESSANDRINI, *Critical points of solutions of elliptic equations in two variables*. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **14** (1987), 229-256.

[2] D. GILBARG - N.S. TRUDINGER, Elliptic Partial Differential Equations of Second Order. Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1983.

[3] P. HARTMAN - A. WINTNER, *On the local behaviour of solutions of non-parabolic partial differential equations*. Amer. J. Math. **75** (1953), 449-476.

[4] B. KAWOHL, *Starshapedness of level sets for the obstacle problem and for the capacitory potential problem*. Proc. Amer. Math. Soc. **89** (1983), 637-640.

[5] D. KINDERLEHRER - G. STAMPACCHIA, An Introduction to Variational Inequalities and Their Applications. Academic Press, New York, London, Toronto, Sydney, San Francisco, 1980.

[6] R. SCHNEIDER, *A note on branch points of minimal surfaces*. Proc. Amer. Math. Soc. **17** (1966), 1254-1257.

Department of Mathematics
Tokyo Institute of Technology
Oh-okayama, Meguro-ku
Tokyo 152,
Japan

Current address:

Department of Mathematics
Faculty of Science, Ehime University
2-5 Bunkyo-cho, Matsuyama-shi
Ehime 790,
Japan