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On the Regularity of the Stationary Navier-Stokes Equations

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0. - Introduction

It is an open problem whether weak solutions $\mathbf{u}$ of the three-dimensional instationary Navier-Stokes equations for incompressible fluids on a finite domain are regular provided that the data are smooth. Beside results about partial regularity [2], [10] it is also known that $\mathbf{u} \in L^5(0, T; L^5)$ implies regularity [12]. In the three-dimensional stationary case regularity can be proved by simple bootstrap arguments.

In this situation J.B. Serrin suggested to study the stationary problem in higher dimensions in order to develop stronger analytical methods which may finally help in the instationary case.

In fact, the regularity problem in the four-dimensional stationary case was solved by Gerhardt [4], Giaquinta-Modica [7], and for the five-dimensional stationary case partial regularity is available [6], [11].

The five-dimensional stationary case has some similarity with the instationary case $N = 3$. For example, in both cases $\mathbf{u} \in L^{10/3}$ and $p \in L^{5/3}$, $\nabla p \in L^{5/4}$ is immediately available.

In this paper we consider the five-dimensional stationary equations

$$\left.\begin{aligned} -\Delta \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} &= -\nabla p + \mathbf{f} \\ \operatorname{div} \mathbf{u} &= 0 \end{aligned}\right\} \text{ in } \Omega$$

and prove the existence of a weak solution $\mathbf{u} \in W_0^{1,2}(\Omega)$ such that, for all $\varepsilon > 0$,

$$u \in L^{4-\varepsilon}(\Omega)$$

on interior domains (Theorem 2.11). Furthermore we show for all $\varepsilon > 0$, $\delta > 0$

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the Morrey condition (Theorem 2.26)

$$\int_\Omega |\mathbf{u}|^{4-\varepsilon} \frac{\xi}{|x-x_0|^{1-\delta}}\,dx < \infty, \qquad \int_\Omega |\nabla \mathbf{u}|^2 \frac{\xi}{|x-x_0|^{1-\delta}}\,dx < \infty,$$

where ξ is the usual truncation function.

The proof relies, among other considerations, on the construction of a 'maximum solution', i.e. a solution with the following property (Theorem 1.51):

$$\sup_G \left(\frac{\mathbf{u}^2}{2} + p\right) \le c_G \qquad \forall G \subset\subset \Omega.$$

The maximum principle for the quantity $\frac{\mathbf{u}^2}{2} + p$ follows (at least formally) from the equation

$$-\Delta\left(\frac{\mathbf{u}^2}{2} + p\right) + \mathbf{u}\cdot\nabla\left(\frac{\mathbf{u}^2}{2} + p\right) = \mathbf{u}\cdot\mathbf{f} + \frac{\partial u_j}{\partial x_i}\frac{\partial u_i}{\partial x_j} - |\nabla\mathbf{u}|^2 - \operatorname{div}\mathbf{f}$$

which was observed in [9], [1] and pointed out to us by M. Struwe. However, it takes some care to justify the formal argument due to the lack of regularity. The structure of the convective term $\mathbf{u}\cdot\nabla\mathbf{u}$ is frequently used during the proof.

Finally, we show the full regularity for 'maximum solutions' assuming a Morrey space inclusion for the velocity field $\mathbf{u}$ slightly stronger than what we are able to prove, namely that for an arbitrary small $\delta > 0$

$$\int_\Omega |\mathbf{u}|^{4-\varepsilon} \frac{\xi}{|x-x_0|^{1+\delta}}\,dx < \infty \qquad \forall \varepsilon > 0.$$

For dimensions higher than five, the authors hope to be able to treat the periodic, stationary case for $N \le 9$ and 'maximum solutions' up to dimension $N \le 7$.

1. - Existence of a maximum solution

We are interested in the regularity of solutions $\mathbf{u} = (u_1, \dots, u_N)$ of the stationary Navier-Stokes equations

$$\text{(1.1)} \qquad \left.\begin{aligned} \Delta\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} &= -\nabla p + \mathbf{f} \\ \operatorname{div}\mathbf{u} &= 0 \end{aligned}\right\} \quad \text{in } \Omega$$

$$\mathbf{u} = 0 \qquad \text{on } \partial\Omega,$$

where $\Omega \subseteq \mathbb{R}^N$, $N \geq 5$ is a smooth bounded domain. For convenience we assume that the external force $\mathbf{f}$ satisfies

$$\begin{aligned} &\mathbf{f} \in L^\infty(\Omega, \mathbb{R}^N) \\ &\operatorname{div} \mathbf{f} \in L^{N+\varepsilon}(\Omega). \end{aligned} \tag{1.2}$$

This assumption can be easily weakened, as the reader can check at every step. We also need the following approximation for $\varepsilon > 0$

$$\left.\begin{aligned} \Delta \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \varepsilon \mathbf{u}|\mathbf{u}|^2 &= -\nabla p + \mathbf{f} \\ \operatorname{div} \mathbf{u} &= 0 \end{aligned}\right\} \quad \text{in } \Omega \\ \mathbf{u} = 0 \qquad \text{on } \partial\Omega. \tag{1.3}$$

Due to the internal constraint $\operatorname{div} \mathbf{u} = 0$ we need two kinds of smooth functions, namely the space $C_0^\infty(\Omega)$ of infinitely differentiable functions with compact support in Ω and the space

$$\mathcal{V} = \{\mathbf{u} \in C_0^\infty(\Omega) : \operatorname{div} \mathbf{u} = 0\}.$$

We denote the closure of $\mathcal{V}$ in $W_0^{1,2}(\Omega)$ by V. Even if we are working with vector-valued functions we do not quote explicitely this dependence. Here and in the sequel we use the convention of summation over repeated indices; $\|\cdot\|_{m,p}$ is the usual norm on the Sobolev space $W^{m,p}$.

1.4 LEMMA. *Let* $\mathbf{f}$ *satisfy* (1.2). *Then there exists* $\mathbf{u} \in V$ *satisfying for all* $\varphi \in \mathcal{V}$

$$\int_\Omega \frac{\partial u_i}{\partial x_j} \frac{\partial \varphi_i}{\partial x_j} + u_i \frac{\partial u_j}{\partial x_i} \varphi_j + \varepsilon u_i |\mathbf{u}|^2 \varphi_i \, dx = \int_\Omega f_i \varphi_i \, dx \tag{1.5}$$

and

$$\begin{aligned} &\|\mathbf{u}\|_{1,2} \leq K \\ &\varepsilon \|\mathbf{u}\|_{0,4}^4 \leq K, \end{aligned} \tag{1.6}$$

where K *is independent of* ε.

PROOF. Using the standard Galerkin method this is an easy result to prove. ∎

1.7 COROLLARY. *For the solution of Lemma* 1.4 *it holds*

$$\int_\Omega |\mathbf{u}|^{\frac{2N}{N-2}} \, dx \leq K, \tag{1.8}$$

(1.9)
$$\int_\Omega |\mathbf{u}\cdot\nabla\mathbf{u}|^{\frac{N}{N-1}}\,dx \le K,$$

(1.10)
$$\|\varepsilon\mathbf{u}|\mathbf{u}|^2\|_{0,4/3} \le K\varepsilon^{1/4}.$$

PROOF. This follows immediately from Sobolev's embedding theorem $W^{1,2}(\Omega) \hookrightarrow L^{2N/(N-2)}(\Omega)$, Hölder's inequality and estimate (1.6). ■

1.11 LEMMA. *Let* $\mathbf{f}$ *satisfy* (1.2). *Then there exists* $p \in W^{1,\frac{N}{N-1}}(\Omega)$ *such that for all* $\varphi \in C_0^\infty(\Omega)$

(1.12)
$$\int_\Omega \frac{\partial u_i}{\partial x_j}\frac{\partial \varphi_i}{\partial x_j} + u_i\frac{\partial u_j}{\partial x_i}\varphi_j + \varepsilon u_i|\mathbf{u}|^2\varphi_i\,dx = -\int_\Omega \frac{\partial p}{\partial x_i}\varphi_i + f_i\varphi_i\,dx.$$

Moreover we have the estimates

(1.13)
$$\|\mathbf{u}\|_{2,\frac{N}{N-1}} \le K, \qquad \|p\|_{1,\frac{N}{N-1}} \le K$$

and

(1.14)
$$\|\nabla^2\mathbf{u}\|_{0,4/3} \le c(\varepsilon), \qquad \|p\|_{1,4/3} \le c(\varepsilon).$$

PROOF. Using Corollary 1.7 we can write equation (1.3) in the form

(1.15)
$$\left.\begin{aligned} -\Delta\mathbf{u} + \nabla p &= \mathbf{g} \\ \operatorname{div}\mathbf{u} &= 0 \end{aligned}\right\} \text{ in } \Omega$$

where the right-hand side belongs to $L^{\frac{N}{N-1}}(\Omega)$. Now the linear theory of the Stokes system gives (1.12) and (1.13). Using $\|\mathbf{u}\|_{0,4} \le c(\varepsilon)$ one sees that $\mathbf{u}\cdot\nabla\mathbf{u}$ belongs to $L^{4/3}(\Omega)$ and therefore the right-hand side of (1.15) belongs to $L^{4/3}(\Omega)$. This immediately gives (1.14). ■

1.16 REMARK. (i) From the regularity proved in Lemma 1.11 it is clear that equation (1.3) holds almost everywhere in Ω and also as an equation in $L^{\frac{N}{N-1}}(\Omega)$ (resp. $L^{4/3}(\Omega)$).

(ii) Already here one could prove that for all $\delta > 0$

$$\|\nabla^2\mathbf{u}\|_{0,4/3-\delta,\mathrm{loc}} \le K.$$

Using the test function

$$\operatorname{rot}\left(\xi\,\frac{\operatorname{rot}\mathbf{u}}{(1+|\operatorname{rot}\mathbf{u}|^s)^{1/s}}\right),$$

where ξ has compact support in Ω. We do not give the details here, because later on the result will be an easy consequence of other facts.

(iii) Note that from (1.14) and (1.6) it follows that

$$\nabla(\mathbf{u} \cdot \nabla \mathbf{u}) \in L^1(\Omega).$$

Indeed, formal derivation gives

$$\frac{\partial}{\partial x_j}\left(u_i \frac{\partial u_k}{\partial x_i}\right) = \frac{\partial u_i}{\partial x_j}\frac{\partial u_k}{\partial x_i} + u_i \frac{\partial^2 u_k}{\partial x_i \partial x_j}$$

and both terms on the right-hand side are bounded in $L^1(\Omega)$ by means of Hölder's inequality. Of course the $L^1(\Omega)$-norm depends on ε. The statement can be made precise by an approximation argument.

Now we will study the properties of the Green function G_h solving

$$\begin{aligned} \Delta G_h - \mathbf{u} \cdot \nabla G_h &= \delta_h(x_0) && \text{in } \Omega \\ G_h &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{1.17}$$

where $\mathbf{u}$ is the solution of (1.3) and δ_h, for $0 < h < \operatorname{dist}(x_0, \partial\Omega)$, is a smooth approximation of the Dirac distribution satisfying

$$\begin{gathered} \operatorname{supp} \delta_h(x_0) \subseteq B_h(x_0) \\ \delta_h(x_0) \geq 0 \\ \int_\Omega \delta_h(x_0)\, dx = 1. \end{gathered} \tag{1.18}$$

In order to simplify the investigations we approximate $\mathbf{u}$ by a sequence $\mathbf{u}^k \in \mathcal{V}$ such that

$$\mathbf{u}^k \to \mathbf{u} \qquad \text{in } V. \tag{1.19}$$

Thus we will investigate the problem

$$\begin{aligned} \Delta G_{hk} - \mathbf{u}^k \cdot \nabla G_{hk} &= \delta_h(x_0) && \text{in } \Omega \\ G_{hk} &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{1.20}$$

In the following we drop the dependence on h, k and x_0 and denote $G = G_{hk}$, $\mathbf{u} = \mathbf{u}^k$, $\delta_h = \delta_h(x_0)$. The weak formulation of (1.20) reads

$$\int_\Omega \nabla G \nabla \varphi\, dx - \int_\Omega \mathbf{u} \cdot \nabla G \varphi\, dx = \int_\Omega \delta_h \varphi\, dx \qquad \forall \varphi \in W_0^{1,2}(\Omega). \tag{1.21}$$

1.22 LEMMA. *For all $h > 0$ and $k \in \mathbb{N}$ there exists a solution $G \in C^\infty(\Omega) \cap W_0^{1,2}(\Omega)$ of* (1.20).

PROOF. The left-hand side of $(1.20)_1$ defines a coercive operator on $W_0^{1,2}(\Omega)$. Thus the Lax-Milgram theorem immediately gives the existence of a weak solution of (1.20). The right-hand side of $(1.20)_1$ is smooth and therefore from the regularity of linear elliptic equations, we obtain the conclusion (see [8, Chap. 8]). ■

1.23 LEMMA. *For all $h > 0$ and $k \in \mathbb{N}$ we have*

$$G \geq 0, \tag{1.24}$$

$$\|G\|_{0,\infty} \leq c(h), \tag{1.25}$$

where $c(h)$ is independent of ε and $\mathbf{u}^k$.

PROOF. Inequality (1.24) is an easy consequence of (1.18), Lemma 1.22 and the maximum principle (see [8, Chap. 3]). In order to prove (1.25) we use the Moser iteration method. Let us use the test function $G|G|^s$ in the weak formulation of (1.20). We get

$$\int_\Omega \nabla G \nabla (G|G|^s) dx - \int_\Omega \mathbf{u} \cdot \nabla G G|G|^s \, dx = \int_\Omega \delta_h G|G|^s \, dx$$

and therefore

$$\int_\Omega |\nabla G|^2 |G|^s \, dx + s \int_\Omega |\nabla G|^2 \, |G|^s \, dx = \int_\Omega \delta_h G|G|^s \, dx, \tag{1.26}$$

where we have used the fact that $\nabla G G|G|^s = \nabla \left(\frac{1}{s+2} |G|^{s+2} \right)$. Using

$$|\nabla G|^2 \, |G|^s = \left| |G|^{s/2} \nabla G \right|^2 = \frac{1}{\left(\frac{s}{2}+1\right)^2} |\nabla |G|^{(s+2)/2}|^2$$

and Sobolev's embedding theorem we obtain

$$\left(\int_\Omega |G|^{(s+2)\cdot \frac{N}{N-2}} \, dx \right)^{\frac{N-2}{N}} \leq c\|\delta_h\|_{0,\infty}(s+1) \int_\Omega |G|^{s+1} \, dx. \tag{1.27}$$

Replacing $s+1$ by s and setting

$$R_s = \max(\|\delta_h\|_{0,\infty}, \|G\|_{0,s})$$

we see that

$$R^{s+1}_{\frac{N}{N-2}(s+1)} \le c(s+1) R^{s+1}_{s+1}. \tag{1.28}$$

Indeed, if $R_{\frac{N}{N-2}(s+1)} = \|G\|_{\frac{N}{N-2}(s+1)}$, (1.28) follows immediately from (1.27), otherwise, if $R_{\frac{N}{N-2}(s+1)} = \|\delta_h\|_{0,\infty}$, we get $R_{\frac{N}{N-2}(s+1)} \le R_{s+1}$ and thus (1.28) holds if $c(s+1) > 1$. Now we set

$$s_0 = 2$$

$$s_i = 2\left(\frac{N}{N-2}\right)^i$$

which yields

$$R_{s+1} \le \prod_{k=0}^{i} c_0^{1/s_k} \prod_{k=0}^{i} s_k^{1/s_k} R_{s_0}. \tag{1.29}$$

Letting i tend to infinity in (1.29) and using the fact that the series in (1.29) are finite and that $\lim_{i\to\infty} s_i = \infty$, we have proved that

$$R_\infty \le c(h) R_{s_0}$$

which immediately gives (1.25). ∎

1.30 COROLLARY. *For all $h > 0$, and $k \in \mathbb{N}$ we have*

$$\|\nabla G\|_{0,2} \le c(h). \tag{1.31}$$

PROOF. We have

$$\int_\Omega |\nabla G|^2\,dx + \int_\Omega \mathbf{u}\cdot\nabla G G\,dx = \int_\Omega \delta_h G\,dx$$

and thus

$$\int_\Omega |\nabla G|^2\,dx \le \|G\|_{0,\infty} \int_\Omega \delta_h\,dx \le c(h).$$ ∎

1.32 LEMMA. *For all $h > 0$ and $k \in \mathbb{N}$ we have*

$$\|\nabla^2 G\|_{0,2,\mathrm{loc}} \le c(h), \tag{1.33}$$

$$\|\nabla G\|_{0,4,\mathrm{loc}} \le c(h). \tag{1.34}$$

PROOF. Using in (1.21)

$$\varphi = -\frac{\partial}{\partial x_j}\left(\xi^2 \frac{\partial}{\partial x_j} G\right)$$

we obtain

$$\int_\Omega \frac{\partial^2 G}{\partial x_i \partial x_j}\frac{\partial}{\partial x_i}\left(\xi^2 \frac{\partial G}{\partial x_j}\right) dx + \int_\Omega u_i \frac{\partial G}{\partial x_i}\frac{\partial}{\partial x_j}\left(\xi^2 \frac{\partial G}{\partial x_j}\right) dx = -\int_\Omega \delta_h \frac{\partial}{\partial x_j}\left(\xi^2 \frac{\partial G}{\partial x_j}\right) dx. \tag{1.35}$$

After some computations we end up with the inequality

$$\begin{aligned}\int_\Omega |\nabla^2 G|^2 \xi^2\, dx \le c\Bigg(&\int_\Omega |\nabla^2 G| \xi |\nabla G||\nabla \xi| dx + \int_\Omega |\nabla \mathbf{u}|\, |\nabla G|^2 \xi^2\, dx. \\ &+ \int_\Omega |\mathbf{u}|\, |G||\nabla^2 G||\nabla \xi| \xi\, dx + \int_\Omega |\mathbf{u}||G||\nabla G||\nabla^2 \xi|\xi\, dx \\ &+ \int_\Omega |\mathbf{u}||G||\nabla G||\nabla \xi|^2\, dx\Bigg)\end{aligned}$$

and thus using (1.25), (1.31), Young's and Hölder's inequalities we obtain

$$\int_\Omega |\nabla^2 G|^2 \xi^2\, dx \le c(h, \xi, \|\mathbf{u}\|_{1,2}) \left\{1 + \left(\int_\Omega |\nabla G|^4 \xi^4\, dx\right)^{1/2}\right\}. \tag{1.36}$$

The integral on the right-hand side can be bounded by means of the left-hand side. Indeed

$$\begin{aligned}\int_\Omega |\nabla G|^4 \xi^4\, dx &\le -\int_\Omega |G||\Delta G||\nabla G|^2 \xi^4 + 2|G||\nabla G|^2 |\nabla^2 G| \xi^4\, dx \\ &\quad - \int_\Omega 4|G||\nabla G|^3 \xi^3 |\nabla \xi|\, dx \\ &\le c(h) \int_\Omega |\nabla^2 G|^2 \xi^2\, dx + \frac{1}{2}\int_\Omega |\nabla G|^4 \xi^4\, dx + c(h, \xi).\end{aligned} \tag{1.37}$$

This together with (1.36) completes the proof. ∎

Since the estimates (1.25), (1.31), (1.33) and (1.34) are independent of the chosen approximation $\mathbf{u}^k$ we can let k tend to infinity in (1.21)

$$\int_\Omega \nabla G \nabla \varphi \, dx - \int_\Omega \mathbf{u} \cdot \nabla G \varphi \, dx = \int_\Omega \delta_h \varphi \, dx \qquad \forall \varphi \in C_0^\infty(\Omega)$$

where $G = G_h$ and $\mathbf{u}$ is the solution of (1.3). Of course the proved estimates remain valid also for G_h and equation (1.17) holds almost everywhere in Ω and also in $L^2_{\mathrm{loc}}(\Omega)$. In the sequel we will need a local version of (1.17). Therefore we multiply (1.17) by $\xi^2, \xi \in C_0^\infty(\Omega)$, use the product rule and after that we multiply the result by a smooth test function $\varphi \in C_0^\infty(\Omega)$ and integrate over Ω. We obtain

$$\begin{aligned} &\int_\Omega \nabla(G\xi^2)\nabla\varphi \, dx + \int_\Omega \mathbf{u} \cdot \nabla \varphi G \xi^2 \, dx \\ &= \int_\Omega \delta_h \xi^2 \varphi \, dx - \int_\Omega \mathbf{u} \cdot \nabla \xi^2 G \varphi \, dx - 4 \int_\Omega \nabla G \nabla \xi \xi \varphi \, dx \\ &- \int_\Omega \Delta \xi^2 G \varphi \, dx \end{aligned} \tag{1.38}$$

which by continuity holds for all $\varphi \in W^{1,4/3}(\Omega)$. Notice that by continuity the weak formulation (1.12) holds for all $\varphi \in W_0^{1,2}(\Omega) \cap L^4(\Omega)$. This allows us to use $\mathbf{u}G\xi^2$ as a test function in (1.12), which yields

$$\begin{aligned} &\int_\Omega |\nabla \mathbf{u}|^2 G\xi^2 + \nabla \frac{\mathbf{u}^2}{2} \nabla(G\xi^2) + \mathbf{u} \cdot \nabla \frac{\mathbf{u}^2}{2} G\xi^2 + \varepsilon |\mathbf{u}|^4 G \xi^2 \, dx \\ &= -\int_\Omega \mathbf{u} \cdot \nabla p G \xi^2 \, dx + \int_\Omega \mathbf{f} \cdot \mathbf{u} G \xi^2 \, dx. \end{aligned} \tag{1.39}$$

On the other hand we can derive the pressure equation from (1.12) using $\varphi = \nabla \psi$ where ψ is a smooth scalar function. After some calculations involving the estimates proved before and mainly $(1.3)_2$ we arrive at

$$\begin{aligned} &\int_\Omega \nabla p \nabla \varphi \, dx = \int_\Omega \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i} \varphi - \varepsilon \mathbf{u} \cdot \nabla \varphi |\mathbf{u}|^2 \, dx + \int_\Omega \mathbf{f} \cdot \nabla \varphi \, dx \\ &\qquad \forall \varphi \in L^\infty(\Omega) \cap W_0^{1,4}(\Omega) \end{aligned} \tag{1.40}$$

and therefore $G\xi^2$ is a possible test function. Replacing φ by $G\xi$ in (1.40) and adding the result to (1.39) we end up with

$$
\begin{aligned}
&\int_\Omega \nabla\left(\frac{\mathbf{u}^2}{2}+p\right)\nabla(G\xi^2)+\mathbf{u}\cdot\nabla\left(\frac{\mathbf{u}^2}{2}+p\right)G\xi^2+\varepsilon|\mathbf{u}|^4G\xi^2\,dx \\
&=\int_\Omega(\nabla\mathbf{u}\circ\nabla\mathbf{u}-|\nabla\mathbf{u}|^2)G\xi^2\varepsilon\mathbf{u}\cdot\nabla(G\xi^2)|\mathbf{u}|^2\,dx \\
&+\int_\Omega \mathbf{f}\cdot\nabla(G\xi^2)+\mathbf{f}\cdot\mathbf{u}G\xi^2\,dx,
\end{aligned}
\tag{1.41}
$$

where $\nabla\mathbf{u}\circ\nabla\mathbf{u}=\frac{\partial u_i}{\partial x_j}\cdot\frac{\partial u_j}{\partial x_i}$. Using

$$
\begin{aligned}
&G\geq 0, \qquad \xi^2\geq 0 \\
&\int_\Omega(\nabla\mathbf{u}\circ\nabla\mathbf{u}-|\nabla\mathbf{u}|^2)G\xi^2\,dx\leq 0
\end{aligned}
$$

we finally obtain

$$
\begin{aligned}
&\int_\Omega \nabla\left(\frac{\mathbf{u}^2}{2}+p\right)\nabla(G\xi^2)+\mathbf{u}\cdot\nabla\left(\frac{\mathbf{u}^2}{2}+p\right)G\xi^2\,dx \\
&\leq-\int_\Omega\varepsilon\mathbf{u}\cdot\nabla(G\xi^2)|\mathbf{u}|^2\,dx+\int_\Omega\mathbf{f}\cdot\nabla(G\xi^2)+\mathbf{f}\cdot\mathbf{u}G\xi^2\,dx.
\end{aligned}
\tag{1.42}
$$

Note that $\frac{\mathbf{u}^2}{2}+p$ belongs to the space $W^{1,4/3}(\Omega)$ and thus we can use formula (1.38) with $\varphi=\frac{\mathbf{u}^2}{2}+p$. We arrive at

$$
\begin{aligned}
&\int_\Omega \delta_h\xi^2\left(\frac{\mathbf{u}^2}{2}+p\right)dx \\
&\leq\int_\Omega\mathbf{u}\cdot\nabla\xi G\left(\frac{\mathbf{u}^2}{2}+p\right)\xi\,dx+\int_\Omega\nabla G\nabla\xi^2\left(\frac{\mathbf{u}^2}{2}+p\right)dx \\
&+\int_\Omega\Delta\xi^2G\varphi\left(\frac{\mathbf{u}^2}{2}+p\right)dx-\int_\Omega\varepsilon\mathbf{u}\cdot\nabla(G\xi^2)|\mathbf{u}|^2\,dx \\
&+\int_\Omega\mathbf{f}\cdot\nabla(G\xi^2)+\mathbf{f}\cdot\mathbf{u}G\xi^2\,dx.
\end{aligned}
\tag{1.43}
$$

Here of course we have $\mathbf{u} = \mathbf{u}_\varepsilon$, $G = G_{h\varepsilon}$. Now let ε tend to 0 for fixed but arbitrary $h > 0$. In order to do this we re-write the second and fourth integral on the right-hand side as follows:

$$\int_\Omega \nabla G \nabla \xi^2 \left(\frac{\mathbf{u}^2}{2} + p\right) dx = -\int_\Omega G\Delta\xi^2 \left(\frac{\mathbf{u}^2}{2} + p\right) + G\nabla\xi^2\nabla\left(\frac{\mathbf{u}^2}{2} + p\right) dx,$$

$$\varepsilon \int_\Omega |\mathbf{u}|^2\mathbf{u}\cdot\nabla(G\xi^2)dx = \varepsilon\int_\Omega |\mathbf{u}|^2\mathbf{u}\cdot\nabla G\xi^2 + |\mathbf{u}|^2\mathbf{u}\cdot\nabla\xi^2 G\,dx.$$

The process of taking the limit is possible because of the estimates proved before, in particular relation (1.10). We finally get

$$\begin{aligned}
&\int_\Omega \delta_h \xi^2 \left(\frac{\mathbf{u}^2}{2} + p\right) dx \\
&\le \int_\Omega 2\mathbf{u}\cdot\nabla\xi G\xi \left(\frac{\mathbf{u}^2}{2} + p\right) dx - 4\int_\Omega \xi G\Delta\xi \left(\frac{\mathbf{u}^2}{2} + p\right) dx \\
&\quad - 4\int_\Omega \xi G\nabla\xi\nabla\left(\frac{\mathbf{u}^2}{2} + p\right) dx + \int_\Omega \Delta\xi^2 G\left(\frac{\mathbf{u}^2}{2} + p\right) dx \\
&\quad + \int_\Omega \mathbf{f}\cdot\nabla(G\xi^2) + \mathbf{f}\cdot\mathbf{u}G\xi^2\,dx.
\end{aligned} \tag{1.44}$$

Here $\mathbf{u}$ does not depend on ε and solves (1.1), but $G = G_h$ solves (1.17). For the process of taking the limit as h tends to 0 we need some estimates on G independent of h. We prove that for G we have the same estimates as for the Green function for the Laplace operator.

1.45 LEMMA. *For the solution G_h of* (1.17) *we have the following estimates independent of h:*

$$\int_\Omega |G_h|^{\frac{N}{N-2}-\delta}\,dx \le K \qquad \forall\delta > 0 \tag{1.46}$$

$$\int_\Omega |\nabla G_h|^{\frac{N}{N-1}-\delta}\,dx \le K \qquad \forall\delta > 0. \tag{1.47}$$

PROOF. Choose $\varphi = \dfrac{G}{(1 + |G|^q)^{1/q}}$, $q > 0$ in (1.21). Notice that $\nabla G\varphi = \nabla F(G)$ with a function F which has linear growth. Using partial integration in the convective term this integral becomes zero. Thus

$$\int_\Omega \frac{|\nabla G|^2}{(1+|G|^q)^{1+1/q}}\,dx = \int_\Omega \delta_h \frac{G}{(1+|G|^q)^{1/q}}\,dx \le 1.$$

Further we have

$$\|G\|_{0,\alpha} \le \|\nabla G\|_{0,\frac{N\alpha}{N+\alpha}}$$

$$\le \left(\int_\Omega \frac{|\nabla G|^2}{(1+|G|^q)^{1+\frac{1}{q}}}\,dx\right)^{\frac12} \left(\int_\Omega (1+|G|^q)^{\left(1+\frac1q\right)\frac{N\alpha}{2N-(N-2)\alpha}}\,dx\right)^{\frac{2N-(N-2)\alpha}{2N\alpha}}.$$

This is finite for $\alpha = \dfrac{N-Nq}{N-2}$; (1.46) follows immediately choosing q arbitrary small. Similarly one can prove (1.47). ∎

1.48 LEMMA. *Let $N = 5$ and let $B_{2R} \subseteq \Omega$ be a ball such that $B_h(x_0) \cap B_R = \emptyset$. Then*

$$\|G_h\|_{0,\infty,B_R} \le c(R)\|G_h\|_{\frac{N}{N-2}-\delta}. \tag{1.49}$$

PROOF. We will use Moser's iteration technique again. In order to do this we choose in (1.21) $\varphi = G^s\xi^2$ and $s > 0$, where $\xi(x) = 1$ for $x \in B_R$, $\xi(x) = 0$ for $x \in \Omega \backslash B_{R+\rho}$, and $0 < \rho < R$. Therefore we get

$$\frac{4s}{(s+1)^2}\int_\Omega |\nabla G^{\frac{s+1}{2}}|^2\xi^2\,dx \le \frac{c_0}{s+1}\left(\int_{B_{R+\rho}} G^{(s+1)\frac{10}{7}}\,dx\right)^{\frac{7}{10}}$$

and consequently, using the embedding theorem, we obtain

$$\left(\int_{B_R} G^{(s+1)\frac53}\,dx\right)^{\frac35} \le c_0\frac{1}{\rho^2}\left(\int_{B_{R+\rho}} G^{(s+1)\frac{10}{7}}\,dx\right)^{\frac{7}{10}}. \tag{1.50}$$

But (1.50) is the starting inequality for the Moser technique, which gives the assertion (see [8, Chap. 8]). ∎

We are now in a position to take the limit as h tends to 0.

1.51 THEOREM. *Let $N = 5$. Then there exists a solution of* (1.1) *such that*

$$\sup_{x\in\Omega_\rho} \frac{\mathbf{u}^2}{2}(x) + p(x) \le c(\rho) \equiv c_0, \tag{1.52}$$

where $\Omega_\rho = \{x \in \Omega : \text{dist}(x, \partial\Omega) \geq \rho\}$.

PROOF. We want to take the limit as $h \to 0$ in inequality (1.44). Let $B_\rho(x_0)$ be such that $B_{2\rho}(x_0) \subseteq \Omega$. Choose ξ such that $\xi(x) = 1$ for $x \in B_\rho(x_0)$ and $\xi(x) = 0$ for $x \in \Omega \backslash B_{2\rho}(x_0)$. Then for $0 < h < \rho$ we have

$$\int_{B_\rho(x_0)} \delta_h(x_0)\left(\frac{\mathbf{u}^2}{2}+p\right)dx$$

$$\leq \int_{B_{2\rho}(x_0)} 2\mathbf{u}\nabla\xi G_h\xi\left(\frac{\mathbf{u}^2}{2}+p\right)dx - 4\int_{B_{2\rho}(x_0)} G_h\xi\Delta\xi\left(\frac{\mathbf{u}^2}{2}+p\right)dx$$

$$-4\int_{B_{2\rho}(x_0)} G_h\xi\nabla\xi\nabla\left(\frac{\mathbf{u}^2}{2}+p\right)dx + \int_{B_{2\rho}(x_0)} \Delta\xi^2 G_h\left(\frac{\mathbf{u}^2}{2}+p\right)dx$$

$$+\int_{B_{2\rho}(x_0)} \mathbf{f}\cdot\nabla(G\xi^2) + \mathbf{f}\cdot\mathbf{u}G\xi^2\,dx.$$

In all the integrals on the right-hand side involving $\nabla\xi$ or $\nabla^2\xi$ we use for G_h estimate (1.49); this implies that these integrals are bounded from above, as the remainder is at least a $L^1(\Omega)$ function. The last integral on the right-hand side is bounded due to the assumption on $\mathbf{f}$. The limiting process as $h \to 0$ completes the proof. ■

1.53 DEFINITION. *In the following we shall refer to solutions satisfying* (1.52) *as maximum solutions.*

2. - Regularity in weighted spaces

From now on we restrict ourselves to the case $N = 5$. Let $\xi \in C_0^\infty(\Omega)$ and let $x_0 \in \Omega$, $\rho > 0$ be such that $B_\rho(x_0) \subseteq\subseteq \text{supp}\,\xi$. Suppose that $\xi(x) = 1$ for $x \in B_\rho(x_0)$.

2.1 THEOREM. *Let* $\mathbf{u}$ *be a maximum solution of* (1.1). *Then for all* $x_0 \in \Omega$ *we have*

$$\int_\Omega \left|\frac{\mathbf{u}^2}{2}+p\right|\frac{\xi^2}{|x-x_0|^3}\,dx \leq K \tag{2.2}$$

$$\int_\Omega |\mathbf{u}\cdot(x-x_0)|^2\frac{\xi^2}{|x-x_0|^5}\,dx \leq K, \tag{2.3}$$

where $K = K(\xi)$ *does not depend on* x_0.

PROOF. On account of Fatou's lemma it suffices to prove the estimates for almost every $x_0 \in \Omega$. Multiplying (1.1) by $\xi^2 \dfrac{\partial}{\partial x_i} \dfrac{1}{|x-x_0|}$ we obtain

$$
\begin{aligned}
&\int_\Omega \Delta u_i \frac{1}{|x-x_0|} \frac{\partial \xi^2}{\partial x_i}\,dx - \int_\Omega u_i u_j \frac{(x-x_0)_i}{|x-x_0|^3} \frac{\partial \xi^2}{\partial x_j}\,dx \\
&\quad - 3\int_\Omega u_i u_j \frac{(x-x_0)_i (x-x_0)_j}{|x-x_0|^5}\,\xi^2\,dx + \int_\Omega \mathbf{u}^2 \frac{\xi^2}{|x-x_0|^3}\,dx \\
&= \int_\Omega p\,\frac{(x-x_0)_i}{|x-x_0|^3} \frac{\partial \xi^2}{\partial x_i}\,dx - 2\int_\Omega p\,\frac{\xi^2}{|x-x_0|^3}\,dx - \int_\Omega f_i \frac{(x-x_0)_i}{|x-x_0|^3}\,\xi^2\,dx.
\end{aligned}
\tag{2.4}
$$

Notice that all the integrals are defined for almost every $x_0 \in \Omega$ either by means of Hölder's inequality or because they are a convolution of two L^1-functions. This also justifies the following partial integrations. We get

$$
\begin{aligned}
&3\int_\Omega u_i u_j \frac{(x-x_0)_i (x-x_0)_j}{|x-x_0|^5}\,\xi^2\,dx \\
&= \int_\Omega \left(\frac{\mathbf{u}^2}{2} + p\right) \frac{2\xi^2}{|x-x_0|^3}\,dx - \int_\Omega u_i u_j \frac{(x-x_0)_i}{|x-x_0|^3} \frac{\partial \xi^2}{\partial x_j}\,dx \\
&\quad + \int_\Omega \Delta u_i \frac{1}{|x-x_0|} \frac{\partial \xi^2}{\partial x_i}\,dx + \int_\Omega p\,\frac{(x-x_0)_i}{|x-x_0|^3} \frac{\partial \xi^2}{\partial x_i}\,dx \\
&\quad + \int_\Omega f_i \frac{(x-x_0)_i}{|x-x_0|^3}\,\xi^2\,dx.
\end{aligned}
\tag{2.5}
$$

Remark that the left-hand side of (2.5) is non-negative. We denote the integrals on the right-hand side by $I_1, \dots, I_5$. Using the semi-boundedness condition (1.52) together with (1.8), (1.14) and (1.2) we find

$$
I_1 \le 2c_0 \int_\Omega \frac{\xi^2}{|x-x_0|^3}\,dx \le c,
$$

$$
\begin{aligned}
|I_2| &\le \int_{\Omega\setminus B_\rho(x_0)} \mathbf{u}^2 |\nabla \xi^2| \frac{1}{|x-x_0|^2}\,dx \\
&\le c\|\mathbf{u}\|_{0,2}^2 \|\nabla \xi\|_{0,\infty} \left\| \frac{1}{|x-x_0|^2} \right\|_{0,\infty,\Omega\setminus B_\rho(x_0)},
\end{aligned}
$$

$$
\begin{aligned}
|I_3| &\leq \int\limits_{\Omega\setminus B_\rho(x_0)} |\Delta \mathbf{u}||\nabla \xi^2| \frac{1}{|x-x_0|}\,dx \\
&\leq c\|\mathbf{u}\|_{2,5/4}^{5/4}\|\nabla\xi\|_{0,\infty}\left\|\frac{1}{|x-x_0|}\right\|_{0,\infty,\Omega\setminus B_\rho(x_0)}, \\
|I_4| &\leq \int\limits_{\Omega\setminus B_\rho(x_0)} |p||\nabla \xi^2| \frac{1}{|x-x_0|^2}\,dx \\
&\leq c\|p\|_{0,5/3}^{5/3}\|\nabla\xi\|_{0,\infty}\left\|\frac{1}{|x-x_0|^2}\right\|_{0,\infty,\Omega\setminus B_\rho(x_0)}, \\
|I_5| &\leq \int\limits_{\Omega} |\mathbf{f}| \frac{\xi^2}{|x-x_0|^2}\,dx \\
&\leq c\|\mathbf{f}\|_{0,\infty}\left\|\frac{1}{|x-x_0|}\right\|_{0,2}^2.
\end{aligned}
$$

Therefore we conclude that

$$
\int\limits_{\Omega}\left|u_i u_j \frac{(x-x_0)_i(x-x_0)_j}{|x-x_0|^5}\xi^2\right| dx \leq c, \tag{2.6}
$$

which proves (2.3). Now we carry the integrals $I_1 \dots I_5$ to the left-hand side and obtain also using (2.6)

$$
\left|\int\limits_{\Omega}\left(\frac{\mathbf{u}^2}{2}+p\right)\frac{\xi^2}{|x-x_0|^3}\,dx\right| \leq c.
$$

This together with the crucial estimate (1.52) yields (2.2). ■

2.7 COROLLARY. *For a maximum solution of* (1.1) *we have for all* $s \in (2,3)$ *and all* $x_0 \in \Omega$

$$
\begin{aligned}
&\int\limits_{\Omega} |p| \frac{\xi^2}{|x-x_0|^s}\,dx \leq K \\
&\int\limits_{\Omega} \mathbf{u}^2 \frac{\xi^2}{|x-x_0|^s}\,dx \leq K,
\end{aligned} \tag{2.8}
$$

where $K = K(\xi)$ *does not depend on* x_0.

PROOF. Again we multiply (1.1) by $\xi^2 \dfrac{\partial}{\partial x_i}\dfrac{1}{|x-x_0|^{s-2}}$. After rearrangements similar to those of the proof of (2.2) we arrive at:

$$
\begin{aligned}
&(3-s)(s-2)\int_\Omega p\,\frac{\xi^2}{|x-x_0|^s}\,dx\\
&= 2(2-s)\int_\Omega\left(\frac{\mathbf{u}^2}{2}+p\right)\frac{2\xi^2}{|x-x_0|^s}\,dx-\int_\Omega \Delta u_i\,\frac{\partial\xi^2}{\partial x_i}\,\frac{1}{|x-x_0|^{s-2}}\,dx\\
&+(2-s)\int_\Omega u_iu_j\,\frac{(x-x_0)_i}{|x-x_0|^s}\,\frac{\partial\xi^2}{\partial x_j}\,dx \qquad (2.9)\\
&+s(s-2)\int_\Omega u_iu_j\,\frac{(x-x_0)_i(x-x_0)_j}{|x-x_0|^{s+2}}\,\xi^2\,dx\\
&+(2-s)\int_\Omega p\,\frac{(x-x_0)_i}{|x-x_0|^s}\,\frac{\partial\xi^2}{\partial x_i}\,dx+(2-s)\int_\Omega f_i\,\frac{(x-x_0)_i}{|x-x_0|^s}\,\xi^2\,dx.
\end{aligned}
$$

The first and the fourth term on the right-hand side are bounded because of (2.2) and (2.3), while the other terms can be treated in the same way as in the above proof. Thus we have

$$
\left|\int_\Omega p\,\frac{\xi^2}{|x-x_0|^s}\,dx\right|\le c \tag{2.10}
$$

but from (1.52) we infer that

$$
p\le p+\frac{\mathbf{u}^2}{2}\le c_0,
$$

which together with (2.10) gives $(2.8)_1$. The estimate $(2.8)_2$ follows immediately. ∎

2.11 THEOREM. *Let* $\mathbf{u}$ *be a maximum solution of* (1.1). *Then we have for all* $r\in(1,2)$ *and all* $x_0\in\Omega$

$$
\begin{aligned}
&\int_\Omega |p|^r\,\xi^2\,dx\le K\\
&\int_\Omega |\mathbf{u}|^{2r}\,\xi^2\,dx\le K,
\end{aligned} \tag{2.12}
$$

where $K=K(\xi)$ *does not depend on* x_0.

PROOF. First, remark that φ_o given by

$$\varphi_o(x_0) = \int_\Omega \frac{1}{|x - x_0|^3} |p|^{r-1} \operatorname{sgn} p_\rho \xi \, dx$$

satisfies

$$-\Delta \varphi_o = |p|^{r-1} \operatorname{sgn} p_\rho \xi.$$

Using $(2.8)_1$ and Hölder's inequality we obtain

$$\int_\Omega \frac{1}{|x - x_0|^3} |p|^{r-1} \operatorname{sgn} p_\rho \xi \, dx \leq K$$

and hence

$$\varphi_o \in L^\infty(\Omega). \tag{2.13}$$

As in (1.40), we derive the weak formulation of the pressure equation:

$$\int_\Omega \nabla p \nabla \psi \, dx = \int_\Omega \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i} \psi \, dx + \int_\Omega f_i \frac{\partial \psi}{\partial x_i} \, dx \qquad \forall \psi \in C_0^\infty(\Omega). \tag{2.14}$$

Inserting $\psi = (\varphi_o \xi)_\rho$, where $(g)_\rho$ denotes the mollifer of the function g, we obtain

$$\begin{aligned} &\int_\Omega p_\rho |p|^{r-1} \operatorname{sgn} p_\rho \xi \, dx \\ &= -\int_\Omega \varphi_o \Delta \xi \, dx - 2 \int_\Omega \nabla p_\rho \nabla \xi \varphi_o \, dx \\ &\quad + \int_\Omega \left(\frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i} \right)_\rho \varphi_o \xi \, dx - \int_\Omega \frac{\partial f_{i\rho}}{\partial x_i} \varphi_o \xi \, dx. \end{aligned}$$

Letting ρ tend to 0^+ and using Fatou's lemma we get

$$\begin{aligned} \int_\Omega |p|^r \xi^2 \, dx \leq \int_\Omega |\varphi_o||\Delta \xi| dx + 2 \int_\Omega |\nabla p||\nabla \xi||\varphi_o| dx \\ + \int_\Omega |\nabla \mathbf{u}|^2 |\varphi_o| \xi \, dx + \int_\Omega |\operatorname{div} \mathbf{f}||\varphi_o| \xi \, dx, \end{aligned}$$

which immediately gives $(2.12)_1$. Estimate $(2.12)_2$ follows using (1.52). ∎

2.15 COROLLARY. *For a maximum solution of* (1.1) *we have*

$$\int_\Omega |p| \left| \frac{\mathbf{u}^2}{2} + p \right| \xi^2 \, dx \leq K \tag{2.16}$$

$$\int_\Omega \left(\frac{\mathbf{u}^2}{2} + p \right)^2 \xi^2 \, dx \leq K. \tag{2.17}$$

PROOF. With the same methods as in the proof of Theorem 2.11, now using a function φ_o which solves

$$-\Delta\varphi_o = \left(\frac{\mathbf{u}^2}{2} + p \right) \operatorname{sgn} \left(p \left(p + \frac{\mathbf{u}^2}{2} \right) \right) \xi,$$

we easily obtain (2.16). Estimate (2.17) follows from the remark that

$$\left(\frac{\mathbf{u}^2}{2} + p \right)^2 \leq |p| \left| \frac{\mathbf{u}^2}{2} + p \right| + c_0 \frac{\mathbf{u}^2}{2}. \qquad \blacksquare$$

2.18 PROPOSITION. *Let* $\mathbf{u}$ *be the maximum solution of* (1.1) *constructed in Section* 1. *Then we have for all* $r \in \left(\frac{1}{4}, \frac{1}{2} \right)$ *and all* $x_0 \in \Omega$

$$\int_\Omega |\nabla \mathbf{u}|^2 \frac{\xi^2}{|x - x_0|^r} \, dx \leq K, \tag{2.19}$$

where $K = K(\xi)$ *does not depend on* x_0.

PROOF. Let $h > 0$ be fixed and let us use in the weak formulation (1.12) the test function $\varphi = \mathbf{u} \dfrac{\xi^2}{\big((x-x_0)^2+h^2\big)^{r/2}}$. Thus we obtain, denoting $\mathbf{u} = \mathbf{u}_\varepsilon$:

$$
\begin{aligned}
&\int_\Omega \frac{|\nabla \mathbf{u}|^2}{\big((x-x_0)^2+h^2\big)^{r/2}}\,\xi^2\,dx + \int_\Omega u_i \frac{\partial u_i}{\partial x_j} \frac{1}{\big((x-x_0)^2+h^2\big)^{r/2}} \frac{\partial \xi^2}{\partial x_j}\,dx \\
&+ r\int_\Omega \frac{\mathbf{u}^2}{2} \frac{5h^2+(3-r)(x-x_0)^2}{\big((x-x_0)^2+h^2\big)^{(r+4)/2}}\,\xi^2\,dx \\
&- \int_\Omega u_i \left(\frac{\mathbf{u}^2}{2}+p\right) \frac{1}{\big((x-x_0)^2+h^2\big)^{r/2}} \frac{\partial \xi^2}{\partial x_i}\,dx \\
&+ r\int_\Omega u_i \left(\frac{\mathbf{u}^2}{2}+p\right) \frac{(x-x_0)_i}{\big((x-x_0)^2+h^2\big)^{(r+2)/2}}\,\xi^2\,dx \\
&+ \varepsilon\int_\Omega \mathbf{u}^4 \frac{1}{\big((x-x_0)^2+h^2\big)^{r/2}}\,\xi^2\,dx \\
&= \int_\Omega f_i u_i \frac{1}{\big((x-x_0)^2+h^2\big)^{r/2}}\,\xi^2\,dx.
\end{aligned}
$$

For fixed $h > 0$ the third term and the last term on the left-hand side are non-negative. While taking the limit as $\varepsilon \to 0^+$ in the other integrals we use the fact that $\nabla\xi$, $\dfrac{1}{(x-x_0)+h^2}$ and $\dfrac{1}{\big((x-x_0)^2+h^2\big)^{(r+2)/2}}$ are L^∞-functions and the embedding $W^{1,2}(\Omega) \hookrightarrow L^{10/3}(\Omega)$. Thus all the integrals can be treated by means of weak and strong convergence. We arrive at

$$
\begin{aligned}
&\int_\Omega \frac{|\nabla \mathbf{u}|^2}{\big((x-x_0)^2+h^2\big)^{r/2}}\,\xi^2\,dx \\
&\le \int_\Omega u_i \frac{\partial u_i}{\partial x_j} \frac{1}{\big((x-x_0)^2+h^2\big)^{r/2}} \frac{\partial \xi^2}{\partial x_j}\,dx \\
&+ \int_\Omega u_i \left(\frac{\mathbf{u}^2}{2}+p\right) \frac{1}{\big((x-x_0)^2+h^2\big)^{r/2}} \frac{\partial \xi^2}{\partial x_i}\,dx \\
&- r\int_\Omega u_i \left(\frac{\mathbf{u}^2}{2}+p\right) \frac{(x-x_0)_i}{\big((x-x_0)^2+h^2\big)^{(r+2)/2}}\,\xi^2\,dx \\
&+ \int_\Omega f_i u_i \frac{1}{\big((x-x_0)^2+h^2\big)^{r/2}}\,\xi^2\,dx.
\end{aligned}
\tag{2.20}
$$

Denoting the integrals on the right-hand side by $I_1, \dots, I_4$ we have

$$|I_1| \le \int_\Omega |\mathbf{u}||\nabla \mathbf{u}||\nabla \xi| \frac{\xi}{|x-x_0|^r}\, dx$$

$$\le \|\nabla \xi\|_{0,\infty} \|\nabla \mathbf{u}\|_{0,2} \left(\int_\Omega \mathbf{u}^2 \frac{1}{|x-x_0|^{2r}} \xi^2\, dx \right)^{1/2},$$

$$|I_2| \le \int_{\Omega \setminus B_\rho(x_0)} |\mathbf{u}| \left| \frac{\mathbf{u}^2}{2} + p \right| |\nabla \xi| \frac{\xi}{|x-x_0|^r}\, dx$$

$$\le \left\| \frac{1}{|x-x_0|^r} \right\|_{0,\infty,\Omega \setminus B_\rho(x_0)} \|\nabla \xi\|_{0,\infty} \left\| \frac{\mathbf{u}^2}{2} + p \right\|_{0,q} \|\mathbf{u}\|_{0,\frac{q}{q-1}},$$

where $q < 2$ and $10/3 > q/(q-1)$,

$$|I_4| \le c\|\mathbf{f}\|_{0,\infty} \left(\int_\Omega \mathbf{u}^2 \frac{1}{|x-x_0|^{2r}} \xi^2\, dx \right)^{1/2},$$

$$|I_3| \le \int_\Omega |\mathbf{u}| \left| \frac{\mathbf{u}^2}{2} + p \right| \frac{1}{|x-x_0|^{r+1}} \xi^2\, dx$$

$$\le \left(\int_\Omega \left| \frac{\mathbf{u}^2}{2} + p \right|^{q/(q-1)} \xi^2\, dx \right)^{(q-1)/q} \left(\int_\Omega |\mathbf{u}|^q \frac{1}{|x-x_0|^{(r+1)q}} \xi^2\, dx \right)^{1/q},$$

where $q > 2$. Since $(r+1)q < 3$ for appropriate q it is possible to choose $\gamma < 1$ and $s < 3$ such that $(r+1)q = s\gamma$. Thus we can write

$$\int_\Omega |\mathbf{u}|^q \frac{1}{|x-x_0|^{(r+1)q}} \xi^2\, dx = \int_\Omega \frac{|\mathbf{u}|^{2\gamma}}{|x-x_0|^{s\gamma}} |\mathbf{u}|^{q-2\gamma} \xi^2\, dx.$$

Due to $(2.8)_1$ the first factor belongs to $L^{1/\gamma}(\Omega)$ and the second one to $L^{10/(3(q-2\gamma))}(\Omega)$. Therefore the product belongs to $L^1(\Omega)$ if $10 = 3q + 4\gamma$.

In fact this implies that $\gamma = \frac{1}{4}(10 - 3q) < 1$ and $s = s(q) = \frac{4(r+1)q}{10-3q}$. Since $\lim_{q \to 2^+} s(q) < 3$ we may find s and q such that this relation holds. Therefore the left-hand side of (2.20) is bounded independently of $h > 0$. Taking the limit as h tends to 0^+ gives the assertion. ∎

2.21 PROPOSITION. *Let* $\mathbf{u}$ *be the maximum solution of* (1.1) *constructed in Section* 1. *Then for all* $x_0 \in \Omega$ *and all* $r \in \left(\frac{1}{4}, \frac{1}{2}\right)$ *and* $q \in [1, 2)$ *we have*

$$\begin{aligned} &\int_\Omega |p|^q \frac{\xi^2}{|x - x_0|^r}\, dx \le K \\ (2.22)\qquad &\int_\Omega |\mathbf{u}|^{2q} \frac{\xi^2}{|x - x_0|^r}\, dx \le K, \end{aligned}$$

where $K = K(\xi)$ *does not depend on* x_0.

PROOF. Put

$$\varphi_o(x_0) = c \int_\Omega |p|^{q-1} \operatorname{sgn} p \frac{\xi}{|x - x_0|^3}\, dx.$$

This function satisfies (see also (2.13))

$$-\Delta \varphi_o = |p|^{q-1} \operatorname{sgn} p \xi \tag{2.23}$$

$$\begin{aligned} &\varphi_o \in L^\infty(\Omega) \cap W^{2,s}_{\text{loc}}(\Omega) \qquad \text{for some } s = s(q) > 2 \\ (2.24)\qquad &\varphi_o \in W^{1,2s}_{\text{loc}}(\Omega) \qquad s > 2, \end{aligned}$$

which follows from the theory of regularity for linear elliptic equations since $|p|^{q-1} \in L^s(\Omega)$ and $s = s(q) > 2$ using simple interpolation argument between $L^\infty(\Omega)$ and $W^{2,s}(\Omega)$ (see also (1.37)).

Now we use $\psi = \varphi_o \frac{1}{\left((x - x_0)^2 + h^2\right)^{r/2}} \xi^3$, $h > 0$ in the weak formulation for the pressure (2.14). After some rearrangement we arrive at

$$
\begin{aligned}
&\int\limits_\Omega |p|^q \frac{1}{\big((x-x_0)^2+h^2\big)^{r/2}}\,\xi^4\,dx \\
&= -2r\int\limits_\Omega p\,\frac{\partial\varphi_o}{\partial x_i}\,\frac{(x-x_0)_i}{\big((x-x_0)^2+h^2\big)^{(r+2)/2}}\,\xi^3\,dx \\
&\quad -2\int\limits_\Omega \frac{\partial\varphi_o}{\partial x_i}\,\frac{1}{\big((x-x_0)^2+h^2\big)^{r/2}}\,p\,\frac{\partial\xi^3}{\partial x_i}\,dx \\
&\quad -r\int\limits_\Omega \varphi_o\,\frac{5h^2+(3-r)(x-x_0)^2}{\big((x-x_0)^2+h^2\big)^{(r+4)/2}}\,p\xi^3\,dx \\
&\quad -2r\int\limits_\Omega \varphi_o p\,\frac{(x-x_0)_i}{\big((x-x_0)^2+h^2\big)^{(r+2)/2}}\,\frac{\partial\xi^3}{\partial x_i}\,dx \\
&\quad -\int\limits_\Omega \varphi_o p\,\frac{1}{\big((x-x_0)^2+h^2\big)^{r/2}}\,\Delta\xi^3\,dx \\
&\quad +\int\limits_\Omega \frac{\partial u_i}{\partial x_j}\,\frac{\partial u_j}{\partial x_i}\,\varphi_o\,\frac{1}{\big((x-x_0)^2+h^2\big)^{r/2}}\,\xi^3\,dx \\
&\quad -\int\limits_\Omega \frac{\partial f_i}{\partial x_i}\,\varphi_o\,\frac{1}{\big((x-x_0)^2+h^2\big)^{r/2}}\,\xi^3\,dx.
\end{aligned}
\tag{2.25}
$$

Denote the integrals on the right-hand side by $J_1, \dots, J_7$. We have

$$
\begin{aligned}
|J_1| &\le c\int\limits_\Omega |p|\,\frac{1}{|x-x_0|^{r+1}}\,|\nabla\varphi_o|\xi^3\,dx \\
&\le c\int\limits_\Omega |p|\,\frac{\xi^2}{|x-x_0|^{2(r+1)}} + |p|\,|\nabla\varphi_o|^2\xi^4\,dx, \\
|J_2| &\le c\int\limits_{\Omega\setminus B_\rho(x_0)} |\nabla\varphi_o|\,|p|\,\frac{\xi^2}{|x-x_0|^r}\,|\nabla\xi|\,dx \\
&\le c\left\|\frac{1}{|x-x_0|^r}\right\|_{0,\infty,\Omega\setminus B_\rho(x_0)} \|\nabla\varphi_o\|_{0,4,\mathrm{loc}}\,\|p\|_{0,4/3,\mathrm{loc}}\,\|\nabla\xi\|_{0,\infty},
\end{aligned}
$$

$$|J_3| \le \int_\Omega |\varphi_o|\,|p|\,\frac{1}{|x-x_0|^{r+2}}\,\xi^3\,dx$$

$$\le c\|\varphi_o\|_{0,\infty}\int_\Omega |p|\,\frac{1}{|x-x_0|^{r+2}}\,\xi^3\,dx,$$

$$|J_4| \le c\int_\Omega |\varphi_o|\,|p|\,|\nabla\xi|\,\frac{\xi^2}{|x-x_0|^{r+1}}\,dx$$

$$\le c\|\varphi_o\|_{0,\infty}\,\|\nabla\xi\|_{0,\infty}\int_\Omega |p|\,\frac{\xi^2}{|x-x_0|^{r+1}}\,dx,$$

$$|J_5| \le c\|\xi\|_{2,\infty}\int_\Omega |\varphi_o|\,|p|\,\frac{\xi}{|x-x_0|^{r}}\,dx$$

$$\le c\|\xi\|_{2,\infty}\,\|\varphi_o\|_{0,\infty}\int_\Omega |p|\,\frac{1}{|x-x_0|^{r}}\,\xi\,dx,$$

$$|J_6| \le c\int_\Omega |\nabla\mathbf{u}|^2\,|\varphi_o|\,\frac{1}{|x-x_0|^{r}}\,\xi^3\,dx$$

$$\le c\|\varphi_o\|_{0,\infty}\int_\Omega |\nabla\mathbf{u}|^2\,\frac{1}{|x-x_0|^{r}}\,\xi^2\,dx,$$

$$|J_7| \le c\int_\Omega |\operatorname{div}\mathbf{f}|\,|\varphi_o|\,\frac{1}{|x-x_0|^{r}}\,\xi^2\,dx$$

$$\le c\|\varphi_o\|_{0,\infty}\,\|\operatorname{div}\mathbf{f}\|_{0,2}\int_\Omega \frac{1}{|x-x_0|^{2r}}\,\xi^2\,dx.$$

Thus the left-hand side of (2.25) is bounded independently of $h > 0$. The assertions follow immediately. ∎

2.26 THEOREM. *Let* $\mathbf{u}$ *be the maximum solution constructed in Section* 1. *Then for all* $x_0 \in \Omega$ *and all* $r \in \left(\frac{1}{4}, 1\right)$ *we have*

$$\int_\Omega \frac{|\nabla\mathbf{u}|^2}{|x-x_0|^{r}}\,\xi^2\,dx \le K \tag{2.27}$$

and for all $q \in (1,2)$

$$\int_\Omega |p|^q \frac{\xi^2}{|x-x_0|^r}\,dx \le K \tag{2.28}$$
$$\int_\Omega |\mathbf{u}|^{2q} \frac{\xi^2}{|x-x_0|^r}\,dx \le K,$$

where $K = K(\xi)$ *does not depend on* x_0.

PROOF. We will show the assertions by induction, namely that for all k and

$$r \in \left(\frac{1}{4}, \sum_{n=1}^{k} \frac{1}{2^n}\right)$$

the estimates (2.27) and (2.28) are true. For $k = 1$ this has already been proved in Propositions 2.18 and 2.20. Thus let (2.27), (2.28) hold for all

$$r \in \left(\frac{1}{4}, \sum_{n=1}^{k-1} \frac{1}{2^n}\right).$$

In order to prove (2.27) we completely repeat the proof of Proposition 2.18, with the only difference that we choose $\varphi = \mathbf{u}\,\dfrac{\xi^2}{\big((x-x_0)^2+h^2\big)^{r/2}}$, for

$$r \in \left(\sum_{n=1}^{k-1} \frac{1}{2^n}, \sum_{n=1}^{k} \frac{1}{2^n}\right).$$

The only resulting difference is the estimate of the integral I_3. We have now

$$|I_3| \le c\int_\Omega |\mathbf{u}|\left|\frac{\mathbf{u}^2}{2}+p\right| \frac{\xi^2}{|x-x_0|^{r+1}}\,dx$$
$$\le c\left(\int_\Omega \left|\frac{\mathbf{u}^2}{2}+p\right|^{\sigma/(\sigma-1)} \frac{\xi^2}{|x-x_0|^t}\,dx\right)^{(\sigma-1)/\sigma}$$
$$\left(\int_\Omega |\mathbf{u}|^\sigma \frac{\xi^2}{|x-x_0|^{(r+1)\sigma-t(\sigma-1)}}\,dx\right)^{1/\sigma},$$

where $t \in \left(\sum_{n=1}^{k-2} \frac{1}{2^n}, \sum_{n=1}^{k-1} \frac{1}{2^n}\right)$ and thus the first integral is bounded by the induction hypothesis. On the other hand $\sigma > 2$ and t can be choosen such that $(r+1)\sigma - t(\sigma - 1) < 3$. Indeed, choosing $t = r - 2^{-k}\gamma$, $\gamma < 1$ and $2 < \sigma < \sigma_0$, where $\sigma_0 = \dfrac{2^{k+1} + 1 + \gamma}{2^k + \gamma}$ we have

$$(r+1)\sigma - t(\sigma - 1) < (r+1)\sigma_0 - t(\sigma_0 - 1) = 3 - \delta_0$$

where $\delta_0 = \sum_{n=1}^{k-1} \dfrac{1}{2^n} - r$. Thus also the second integral is bounded by the arguments of Proposition 2.18. Now we will show $(2.28)_1$ for $r \in \left(\sum_{n=1}^{k-1} \dfrac{1}{2^n}, \sum_{n=1}^{k} \dfrac{1}{2^n}\right)$. Again we can repeat the proof of Lemma 2.21. But in order to estimate the integral J_1 we must investigate equation (2.23) a little bit more carefully. First, multiply (2.23) by $\varphi_o \xi^2 |x - x_0|^{-s}$, $s < 3$. After some rearrangement we arrive at

$$\begin{aligned} &\int_\Omega |\nabla \varphi_o|^2 \frac{\xi^2}{|x - x_0|^s}\, dx \\ &\le c \int_\Omega |\nabla \varphi_o|\, |\varphi_o|\, |\nabla \xi^2| \frac{1}{|x - x_0|^s}\, dx \\ &\quad + c \int_\Omega |\nabla \varphi_o|\, |\varphi_o| \xi^2 \frac{1}{|x - x_0|^{s+1}}\, dx \\ &\quad + \int_\Omega |p|^{q-1} \frac{1}{|x - x_0|^s} \varphi_o \xi^3\, dx. \end{aligned} \tag{2.29}$$

Using (2.24) and $(2.8)_1$ the first and last integral on the right-hand side can be easily estimated. Further we have that the second integral on the right-hand side of (2.29) is bounded by

$$\|\varphi_o\|_{0,\infty} \left(\varepsilon \int_\Omega |\nabla \varphi_o|^2 \frac{\xi^2}{|x - x_0|^s}\, dx + \frac{c}{\varepsilon} \int_\Omega \frac{\xi^2}{|x - x_0|^{s+2}}\, dx \right).$$

For an appropriate ε we move the first term to the left-hand side; the second term is bounded as $s + 2 < 5$. Now we are ready to estimate the integral J_1 of (2.25):

$$\begin{aligned} |J_1| &\le \varepsilon \int_\Omega |p|^q \frac{1}{\left((x - x_0)^2 + h^2\right)^{r/2}} \xi^4\, dx \\ &\quad + \frac{c}{\varepsilon} \int_\Omega |\nabla \varphi_0|^{q/(q-1)} \frac{1}{|x - x_0|^{(r(q-1)+q)/(q-1)}} \xi^{(3q-4)/(q-1)}\, dx. \end{aligned}$$

We move the first term to the left-hand side; the second term J_8 can be estimated as follows (by Hölder's inequality):

$$|J_8| \leq c \left(\int_\Omega \frac{|\nabla \varphi_0|^2}{|x - x_0|^s} \xi^2 \, dx \right)^{(r(q-1)+q)/(q-1)} \left(\int_\Omega |\nabla \varphi|^\gamma \xi^\beta \, dx \right)^{(s-2q-2r(q-1))/(s(q-1))}$$

where $\gamma = \dfrac{q(s-2) - 2r(q-1)}{s - 2q - 2r(q-1)}$, but $\gamma \leq 4$ for all $r < 1$ and all $s < 3$ and $q \in (1, 2)$ and thus we can use (2.29) and $(2.24)_2$. This completes the proof. ∎

Unfortunately the estimates proved in Theorem 2.26 are not enough to obtain full regularity of a maximum solution. In order to prove full regularity we have to assume that for an arbitrary small $\varepsilon > 0$ and all $q < 4$

$$\int_\Omega |\mathbf{u}|^q \frac{\xi^2}{|x - x_0|^{1+\varepsilon}} \, dx \leq K,$$

which is a little bit more than proved in $(2.28)_2$.

3. - Full regularity

In this section we will show that one obtains full regularity of a maximum solution of system (1.1) under assumptions slightly stronger than those we were able to establish in the previous Section. First, let us give two useful results on weighted L^p-spaces.

3.1 PROPOSITION. *Let* $f \in W_0^{1,r}(\Omega)$, $1 < r < \infty$, *satisfy*

$$\int_\Omega \frac{|\nabla f|^r}{|x - x_0|^{N-\lambda}} \xi^2 \, dx \leq K \qquad 0 < \lambda < N, \tag{3.2}$$

where K *does not depend on* x_0 *and* ξ *is the usual truncation function. Then we have for all* $0 < \alpha < N - \lambda$ *the inequality*

$$\sup_{x_0 \in \Omega} \int_\Omega \frac{|f|^q}{|x - x_0|^\alpha} \xi^2 \, dx \leq c \left(\sup_{x_0 \in \Omega} \int_\Omega \frac{|\nabla f|^r}{|x - x_0|^{N-\lambda}} \xi^2 \, dx \right)^{q/r}, \tag{3.3}$$

where

$$\frac{1}{q} = \frac{1}{r} - \frac{1}{\lambda}. \tag{3.4}$$

PROOF. See Chiarenza-Frasca [3]. The result there is stated in the framework of Morrey spaces. ∎

Let v be the solution of

$$-\Delta v = f \qquad \text{a.e. in } \Omega, \tag{3.5}$$

where $f \in L^p(\Omega)$, $1 < p < \infty$.

3.6 PROPOSITION. *Let* $f \in L^p(\Omega)$, $1 < p < \infty$, *satisfy*

$$\int_\Omega \frac{|f|^p}{|x - x_0|^{N-\lambda}} \xi^2 \, dx \le K \quad 0 < \lambda < N, \tag{3.7}$$

where the constant K does not depend on $x_0 \in \Omega$. Then the solution v of (3.5) *satisfies for all $x_0 \in \Omega$ and all $0 < \alpha < N - \lambda$*

$$\int_\Omega \frac{|\nabla^2 v|^p}{|x - x_0|^\alpha} \xi^2 \, dx \le K. \tag{3.8}$$

PROOF. See Chiarenza-Frasca [3]. Also this result is stated in the framework of Morrey spaces. ∎

3.9 LEMMA. *Let $\varepsilon_0 > 0$ be arbitrarily small but fixed and let a maximum solution of* (1.1) *satisfy for all* $1 \le q < 4$

$$\int_\Omega \frac{|\mathbf{u}|^q}{|x - x_0|^{1+\varepsilon_0}} \xi^2 \, dx \le K, \tag{3.10}$$

where $K = K(\xi)$ does not depend on x_0. Then we have for all $0 < \alpha < \varepsilon_0/8$

$$\int_\Omega \frac{|\nabla \mathbf{u}|^2}{|x - x_0|^{1+\alpha}} \xi^2 \, dx \le K. \tag{3.11}$$

PROOF. We repeat the proof of Proposition 2.18, now with $r = 1 + \alpha$. The only difference consists in the treatment of the integral I_3. The following computations become simple to understand if we assume in (3.10) that $q = 4$.

We have, using (3.10) and $(2.8)_1$:

$$\begin{aligned}
&\int_\Omega |\mathbf{u}|\,|p|\,\frac{\xi^2}{|x-x_0|^{2+\alpha}}\,dx \\
&\leq \int_\Omega \frac{|\mathbf{u}|^q}{|x-x_0|^{1+\varepsilon_0}}\,\xi^2\,dx + \int_\Omega \frac{|p|^{q/(q-1)}}{|x-x_0|\chi}\,\xi^2\,dx \\
&\leq c + \int_\Omega \frac{|p|}{|x-x_0|^s}\,\xi^2\,dx + \int_\Omega \frac{|p|^{q(r-1)/(q(r-1)-r)}}{|x-x_0|^\beta}\,\xi^2\,dx,
\end{aligned}$$

where $s < 3$, $1 < r < \infty$, and

$$\begin{aligned}
\chi &= \frac{2q-1}{q-1} - \varepsilon_0\,\frac{1}{q-1} + \alpha\,\frac{q}{q-1}, \\
\beta &= \frac{(2q-1)r}{r(q-1)-q} - \varepsilon_0\,\frac{r}{r(q-1)-q} + \left(\alpha - \frac{s}{r}\right)\frac{rq}{r(q-1)-q}.
\end{aligned}$$

In order to obtain the last inequality we have used the relations

$$\frac{q}{q-1} = \frac{q}{q-1}\left(\frac{1}{r} + \frac{r-1}{r}\right),$$

$$\frac{2q-1}{q-1} - \varepsilon_0\,\frac{1}{q-1} + \alpha\,\frac{q}{q-1} = s\,\frac{q}{q-1}\,\frac{1}{r} + \frac{2q-1}{q-1} + \varepsilon_0\,\frac{1}{q-1} + \left(\alpha - \frac{s}{r}\right)\frac{q}{q-1}$$

and Hölder's inequality with exponent $r(q-1)/q$. Now the second term in the last inequality is bounded thanks to (2.8) and for the third one we use $(2.28)_1$ which causes restrictions on s, q and r. Let us choose now $\alpha = \varepsilon_0/8$. The following must be fulfilled:

$$\text{(3.12)}\qquad \begin{aligned}
&r > \frac{q}{q-2} \\
&r\left(1 - \varepsilon_0\,\frac{8-q}{8q}\right) < s - 1.
\end{aligned}$$

Choose now for $\beta < 1$ and $1 < \gamma$ numbers q and r as follows:

$$q = 4\beta, \qquad r = \gamma\,\frac{q}{q-2}.$$

Formula $(3.12)_2$ can be re-written as

$$\gamma\left(\frac{2\beta}{2\beta-1} - \varepsilon_0\,\frac{2-\beta}{4(2\beta-1)}\right) < s - 1$$

or

$$\gamma(2-\delta) < s-1 \tag{3.13}$$

for some $\delta = \delta(\varepsilon_0)$ if $\beta \in (\beta_0(\varepsilon_0), 1)$. Now γ can be specified such that for an appropriate $s < 3$ inequality (3.13) is satisfied. Let us now treat the integral

$$\begin{aligned}&\int\limits_\Omega \frac{|\mathbf{u}|^3}{|x-x_0|^{2+\alpha}}\,\xi^2\,dx\\ &\le \int\limits_\Omega \frac{|\mathbf{u}|^q}{|x-x_0|^{1+\varepsilon_0}}\,\xi^2\,dx + \int\limits_\Omega \frac{\xi^2}{|x-x_0|^\gamma}\,dx,\end{aligned}$$

where

$$\gamma = (1+\alpha-\varepsilon_0)\frac{q}{q-3} + 1 + \varepsilon_0 < 5.$$

Now we can complete the proof as in Proposition 2.18. ∎

Relation (3.11) will be the starting point for our subsequent investigations. But before proceeding we will show that the behaviour of the gradient of the pressure and of the convective term are the same.

3.14 REMARK. Consider the solution φ_i of

$$-\Delta\varphi_i = \frac{\partial u_i}{\partial x_j}u_j - f_i \qquad \text{a.e. in } \Omega, \quad i = 1, \dots, N. \tag{3.15}$$

Let us assume that

$$\int\limits_\Omega |\nabla\mathbf{u}|^r\,|\mathbf{u}|^r\,\frac{1}{|x-x_0|^{s_0}}\,\xi^2\,dx \le c, \tag{3.16}$$

where $1 < r < \infty$ and $0 < s_0 < N$. Thanks to Proposition 3.6 we have for all $0 < s < s_0$

$$\int\limits_\Omega |\nabla^2\varphi_i|^r\,\frac{1}{|x-x_0|^s}\,\xi^2\,dx \le c.$$

Differentiation of (3.15) with respect to x_i and summation of the result give in the weak formulation

$$\int\limits_\Omega \nabla\frac{\partial\varphi_i}{\partial x_i}\nabla\psi\,dx = \int\limits_\Omega \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}\psi\,dx + \int\limits_\Omega f_i\frac{\partial\psi}{\partial x_i}\,dx \qquad \forall\psi \in C_0^\infty(\Omega).$$

This and (2.14) yield

$$\int_\Omega \nabla \left(\frac{\partial \varphi_i}{\partial x_i} - p \right) \nabla \psi \, dx = 0 \qquad \forall \psi \in C_0^\infty(\Omega).$$

Therefore $\operatorname{div} \varphi - p$ is a harmonic function and thus $\operatorname{div} \varphi$ and p have the same behaviour in weighted L^p-spaces. Remark that from the representation theorem one gets

$$\|\operatorname{div} \varphi - p\|_{1,\infty,\mathrm{loc}} \le c \|\operatorname{div} \varphi - p\|_{1,1,\mathrm{loc}}.$$

In particular we have for all $0 < s < s_0$

$$\int_\Omega |\nabla p|^r \frac{1}{|x - x_0|^s} \xi^2 \, dx \le c. \tag{3.17}$$

Relations (3.16) and (3.17) show that the convective term and the gradient of the pressure have the same regularity. Therefore we will treat in the sequel only the convective term.

3.18 LEMMA. *Let a maximum solution of* (1.1) *satisfy for all* $x_0 \in \Omega$ *and all* $\alpha < \alpha_0$, $\alpha_0 > 0$

$$\int_\Omega \frac{|\nabla \mathbf{u}|^2}{|x - x_0|^{1+\alpha}} \xi^2 \, dx \le K. \tag{3.19}$$

Then for all $\alpha < \alpha_0$ *and all* $x_0 \in \Omega$

$$\int_\Omega \frac{|\nabla \mathbf{u}|^l}{|x - x_0|^{1+\alpha}} \xi^2 \, dx \le K, \tag{3.20}$$

where $l > 2$ *and* $K = K(\xi)$ *does not depend on* x_0.

PROOF. Let us denote $l_0 \equiv 2$. From (3.19) and Proposition 3.1 it follows that for all $\alpha < \alpha_0$, $1 < q < q_0$ and almost every $x_0 \in \Omega$

$$\int_\Omega \frac{|\mathbf{u}|^q}{|x - x_0|^{1+\alpha}} \xi^2 \, dx \le c, \tag{3.21}$$

where

$$\frac{1}{q_0} = \frac{1}{l_0} - \frac{1}{4 - \alpha_0}. \tag{3.22}$$

By means of Hölder's inequality it is clear that for all $1 < r < r_0$

$$\int_\Omega \frac{|\mathbf{u}|^r \, |\nabla \mathbf{u}|^r}{|x - x_0|^{1+\alpha}} \xi^2 \, dx \le c, \tag{3.23}$$

where

$$\frac{1}{r_0} = \frac{1}{q_0} + \frac{1}{l_0} = \frac{2}{l_0} - \frac{1}{4-\alpha_0}. \tag{3.24}$$

Remark 3.14 and Proposition 3.6 imply that

$$\int_\Omega \frac{|\nabla^2 \mathbf{u}|^r}{|x-x_0|^{1+\alpha}} \xi^2 \, dx \le c, \qquad 1 \le r < r_0, \ \alpha < \alpha_0. \tag{3.25}$$

Now we apply again Proposition 3.1 to (3.25), which yields

$$\int_\Omega \frac{|\nabla \mathbf{u}|^l}{|x-x_0|^{1+\alpha}} \xi^2 \, dx \le c, \qquad 1 \le r < r_0, \ \alpha < \alpha_0 \tag{3.26}$$

for almost every $x_0 \in \Omega$, $1 \le l < l_1$, where l_1 is given by

$$\frac{1}{l_1} = \frac{1}{r_0} - \frac{1}{4-\alpha_0} = \frac{2}{l_0} - \frac{2}{4-\alpha_0}. \tag{3.27}$$

Clearly, for $\alpha_0 > 0$ and $l_0 = 2$ we get

$$l_1 > l_0$$

which concludes the proof. ∎

3.28 THEOREM. *A maximum solution of* (1.1) *which satisfies* (3.10) *has full regularity, i.e.*

$$\begin{aligned} &\mathbf{u} \in W^{2,q}_{\text{loc}}(\Omega) \\ &p \in W^{1,q}_{\text{loc}}(\Omega) \end{aligned} \tag{3.29}$$

where $q \in [1, \infty)$.

PROOF. From the proof of Lemma 3.18 it is clear that the whole procedure can be repeated. We obtain recursive formulae for l_n and r_n given by

$$\begin{aligned} \frac{1}{l_n} &= \frac{2^n}{l_0} - \frac{2}{4-\alpha_0}(2^n - 1) \qquad &\text{for } \frac{4}{2^n} > \alpha_0 \\ \frac{1}{r_n} &= \frac{2^{n+1}}{l_0} - \frac{2^{n+2}-3}{4-\alpha_0} \qquad &\text{for } \frac{3}{2^n} > \alpha_0. \end{aligned} \tag{3.30}$$

Remark that both $\frac{1}{l_n}$ and $\frac{1}{r_n}$ tend to $-\infty$, but formulae (3.30) have only sense if the right-hand sides are positive. Remark further that the condition in $(3.30)_2$ is violated at first, but this means that (3.23) holds for all $r < \infty$ and therefore also (3.26) holds for all $l < \infty$. Thus we have established:

$$\int_\Omega \frac{|\nabla \mathbf{u}|^l}{|x-x_0|^{1+\alpha}}\, \xi^2\, dx \leq c, \qquad 1 \leq l < \infty$$

$$\int_\Omega \frac{|\nabla^2 \mathbf{u}|^r}{|x-x_0|^{1+\alpha}}\, \xi^2\, dx \leq c, \qquad 1 \leq r < \infty,$$

which is more than what is stated in (3.29). ∎

3.31 REMARK. Let us note that the argument of Section 3 and of the first part of Section 2 (Theorem 2.1 - Corollary 2.15) works in arbitrary dimension. We have used only existence of a maximum solution, integrability of $\dfrac{1}{|x-x_0|^r}$ for $r < N$ and Hölder's inequality. Therefore, if we replace in Theorem 2.1

$$\int_\Omega \left|\frac{u^2}{2} + p\right| \frac{\xi^2}{|x-x_0|^3}\, dx \leq K$$

by

$$\int_\Omega \left|\frac{u^2}{2} + p\right| \frac{\xi^2}{|x-x_0|^{N-2}}\, dx \leq K$$

all the statements remain true. Assumption (3.10) has to be replaced by

$$\int_\Omega \frac{\mathbf{u}^q}{|x-x_0|^{N-4+\varepsilon}}\, \xi^2\, dx \leq K, \qquad 1 \leq q < 4.$$

The only reason why we did not state the assertions in the general setting is that we are not able to prove existence of a maximum solution in arbitrary dimension.

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