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Convexly totally bounded and strongly totally bounded sets. Solution of a problem of Idzik

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Convexly Totally Bounded and Strongly Totally Bounded Sets. Solution of a Problem of Idzik

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0. - Introduction

A long outstanding problem in fixed point theory is the following

PROBLEM OF SCHAUDER (Problem 54, [G]). *Does every continuous function $f : C \to C$ defined on a convex compact subset of a Hausdorff topological linear space have a fixed point?*

Schauder [S] gave a positive answer to this problem if the linear space is a Banach space. Thychonoff [T] generalized Schauder's theorem for locally convex spaces. Schauder's problem is still open even for metrizable (nonlocally convex) spaces. Idzik [I1] proved that the answer to Schauder's problem is "yes" if C is convexly totally bounded. This notion was introduced by Idzik [I1]: A subset K of a topological linear space E is called *convexly totally bounded* (*ctb* for short) if for every 0-neighbourhood U there are $x_1, \dots, x_n \in E$ and convex subsets $C_1, \dots, C_n$ of U such that $K \subset \bigcup_{i=1}^{n}(x_i + C_i)$. Idzik formulated – comparing his theorem with Schauder's problem – the following

PROBLEM (cf. Problem 4.7 of [I2]). *Is every convex compact subset of a Hausdorff topological linear space ctb?*

A positive answer to this problem would imply a positive answer to Schauder's problem.

In the first section of this paper we give a negative answer to Idzik's problem. In Section 2, we introduce the notion of strongly convexly totally bounded (sctb) sets and in Section 3 a parameter, which measures "the lack of strongly convexly total boundedness". This notion and this parameter is – in contrast to convexly total boundedness – invariant when one passes to the convex hull of a set. That admits the formulation of a fixed point theorem of

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Darbo type [D] in nonlocally convex spaces. In Sections 4 and 5, we examine the mentioned parameter in the space l_p $(0 < p < 1)$ and in the space L_0 of measurable functions.

The notion of sctb sets and the corresponding noncompactness measure is the main tool in [DP/T2] to get a best approximation result of Fan type in nonlocally convex linear spaces. In [W] strongly convexly total boundedness is linked with affine embeddability in locally convex spaces.

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In this paper we use the terminology of [J]. $\mathbb{N}$ and $\mathbb{R}$ stands for the sets of all natural and real numbers, respectively.

1. - A compact convex set not convexly totally bounded

In this section we construct a compact convex set, which is not ctb; this solves Problem 4.7 of Idzik [I2]. We obtain such a set by a modification of Robert's example for a compact convex set without extreme point, see [R], [Ro, Section 5.6]. Robert's example is based on his notion of *needle point*; that is a point $x_0 \neq 0$ of an (in the sense of [J]) F-normed linear space $(E, \|\cdot\|)$ with the following property: for every $\varepsilon > 0$, the ball $B_\varepsilon := \{x \in B : \|x\| \leq \varepsilon\}$ contains a finite set F such that $\operatorname{co} F \subset \operatorname{co}\{0, x_0\} + B_\varepsilon$ and $x_0 \in \operatorname{co} F + B_\varepsilon$. (Hereby $\operatorname{co} F$ denotes the convex hull of F). In our construction, a stronger property plays an important role:

DEFINITION 1.1. We call a point x_0 of an F-normed linear space $(E, \|\cdot\|)$ a *strong needle point*, if $x_0 \neq 0$ and, for every $\varepsilon > 0$, there is a natural number k and an infinite subset M of B_ε such that $\operatorname{co} M \subset \operatorname{co}\{0, x_0\} + B_\varepsilon$ and $x_0 \in \operatorname{co} F + B_\varepsilon$ for every finite subset F of M, with $|F| \geq k$. (Hereby we denote by $|F|$ the cardinality of F).

LEMMA 1.2. *For each $n \in \mathbb{N} \cup \{0\}$, let F_n be a finite subset of an F-normed linear space E and $\varepsilon_n > 0$ such that $\operatorname{co} F_n \subset \operatorname{co} F_0 + B_{\varepsilon_n}$. Assume that $\sum_{n=1}^{\infty} \varepsilon_n < +\infty$. Then $C := \operatorname{co} \bigcup_{n=0}^{\infty} F_n$ is totally bounded.*

PROOF. Since $\bigcup_{i=0}^{n} F_i$ is a compact subset of a finite dimensional subspace of E, also $C_n := \operatorname{co} \bigcup_{i=0}^{n} F_i$ is compact. Furthermore, we have $r_n := \sum_{i>n} \varepsilon_i \to 0$ $(n \to \infty)$. Therefore it is enough to prove that $C \subset C_n + B_{r_n}$ for all $n \in \mathbb{N}$.

For α, $\beta \in [0,1]$, with $\alpha+\beta = 1$ and $n \in \mathbb{N}$, we have

$$\alpha \cdot C_n + \beta \cdot \operatorname{co} F_{n+1} \subset \alpha \cdot C_n + \beta \cdot \operatorname{co} F_0 + \beta \cdot B_{\varepsilon_{n+1}}$$
$$\subset \alpha \cdot C_n + \beta \cdot C_n + B_{\varepsilon_{n+1}} \subset C_n + B_{\varepsilon_{n+1}},$$

hence $C_{n+1} = \bigcup_{\alpha\in[0,1]} \alpha \cdot C_n + (1-\alpha)\cdot \operatorname{co} F_{n+1} \subset C_n + B_{\varepsilon_{n+1}}$. It follows inductively that $C_{n+k} \subset C_n + \sum_{i=n+1}^{n+k} B_{\varepsilon_i}$ for n, $k \in \mathbb{N}$; therefore $C_{n+k} \subset C_n + B_{r_n}$ and $C = \bigcup_{k=1}^{\infty} C_{n+k} \subset C_n + B_{r_n}$.

THEOREM 1.3. *Let E be a complete F-normed linear space, which contains a strong needle point. Then E contains a compact absolutely convex set, which is not ctb.*

PROOF. Let $\varepsilon_n > 0$, with $\sum_{n=1}^{\infty} \varepsilon_n < +\infty$, and x_0 be a strong needle point of E. Choose infinite subsets M_n of B_{ε_n} and $k_n \in \mathbb{N}$ such that $\operatorname{co} M_n \subset \operatorname{co}\{0, x_0\} + B_{\varepsilon_n}$, and $x_0 \in \operatorname{co} F + B_{\varepsilon_n}$, for every finite subset F of M_n, with $|F| \geq k_n$. Let $F_0 := \{0, x_0\}$ and F_n be finite subsets of M_n with $|F_n| \geq n \cdot k_n$ for $n \in \mathbb{N}$. Since $C := \operatorname{co} \bigcup_{n=0}^{\infty} F_n$ is totally bounded by Lemma 1.2, $\overline{C}$ and $\overline{C}$-$\overline{C}$ are compact. Therefore the closed absolutely convex hull $K := \overline{\operatorname{aco}} \bigcup_{n=0}^{\infty} F_n$ of C is a compact subset of $\overline{C}$-$\overline{C}$ containing C. We now show that C is not ctb; then neither is K. Let $\varepsilon > 0$, with $x_0 \notin B_{4\varepsilon}$. Suppose that C is ctb. Then there are convex subsets C_i of B_ε and $y_i \in E$ such that $C \subset \bigcup_{i=1}^{m} (y_i + C_i)$. Choose $n \geq m$ with $\varepsilon_n \leq \varepsilon$. Since $F_n \subset C \subset \bigcup_{i=1}^{m} (y_i + C_i)$ and consequently

$$\sum_{i=1}^{m} |F_n \cap (y_i + C_i)| \geq |F_n| \geq n \cdot k_n \geq m \cdot k_n,$$

for some $j \in \{1, \ldots, m\}$, the set $F := F_n \cap (y_j + C_j)$ has at least k_n elements. Therefore

$$x_0 \in \operatorname{co} F + B_{\varepsilon_n} \subset y_j + C_j + B_\varepsilon \subset y_j + B_{2\varepsilon}.$$

Let $y \in F$. Then $y_j \in y - C_j \subset F_n - C_j \subset B_{\varepsilon_n} + B_\varepsilon \subset B_{2\varepsilon}$. It follows that $x_0 \in y_j + B_{2\varepsilon} \subset B_{2\varepsilon} + B_{2\varepsilon} \subset B_{4\varepsilon}$, a contradiction.

We now show the existence of strong needle points in the same way as Roberts has shown the existence of needle points; see [R], [Ro, Section 5.6].

In the following, let $\varphi : [0,+\infty[\to [0,+\infty[$ *be a continuous increasing concave function such that*
$\varphi(x) = 0$ *iff* $x = 0$ *and*
$\varphi(x)/x \to 0$ *for* $0 < x \to +\infty$.

If $\mathcal{U}$ is an algebra of sets and μ a $[0,1]$-valued measure (:= finitely additive set function) on $\mathcal{U}$, we denote by $S(\mathcal{U})$ the space of real-valued $\mathcal{U}$-simple functions, i.e. the linear hull of the system of all characteristic functions of sets of $\mathcal{U}$; furthermore we put

$$\|f\|_\varphi = \int \varphi\circ|f|\,\mathrm{d}\mu \quad \text{and} \quad \|f\|_p := \left(\int |f|^p\,\mathrm{d}\mu\right)^{1/p}$$

for $f \in S(\mathcal{U})$ and $1 \le p < +\infty$. $\|\cdot\|_\varphi$ is then a Riesz pseudonorm in the sense of [A/B, p. 39] and $\|f\|_\varphi \le \varphi(\|f\|_1)$ for $f \in S(\mathcal{U})$.

In the following let $\mathcal{U}_0$ *be an algebra in the power set* $\mathbb{P}(\Omega_0)$ *of a non-empty set* Ω_0,
$\mu_0 : \mathcal{U}_0 \to [0,1]$ *a measure with* $\mu_0(\Omega_0) = 1$,
(A_n) *a sequence in* $\mathcal{U}_0$ *with* $0 < \mu_0(A_n) \to 0$ $(n \to \infty)$,
$(\Omega, \mathcal{U}, \mu) = \bigotimes_{n\in\mathbb{N}}(\Omega_n, \mathcal{U}_n, \mu_n)$, *where* $(\Omega_n, \mathcal{U}_n, \mu_n) = (\Omega_0, \mathcal{U}_0, \mu_0)$ *for* $n \in \mathbb{N}$.

For $f : \Omega_0 \to \mathbb{R}$ and $i \in \mathbb{N}$, we define $S_i(f) : \Omega \to \mathbb{R}$ by $(S_i(f))((x_n)) := f(x_i)$.

LEMMA 1.4. *Let* $f \in S(\mathcal{U}_0)$, *with* $\int f\,\mathrm{d}\mu_0 = 1$, *and* $0 \le t_i \le t < +\infty$, *with* $\sum_{i=1}^{n} t_i \le 1$. *Then* $g := \sum_{n=1}^{n} t_i(S_i(f) - 1) \in S(\mathcal{U})$, $\|g\|_1 \le \sqrt{t}\cdot\|f-1\|_2$, $\|g\|_\varphi \le \varphi\left(\sqrt{t}\cdot\|f-1\|_2\right)$.

For the proof, see [Ro, pp. 244-245] or [R, Lemma 3.8].

LEMMA 1.5. *Let* $0 < b \le 1$ *and* $\delta > 0$. *Then there are a non-negative function* $f \in S(\mathcal{U}_0)$ *and a number* $a \in]0, b[$ *with the following properties:*

(i) $\int f\,\mathrm{d}\mu_0 = 1$;

(ii) $\|f\|_\varphi \le \delta$;

(iii) *if* $n \in \mathbb{N}$ *and* $t_i \ge b$, *with* $\sum_{i=1}^{n} t_i \le 1$, *then*

$$\left\|\sum_{i=1}^{n} t_i S_i(f)\right\|_\varphi \le \delta;$$

(iv) *if* $n \in \mathbb{N}$ *and* $0 \le t_i \le a$, *with* $\sum_{i=1}^{n} t_i \le 1$, *then*

$$\left\| \sum_{i=1}^{n} t_i(S_i(f) - 1) \right\|_{\varphi} \le \delta.$$

For the proof, see [Ro, Lemma 5.6.3] or [R, Lemma 3.10]. Hereby, 1.4 is used to prove (iv) of 1.5. With the aid of 1.5, one can prove the following proposition as [Ro, Proposition 5.6.4].

PROPOSITION 1.6. *Denote by* f_0 *the constant function equal to* 1 *on* Ω. *Let* $\varepsilon > 0$. *Then there is a non-negative function* $f \in S(\mathfrak{U}_0)$ *with the following properties:*

(i) $\int f \, d\mu_0 = 1$;

(ii) $M := \{S_i(f) : i \in \mathbb{N}\}$ *is contained in* $B_\varepsilon := \{h \in S(\mathfrak{U}) : \|h\|_\varphi \le \varepsilon\}$;

(iii) *there is a* $k \in \mathbb{N}$ *such that, for every* $I \subset \mathbb{N}$, *with* $|I| \ge k$, *we have* $f_0 \in \operatorname{co}\{S_i(f) : i \in I\} + B_\varepsilon$;

(iv) $\operatorname{co} M \subset \operatorname{co}\{0, f_0\} + B_\varepsilon$.

For small ε, a function f satisfying (i) and (ii) of 1.6 cannot be constant. The functions $S_i(f)$ are then different for different indices i; therefore M is an infinite set. It follows:

THEOREM 1.7. *Denote by* N *the space* $N := \{f \in S(\mathfrak{U}) : \|f\|_\varphi = 0\}$ *of all* $\mathfrak{U}$*-simple null functions. Then* $(S(\mathfrak{U}), \|\cdot\|_\varphi)/N$ *contains strong needle points.*

COROLLARY 1.8. *Let* λ *be the Lebesgue measure on* $[0, 1]$ *and* $\varphi : [0, +\infty[\to [0, +\infty[$ *be a continuous increasing concave function such that* $\varphi(x) = 0$ *iff* $x = 0$ *and* $\varphi(x)/x \to 0$ $(0 < x \to +\infty)$. *Then the Orlicz space* $L_\varphi(\lambda)$ *contains a compact absolutely convex set, which is not ctb.*

PROOF. Since the measure algebras induced by λ and by $\lambda \otimes \lambda \otimes \lambda \otimes \cdots$ are isomorphic, the space $L_\varphi(\lambda)$ is isomorphic to the completion of a space $(S(\mathfrak{U}), \|\cdot\|_\varphi)/N$ considered in 1.7. Since the last-named space contains strong needle points, so does $L_\varphi(\lambda)$. Now the assertion follows from 1.3.

COROLLARY 1.9. *Let* λ *be the Lebesgue measure on* $[0, 1]$ *and* $0 \le p < 1$. *Then* $L_p(\lambda)$ *contains a compact absolutely convex set, which is not ctb.*

PROOF. Apply 1.8 with $\varphi(x) = x^p$ for $0 < p < 1$ and $\varphi(x) = x/(1 + x)$ for $p = 0$.

2. - Strongly convexly totally bounded sets

In this section, let (E, τ) be a topological linear space. An important question in fixed point theory is: under which condition for a set $K \subset E$, the convex hull of any totally bounded subset of K is totally bounded, see [K, p. 10], [H1, p. 31], cf. also 3.1 (ii). Easy examples show that the convex hull of a ctb set does not need to be ctb, see 5.3 (b). Therefore we introduce in 2.1 a property stronger than convexly total boundedness, which is conserved passing to the convex hull, see 2.2 (b).

DEFINITION 2.1. A subset K of E is said to be *strongly convexly totally bounded* (*sctb* for short) if, for every 0-neighbourhood U, there is a convex subset C of U and a finite subset F of E such that $K \subset F + C$.

A subset of a *locally convex* linear space is sctb iff it is totally bounded.

PROPOSITION 2.2. *(a) If K_1 and K_2 are sctb subsets of E, then the sets $K_1 \cup K_2$, $K_1 + K_2$ and $\alpha \cdot K_1$ for any $\alpha \in \mathbb{R}$ are sctb. (b) If K is a sctb subset of E, then the closed absolutely convex hull $\overline{\text{aco}}\, K$ of K is sctb.*

PROOF. Let U be a circled closed 0-neighbourhood in E.

(i) There are finite sets $F_i \subset E$ and convex sets $C_i \subset U$ such that $K_i \subset F_i + C_i$ for $i = 1, 2$. Since $\text{co}(C_i \cup \{0\}) \subset U$, we may assume that $0 \in C_i$. Then the convex set $C := C_1 + C_2$ contains C_1 and C_2. It follows that $K_1 \cup K_2 \subset (F_1 \cup F_2) + C$, $K_1 + K_2 \subset (F_1 + F_2) + C$, $C \subset U + U$. Hence $K_1 \cup K_2$ and $K_1 + K_2$ are sctb.

(ii) Since $\alpha \cdot K_1 \subset \alpha \cdot F_1 + \alpha \cdot C_1$, $\alpha \cdot C_1 \subset \alpha \cdot U$ and $\overline{K}_1 \subset F_1 + \overline{C}_1$, $\overline{C}_1 \subset \overline{U} = U$, the sets $\alpha \cdot K_1$ and $\overline{K}_1$ are sctb.

(iii) Let K be sctb; replacing K by $K \cup \{0\}$, we may assume that $0 \in K$. Let F be a finite subset of E and C a convex subset of U such that $K \subset F + C$, and V be a 0- neighbourhood in E. Since $E_0 := \text{span}\, F$ is finite dimensional, V contains a convex 0-neighbourhood V_0 in E_0; morover $\text{co}\, F$ is a compact subset of E_0. It follows that there is a finite subset F_0 of E_0 such that $\text{co}\, F \subset F_0 + V_0$. Since K is a subset of the convex set $\text{co}\, F + C$, we get $\text{co}\, K \subset \text{co}\, F + C \subset F_0 + (V_0 + C)$, where $V_0 + C$ is a convex subset of $V + U$. We have proved that $\text{co}\, K$ is sctb. Since $\text{aco}\, K \subset \text{co}\, K - \text{co}\, K$, the set $\text{aco}\, K$ is sctb by (i). It now follows from (ii) that the closure $\overline{\text{aco}}\, K$ is sctb.

Obviously every sctb set is ctb. In 5.3 (b), we give an example for a ctb set, the convex hull of which is not ctb; by 2.2 (b), such a set is ctb, but not sctb. But we do not know whether there are also *convex* ctb sets, which are not sctb.

In [W, Section 2] it is proved that a compact convex subset K of E is sctb iff there is a locally convex linear topology σ on E such that the relative topologies $\sigma|K$ and $\tau|K$ coincide. Using the easy implication ($\Leftarrow$) of this equivalence for $\sigma = \sigma(E, E')$ one obtains:

PROPOSITION 2.3. *If the continuous dual E' of E separates the points, then every compact convex subset of E is sctb.*

Other examples for sctb sets are the compact convex order-bounded subsets of Orlicz function spaces, as mentioned at the end of [W, Section 2].

In 3.5 we will use the following fact:

PROPOSITION 2.4. *Let $(\tau_i)_{i\in I}$ be a family of linear topologies on E and $\tau = \sup_{i\in I} \tau_i$. Then a set $K \subset E$ is sctb, ctb or totally bounded in (E,τ) iff K is sctb, ctb or totally bounded in (E,τ_i), respectively, for every $i \in I$.*

PROOF (of the non-obvious implication ($\Leftarrow$) in the sctb case). Since every 0-neighbourhood in (E,τ) is a 0-neighbourhood in $(E, \sup_{i\in J} \tau_i)$ for some finite subset J of I, we may assume that I is finite. Inductively we can reduce the assertion to the case that τ is the supremum of two linear topologies τ_1 and τ_2.

Let U be a 0-neighbourhood in (E,τ) and V_i be a 0-neighbourhood in (E,τ_i) such that $(V_1 - V_1) \cap (V_2 - V_2) \subset U$. By assumption, there are convex sets $C_i \subset V_i$ and elements x_r, $y_s \in E$ such that $K \subset \bigcup_{r=1}^{m}(x_r + C_1)$ and $K \subset \bigcup_{s=1}^{n}(y_s + C_2)$. Choose $z_{rs} \in C_{rs} := (x_r + C_1) \cap (y_s + C_2)$ if $C_{rs} \neq \emptyset$ and z_{rs} arbitrary of E if $C_{rs} = \emptyset$. Then $C_{rs} \subset z_{rs} + C$, where $C := (C_1 - C_1) \cap (C_2 - C_2)$ is a convex subset of U. Moreover $K \subset \bigcup_{r,s} C_{rs} \subset \bigcup_{r,s} z_{rs} + C$.

3. - Noncompactness measures

One of the main tools in fixed point theory, after the pioneering work of Darbo [D], is the noncompactness measure. There are many axiomatic approaches to this concept (cf. [B/G], [B/R], [Rz], [H1] and references therein). All the approaches try to describe the minimal properties for a fixed point result. An axiomatic approach can be useful also in the frame of nonlocally convex spaces. Let I be a non-empty set and $V = [0, +\infty]^I$ the set of all functions from I to $[0, +\infty]$. V will be equipped with the usual agebraic operations, the usual order, and with the topology of pointwise convergence.

Let E be a Hausdorff topological linear space.

DEFINITION 3.1. We call a set function $\varphi : 2^E \to V$ a noncompactness measure in E, if φ has the following properties.

(1) If A is a convex complete subset of E and $\varphi(A) = 0$, then A has the fixed point property.

(2) $\varphi(A) = \varphi(\overline{\mathrm{co}}\, A)$.

(3) $A \subseteq B$ implies $\varphi(A) \leq \varphi(B)$.

(4) If (A_n) is a decreasing sequence of complete non-empty subsets of E with $\varphi(A_n) \to 0$ $(n \to \infty)$, then $\bigcap_{n\in\mathbb{N}} A_n$ is non-empty.

Classical examples of noncompactness measures with $V = [0, +\infty]$ are the Hausdorff measure [G/G/M] and the Kuratowski measure [Ku].

DEFINITION 3.2. Let C be a non-empty subset of E and φ a noncompactness measure in E. A continuous function $f : C \to C$ is said to be a *φ-contraction* if there exists $0 \leq q < 1$ such that $\varphi(f(A)) \leq q \cdot \varphi(A)$ for every subset A of C.

THEOREM 3.3. *Let C be a non-empty complete convex subset of E, φ a noncompactness measure and $f : C \to C$ a φ-contraction. If $\varphi(C) \in [0, +\infty[^I$, then f has a fixed point.*

PROOF. It is classic. We define inductively a sequence of sets by $C_0 := C$ and $C_{n+1} := \overline{\text{co}}\, f(C_n)$. We have

$$\varphi(C_n) \leq q^n \cdot \varphi(C) \to 0 \quad (n \to \infty) \quad \text{in } V.$$

By Axiom 4 in Definition 3.1, the set $C_\infty = \bigcap_{n\in\mathbb{N}} C_n$ is non-empty, moreover complete and convex. Since $f(C_\infty) \subseteq C_\infty$ and $\varphi(C_\infty) = 0$, the statement follows by Axiom 1 in Definition 3.1.

The crucial problem in the previous approach is that Axiom 2 in Definition 3.1 in general does not hold in the nonlocally convex case for the noncompactness measures usually used in locally convex spaces, cf. [H2], [DP/T1].

We will see in 3.8 that the set function $\overline{\gamma}_s$ introduced below, which measures the "nonstrongly convexly total boundedness", is a noncompactness measure in the sense of Definition 3.1.

In the following, let $(\|\cdot\|_i)_{i\in I}$ be a family of F-seminorms inducing the topology in E.

NOTATION 3.4. For $i \in I$ and $A \subset E$ we put

$B_{i,\varepsilon} := \{x \in E : \|x\|_i \leq \varepsilon\}$ for $\varepsilon > 0$,

$\gamma_i(A) := \inf\{\varepsilon \in]0, +\infty]$: there is a finite subset $F \subset E$ such that $A \subset F + B_{i,\varepsilon}\}$,

$\overline{\gamma}_i(A) := \inf\{\varepsilon \in]0, +\infty]$: there are $x_1, \ldots, x_n \in E$ and convex subsets $C_1, \ldots, C_n$ of $B_{i,\varepsilon}$ such that $A \subset \bigcup_{i=1}^{n} x_i + C_i\}$,

$\overline{\gamma}_{i,s}(A) := \inf\{\varepsilon \in]0, +\infty]$: there is a finite subset $F \subset E$ and a convex subset C of $B_{i,\varepsilon}$ such that $A \subset F + C\}$.

Furthermore, $\gamma := (\gamma_i)_{i\in I}$, $\overline{\gamma} := (\overline{\gamma}_i)_{i\in I}$, $\overline{\gamma}_s := (\overline{\gamma}_{i,s})_{i\in I}$.

γ is the noncompactness measure of Hausdorff. The set function $\overline{\gamma}$ is in some sense equivalent to the "measure of the lack of convex precompactness" introduced in the [DP/T1], Definition 2.1].

PROPOSITION 3.5. *Let $A \subset E$.*

(a) $\gamma(A) \leq \overline{\gamma}(A) \leq \overline{\gamma}_s(A)$. *The three measures coincide if the F-seminorms $\|\cdot\|_i$ are even seminorms.*

(b) *A is sctb, ctb or totally bounded iff $\overline{\gamma}_s(A) = 0$, $\overline{\gamma}(A) = 0$ or $\gamma(A) = 0$, respectively.*

PROOF. (a) of 3.5 is obvious; (b) follows from 2.4. Hereby observe that e.g. $\overline{\gamma}_{i,s}(A) = 0$ iff A is sctb in the $\|\cdot\|_i$-topology.

LEMMA 3.6. *γ, $\overline{\gamma}$ and $\overline{\gamma}_s$ have property* (3) *and* (4) *of* 3.1.

PROOF. Obviously, γ, $\overline{\gamma}$ and $\overline{\gamma}_s$ are monotone. By 3.5 (a), it is enough to show that γ has property (4). Let (A_n) be a decreasing sequence of complete non-empty subsets of E with $\varphi(A_n) \to 0$ $(n \to \infty)$ and $x_n \in A_n$ for $n \in \mathbb{N}$. Then

$$\gamma(\{x_i : i \in \mathbb{N}\}) = \gamma(\{x_i : i \geq n\}) \leq \gamma(A_n) \to 0 \quad (n \to +\infty),$$

hence $\gamma(\{x_i : i \in \mathbb{N}\}) = 0$ and $\{x_i : i \in \mathbb{N}\}$ is relatively compact. Therefore $\{x_i : i \in \mathbb{N}\}$ has cluster points, which obviously belong to the intersection $\bigcap_{n\in\mathbb{N}} A_n$.

REMARK. 3.7. An obvious modification of a part of the proof of 2.2 shows that $\overline{\gamma}_s$ has property (2) of 3.1, whereas γ and $\overline{\gamma}$ do not as Example 5.3 (b) shows. $\overline{\gamma}$ and therefore $\overline{\gamma}_s$ satisfy (1) of 3.1 by Idzik's fixed point theorem [I1, Theorem 2.4]. Whether γ satisfies (1) of 3.1, is exactly the problem of Schauder mentioned in the introduction.

COROLLARY 3.8. *$\overline{\gamma}_s$ is a noncompactness measure in the sense of* 3.1.

An obvious generalization of the proof of 2.2 (a) yields:

PROPOSITION 3.9. *Let A_1, A_2, $A \subset E$ and $\alpha \in \mathbb{R}$. Then*

$$\overline{\gamma}_s(A_1 \cup A_2) \leq \overline{\gamma}_s(A_1) + \overline{\gamma}_s(A_2),$$

$$\overline{\gamma}_s(A_1 + A_2) \leq \overline{\gamma}_s(A_1) + \overline{\gamma}_s(A_2),$$

$$\overline{\gamma}_s(\alpha A) \leq n \cdot \overline{\gamma}(A) \text{ if } |\alpha| \leq n \in \mathbb{N}.$$

4. - Noncompactness measures in l_p, $0 < p < 1$

Let $0 < p < 1$. For any real sequence $x = (x_n)$, we put $\|x\|_p := \sum_{n \in \mathbb{N}} |x_n|^p$. The space $(l_p, \|\cdot\|_p)$ is an example for an F- normed, nonlocally convex space, in which $\gamma(A) = \overline{\gamma}(A) = \overline{\gamma}_s(A)$ holds for every convex subset A.

THEOREM 4.1. *(a) If A is a pointwise bounded subset of l_p, then*

$$\gamma(A) = \inf_{n \in \mathbb{N}} \sup_{(x_k) \in A} \sum_{i=n}^{\infty} |x_i|^p.$$

(b) If A is a convex subset of l_p, then $\gamma(A) = \overline{\gamma}(A) = \overline{\gamma}_s(A)$.

PROOF. (a) For $n \in \mathbb{N}$ denote by $P_n : l_p \to l_p$ the projection $(x_i)_{i \in \mathbb{N}} \to (x_1, \ldots, x_n, 0, 0, \ldots)$. Let $Q_n = I - P_n$, where I is the identity. Put $\sigma(A) := \sup_{x \in A} \|x\|_p$ and

$$\tau(A) := \inf_{n \in \mathbb{N}} \sigma(Q_n(A)) = \inf_{n \in \mathbb{N}} \sup_{(x_k) \in A} \sum_{i=n}^{\infty} |x_i|^p \qquad \text{for } A \subset l_p.$$

(i) $\tau(A) \leq \gamma(A)$: let $\alpha > \gamma(A)$ and F be a finite subset of l_p with $A \subset F + B_\alpha$. Then $\tau(A) \leq \tau(F) + \tau(B_\alpha) \leq 0 + \sigma(B_\alpha) = \alpha$.

(ii) Let A be pointwise bounded. Then $\gamma(A) \leq \tau(A)$. Let $\alpha > \tau(A)$.

Then $\alpha > \sigma(Q_n(A))$ for some $n \in \mathbb{N}$. Since $P_n(A)$ is a bounded subset of a finite dimensional space, $\gamma(P_n(A)) = 0$ and from $A \subset P_n(A) + Q_n(A)$ follows $\gamma(A) \leq \gamma(P_n(A)) + \gamma(Q_n(A)) \leq 0 + \sigma(Q_n(A)) < \alpha$.

(b) By (a) and 3.5 (a), it is enough to prove that $\overline{\gamma}_s(A) \leq \tau(A)$, if A is convex and pointwise bounded. Let $\alpha > \tau(A)$. As in (ii), we get $\overline{\gamma}_s(A) \leq \overline{\gamma}_s(P_n(A)) + \overline{\gamma}_s(Q_n(A))$ for some $n \in \mathbb{N}$. Morover, $\overline{\gamma}_s(P_n(A)) = \gamma(P_n(A)) = 0$ since $P_n(A)$ is contained in a finite dimensional space and $\overline{\gamma}_s(Q_n(A)) \leq \sigma(Q_n(A)) < \alpha$, since $Q_n(A)$ is convex.

It follows from 4.1 (b) and 2.2 (b):

COROLLARY 4.2. *A subset of l_p is sctb iff its convex hull if totally bounded.*

By 4.1 (b) or by 2.3, a convex subset of l_p is totally bounded iff it is ctb iff it is sctb. In [W] it is proved that any subset of l_p is ctb iff it is sctb. On the other hand l_p contains totally bounded subsets, the convex hull of which is not totally bounded; such a set is totally bounded, but by 2.2 (b) not sctb.

A statement analogous to 4.1 (a) is given in [B/G] if $1 \leq p < \infty$.

5. - The noncompactness measure $\overline{\gamma}_s$ in L_0

In the following, let Ω be a non-empty set, $\mathcal{U}$ an algebra in the power set $\mathbb{P}(\Omega)$ of Ω and $\eta : \mathbb{P}(\Omega) \to [0, +\infty[$ a submeasure, i.e. a monotone, subadditive function with $\eta(\emptyset) = 0$. Then $\|f\| := \inf\{a > 0 : \eta(\{|f| \geq a\}) \leq a\}$ defines a group seminorm on $\mathbb{R}^\Omega$. Let L_0 be the closure of the space $S := \operatorname{span}\{\chi_A : A \in \mathcal{U}\}$ of $\mathcal{U}$-simple functions in $(\mathbb{R}^\Omega, \|\cdot\|)$. We will identify functions f, $g \in \mathbb{R}^\Omega$, for which $\|f - g\| = 0$. Then the space $(L_0, \|\cdot\|)$ of "measurable functions" becomes an F-normed linear space.

In [A/DP], [T/W], the following two parameters λ and ω are used to estimate the noncompctness measure γ in L_0:

$\lambda(M) := \inf\{\varepsilon > 0$: there is an $a \in [0, +\infty[$ such that $\eta(\{|f| \geq a\}) \leq \varepsilon$ for all $f \in M\}$,

$\omega(M) := \inf\{\varepsilon > 0$: there is a partition $A_1, \ldots, A_n \in \mathcal{U}$ of Ω such that for every $f \in M$ there is a set $D \subset \Omega$ with $\eta(D) \leq \varepsilon$ and $\sup\{|f(s) - f(t)| : s, t \in A_i \backslash D\} \leq \varepsilon$ for $i = 1, \ldots, n\}$.

By [T/W, 2.2.9] we have $\max\{\lambda, \omega/2\} \leq \gamma \leq \lambda + \omega$. To estimate $\overline{\gamma}_s$ we use the following parameters:

$\overline{\lambda}(M) := \inf\{\varepsilon > 0$: there is a set $D \subset \Omega$ such that $\eta(D) \leq \varepsilon$ and $\sup\{|f(x)| : f \in M, x \in \Omega \backslash D\} < +\infty\}$,

$\overline{\omega}(M) := \inf\{\varepsilon > 0$: there is a partition $A_1, \ldots, A_n \in \mathcal{U}$ of Ω and a set $D \subset \Omega$ such that $\eta(D) \leq \varepsilon$ and $\sup\{|f(s) - f(t)| : s, t \in A_i \backslash D\} \leq \varepsilon$, for $i = 1, \ldots, n$ and $f \in M\}$.

Obviously, $\lambda \leq \overline{\lambda}$ and $\omega \leq \overline{\omega}$.

THEOREM 5.1. *$\overline{\gamma}_s(M) \leq \overline{\gamma}(M) + \overline{\omega}(M)$ holds for $M \subset L_0$.*

PROOF. It is similar to that of the known inequality $\gamma \leq \lambda + \omega$. Let $\alpha > \overline{\lambda}(M)$ and $\beta > \overline{\omega}(M)$. There are sets D_1, $D_2 \subset \Omega$ and a partition $A_1, \ldots, A_n \in \mathcal{U}$ of Ω such that

$\eta(D_1) < \alpha$ and $s := \sup\{|f(x)| : f \in M, x \in \Omega \backslash D_1\} < +\infty$,

$\eta(D_2) < \beta$ and $\sup\{|f(s) - f(t)| : s, t \in A_i \backslash D_2\} \leq \beta$ for $i = 1, \ldots, n$.

Let k, $m \in \mathbb{N}$ such that $1/m < \alpha$ and $-s + k/m > s$. We put $Y := \{-s + i/m : i = 0, \ldots, k\}$ and $F := \left\{\sum_{i=1}^{n} y_i \cdot \chi_{A_i} : y_i \in Y\right\}$. For $f \in M$, there is a $g \in F$ such that $|f(x) - g(x)| \leq 1/m + \beta/2 \leq \alpha + \beta$ for every $x \in \Omega \backslash (D_1 \cup D_2)$; therefore $f - g \in C := \{h \in L_0 : \sup\{|h(x)| : x \in \Omega \backslash (D_1 + D_2)\} \leq \alpha + \beta\}$. Since F is finite and C a convex subset of $B_{\alpha+\beta}$, it follows that $\overline{\gamma}_s(M) \leq \alpha + \beta$.

One immediately sees that $\overline{\lambda}(\operatorname{co} M) = \overline{\lambda}(M)$ and $\overline{\omega}(\operatorname{co} M) = \overline{\omega}(M)$ for $M \subset L_0$ and that $\overline{\lambda}$ and $\overline{\omega}$ are monotone. Therefore it follows from 5.1 and 3.8:

COROLLARY 5.2. *$\overline{\lambda} + \overline{\omega}$ is a noncompactness measure in L_0 (in the sense of* 3.1*).*

We see in 5.3 (a) that in the inequality $\max\{\lambda, \omega/2\} \le \gamma$ one cannot replace λ, ω, γ by $\overline{\lambda}$, $\overline{\omega}$, $\overline{\gamma}_s$.

EXAMPLE 5.3. Let $\mathcal{U}$ be the Borel algebra of $\Omega = [0,1]$ and $\mu = \eta|\mathcal{U}$ the Lebesgue measure. Let $A_1, A_2, \dots$ be an enumeration of the intervals $[(i-2)/2^n, i/2^n[$ $(i, n \in \mathbb{N}; i \le 2^n)$, (a_n) a sequence of positive numbers such that $a_n \to +\infty$ $(n \to \infty)$, $f_n = a_n \chi_{A_n}$ for $n \in \mathbb{N}$ and $M = \{f_n : n \in \mathbb{N}\}$.

(a) $\overline{\lambda}(M) = \overline{\omega}(M) = 1$. But $\overline{\gamma}_s(M) = 0$ if $a_n \mu(A_n) \to 0$ $(n \to \infty)$.

(b) M is ctb. But $\operatorname{co} M$ is not bounded and therefore not ctb if $a_n \mu(A_n) \to +\infty$ $(n \to \infty)$.

(a) We prove the last assertion. If $n_0 \in \mathbb{N}$, $\varepsilon > 0$ and $a_n \mu(A_n) \le \varepsilon^2$ for $n \ge n_0$, then $f_n \in C := \left\{ f \in S : \int |f| \, d\mu \le \varepsilon^2 \right\}$ for $n \ge n_0$ and C is a convex subset of B_ε. Hence M is sctb.

(b) Let $n \in \mathbb{N}$, $\varepsilon = 2^{-n}$ and $B_i = [(i-1)/2^n, i/2^n[$ for $i \le 2^n$. Then $C_i := \{f \chi_{B_i} : f \in L_0\}$ are convex subsets of B_ε and $M \setminus \bigcup_{1 \le i \le 2^n} C_i$ is finite. Hence M is ctb.

Let $b_i := a_k$ if $B_i = A_k$. Then

$$\operatorname{co} M \ni \sum_{i \le i \le 2^n} 2^{-n} b_i \chi_{B_i} \ge \min_{1 \le i \le 2^n} b_i \mu(B_i) \cdot \chi_{[0,1[}.$$

Therefore $\operatorname{co} M$ is not bounded (in the linear topological sense), if $a_n \mu(A_n) \to +\infty$ $(n \to \infty)$.

EXAMPLE 5.4. Let $\mathcal{U}$, Ω, μ, η be chosen as in 5.3 and $n \in \mathbb{N}$.

(a) For $A_{i,n} := \{f \chi_{[(i-1)/n, i/n[} : 0 \le f \le 2n\}$, $A_n := \bigcup_{i=1}^{n} A_{i,n}$, we have $\overline{\lambda}(A_n) = 0$, $\omega(A_n) \le 1/n$, but $\overline{\gamma}_s(A_n) \ge 1/2$ since $A := \{f \in L_0 : 0 \le f \le 2\} \subset \operatorname{co} A_n$ and therefore

$$1/2 = \gamma(A) \le \overline{\gamma}_s(\operatorname{co} A_n) = \overline{\gamma}_s(A_n).$$

(b) For $B_{i,n} := \{c \chi_{[(i-1)/n, i/n[} : c \in \mathbb{R}\}$, $B_n := \bigcup_{i=1}^{n} B_{i,n}$ we have $\lambda(B_n) = 1/n$, $\overline{\omega}(B_n) = 0$, but $\overline{\gamma}_s(B_n) = 1$ since $B := \{c \chi_{[0,1[} : c \in \mathbb{R}\} \subset \operatorname{co} B_n$ and therefore

$$1 = \lambda(B) \le \gamma(B) \le \overline{\gamma}_s(\operatorname{co} B_n) = \overline{\gamma}_s(B_n) \le 1.$$

In contrast to 5.4, we will see that, under the assumptions of 5.4, $\overline{\lambda}(M) = \omega(M) = 0$ or $\lambda(M) \doteq \overline{\omega}(M) = 0$ implies $\overline{\gamma}_s(M) = 0$.

PROPOSITION 5.5. *Let $M \subset L_0$ and $\lambda(M) = 0$. Then $\overline{\lambda}(M) \leq \overline{\omega}(M)$ and therefore $\overline{\gamma}_s(M) \leq 2 \cdot \overline{\omega}(M)$.*

PROOF. Let $\lambda(M) = 0$. By 5.1, it is enough to show that $\overline{\lambda}(M) \leq \overline{\omega}(M)$. Let $\alpha > \overline{\omega}(M)$. Then there is a set $D \subset \Omega$ with $\eta(D) < \alpha$ and a partition $A_1, \dots, A_n \in \mathcal{U}$ of Ω such that

$$\sup\{|f(s) - f(t)| : s, t \in A_i \backslash D\} < \alpha \text{ for } i = 1, \dots, n.$$

We may assume that $\eta(A_i \backslash D) > 0$ for $i < m$ and $A_i \subset D$ for $i \geq m$, for some $m \in \mathbb{N}$. Since $\lambda(M) = 0$, there is a $b \in [0, +\infty[$ and, for every $f \in M$, a set $D(f) \subset \Omega$ such that $\eta(D(f)) < \min_{i<m} \eta(A_i \backslash D)$ and $|f(x)| \leq b$ for $x \in \Omega \backslash D(f)$. Let $f \in M$ and $x \in \Omega \backslash D$. We show that $|f(x)| \leq \alpha + b$. In fact, if $i < m$ with $x \in A_i$, then $\eta(D(f)) < \eta(A_i \backslash D)$; therefore there is an $y \in (A_i \backslash D) \backslash D(f)$ and $|f(x)| \leq |f(x) - f(y)| + |f(y)| \leq \alpha + b$. It follows that $\overline{\lambda}(M) \leq \eta(D) < \alpha$.

PROPOSITION 5.6. *Assume that $\eta(B) = \inf\{\eta(A) : B \subset A \in \mathcal{U}\}$ for any $B \subset \Omega$ and that $\mu := \eta|\mathcal{U}$ is additive. Let $M \subset L_0$ and $\omega(M) = 0$. Then $\overline{\gamma}_s(M) \leq \overline{\lambda}(M)$.*

PROOF. Let $\omega(M) = 0$ and $\alpha > \overline{\lambda}(M)$. By assumption, there is a set $D \in \mathcal{U}$, with $\eta(D) < \alpha$, and a number $c > 0$ such that $|f(x)| \leq c$ for $f \in M$ and $x \in \Omega \backslash D$. $M_1 := L_0 \cdot \chi_D$ is a convex subset of B_α, hence $\overline{\gamma}_s(M_1) \leq \alpha$. The set $M_2 := M \cdot \chi_{\Omega \backslash D}$ is totally bounded, since $\omega(M_2) = \lambda(M_2) = 0$. On $\{f \in L_0 : |f| \leq c\}$ the $\|\cdot\|$-topology coincides with the $\|\cdot\|_1$-topology, where $\|f\|_1 := \int |f| \, d\mu$. Thereofore M_2 is also totally bounded with respect to the (semi-)norm $\|\cdot\|_1$ and therefore sctb, i.e. $\overline{\gamma}_s(M_2) = 0$. Since $M \subset M_1 + M_2$, it follows $\overline{\gamma}_s(M) \leq \overline{\gamma}_s(M_1) + \overline{\gamma}_s(M_2) \leq \alpha$.

Under the assumption of 5.6, a set $M \subset L_0$ is sctb if $\omega(M) = \overline{\gamma}(M) = 0$, in particular, if M is totally bounded and $\overline{\lambda}(M) = 0$. The next proposition clarifies the meaning of $\overline{\lambda}(M) = 0$.

PROPOSITION 5.7. *Assume that $\eta(B) = \inf\{\eta(A) : B \subset A \in \mathcal{U}\}$ for $B \subset \Omega$. Then for $M \subset L_0$, $\overline{\lambda}(M) = 0$ iff $M \subset [-\varphi, \varphi]$ for some $\varphi \in L_0$, ≥ 0.*

PROOF. $\Rightarrow$: Let $\overline{\lambda}(M) = 0$. By assumption, there are $A_n \in \mathcal{U}$ and $a_n \in [0, +\infty[$ such that $\eta(\Omega \backslash A_n) \leq 1/n$ and $|f(x)| \leq a_n$ for $f \in M$ and $x \in A_n$. Define $B_n := A_n \backslash \bigcup_{i<n} A_i$, $\varphi_n = \sum_{i=1}^{n} a_i \chi_{B_i}$, $\varphi = \sum_{i=1}^{\infty} a_i \chi_{B_i}$. Then $\varphi_n \in S$, $\|\varphi - \varphi_n\| \leq \eta(\Omega - A_n) \leq 1/n$, hence $\varphi \in L_0$. Moreover, $|f(x)| \leq \varphi(x)$ for $f \in M$ and $x \in \Omega \backslash D$, where $D = \Omega \backslash \bigcup_{n=1}^{\infty} A_n$ and $\eta(D) = 0$.

$\Leftarrow$: Let φ be a positive function of L_0, $M \subset [-\varphi, \varphi]$ and $\varepsilon > 0$. Then there is a positive number c such that $\eta(\{\varphi \geq c\}) \leq \varepsilon$, hence $\overline{\lambda}(M) \leq \eta(\{\varphi \geq c\}) \leq \varepsilon$.

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