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# Regularity of Free Boundaries in Two Dimensions

MAKOTO SAKAI

## 1. - Introduction

In this paper we discuss regularity of free boundaries in two dimensions which appear in an obstacle problem. Let  $u$  be a non-negative function defined in the unit disk  $B_1$  of the complex  $z$ -plane such that:

- (i)  $\Gamma(u) = (\partial\Omega(u)) \cap B_1$  contains the origin 0, where  $\Omega(u) = \{z \in B_1 : u(z) > 0\}$ ;
- (ii)  $u$  is of class  $C^1$  in  $B_1$ ;
- (iii)  $\Delta u(z) = 1$  in  $\Omega(u)$  in the sense of distributions.

What can we say about regularity of the free boundary  $\Gamma(u)$ ?

An accurate description of  $\Gamma(u)$  was given by Caffarelli and Rivière in [1] and [2]. They showed that either:

- (1) 0 is a regular point; namely, for a small disk  $B_\delta$  with radius  $\delta > 0$  and center 0,  $\Omega(u) \cap B_\delta$  is simply connected and  $\Gamma(u) \cap B_\delta$  is a regular analytic simple arc passing through 0;

or

- (2)  $B_\delta \setminus \Omega(u)$  is arranged along a straight line for small  $B_\delta$ ; more precisely, there is an increasing function  $\eta$  defined on a half-open interval  $[0, \delta)$  such that  $\eta(0) = 0$  and

$$B_\delta \setminus \Omega(u) \subset e^{i\alpha} \{z = x + iy \in B_\delta : |y| \leq \eta(|x|)\},$$

where  $\alpha$  denotes a real number and  $e^{i\alpha}E$  for a set  $E$  denotes  $\{e^{i\alpha}z : z \in E\}$ .

Furthermore they proved in [1] that in case (2) it follows that:

- ( $\alpha$ ) if 0 is not an isolated point of  $\Gamma(u)$  and if the interior of  $B_1 \setminus \Omega(u)$  is empty, then  $\Gamma(u) \cap B_\delta$  is a real analytic simple arc, and
- ( $\beta$ ) the boundary of each connected component of the interior of  $B_\delta \setminus \Omega(u)$  is the union of a finite number of real analytic simple arcs.

Their results are fairly accurate, but there is still a possibility that an infinite number of connected components of the interior of  $B_1 \setminus \Omega(u)$  exist and cluster around 0.

The purpose of this paper is to give a complete description of the free boundary  $\Gamma(u)$  and to exclude such a possibility. Our main result is:

**THEOREM 1.1.** *Let  $u$ ,  $\Omega = \Omega(u)$  and  $\Gamma = \Gamma(u)$  be as above. Then the origin 0 is either a (1) regular, or a (2a) degenerate, or a (2b) double or a (2c) cusp point of  $\Gamma$ . Namely, there is a small disk  $B = B_\delta$  such that one of the following occurs:*

- (1)  $\Omega \cap B$  is simply connected and  $\Gamma \cap B$  is a regular real analytic simple arc passing through 0;
- (2a)  $\Gamma \cap B = \{0\}$  or  $\Gamma \cap B$  is a regular real analytic simple arc passing through 0.  $\Omega \cap B$  is equal to  $B \setminus \Gamma$ ;
- (2b)  $\Omega \cap B$  consists of two simply connected components  $\Omega_1$  and  $\Omega_2$ .  $(\partial\Omega_1) \cap B$  and  $(\partial\Omega_2) \cap B$  are distinct regular real analytic simple arcs passing through 0. They are tangent to each other at 0;
- (2c)  $\Omega \cap B$  is simply connected and  $\Gamma \cap B$  is a regular real analytic simple arc except for a cusp at 0. The cusp is pointing into  $\Omega \cap B$ . It is a very special one. There is a holomorphic function  $T$  defined on a closed disk  $\overline{B_\varepsilon}$  such that:
  - i.  $T(0) = 0$ ,  $T'(0) = 0$  and  $T''(0) \neq 0$ ;
  - ii.  $T$  is univalent on the closure  $\overline{H}$  of a half disk  $H = \{\tau \in B_\varepsilon : \text{Re } \tau > 0\}$ ;
  - iii.  $T$  satisfies  $\Gamma \cap B \subset T(i(-\varepsilon, \varepsilon))$  and  $T(\overline{H}) \subset \Omega \cup \Gamma$ , where  $i(-\varepsilon, \varepsilon) = \{it : -\varepsilon < t < \varepsilon\}$ .

Furthermore, all the second derivatives of  $u$  are continuous up to  $\Gamma$ , on  $\Omega$  and  $u$  is real analytic up to  $\Gamma$ , on  $\Omega$  except double and cusp points of  $\Gamma$ . If 0 is a double point, then there is a positive number  $\gamma$  such that

$$(1.1) \quad B \setminus \Omega(u) \subset e^{i\alpha} \{z = x + iy \in B : |y| \leq \gamma x^2\},$$

where  $e^{i\alpha}$  denotes the unit vector at 0 tangent to  $\Gamma$ . If 0 is a cusp point, then, for some positive number  $\gamma$ , it follows that

$$(1.2) \quad B \setminus \Omega(u) \subset e^{i\alpha} \{z = x + iy \in B : x \leq 0 \text{ and } |y| \leq \gamma x^2\},$$

where  $e^{i\alpha}$  denotes the unit vector at 0 tangent to  $\Gamma$  and pointing into  $\Omega$ . Namely, we can take a quadratic function  $\gamma t^2$  as a function  $\eta(t)$  in the argument given by Caffarelli and Riviere.

This regularity theorem holds also if we replace the constant function with value 1 in (iii) by a positive real analytic function  $\varphi$  defined in  $B_1$ . This fact is quite interesting when we compare it with an example of the free boundary for the obstacle problem with  $C^\infty$ -obstacle due to Schaeffer [9]: if we replace the

constant function with value 1 in (iii) by some special positive  $C^\infty$ -function  $\varphi$  defined in  $B_1$ , then there is a non-negative function  $u$  satisfying (i) to (iii) such that an infinite number of connected components of the interior of  $B_1 \setminus \Omega(u)$  actually cluster around 0.

**2. - Proof of Theorem 1.1**

We shall first define classes of functions which appear in an obstacle problem. For the free boundary for the obstacle problem, we refer to Chapter V of Kinderlehrer -Stampacchia [4] and Chapter 6 of Rodrigues [6].

DEFINITION 2.1. We say that a non-negative function  $u$  in  $B_\rho$  is of class  $P(1, B_\rho)$  if  $u$  satisfies (i) to (iii) of Section 1 with  $B_\rho$  replacing  $B_1$ . We say that a real-valued function  $u$  in  $B_\rho$  is of class  $R(1, B_\rho)$  if there exists an open subset  $\Omega(u)$  of  $B_\rho$  such that  $u$  and  $\Omega(u)$  satisfy (ii) and (iii) of Section 1 with  $B_\rho$  replacing  $B_1$  and if

- (i') 0 is contained in  $\Gamma(u) = (\partial\Omega(u)) \cap B_\rho$ , and
- (iv)  $\frac{\partial u}{\partial x}(z) = \frac{\partial u}{\partial y}(z) = 0$  on  $\Gamma(u)$ , where  $z = x + iy$ .

If  $u$  is of class  $P(1, B_\rho)$ , then  $u(z) \geq 0$  in  $B_\rho$  and  $u(z) = 0$  on  $\Gamma(u)$ , so  $u$  satisfies (iv). Thus, by taking  $\Omega(u) = \{z \in B_\rho; u(z) > 0\}$ , we see that  $P(1, B_\rho) \subset R(1, B_\rho)$ .

LEMMA 2.1. Let  $u \in R(1, B_\rho)$ . Then  $S(z) = \bar{z} - 4 \frac{\partial u}{\partial z}(z)$ , where  $\bar{z}$  denotes the complex conjugate of  $z$ , is the Schwarz function of  $\Omega(u) \cup \Gamma(u)$  in  $B_\rho$ ; namely,  $S$  is a function defined on  $\Omega(u) \cup \Gamma(u)$  which is holomorphic in  $\Omega(u)$ , is continuous on  $\Omega(u) \cup \Gamma(u)$  and satisfies  $S(\zeta) = \bar{\zeta}$  on  $\Gamma(u)$ .

PROOF. By (ii),  $S$  is continuous on  $\Omega(u) \cup \Gamma(u)$ . Since by (iii)

$$\partial S(z) \partial \bar{z} = 1 - 4(\partial^2 u(z))/(\partial z \partial \bar{z}) = 1 - \Delta u(z) = 0 \text{ in } \Omega(u),$$

$S$  is holomorphic in  $\Omega(u)$ . By (iv),  $\partial u / \partial z = 0$  on  $\Gamma(u)$ , and so  $S(z) = \bar{z}$  on  $\Gamma(u)$ . Hence  $S$  is the Schwarz function of  $\Omega(u) \cup \Gamma(u)$  in  $B_\rho$ . Q.E.D.

The next proposition is just an application of a regularity theorem proved in [8]; nevertheless, it is somewhat surprising when we look at the definition of  $R(1, B_\rho)$ . In particular, all the second derivatives of  $u$  are continuous up to  $\Gamma(u)$ , on  $\Omega(u)$  for every  $u \in R(1, B_\rho)$ .

PROPOSITION 2.2. Let  $u \in R(1, B_\rho)$ ,  $\Omega = \Omega(u)$  and  $\Gamma = \Gamma(u)$ . Then the origin 0 is a regular, double, cusp point of  $\Gamma$  in the sense of Theorem 1.1 or a degenerate point in the sense that:

(2a') for a small disk  $B = B_\delta$ ,  $\Gamma \cap B = \{0\}$  or  $\Gamma \cap B$  is an infinite set accumulating at 0 and is contained in a uniquely determined regular real analytic simple arc passing through 0.  $\Gamma \cap B$  is a proper subset of the arc or the whole arc.  $\Omega \cap B$  is equal to  $B \setminus \Gamma$ .

Furthermore, all the second derivatives of  $u$  are continuous up to  $\Gamma$ , on  $\Omega$  and  $u$  is real analytic up to  $\Gamma$ , on  $\Omega$  except double and cusp points of  $\Gamma$ .

PROOF. The first assertion follows from Lemma 2.1 and the Regularity Theorem for a boundary having a Schwarz function, see [8]. Let  $U(z)$  and  $V(z)$  be the real and imaginary parts of  $S(z) = \bar{z} - 4 \frac{\partial u}{\partial z}(z)$ , respectively. Then  $U(z) = x - 2\partial u/\partial x$ ,  $V(z) = -y + 2\partial u/\partial y$ ,  $\partial U/\partial x = 1 - 2(\partial^2 u)/(\partial x^2) = -1 + 2(\partial^2 u)/(\partial y^2)$ ,  $\partial V/\partial x = 2(\partial^2 u)/(\partial x \partial y)$  and  $S' = (\partial U/\partial x) + i(\partial V/\partial x)$  in  $\Omega$ . Since  $\lim_{z \in \Omega, z \rightarrow \zeta} S'(z)$  exists for every  $\zeta$  on  $\Gamma$  by Corollary 5.4 of [8], we see that all the second derivatives of  $u$  are continuous up to  $\Gamma$ , on  $\Omega$ . The final assertion follows from the Regularity Theorem. Q.E.D.

Now we shall give a proof of our Theorem 1.1. Since  $P(1, B_\rho) \subset R(1, B_\rho)$ , from Proposition 2.2 it follows that the origin 0 falls in one of the situations (1), (2a'), (2b) or (2c). If the origin 0 is in (2a'), then  $S(z) = \bar{z} - 4 \frac{\partial u}{\partial z}(z)$  is holomorphic in  $B$ , and so  $\frac{\partial u}{\partial z}(z) = 0$  on the arc determined by  $\Gamma \cap B$  if 0 is not an isolated point of  $\Gamma$ . This means that  $u(z) = 0$  on the arc. Hence  $\Gamma \cap B$  is the whole arc determined by  $\Gamma \cap B$  and (2a) holds. If (2b) holds, then (1.1) holds for some  $\alpha$  and  $\gamma$  by Corollary 5.3 of [8]. To show that (1.2) holds if (2c) holds, we need the following lemma:

LEMMA 2.3. Let  $u$  and  $S$  be as in Lemma 2.1. Assume that 0 is a cusp point of  $\Gamma$  and let  $z = T(\tau) = \kappa e^{i\alpha}(\tau^2 + a_3\tau^3 + a_4\tau^4 + \dots)$  be a one-to-one conformal mapping of  $\{\tau \in B_\varepsilon : \text{Re } \tau > 0\}$  into  $\Omega$  such that  $\Gamma \cap B \subset T(i(-\varepsilon, \varepsilon))$ , where  $\kappa > 0$ . Then  $\text{Re } a_3 \leq 0$ , and  $\text{Re } a_3 = 0$  if  $u \in P(1, B_\rho)$ .

PROOF. We can assume that  $\kappa e^{i\alpha} = 1$ . Set  $\alpha_j = \text{Re } a_j$  and  $\beta_j = \text{Im } a_j$ . If  $\tau = it$  and  $t$  is real, then  $x = \text{Re } T(\tau) = -t^2 + \beta_3 t^3 + \alpha_4 t^4 + \dots$  and  $y = \text{Im } T(\tau) = -\alpha_3 t^3 + \beta_4 t^4 + \dots$ . Hence  $x < 0$  for  $t$  with small  $|t|$ . If  $\alpha_3 > 0$ , then  $y < 0$  for small  $t > 0$  and  $y > 0$  for  $t < 0$  with small  $|t|$ . This contradicts the univalence of the mapping  $T$ . Therefore  $\alpha_3 \leq 0$ .

Since  $S(z) = S(T(\tau)) = \overline{T(-\bar{\tau})} = \tau^2 - \bar{a}_3 \tau^3 + \bar{a}_4 \tau^4 + \dots$ , we obtain

$$\begin{aligned} \frac{\partial u}{\partial x}(z) - i \frac{\partial u}{\partial y}(z) &= 2 \frac{\partial u}{\partial z}(z) = \frac{1}{2}(\bar{z} - S(z)) \\ &= -i \text{Im } \tau^2 + \bar{a}_3 \text{Re } \tau^3 - i \bar{a}_4 \text{Im } \tau^4 + \dots \\ &= (\alpha_3 r^3 \cos 3\theta + \dots) - i(r^2 \sin 2\theta + \beta_3 r^3 \cos 3\theta + \dots), \end{aligned}$$

where  $\tau = re^{i\theta}$ . For small fixed  $r > 0$ ,  $\frac{\partial u}{\partial y}(z) = 0$  has a unique solution  $\theta = \theta(r)$  in  $(-\pi/2, \pi/2)$  and  $\theta(r)$  is close to 0. Hence  $u(T(re^{i\theta(r)})) - u(0) = \int_J \frac{\partial u}{\partial x}(z) dx$ , where  $J = \{T(se^{i\theta(s)}) : 0 \leq s \leq r\}$ . If  $\alpha_3 < 0$ , then  $\frac{\partial u}{\partial x}(z) < 0$  on  $J$  for small  $r > 0$ , and so  $u(T(re^{i\theta(r)})) < u(0)$ . Thus if  $u \in P(1, B_\rho)$ , namely, if  $u(z) \geq 0$  in  $B_\rho$ , then  $\alpha_3 = 0$ . Q.E.D.

Now we shall show that (1.2) holds if (2c) holds. We use the notation of Lemma 2.3. From the lemma we see that  $\alpha_3 = \text{Re } a_3 = 0$ . Hence, for  $\tau = it$ ,  $y = \beta_4 t^4 + \dots$ . Since  $x = -t^2 + \beta_3 t^3 + \dots$  for  $\tau = it$ ,  $x \leq 0$  and  $|y| \leq 2|\beta_4|x^2$  for  $z \in \Gamma \cap B_\delta$  with small  $\delta > 0$ . This completes the proof of Theorem 1.1.

### 3. - A function associated with a Schwarz function

In Section 2, to each  $u \in R(1, B_1)$ , we have assigned a Schwarz function  $S$  by  $S(z) = \bar{z} - 4 \frac{\partial u}{\partial z}(z)$ . In this section we shall discuss the converse. For a given Schwarz function  $S$  defined on  $\Omega \cup \Gamma$ , we shall construct a function  $u \in R(1, B_\rho)$  such that  $S(z) = \bar{z} - 4 \frac{\partial u}{\partial z}(z)$  on  $(\Omega \cup \Gamma) \cap B_\rho$ .

In contrast with the arguments given in Chapter V of Kinderlehrer-Stampacchia [4], Lewy-Stampacchia [5], Schaeffer [9] and others, we do not assume that the free boundary  $\Gamma$  is a simple arc or a continuum.

Let  $S$  be the Schwarz function of  $\Omega \cup \Gamma$  in  $B_1$ , namely, let  $S$  be a function which is holomorphic in an open subset  $\Omega$  of  $B_1$ , is continuous on  $\Omega \cup \Gamma$  and satisfies  $S(\zeta) = \bar{\zeta}$  on  $\Gamma$ , where  $\Gamma = (\partial\Omega) \cap B_1$  and  $0 \in \Gamma$ . Let  $D$  be a connected component of  $\Omega$ . By the Regularity Theorem of [8], every point of  $(\partial D) \cap B_1$  is an accessible boundary point of  $D$ . Let  $\zeta$  be a fixed point of  $(\partial D) \cap B_1$  and let  $z \in D$ . The integral  $\int_J S(w)dw$  may depend on the choice of path  $J$  in

$D \cup \{\zeta\}$  joining  $\zeta$  and  $z$ . We shall show that  $\text{Re} \int_J S(w)dw$  does not depend on  $J$ . Let  $J'$  be another path in  $D \cup \{\zeta\}$  joining  $\zeta$  and  $z$ . To show that  $\text{Re} \int_{J-J'} S(w)dw = 0$ , we may assume that  $J - J'$  is a simple closed curve.

If  $J - J'$  does not surround any part of  $B_1 \setminus \Omega$ , then, by the Cauchy theorem,  $\int_{J-J'} S(w)dw = 0$ . If  $J - J'$  surrounds a part  $E$  of  $B_1 \setminus \Omega$ , then, by the Regularity Theorem, we may assume that  $\partial E$  consists of a finite number of

analytic simple closed curves having possibly double and cusp points. Hence

$$\int_{J-J'} S(w)dw = \int_{\partial E} S(w)dw = \int_{\partial E} \bar{w}dw = \int_E d\bar{w} \wedge dw = 2i \int_E dm,$$

and therefore the real part of this integral is 0. Thus  $\operatorname{Re} \int_{\gamma} S(w)dw$  is single-valued and harmonic in  $D$ .

Next, for  $\rho$  with  $0 < \rho < 1$ , we shall define a function  $h$  harmonic in  $\Omega \cap B_{\rho}$  and continuous on  $\overline{\Omega \cap B_{\rho}}$ . By the Regularity Theorem, we may assume that  $\partial(\overline{\Omega \cap B_{\rho}})$  consists of a finite number of piecewise analytic simple closed curves. Take a connected component  $F_k$  of  $\overline{\Omega \cap B_{\rho}}$  and let  $\zeta_k$  be a fixed point on  $(\partial F_k) \setminus (\partial B_{\rho})$ . Set

$$h_k(z) = 2 \operatorname{Re} \int_{\zeta_k}^z S(w)dw + |\zeta_k|^2$$

on  $F_k$ . Then  $h_k$  is harmonic in the interior of  $F_k$ , continuous on  $F_k$  and

$$\begin{aligned} (3.1) \quad h_k(z_2) - h_k(z_1) &= 2 \operatorname{Re} \int_{z_1}^{z_2} \bar{w}dw = \int_{z_1}^{z_2} (\bar{w}dw + wd\bar{w}) \\ &= \int_{z_1}^{z_2} d|w|^2 = (|z_2|^2 - |z_1|^2) \end{aligned}$$

if  $z_1$  and  $z_2$  belong to the same connected component of  $(\partial F_k) \setminus (\partial B_{\rho})$ . The function  $h_k$  depends on the choice of  $\zeta_k \in (\partial F_k) \setminus (\partial B_{\rho})$  and is uniquely determined up to a real additive constant. Now we define a function  $h$  on  $\overline{\Omega \cap B_{\rho}}$  by  $h(z) = h_k(z)$  on  $F_k$  and set

$$u(z) = \frac{1}{4} (|z|^2 - h(z))$$

on  $\overline{\Omega \cap B_{\rho}}$ . Then  $\Delta u(z) = 1$  in  $\Omega \cap B_{\rho}$ ,  $u$  is continuous on  $\overline{\Omega \cap B_{\rho}}$  and, by (3.1),  $u$  is constant on each connected component of  $(\partial(\overline{\Omega \cap B_{\rho}})) \setminus \partial B_{\rho}$ . We extend  $u$  onto  $B_{\rho}$  so that the extension, which we denote by  $u$  again, is continuous in  $B_{\rho}$ , is of class  $C^1$  in the interior  $I$  of  $B_{\rho} \setminus \Omega$  and satisfies

$$(3.2) \quad \lim_{z \in I, z \rightarrow \zeta} (\partial u / \partial x)(z) = \lim_{z \in I, z \rightarrow \zeta} (\partial u / \partial y)(z) = 0$$

for every  $\zeta \in (\partial(\overline{\Omega \cap B_{\rho}})) \setminus \partial B_{\rho}$ . This is possible, because  $u$  is constant on each connected component of  $(\partial(\overline{\Omega \cap B_{\rho}})) \setminus \partial B_{\rho}$ .

LEMMA 3.1. *The function  $u$  defined above satisfies  $S(z) = \bar{z} - 4 \frac{\partial u}{\partial z}(z)$  on  $(\Omega \cup \Gamma) \cap B_\rho$  and is of class  $R(1, B_\rho)$  if we set  $\Omega(u) = \Omega \cap B_\rho$ .*

PROOF. Since  $0 \in \Gamma$ , (i') of Definition 2.1 is satisfied. We have already seen that  $\Delta u(z) = 1$  in  $\Omega(u)$ . To show (iv) of Definition 2.1, let  $\Sigma_k(z) = \int_{\zeta_k}^z S(w)dw$  on a connected component  $F_k$  of  $\overline{\Omega \cap B_\rho}$ . Then

$$S(z) = \Sigma'_k(z) = \frac{\partial \Sigma_k(z)}{\partial z} + \frac{\partial \overline{\Sigma_k(z)}}{\partial z} = \frac{\partial 2 \operatorname{Re} \Sigma_k(z)}{\partial z}$$

in  $\Omega \cap B_\rho$ . Hence  $\partial u(z)/\partial z = (1/4)(\bar{z} - S(z))$  in  $\Omega \cap B_\rho$ . Since  $S(\zeta) = \bar{\zeta}$  on  $\Gamma$ ,

$$(3.3) \quad \lim_{z \in \Omega \cap B_\rho, z \rightarrow \zeta} (\partial u / \partial x)(z) = \lim_{z \in \Omega \cap B_\rho, z \rightarrow \zeta} (\partial u / \partial y)(z) = 0$$

for every  $\zeta \in \Gamma \cap B_\rho = \Gamma(u)$ . By (3.2) and (3.3),

$$(3.4) \quad \lim_{z \rightarrow \zeta} (\partial u / \partial x)(z) = \lim_{z \rightarrow \zeta} (\partial u / \partial y)(z) = 0$$

for every  $\zeta \in (\partial(\overline{\Omega \cap B_\rho})) \setminus \partial B_\rho$ . If  $\zeta \in \Gamma(u) \setminus \partial(\overline{\Omega \cap B_\rho})$ , then  $\zeta$  is a degenerate point of  $\Gamma$ . If  $\zeta$  is an isolated degenerate point of  $\Gamma$ , then  $S(\zeta) = \bar{\zeta}$  and (3.4) holds. If  $\zeta$  is a non-isolated degenerate point of  $\Gamma$ , there exists a real analytic simple arc passing through  $\zeta$  such that  $S(z) = \bar{z}$  on the arc. Hence  $u$  is constant on the arc and (3.4) holds. Thus (iv) of Definition 2.1 holds and  $u$  is of class  $C^1$  in  $B_\rho$ . Q.E.D.

To discuss the case  $u(z) \geq 0$  in  $\Omega \cap B_\rho$  for some  $\rho > 0$ , we shall have a more detailed discussion. We first note that, by the Regularity Theorem,  $\overline{\Omega \cap B_\rho}$  is connected for sufficiently small  $\rho > 0$ . For such a small  $\rho$ , we see that  $h(\zeta) = |\zeta|^2$  on  $\Gamma \cap B_\rho$ , and so  $u(\zeta) = 0$  on  $\Gamma \cap B_\rho$ . We set  $u_S(z) = u(z)$  on  $\overline{\Omega \cap B_\rho}$  and  $u_S(z) = 0$  on  $B_\rho \setminus \overline{\Omega \cap B_\rho}$ .

LEMMA 3.2. *Let  $S$  be the Schwarz function of  $\Omega \cup \Gamma$  in  $B_1$ . If  $0$  is a regular, non-isolated degenerate or double point of  $\Gamma$ , then  $u_S \in P(1, B_\rho)$  for sufficiently small  $\rho > 0$ .*

PROOF. If  $0$  is a non-isolated degenerate point of  $\Gamma$ , then  $S$  is holomorphic in  $B_\delta$  for small  $\delta > 0$ . Hence  $u_S$  is real analytic in  $B_\delta$ ,  $\Delta u_S(z) = 1$  in  $B_\delta$  and  $u_S(z) = (\partial u_S / \partial x)(z) = (\partial u_S / \partial y)(z) = 0$  on the arc  $J$  determined by  $\Gamma$ . Let  $(\partial u_S / \partial n)(z)$  be the derivative along the direction normal to  $J$ . Then  $(\partial^2 u_S / \partial n^2)(z) = \Delta u_S(z) = 1$  on  $J$ . Since  $u_S(z) = (\partial u_S / \partial n)(z) = 0$  on  $J$ , we see that  $u_S(z) > 0$  on  $B_\rho \setminus J$  and  $u_S(z) = 0$  on  $J \cap B_\rho$  for some small  $\rho > 0$ . Hence  $u_S \in P(1, B_\rho)$  and  $\Gamma(u_S) = J \cap B_\rho$ .

The same argument works for the case that  $0$  is a regular or double point of  $\Gamma$ . Q.E.D.

REMARK. If 0 is a regular or double point of  $\Gamma$ , then  $\Omega(u_S) \equiv \{z \in B_\rho : u_S(z) > 0\} = \Omega \cap B_\rho$  for some  $\rho > 0$ . If 0 is a non-isolated degenerate point of  $\Gamma$ , then  $\Omega(u_S) = B_\rho \setminus J$  and  $\Omega \cap B_\rho = B_\rho \setminus \Gamma$ . Hence  $\Omega(u_S) \subset \Omega \cap B_\rho$  and the equality does not hold in general.

Next we shall discuss the case that 0 is a cusp point of  $\Gamma$ . To do so, we shall define the index of a cusp point. Let  $S$  be the Schwarz function of  $\Omega \cup \Gamma$  in  $B_1$  and assume that 0 is a cusp point. Let  $e^{i\alpha}$  be the unit vector at 0 tangent to  $\Gamma$  and pointing into  $\Omega$ . Let  $C_\varepsilon$  be a half circle defined by  $C_\varepsilon = \{z \in \mathbb{C} : |z| = \varepsilon, |\arg z - \alpha| \leq \pi/2\}$ .  $C_\varepsilon$  is oriented counterclockwise. Let  $v(z) = S(z) - \bar{z}$ . It is known that  $\int_{C_\varepsilon} d \arg v(z) = \pm\pi + o(\varepsilon)$ .

DEFINITION 3.1. We call the origin 0 a cusp with index  $-1/2$  (resp.  $+1/2$ ) if  $\int_{C_\varepsilon} d \arg v(z) = \pi + o(\varepsilon)$  (resp.  $\pi + o(\varepsilon)$ ).

Let  $J$  be the arc starting from 0 and defined by  $J = \{z \in \Omega : \text{Im } v(z) = 0\}$ . Then  $J$  intersects  $\partial B_\varepsilon$  once for small  $\varepsilon > 0$  and  $\text{Re } v(z) = v(z) \neq 0$  on  $J \cap B_\varepsilon$ . The index of the cusp point 0 is  $-1/2$  if  $\text{Re } v(z) < 0$  on  $J \cap B_\varepsilon$  and  $+1/2$  if  $\text{Re } v(z) > 0$  on  $J \cap B_\varepsilon$ . For further details, see [7]. From arguments in Section 4 of [7] and Section 5 of Kinderlehrer-Nirenberg [3], we obtain:

LEMMA 3.3. Let  $S$  be the Schwarz function of  $\Omega \cup \Gamma$  in  $B_1$ . Let 0 be a cusp point of  $\Gamma$  and let  $u_S$  be the function defined before Lemma 3.2. Then  $u_S \in P(1, B_\rho)$  for sufficiently small  $\rho > 0$  if and only if the index of the cusp point is equal to  $-1/2$ .

PROOF. We can assume that  $e^{i\alpha} = 1$ . Let  $J$  be the arc starting from 0 defined by  $J = \{z \in \Omega : \text{Im } v(z) = 0\}$ . Then  $(\partial u_S / \partial y)(z) = 0$  on  $J$ , because  $(\partial u_S / \partial z)(z) = (1/4)(\bar{z} - S(z)) = -(1/4)v(z)$  is real on  $J$ . Let  $z(s)$  be a point on  $J$ , where  $s$  denotes the arc length of  $J$  from 0 to  $z(s)$ . Then, by the mean value theorem,  $u_S(z(s)) = s(\partial u_S / \partial s)(z(\lambda s))$  for some  $\lambda$  with  $0 < \lambda < 1$ . Since  $(\partial u_S / \partial s)(z(\lambda s)) = (\partial u_S / \partial x)(z(\lambda s))(\partial x / \partial s)(\lambda s)$ , we obtain

$$\begin{aligned} (1/4)\text{Re } v(z(\lambda s)) &= (1/2)(\partial u_S / \partial x)(z(\lambda s)) \\ &= (1/2)((\partial x / \partial s)(\lambda s))^{-1}(\partial u_S / \partial s)(z(\lambda s)) \\ &= (1/2)(s(\partial x / \partial s)(\lambda s))^{-1}u_S(z(s)). \end{aligned}$$

Since  $(\partial x / \partial s)(\lambda s) > 0$  for small  $s > 0$ ,  $\text{Re } v(z(\lambda s)) < 0$  if  $u_S(z) > 0$  in  $\Omega \cap B_\rho$ . Hence the index is equal to  $-1/2$  if  $u_S \in P(1, B_\rho)$ .

Conversely, if the index is equal to  $-1/2$ , then  $\text{Re } v(z(\lambda s)) < 0$ , and so  $u_S(z) > 0$  on  $J$ . Since

$$\lim_{z \in \Omega, z \rightarrow 0} S'(z) = e^{-2i\alpha} = 1$$

by Corollary 5.4 of [8], using the same argument as in the proof of Proposition 2.2 we can see that

$$\lim_{z \in \Omega, z \rightarrow 0} \frac{\partial^2 u_S}{\partial y^2}(z) = 1.$$

Hence the restriction of  $u_S$  to each vertical line  $x = x_0$  is a convex function of  $y$  in a neighborhood of 0. It attains its minimum at  $x_0 + iy_0 \in J$  if  $x_0 > 0$  (because  $(\partial u_S / \partial y)(z) = 0$  on  $J$ ) and at  $x_0 + iy_0 \in \Gamma$  if  $x_0 \leq 0$ , where  $u_S(x_0 + iy_0) = (\partial u_S / \partial y)(x_0 + iy_0) = 0$ . Thus  $u_S(z) > 0$  on  $\Omega \cap B_\rho$  for some  $\rho > 0$ . Q.E.D.

REMARK. Using the same argument as in the proof of Lemma 3.3, we can see that if 0 is an isolated point of  $\Gamma$  then  $u_S \in P(1, B_\rho)$  for small  $\rho > 0$  if and only if the index of  $v(z) = S(z) - \bar{z}$  at 0 is equal to  $-1$ . For the definition of the index of  $v$  at an isolated point of  $\Gamma$ , see [7].

#### 4. - Cusp points

Let  $u \in R(1, B_\rho)$  and assume that the origin 0 is a cusp point of  $\Gamma(u)$ . By Lemmas 2.1 and 3.1, we see that this cusp is the same as the cusp which appears on a boundary having a Schwarz function. It is precisely described by a conformal mapping as that given in (2c) of Theorem 1.1. In contrast with this fact, the cusp point for  $u \in P(1, B_\rho)$  is a very special one. Here we shall have a more detailed discussion about the cusp point for  $u \in P(1, B_\rho)$ , which improves results due to Schaeffer [9] and Kinderlehrer-Nirenberg [3].

Let  $u \in P(1, B_\rho)$ ,  $\Omega = \Omega(u)$  and  $\Gamma = \Gamma(u)$ . Let 0 be a cusp point of  $\Gamma$  and let  $e^{i\alpha} = -1$ , where  $e^{i\alpha}$  denotes the unit tangent vector to  $\Gamma$  at 0 pointing into  $\Omega$ . For small  $\delta > 0$ ,  $\Gamma \cap B_\delta$  is a simple arc with cusp at 0. We divide  $\Gamma \cap B_\delta$  into two regular real analytic simple arcs  $\Gamma_1$  and  $\Gamma_2 : \Gamma \cap B_\delta = \Gamma_1 \cup \Gamma_2$ . We may assume for  $j = 1, 2$  that  $\Gamma_j$  can be represented as the graph of  $y = y_j(x)$  on  $[0, \delta)$ .

PROPOSITION 4.1. *For  $j = 1, 2$  let*

$$y_j(x) = \gamma_j x^{n_j/2} + o(x^{n_j/2}),$$

where  $\gamma_j$  denotes a non-zero constant. Then  $n_1 = n_2$  and if we write  $n$  for  $n_1 = n_2$ , then  $n$  is a natural number such that  $n \geq 4$  and  $n \not\equiv 3 \pmod{4}$ ; it follows that  $\gamma_2 = (-1)^n \gamma_1$ . Conversely, all integers  $n$  with  $n \geq 4$  and  $n \not\equiv 3 \pmod{4}$  actually occur in this situation for some  $u \in P(1, B_\rho)$ .

PROOF. To use the notation as in the proof of Lemma 2.3, we assume again that  $e^{i\alpha} = 1$  and that  $z = T(\tau) = \tau^2 + a_3\tau^3 + a_4\tau^4 + \dots$  is a one-to-one conformal mapping of  $\{\tau \in B_\varepsilon : \text{Re } \tau > 0\}$  into  $\Omega$  such that  $\Gamma \cap B_\delta \subset T(i(-\varepsilon, \varepsilon))$ . For

$t \in (-\varepsilon, \varepsilon)$ ,  $z = T(it) = -t^2 + \sum_{j=3}^{\infty} a_j i^j t^j$ . Hence

$$x = -t^2 + \beta_3 t^3 + \alpha_4 t^4 - \beta_5 t^5 - \alpha_6 t^6 + \dots$$

and

$$y = -\alpha_3 t^3 + \beta_4 t^4 + \alpha_5 t^5 - \beta_6 t^6 - \alpha_7 t^7 + \dots,$$

where  $a_j = \alpha_j + i\beta_j$ . Thus we see that  $n = n_1 = n_2$  is a natural number not less than 3 and  $\gamma_2 = (-1)^n \gamma_1$ .

Assume by contradiction that  $n \equiv 3 \pmod{4}$ ; then  $\alpha_j = 0$  for odd  $j$  with  $3 \leq j < n$ ,  $\beta_j = 0$  for even  $j$  with  $4 \leq j < n$  and  $\alpha_n \neq 0$ . If  $\alpha_n > 0$ , then  $y < 0$  for small  $t > 0$  and  $y > 0$  for  $t < 0$  with small  $|t|$ . This contradicts the univalence of the mapping  $T$  in the half disk. Hence  $\alpha_n < 0$ . Since

$$S(z) - \bar{z} = v(z) = \operatorname{Re} v(z) = 2(-\alpha_n r^n \cos n\theta + \dots)$$

on  $J = \{z \in \Omega \cap B_\delta : \operatorname{Im} v(z) = 0\}$  and  $\theta(r)$  is close to 0 as in the proof of Lemma 2.3, we see that  $v(z) = \operatorname{Re} v(z) > 0$  on  $J$ , and so  $u(z) < 0$  on  $J$ . This contradicts the hypothesis  $u \in P(1, B_\rho)$ , and therefore we have proved that  $n \not\equiv 3 \pmod{4}$  and  $n \geq 4$ .

Now we shall construct examples. Let  $n \equiv 1 \pmod{4}$  and  $n \geq 5$ . Take a small  $\alpha_n > 0$  and set  $T(\tau) = \tau^2 + \alpha_n \tau^n$ . Then  $x = -t^2$ ,  $y = \alpha_n t^n$  for  $\tau = it$  and  $T$  is univalent in  $\{\tau \in B_\varepsilon : \operatorname{Re} \tau > 0\}$  for some  $\varepsilon > 0$ , because  $y > 0$  for  $t > 0$  and  $y < 0$  for  $t < 0$ . Set  $S(z) = \overline{T(-\bar{\tau})}$  and  $v(z) = S(z) - \bar{z}$ . Then  $v(z) = -2\alpha_n r^n \cos n\theta + i2r^2 \sin 2\theta$  and  $v(z) = \operatorname{Re} v(z) = -2\alpha_n r^n < 0$  on  $J = \{\zeta : 0 < \zeta = \operatorname{Re} \zeta < \delta\}$ , where  $\tau = r e^{i\theta}$ . Hence  $u_S \in P(1, B_\rho)$  for some  $\rho > 0$ .

If  $n \equiv 0 \pmod{4}$  and  $n \geq 4$  we take small  $\beta_3, \beta_n > 0$  and set  $T(\tau) = \tau^2 + i\beta_3 \tau^3 + i\beta_n \tau^n$ . It follows that  $x = -t^2 + \beta_3 t^3$  and  $y = \beta_n t^n$  for  $\tau = it$ . Let

$$\Gamma' = \{-t^2 + i\beta_n t^n : t \in (-\varepsilon, 0)\} = \{-t^2 + i\beta_n t^n : t \in (0, \varepsilon)\}.$$

Then  $\Gamma'$  is a regular real analytic simple arc contained in the second quadrant. We see that  $z$  lies on the right of  $\Gamma'$  for small  $t > 0$  and on the left of  $\Gamma'$  for  $t < 0$  with small  $|t|$ . Hence  $T$  is univalent on  $\{\tau \in B_\varepsilon : \operatorname{Re} \tau > 0\}$  for some  $\varepsilon > 0$ . Defining  $S$  and  $v$  as above, we obtain  $v(z) = 2\beta_n r^n \sin n\theta + i2(r^2 \sin 2\theta + \beta_3 r^3 \cos 3\theta)$ . Since  $\beta_3 > 0$  and  $\cos 3\theta(r) > 0$  for  $\theta(r)$  close to 0,  $\operatorname{Im} v(z) = 0$  only if  $\theta(r) < 0$ . Hence  $\operatorname{Re} v(z) = 2\beta_n r^n \sin n\theta(r) < 0$  on  $J = \{z \in \Omega \cap B_\delta : \operatorname{Im} v(z) = 0\}$ , and so  $u_S \in P(1, B_\rho)$  for some  $\rho > 0$ .

If  $n \equiv 2 \pmod{4}$  and  $n \geq 6$ , we take small  $\beta_5, \beta_n > 0$  and set  $T(\tau) = \tau^2 + i\beta_5 \tau^5 + i\beta_n \tau^n$ . We notice that

$$\Gamma' = \{-t^2 - i\beta_n t^n : t \in (0, \varepsilon)\}$$

is contained in the third quadrant in this case. By repeating the same argument as above, we can see that  $u_S \in P(1, B_\rho)$  for some  $\rho > 0$ . Q.E.D.

**5. - Holomorphic functions having real analytic boundary values**

Let  $\varphi(x, y)$  be a complex-valued real analytic function of  $x$  and  $y$  in  $B_\rho$ , where  $z = x + iy$ . We note that

$$\varphi(x, y) = \varphi\left(\frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2i}\right).$$

To discuss the local property of the function  $\varphi(x, y)$ , we may assume that

$$f(z, w) = \varphi\left(\frac{z + w}{2}, \frac{z - w}{2i}\right)$$

is a holomorphic function of  $z$  and  $w$  in  $B_\rho^2 = B_\rho \times B_\rho$ . In what follows we write  $f(z, \bar{z})$  for  $\varphi(x, y)$ .

Let  $F(z, w)$  be a holomorphic function of two variables  $z$  and  $w$  in  $B_\rho^2$ . Let  $\Omega$  be an open subset of  $B_\rho$  such that  $0 \in \partial\Omega$  and let  $\Gamma = (\partial\Omega) \cap B_\rho$ .

DEFINITION 5.1. Let  $S^F$  be a function defined on  $\Omega \cup \Gamma$ . We call  $S^F$  the *holomorphic function of  $\Omega \cup \Gamma$  in  $B_\rho$  having the boundary values  $F(\zeta, \bar{\zeta})$*  if:

- (i)  $S^F$  is holomorphic in  $\Omega$ ;
- (ii)  $S^F$  is continuous on  $\Omega \cup \Gamma$ ;
- (iii)  $S^F(\zeta) = F(\zeta, \bar{\zeta})$  on  $\Gamma$ .

If  $0$  is a non-isolated point of  $\Gamma$  then  $S^F$  is uniquely determined (see remarks after Definition 3.1 of [8]).

If  $F(z, \bar{z})$  is a holomorphic function of  $z$  in  $B_\rho$ , then  $F(z, \bar{z})$  satisfies (i) to (iii) and it is a function  $S^F$  of  $\Omega \cup \Gamma$  in  $B_\rho$  having the boundary values  $F(\zeta, \bar{\zeta})$ . Namely,  $S^F$  exists for any open subset  $\Omega$  of  $B_\rho$  if  $F(z, \bar{z})$  is a holomorphic in  $B_\rho$ . The situation is quite different if  $F(z, \bar{z})$  is not holomorphic in  $B_\rho$ .

We state the following lemma without proof.

LEMMA 5.1. Let  $f$  be a holomorphic function in  $B_\rho^2$ . Then:

- (1)  $f(z, \bar{z})$  is identically equal to zero in  $B_\rho$  if and only if  $f(z, w)$  is identically equal to zero in  $B_\rho^2$ ;
- (2)  $f(z, \bar{z})$  is holomorphic in  $B_\rho$  if and only if  $\partial f / \partial w$  is identically equal to zero as a function of two variables in  $B_\rho^2$ .

In what follows, for the sake of simplicity, we consider the case  $(\partial F / \partial w)(0, 0) \neq 0$ .

PROPOSITION 5.2. *Let  $F$  be a holomorphic function in  $B_1^2$  satisfying  $(\partial F/\partial w)(0, 0) \neq 0$ . Let  $\Omega$  be an open subset of  $B_1$  such that  $0 \in \partial\Omega$  and let  $\Gamma = (\partial\Omega) \cap B_1$ . Then there exists a holomorphic function  $S^F$  of  $(\Omega \cap B_\rho) \cup (\Gamma \cap B_\rho)$  in  $B_\rho$  having the boundary values  $F(\zeta, \bar{\zeta})$  for some  $\rho > 0$  if and only if there exists a Schwarz function of  $(\Omega \cap B_\delta) \cup (\Gamma \cap B_\delta)$  in  $B_\delta$  for some  $\delta > 0$ .*

The proof of this Proposition follows immediately from the following two lemmas.

LEMMA 5.3. *Let  $F$  be a holomorphic function in  $B_1^2$ . If there exists a Schwarz function  $S$  of  $(\Omega \cap B_\delta) \cup (\Gamma \cap B_\delta)$  in  $B_\delta$ , then  $S^F(z) = F(z, S(z))$  is the holomorphic function of  $(\Omega \cap B_\rho) \cup (\Gamma \cap B_\rho)$  in  $B_\rho$  having the boundary values  $F(\zeta, \bar{\zeta})$  for some  $\rho > 0$ .*

PROOF. Take  $\rho > 0$  so that  $\rho < \delta$  and  $|S(z)| < 1$  on  $(\Omega \cup \Gamma) \cap B_\rho$ . Then the function  $F(z, S(z))$  is holomorphic in  $\Omega \cap B_\rho$ , is continuous on  $(\Omega \cap B_\rho) \cup (\Gamma \cap B_\rho)$  and satisfies  $F(\zeta, S(\zeta)) = F(\zeta, \bar{\zeta})$  on  $\Gamma \cap B_\rho$ , because  $S(\zeta) = \bar{\zeta}$  on  $\Gamma \cap B_\rho$ .  
Q.E.D.

LEMMA 5.4. *Let  $S^F$  be the holomorphic function of  $(\Omega \cap B_\rho) \cup (\Gamma \cap B_\rho)$  in  $B_\rho$  having the boundary values  $F(\zeta, \bar{\zeta})$  and assume that  $(\partial F/\partial w)(0, 0) \neq 0$ . Then there exists a Schwarz function of  $(\Omega \cap B_\delta) \cup (\Gamma \cap B_\delta)$  in  $B_\delta$  for some  $\delta > 0$ .*

PROOF. We introduce a new variable  $\tau$  and consider a function  $F(z, w) - \tau$  which is holomorphic in  $B_\rho^2 \times B_\rho(\tau_0)$ , where  $\tau_0 = F(0, 0)$  and  $B_\rho(\tau_0)$  denotes the disk with radius  $\rho$  and center  $\tau_0$ . Since  $(\partial/\partial w)(F(z, w) - \tau)(0, 0, \tau_0) = (\partial F/\partial w)(0, 0) \neq 0$  by the implicit function theorem there exists a unique holomorphic function  $g(z, \tau)$  in  $B_\varepsilon \times B_\varepsilon(\tau_0)$  for some  $\varepsilon > 0$  such that  $g(0, \tau_0) = 0$  and  $F(z, g(z, \tau)) - \tau = 0$  in  $B_\varepsilon \times B_\varepsilon(\tau_0)$ . Take  $\delta > 0$  so that  $\delta < \varepsilon$  and  $|S^F(z) - \tau_0| < \varepsilon$  on  $(\Omega \cup \Gamma) \cap B_\delta$ . Then  $S(z) = g(z, S^F(z))$  is holomorphic in  $\Omega \cap B_\delta$  and continuous on  $(\Omega \cap B_\delta) \cup (\Gamma \cap B_\delta)$ . Since  $F(\zeta, \bar{\zeta}) - S^F(\zeta) = 0$  on  $\Gamma \cap B_\rho$ , we obtain  $\bar{\zeta} = g(\zeta, S^F(\zeta))$  on  $\Gamma \cap B_\delta$ , because  $g(z, \tau)$  is uniquely determined by  $z$  and  $\tau$ . Hence  $S(\zeta) = g(\zeta, S^F(\zeta)) = \bar{\zeta}$  on  $\Gamma \cap B_\delta$  and  $S$  is the Schwarz function of  $(\Omega \cap B_\delta) \cup (\Gamma \cap B_\delta)$  in  $B_\delta$ .  
Q.E.D.

In the previous proof we have applied the implicit function theorem. The same idea can be found in Lewy-Stampacchia [5]. From Proposition 5.2 we see that if  $(\partial F/\partial w)(0, 0) \neq 0$  and if there exists a holomorphic function of  $(\Omega \cap B_\rho) \cup (\Gamma \cap B_\rho)$  in  $B_\rho$  having the boundary values  $F(\zeta, \bar{\zeta})$  for some  $\rho > 0$ , then the origin is a regular, degenerate, double or cusp point in the sense of the Regularity Theorem.

Let  $F(z, w) = F(0, 0) + z^n F_1(z, w)$ , where  $n$  denotes a non-negative integer and  $F_1$  denotes a holomorphic function of  $z$  and  $w$  such that  $F_1(0, w)$  is not identically equal to zero as a function of  $w$ . By applying the Fuchs theorem (see [8, Sec. 2]), we see that there exists a  $S^F$  of  $\Omega \cup \Gamma$  in  $B_\rho$  if and only if there exists a  $S^{F_1}$  of  $\Omega \cup \Gamma$  in  $B_\rho$  and  $S^F(z) = F(0, 0) + z^n S^{F_1}(z)$ . Hence the same conclusion of Proposition 5.2 holds if  $(\partial F_1/\partial w)(0, 0) \neq 0$ . If  $(\partial F_1/\partial w)(0, 0) = 0$ ,

then the function  $g(z, \tau)$  for  $F_1$  in the proof of Lemma 5.4 has a singularity at  $(0, F_1(0, 0))$  and the conclusion of Lemma 5.4 would be complicated.

**6. - A generalization of Theorem 1.1**

We first state the following lemma without proof.

LEMMA 6.1. *Let  $f$  be a holomorphic function in  $B_\rho^2$ . Then:*

- (1)  $f(z, \bar{z})$  is real-valued in  $B_\rho$  if and only if  $\check{f} = f$  in  $B_\rho^2$ , where:

$$\check{f}(z, w) = \overline{f(\bar{w}, \bar{z})}.$$

*In other words,  $a_{jk} = \overline{a_{kj}}$  for every  $j, k \geq 0$  if  $f$  has the Taylor expansion  $\sum a_{jk} z^j w^k$  in  $B_\rho^2$ ;*

- (2)  $f(z, \bar{z})$  is real-valued in  $B_\rho$  if and only if  $\Phi(z, \bar{z})$  is real-valued in  $B_\rho$ , where:

$$\Phi(z, w) = \int_0^z \left\{ \int_0^w f(s, t) dt \right\} ds.$$

Next we define two classes of real-valued functions in  $B_\rho$ .

DEFINITION 6.1. Let  $f$  be a holomorphic function in  $B_\rho^2$  such that  $f(z, \bar{z})$  is real-valued and satisfies  $f(z, \bar{z}) \neq 0$  in  $B_\rho$ . We say that a real-valued function  $u$  in  $B_\rho$  is of class  $R(f, B_\rho)$  if there exists an open subset  $\Omega(u)$  of  $B_\rho$  and if:

- (i)  $0$  is contained in  $\Gamma(u) = (\partial\Omega(u)) \cap B_\rho$ ;
- (ii)  $u$  is of class  $C^1$  in  $B_\rho$ ;
- (iii)  $\Delta u(z) = f(z, \bar{z})$  in  $\Omega(u)$  in the sense of distributions;
- (iv)  $\frac{\partial u}{\partial x}(z) = \frac{\partial u}{\partial y}(z) = 0$  on  $\Gamma(u)$ .

DEFINITION 6.2. Let  $f$  be a holomorphic function in  $B_\rho^2$  such that  $f(z, \bar{z})$  is real-valued and satisfies  $f(z, \bar{z}) > 0$  in  $B_\rho$ . Let  $u$  be a non-negative function in  $B_\rho$  and set  $\Omega(u) = \{z \in B_\rho : u(z) > 0\}$ . We say that  $u$  is of class  $P(f, B_\rho)$  if  $u$  and  $\Omega(u)$  satisfy (i) to (iii) of Definition 6.1.

In Definition 6.1, the set  $\Omega(u)$  may not be uniquely determined. For  $u \in P(f, B_\rho)$ , we take  $\Omega(u) = \{z \in B_\rho : u(z) > 0\}$ ; then  $u$  satisfies (iv), and so  $u$  is of class  $R(f, B_\rho)$ .

To prove a regularity theorem for functions of class  $P(f, B_\rho)$ , we start with two lemmas.

LEMMA 6.2. Let  $u$  be a function of class  $R(f, B_\rho)$  and set

$$(6.1) \quad F(z, w) = \int_0^w f(z, t) dt.$$

Then

$$S^F(z) = F(z, \bar{z}) - 4 \frac{\partial u}{\partial z}(z)$$

is the holomorphic function of  $\Omega(u) \cup \Gamma(u)$  in  $B_\rho$  having the boundary values  $F(\zeta, \bar{\zeta})$ .

PROOF. By the chain rule, we obtain

$$\begin{aligned} \frac{\partial S^F}{\partial \bar{z}}(z) &= \frac{\partial F}{\partial z}(z, \bar{z}) \frac{\partial z}{\partial \bar{z}} + \frac{\partial F}{\partial w}(z, \bar{z}) \frac{\partial \bar{z}}{\partial \bar{z}} - 4 \frac{\partial^2 u}{\partial z \partial \bar{z}}(z) \\ &= f(z, \bar{z}) - \Delta u(z) \end{aligned}$$

in  $\Omega(u)$ . Hence, by (iii) of Definition 6.1,  $S^F$  is holomorphic in  $\Omega(u)$ . It is continuous on  $\Omega(u) \cup \Gamma(u)$ , because  $u$  is of class  $C^1$  in  $B_\rho$ . On  $\Gamma(u)$ , by (iv) of Definition 6.1, we obtain  $S^F(\zeta) = F(\zeta, \bar{\zeta})$ . This completes the proof. Q.E.D.

LEMMA 6.3. Let  $S^F$  be the holomorphic function of  $\Omega \cup \Gamma$  in  $B_1$  having the boundary values  $F(\zeta, \bar{\zeta})$  and set  $f = \partial F / \partial w$ . If  $F$  satisfies (6.1) and if  $f(z, \bar{z})$  is real-valued in  $B_1$  and  $f(0, 0) \neq 0$ , then

$$u(z) = \begin{cases} \frac{1}{4} \left( \Phi(z, \bar{z}) - 2 \operatorname{Re} \int_0^z S^F(\tau) d\tau \right) & \text{in } \Omega \cap B_\rho \\ 0 & \text{on } B_\rho \setminus \Omega \end{cases}$$

is of class  $R(f, B_\rho)$  for some  $\rho > 0$ , where

$$\Phi(z, w) = \int_0^z F(s, w) ds.$$

PROOF. Since  $(\partial F) / (\partial w)(0, 0) = f(0, 0) \neq 0$ , by Proposition 5.2 and the Regularity Theorem in [8], the origin 0 is a regular, degenerate, double or cusp point in the sense of the Regularity Theorem. If 0 is not a degenerate point, then we can take a small  $\rho > 0$  such that each connected component of  $\Omega \cap B_\rho$  is simply connected. If 0 is a degenerate point, then we can take  $\rho$  such that  $S^F$  is holomorphic in  $B_\rho$ . In any case  $\int_0^z S^F(\tau) d\tau$  is well-defined and

single-valued in  $\Omega \cap B_\rho$ . By (2) of Lemma 6.1,  $\Phi(z, \bar{z})$  is real-valued in  $B_1$ . Thus  $u$  is well-defined and real-valued in  $B_\rho$ . Set  $\Omega(u) = \Omega \cap B_\rho$ .

By definition, (i) of Definition 6.1 is satisfied. By the chain rule, we obtain

$$\frac{\partial \Phi(z, \bar{z})}{\partial z} = \frac{\partial \Phi}{\partial z}(z, \bar{z}) + \frac{\partial \Phi}{\partial w}(z, \bar{z}) \frac{\partial \bar{z}}{\partial z} = F(z, \bar{z}),$$

and so

$$4 \frac{\partial u}{\partial z}(z) = F(z, \bar{z}) - S^F(z)$$

in  $\Omega(u)$ . Since  $u$  is real-valued, this equality implies (iv) of Definition 6.1. From this fact and condition  $u(0) = 0$ , we see that  $u(z) = 0$  on  $\Gamma(u) = \Gamma \cap B_\rho$  and hence (ii) of Definition 6.1 holds. Using the chain rule again, we obtain

$$\Delta u(z) = 4 \frac{\partial^2 u}{\partial \bar{z} \partial z}(z) = \frac{\partial F}{\partial w}(z, \bar{z}) = f(z, \bar{z})$$

in  $\Omega(u)$ . Hence  $u$  satisfies (iii) of Definition 6.1 and therefore it is of class  $R(f, B_\rho)$ . Q.E.D.

REMARK. We can construct a (complex-valued)  $u$  as above also in case  $f(z, \bar{z})$  is complex-valued. In fact,  $f(z, w)$  can be uniquely decomposed as  $f_1(z, w) + i f_2(z, w)$ , where  $f_j, j = 1, 2$ , are holomorphic functions in  $B_1^2$  satisfying  $\bar{f}_j = f_j$ .

Lemmas 6.2 and 6.3 together with Proposition 5.2 imply that there exists a function  $u \in R(f, B_\rho)$  with  $\Omega(u) = \Omega \cap B_\rho$  for some  $\rho > 0$  if and only if there exists a Schwarz function of  $(\Omega \cap B_\delta) \cup (\Gamma \cap B_\delta)$  in  $B_\delta$  for some  $\delta > 0$ , where  $\Omega$  denotes an open subset of  $B_1$  such that  $0 \in \partial\Omega$  and  $\Gamma = (\partial\Omega) \cap B_1$ . As a consequence, Proposition 2.2 with  $R(f, B_\rho)$  replacing  $R(1, B_\rho)$  holds.

Finally we shall show the following theorem which is a generalization of our Theorem 1.1.

THEOREM 6.4. *Let  $u \in P(f, B_\rho)$ ,  $\Omega = \Omega(u)$  and  $\Gamma = \Gamma(u)$ . Then the same assertion as in Theorem 1.1 holds.*

PROOF. Since  $P(f, B_\rho) \subset R(f, B_\rho)$ , what we have to prove is that if 0 is a degenerate point in the sense of the Regularity Theorem, then (2a) of Theorem 1.1 holds and if 0 is a cusp point, then (1.2) holds.

From Lemmas 5.3 and 6.2 it follows that

$$(6.2) \quad 4 \frac{\partial u}{\partial z}(z) = F(z, \bar{z}) - S^F(z) = F(z, \bar{z}) - F(z, S(z)),$$

in  $\Omega \cap B_\delta$  for some  $\delta > 0$ , where  $F$  is the function defined by (6.1) and  $S$  denotes the Schwarz function of  $\Omega \cup \Gamma$  in  $B_\delta$ . Hence  $u$  is real analytic up to  $\Gamma$  on  $\Omega$  except double and cusp points of  $\Gamma$ .

If 0 is a non-isolated degenerate point in the sense of the Regularity Theorem, then  $u(\zeta) = 0$  on the arc determined by  $\Gamma \cap B_\delta$ , because  $u(0) = 0$  and  $\frac{\partial u}{\partial z}(\zeta) = 0$  on the arc. Hence  $\Gamma \cap B_\delta = \{\zeta \in B_\delta : u(\zeta) = 0\}$  and  $\Gamma \cap B_\delta$  is the whole arc.

Next assume that 0 is a cusp point and that  $z = T(\tau) = \tau^2 + a_3\tau^3 + \dots$  is a one-to-one conformal mapping of  $\{\tau \in B_\varepsilon : \operatorname{Re} \tau > 0\}$  into  $\Omega$  such that  $\Gamma \cap B_\delta \subset T(i(-\varepsilon, \varepsilon))$  for some  $\delta$  and  $\varepsilon$ . Let  $F(z, w) = \sum_{j,k \geq 0} (a_{jk}/(k+1))z^j w^{k+1}$ .

Since  $S(z) = \overline{T(-\bar{\tau})} = \tau^2 - \bar{a}_3\tau^3 + \dots$ , we obtain

$$\begin{aligned} F(z, \bar{z}) - F(z, S(z)) &= (a_{00}\bar{z} + O(|\tau|^4)) - (a_{00}S(z) + O(|\tau|^4)) \\ &= a_{00}(\bar{z} - S(z)) + O(|\tau|^4). \end{aligned}$$

From Definition 6.2, it follows that

$$a_{00} = \frac{\partial F}{\partial w}(0, 0) = f(0, 0) > 0.$$

By recalling (6.2) and applying the same argument as in the proof of Lemma 2.3, we see that  $\operatorname{Re} a_3 = 0$  if  $u(z) > 0$  in  $\Omega$ , and therefore (1.2) holds. Q.E.D.

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