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H. P. SCHLICKEWEI
W. M. SCHMIDT

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Linear Equations in Members of Recurrence Sequences

H.P. SCHLICKEWEI - W.M. SCHMIDT †

We begin by reviewing the equation $u_n = v_m$ in two variables n, m where $\{u_n\}, \{v_m\}$ are given linear recurrences. We then turn to the equation

$$Au_n + Bu_m + Cu_\ell = 0$$

in three variables n, m, ℓ, where $\{u_n\}$ is a given recurrence. It turns out that the solutions, with finitely many exceptions, lie in a finite number of linear or exponential one-parameter families.

1. - Introduction

A *linear recurrence sequence*, briefly *recurrence sequence*, is a sequence $\{u_n\}_{n\in\mathbb{Z}}$ of complex numbers satisfying a relation

$$u_{n+k} = \nu_{k-1}u_{n+k-1} + \cdots + \nu_1 u_{n+1} + \nu_0 u_n \qquad (n \in \mathbb{Z}) \tag{1.1}$$

with given $k > 0$ and given coefficients ν_i. Note that we understand u_n to be defined for positive as well as negative or zero subscripts. The equation $u_n = 0$ or more generally $u_n = c$ in the unknown n, as well as the equation $u_n = u_m$ or more generally

$$u_n = v_m \tag{1.2}$$

in unknowns n, m where $\{u_n\}, \{v_m\}$ are given recurrence sequences, have been the subject of much recent work (see, e.g., [2], [4], [5], [6], [8]). In the present treatise we will study the equation

$$Au_n + Bu_m + Cu_\ell = 0 \tag{1.3}$$

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in unknowns n, m, ℓ. But before turning to (1.3), we will reformulate and augment the results on (1.2), which are essentially due to Laurent [4], [5]. This will bring out the analogy between (1.2) and (1.3).

The *companion polynomial* of the relation (1.1) is

$$P(z) = z^k - \nu_{k-1} z^{k-1} - \cdots - \nu_0 = \prod_{i=0}^{r} (z - \alpha_i)^{\sigma_i},$$

say, with distinct $\alpha_0, \ldots, \alpha_r$. It will be practical to make the following convention. If some root of P is a root of unity, let α_0 be such a root, and $\alpha_1, \ldots, \alpha_r$ the other roots of P. If no root of P is a root of unity, set $\alpha_0 = 1$, $\sigma_0 = 0$, and let $\alpha_1, \ldots, \alpha_r$ be the roots of P. We will suppose throughout that $\nu_0 \neq 0$, so that $\alpha_0, \ldots, \alpha_r$ are nonzero. It is well known that $u_n = F(n)$ where $F(x)$ is a function of polynomial-exponential type, more precisely

$$F(x) = \sum_{i=0}^{r} f_i(x) \alpha_i^x, \tag{1.4}$$

where f_i is a polynomial of degree $< \sigma_i$. Here a polynomial of degree < 0 is understood to be zero. Conversely, with $F(x)$ given by (1.4), the sequence $u_n = F(n)$ satisfies (1.1).

We will suppose throughout that our sequences are *non-degenerate*, i.e. that α_i/α_j for any $i \neq j$ is not a root of 1. Thus only α_0 is a root of 1. We will also suppose that $r \geq 1$ and that $f_i \neq 0$ for $1 \leq i \leq r$, so that $F(x)$ is not an "almost polynomial" of the type $f_0(x)\alpha_0^x$.

We will say that F is *defined over the algebraic numbers* if each α_i is algebraic and the coefficients of the polynomials f_i are algebraic.

Similarly, the sequence $v_m = G(m)$ with

$$G(y) = \sum_{i=0}^{r'} g_i(y) \beta_i^y. \tag{1.5}$$

Equation (1.2) becomes

$$F(x) = G(y), \tag{1.6}$$

to be solved in integers x, y.

DEFINITION. F, G are called *related* if $r' = r$ and if after a reordering of $\beta_1, \ldots, \beta_r$ we have

$$\alpha_i^p = \beta_i^q \qquad (i = 1, \ldots, r) \tag{1.7}$$

with certain nonzero integers p, q. They are *doubly related* if there is a second reordering of $\beta_1, \ldots, \beta_r$ with this property, i.e., if there is a non-trivial permutation π of $\{1,\dots,r\}$ such that we have both (1.7), and

$$\alpha_i^{p'} = \beta_{\pi(i)}^{q'} \qquad (i = 1,\dots,r) \tag{1.8}$$

with nonzero integers p', q'. They are called *simply related* if they are related but not doubly related.

In section 2 we will establish the following Lemma, which essentially is due to Laurent [4, Lemme 2], and whose proof has its roots in Evertse [3].

LEMMA 1. *When F, G are doubly related, then r is even, $p'/q' = -p/q$, and after a proper ordering of $\alpha_1,\dots,\alpha_r$ and of $\beta_1,\dots,\beta_r$, we have* (1.7) *and*

$$\alpha_i^{p'} = \beta_{i+1}^{q'},\ \alpha_{i+1}^{p'} = \beta_i^{q'} \qquad \text{when } 1 \le i < r,\ i \text{ odd}. \tag{1.9}$$

(Thus (1.8) *holds with the permutation π having $\pi(i) = i+1$, $\pi(i+1) = i$ for i odd). The products $\alpha_i\alpha_{i+1}$ and $\beta_i\beta_{i+1}$ for i odd are roots of* 1. *Conversely, if F, G are related, if r is even and the products $\alpha_i\alpha_{i+1}$ for i odd are roots of* 1, *then F, G are doubly related.*

There cannot be a third permutation of $\beta_1,\dots,\beta_r$ with a property like (1.8).

From now on, when F, G are related or doubly related, we will suppose that we have (1.7), or both (1.7), (1.9), respectively. Since α_0, β_0 are roots of 1, for related F, G we may pick p, q such that (1.7) holds also for $i = 0$, i.e., we have

$$\alpha_0^p = \beta_0^q. \tag{1.10}$$

The pairs (p, q) with (1.7), (1.10) are integer multiples of a "minimal" pair (p_0, q_0) which we may choose with $p_0 > 0$. (Note that p_0, q_0 need not be coprime). In what follows, (p, q) will be this minimal pair. For doubly related F, G we can choose p', q' such that

$$\alpha_0^{p'} = \beta_0^{q'}$$

holds in addition to (1.9), and in the sequel we will take (p', q') minimal with this property.

PROPOSITION 1. *Equation* (1.6) *has only finitely many solutions unless F, G are related. When F, G are simply related, all but finitely many solutions satisfy the system of equations*

$$f_i(x)\alpha_i^x = g_i(y)\beta_i^y \qquad (i = 0, 1, \dots, r). \tag{1.11}$$

If F, G are doubly related, all but finitely many solutions satisfy either (1.11), *or the system*

$$f_i(x)\alpha_i^x = g_{i+1}(y)\beta_{i+1}^y,\quad f_{i+1}(x)\alpha_{i+1}^x = g_i(y)\beta_i^y \quad (1 \le i < r,\ i \text{ odd}), \tag{1.12a}$$

(1.12b) $$f_0(x)\alpha_0^x = g_0(y)\beta_0^y.$$

This Proposition is essentially a reformulation of Laurent [4, Théorème 3].

There remains the question of the solutions of (1.11) or of (1.12a), (1.12b). Since (1.11) and the system (1.12a), (1.12b) are of the same nature (disguised by the notation), it will suffice to deal with (1.11). Call an *ordered* pair F, G *exceptional* if

(a) F, G are simply related;
(b) there is a natural number $N > 1$ which is an integer power of each α_i and each β_i with $1 \le i \le r$;
(c) either $|\alpha_i| > 1$ for $1 \le i \le r$, or $|\alpha_i| < 1$ for $1 \le i \le r$;
(d) f_0, g_0 are constant;
(e) each g_i is constant and for $1 \le i \le r$, $f_i(x) = \gamma_i(x-\gamma)^{e_i}$ where γ is rational and $e_i > 0$.

Note that by (1.7), (c) implies that either $|\beta_i| > 1$ for $1 \le i \le r$, or $|\beta_i| < 1$ for $1 \le i \le r$. Therefore only condition (e) is not symmetric in F, G.

PROPOSITION 2. *If F, G are related, but neither F, G nor G, F is exceptional, then* (1.11) *either has only finitely many solutions, or it has finitely many solutions together with a* 1*-parameter linear family of solutions*

$$x(t) = pt + a, \quad y(t) = qt + b \qquad (t \in \mathbb{Z})$$

with certain $a, b \in \mathbb{Z}$.

Similar results are described in Laurent [5, Théorème 2, Lemme 7 and 8].

It follows that if F, G are as in the proposition, the solutions of (1.6) will consist of a finite number, plus possibly a family $\mathcal{F} : x = pt + a,\ y = qt + b$ $(t \in \mathbb{Z})$, and in the case when F, G are doubly related also possibly a family $\mathcal{F}' : x = p't + a',\ y = q't + b'\ (t \in \mathbb{Z})$.

EXAMPLE. $u_n = F(n)$, $v_m = G(m)$ with

$$F(x) = 3x((7+5\sqrt{2})^x + (7-5\sqrt{2})^x),$$
$$G(y) = (2y+1)((1+\sqrt{2})(3+2\sqrt{2})^y + (1-\sqrt{2})(3-2\sqrt{2})^y).$$

Here α_1, α_2 are $7 \pm 5\sqrt{2} = (1 \pm \sqrt{2})^3$; further $f_0(x) = 0$, $f_1(x) = f_2(x) = 3x$. We have $u_0 = 0$, $u_1 = 42, \ldots$, and the companion polynomial is $(z^2 - 14z - 1)^2$. On the other hand, β_1, β_2 are $3 \pm 2\sqrt{2} = (1 \pm \sqrt{2})^2$, and $g_0(y) = 0$ and $g_1(y), g_2(y)$ are $(1 \pm \sqrt{2})(2y+1)$. We have $v_0 = 2$, $v_1 = 42, \ldots$, and the companion polynomial of $\{v_m\}$ is $(z^2 - 6z + 1)^2$. Here $\alpha_i^2 = \beta_i^3$, so that (1.7) holds with $p = 2$, $q = 3$. Also $\alpha_1^2 = \beta_2^{-3}$, $\alpha_2^2 = \beta_1^{-3}$, so that (1.9) holds with $p' = 2$, $q' = -3$. Therefore

F, G are doubly related. Note that $\alpha_1\alpha_2 = -1$, $\beta_1\beta_2 = 1$, so that $\alpha_1\alpha_2$ and $\beta_1\beta_2$ are roots of 1. System (1.11) becomes

$$3x(1 \pm \sqrt{2})^{3x} = (2y+1)(1 \pm \sqrt{2})^{1+2y},$$

which for $x, y \in \mathbb{Z}$ leads to $3x = 1 + 2y$, therefore to the family

$$x = 2t+1, \quad y = 3t+1 \qquad (t \in \mathbb{Z}).$$

Similarly, (1.12a) is solved by

$$x = 2t+1, \quad y = -3t-2 \qquad (t \in \mathbb{Z}).$$

PROPOSITION 3. *Suppose the ordered pair F, G is exceptional. Then in addition to finitely many solutions, the solutions to* (1.11) (*and therefore to* (1.6)) *will comprise a finite number* (*possibly zero*) *of exponential families $\mathcal{G}_1, \ldots, \mathcal{G}_\ell$ of the type*

$$\mathcal{G}_j: \qquad x(s) = pc_jR^s + \gamma, \;\; y(s) = qc_jR^s + as + b_j \qquad (s \in \mathbb{N}).$$

Here $R \in \mathbb{Z}$, $R > 1$, R is a rational power of each α_i and each β_i with $1 \le i \le r$. Further $a \in \mathbb{Z}\setminus\{0\}$, more precisely $a > 0$ or $a < 0$ depending on whether each $|\beta_i| > 1$ or each $|\beta_i| < 1$ $(1 \le i \le r)$. Finally, $c_j \neq 0$, b_j lie in $\mathbb{Q}$, with the property that $x(s), y(s) \in \mathbb{Z}$ for each $s \in \mathbb{N}$ $(1 \le j \le \ell)$.

Note that R has to be a rational power of the number N in the definition of exceptional pairs.

EXAMPLE. $F(x) = (2x-1)^2 \cdot 9^x - (2x-1)^3 \cdot 27^x$, $G(y) = 9^y - 27^y$. Here $\alpha_1 = \beta_1 = 9$, $\alpha_2 = \beta_2 = 27$, and F, G is exceptional with $N = 3^6$ and $\gamma = 1/2$. Now (1.11) becomes

$$(2x-1)^2 \cdot 9^x = 9^y, \quad (2x-1)^3 \cdot 27^x = 27^y,$$

which yields $(2x-1) \cdot 3^x = 3^y$. Setting $y - x = s - 1$, we have $2x - 1 = 3^{s-1}$, therefore

$$x(s) = \frac{1}{6} \cdot 3^s + \frac{1}{2}, \quad y(s) = \frac{1}{6} \cdot 3^s + s - \frac{1}{2},$$

and here $x(s), y(s) \in \mathbb{Z}$ for $s \in \mathbb{N}$.

A function F of polynomial-exponential type is related to itself with $p = q = 1$. Call F *symmetric* if it is doubly related with itself. By Lemma 1 this happens when r is even and after suitable ordering, $\alpha_i\alpha_{i+1}$ with i odd in $1 \le i < r$ is a root of 1. Here $q' = -p'$. Further if F, G are doubly related, then F and G are symmetric, and conversely when F, G are related and F

is symmetric, then F, G are doubly related. When F is symmetric, then the equation

$$F(x) = F(y) \tag{1.13}$$

has the solutions $y = x$, as well as possibly a linear family $x = p't+a$, $y = -p't+b$ ($t \in \mathbb{Z}$), as well as perhaps finitely many further solutions. This recovers a result of Laurent [5]. For instance, the Fibonacci sequence has $u_n = F(n)$ with

$$F(x) = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^x - \left(\frac{1-\sqrt{5}}{2}\right)^x\right),$$

so that α_1, α_2 are $\frac{1}{2}(1 \pm \sqrt{5})$, therefore $(\alpha_1\alpha_2)^2 = 1$, $\alpha_1^2 = \alpha_2^{-2}$, and F is symmetric with $p' = 2$, $q' = -2$. Equation (1.13) has the trivial solutions, as well as $x = 2t + 1$, $y = -2t - 1$ ($t \in \mathbb{Z}$).

We now turn to (1.3), i.e. an equation

$$AF(x) + BF(y) + CF(z) = 0 \tag{1.14}$$

with nonzero coefficients A, B, C, and in unknowns $x, y, z \in \mathbb{Z}$. There may be solutions with $F(x) = 0$, $BF(y) + CF(z) = 0$. For these, there are only finitely many possibilities for x (according to the Skolem-Mahler-Lech Theorem, but see also §4), and the relation in y, z is of the type already studied in the propositions. Therefore it will be enough to study *proper* solutions, i.e., solutions with $F(x)F(y)F(z) \neq 0$.

THEOREM 1. *Suppose $f_0(x) = 0$. If F is not symmetric, then all but finitely many proper solutions of* (1.14) *satisfy the system*

$$Af_i(x)\alpha_i^x + Bf_i(y)\alpha_i^y + Cf_i(z)\alpha_i^z = 0 \qquad (1 \leq i \leq r). \tag{1.15}$$

If F is symmetric, all but finitely many solutions satisfy (1.15) *or*

$$\begin{aligned} Af_i(x)\alpha_i^x + Bf_i(y)\alpha_i^y + Cf_{i+1}(z)\alpha_{i+1}^z &= 0 \\ Af_{i+1}(x)\alpha_{i+1}^x + Bf_{i+1}(y)\alpha_{i+1}^y + Cf_i(z)\alpha_i^z &= 0 \end{aligned} \qquad \begin{pmatrix} 1 \leq i < r \\ i \text{ odd} \end{pmatrix}, \tag{1.16z}$$

or systems (1.16x) *or* (1.16y) *obtained from* (1.16z) *by a permutation of the roles of the variables.*

Theorem 1 would be wrong without the hypothesis that $f_0(x) = 0$. An example for this is provided by the equation

$$F(x) - F(y) + F(z) = 0$$

with

$$F(x) = x + (3+2\sqrt{2})^x + (3-2\sqrt{2})^x + (2+\sqrt{3})^x + (2-\sqrt{3})^x,$$

having solutions

$$x = 2s, \ y = \frac{1}{2}\,F(2s), \ z = -\frac{1}{2}\,F(2s) \qquad (s \in \mathbb{N}).$$

For if we set $F(x) = x + F_1(x)$, these values have $F_1(y) = F_1(z)$ and $F_1(x) + x - y + z = 0$.

Call F (with $f_0 = 0$) *exceptional* if there is a natural number $N > 1$ which is an integral power of each α_i, if each $|\alpha_i| > 1$ or each $|\alpha_i| < 1$ $(i = 1, \dots, r)$, and if $f_i = \gamma_i(x - \gamma)$ with $\gamma \in \mathbb{Q}$ $(1 \le i \le r)$. F cannot be both symmetric and exceptional.

THEOREM 2. *Let F be given by* (1.4) *with $f_0 = 0$, and defined over the algebraic numbers. Then the proper solutions of* (1.14) *are made up of a finite set, plus a finite number of linear families*

$$\mathcal{F}_j: \qquad y = x + h_j, \ z = x + k_j \qquad (x \in \mathbb{Z}),$$

plus a finite number of linear families $\mathcal{F}_j^x$ or $\mathcal{F}_j^y$ or $\mathcal{F}_j^z$, where, e.g.,

$$\mathcal{F}_j^z: \qquad x = mt + a_j, \ y = mt + b_j, \ z = -mt + c_j \qquad (t \in \mathbb{Z}),$$

plus a finite number of exponential families $\mathcal{G}_j^x$ or $\mathcal{G}_j^y$ or $\mathcal{G}_j^z$, where e.g.

$$\mathcal{G}_j^x: \qquad x = c_j R^s + \gamma, \ y = c_j R^s + as + b_j, \ z = c_j R^s + a's + b'_j \qquad (s \in \mathbb{N}).$$

Now let us be more precise:

SUPPLEMENT 1. *The family $\mathcal{F}_j$ will occur precisely if we have the polynomial identities*

$$(1.17) \qquad A f_i(x) + B f_i(x + h_j)\alpha_i^{h_j} + C f_i(x + k_j)\alpha_i^{k_j} = 0 \qquad (1 \le i \le r).$$

This is possible only if either $A + B + C = 0$, $h_j = k_j = 0$, or if all the f_i are constant, with at most one exception which may be linear.

EXAMPLE. $F(x) = (x+1)(1/3)^x + (-2/3)^x$. For the equation $2F(x) - 9F(y) + 27F(z) = 0$, system (1.17) becomes

$$2(x+1) - 9(x+h+1)(1/3)^h + 27(x+k+1)(1/3)^k = 0,$$
$$2 - 9(-2/3)^h + 27(-2/3)^k = 0,$$

with the particular solution $h = 1$, $k = 3$, so that $y = x + 1$, $z = x + 3$ is a solution family $\mathcal{F}$.

SUPPLEMENT 2. *A family $\mathcal{F}^x$ or $\mathcal{F}^y$ or $\mathcal{F}^z$ may occur only if F is symmetric. The parameter m in these families may be taken to be the least $m > 0$ with $(\alpha_i\alpha_{i+1})^m = 1$ (i odd, $1 \le i < r$).*

EXAMPLE. $F(x) = (1+\sqrt{2})^x + (1-\sqrt{2})^x$. For the equation $F(x) - 5F(y) + 2F(z) = 0$, system (1.15) becomes

$$(1 \pm \sqrt{2})^x - 5(1 \pm \sqrt{2})^y + 2(1 \pm \sqrt{2})^z = 0,$$

and this has, e.g., the solutions $y = x+2$, $z = x+3$ of type $\mathcal{F}$. System (1.16y) becomes

$$(1 \pm \sqrt{2})^x - 5(1 \mp \sqrt{2})^y + 2(1 \pm \sqrt{2})^z = 0,$$

and this has, e.g., the solutions $x = 2t$, $y = -2t-2$, $z = 2t+3$ ($t \in \mathbb{Z}$) of type $\mathcal{F}^y$.

SUPPLEMENT 3. *A family $\mathcal{G}_j^x$ or $\mathcal{G}_j^y$ or $\mathcal{G}_j^z$ may occur only if F is exceptional. Here $R > 1$, $R \in \mathbb{Z}$, and R is a rational power of each α_i $(1 \le i \le r)$, and γ is the number with $f_i = \gamma_i(x-\gamma)$. Further $c_j \in \mathbb{Q}^*$, $b_j, b_j' \in \mathbb{Q}$ such that $(x,y,z) = (x_j(s), y_j(s), z_j(s)) \in \mathbb{Z}^3$ for each $s \in \mathbb{N}$.*

EXAMPLE. $F(x) = x \cdot 2^x$, so that F is exceptional with $r = 1$, $\alpha_1 = 2$, $\gamma = 0$. The equation $F(x) + 16F(y) - 8F(z) = 0$ has, e.g., the linear solution family

$$\mathcal{F}: \qquad x, \quad y = x-4, \quad z = x-2 \qquad (x \in \mathbb{Z}),$$

and the exponential family

$$\mathcal{G}^x: \qquad x = 2^s, \quad y = 2^s + s - 4, \quad z = 2^s + s - 3 \qquad (s \in \mathbb{N}).$$

Consider an equation

$$F_1(x_1) + \cdots + F_n(x_n) = 0, \tag{1.18}$$

where each $F_j(x)$ is of type (1.4), with a sum over i in $0 \le i \le r_j$ with $r_j > 0$. In view of our theorems, we make the following:

CONJECTURE. The solutions $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{Z}^n$ of (1.18) consist of a finite set, together with a finite set of families $\mathcal{G}_1, \ldots, \mathcal{G}_\ell$, with

$$\mathcal{G}_j: \qquad \mathbf{x}(\mathbf{s}) = \mathbf{p}_j(\mathbf{s}) = (p_{j1}(\mathbf{s}), \ldots, p_{jn}(\mathbf{s})) \qquad (\mathbf{s} \in \mathbb{N}^{m(j)})$$

where $m(j) \ge 1$ and where each p_{jk} is of polynomial-exponential type, i.e., of the form

$$\sum_{i=1}^{w} u_i(\mathbf{s})\alpha_i^{\mathbf{s}},$$

where u_i is a polynomial and $\mathbf{a}_i^{\mathbf{s}} = \alpha_{i1}^{s_1} \cdots \alpha_{im}^{s_m}$ with $\alpha_{ih} \neq 0$. Moreover, in the case when no root belonging to any F_i is a radical, i.e. satisfies an equation $x^t - a = 0$ with $t \in \mathbb{Z}\backslash\{0\}$ and $a \in \mathbb{Z}$, we conjecture that we may take the families $\mathcal{G}_1, \ldots, \mathcal{G}_\ell$ to be linear.

No such simple conjecture may be made for general equations of polynomial-exponential type. The equation

$$\sum_{i=2}^{n} x_i^2 - 2\sum_{i=1}^{n-1} x_{i+1} 2^{x_i} + \sum_{i=1}^{n-1} 4^{x_i} = 0$$

is a disguised form of

$$\sum_{i=1}^{n-1} (x_{i+1} - 2^{x_i})^2 = 0,$$

with solutions

$$x_1 = s, \;\; x_2 = 2^s, \;\; x_3 = 2^{2^s}, \ldots, x_n = 2^{2^{\cdot^{\cdot^{2^s}}}} \qquad (s \in \mathbb{Z}, \;\; s \geq 0).$$

2. - Proof of Lemma 1

Let F, G be doubly related with (1.7), (1.8). The permutation π of $\{1, \ldots, r\}$ is a product of disjoint cycles

$$(i_0, i_1, \ldots, i_{t-1}), \tag{2.1}$$

where we allow cycles of length $t = 1$. We have $\alpha_{i_j}^{p'} = \beta_{i_{j+1}}^{q'}$ for $0 \leq j < t$, with the notation $i_{j+t} = i_j$, and therefore

$$\alpha_{i_j}^{p'q} = \beta_{i_{j+1}}^{q'q} = \alpha_{i_{j+1}}^{q'p},$$

so that

$$\alpha_{i_j}^{(p'q)^t} = \alpha_{i_j}^{(q'p)^t}.$$

This yields $(p'q)^t = (q'p)^t$, therefore $p'q = \sigma q'p$ with $\sigma = \pm 1$. Setting $p'q = u$, we have $\alpha_{i_j}^{u} = \alpha_{i_{j+1}}^{\sigma u} = \alpha_{i_{j+2}}^{u} = \cdots$. When $\sigma = 1$, this shows that $\alpha_{i_j}/\alpha_{i_{j+1}}$ is a root of 1, so that $i_{j+1} = i_j$ (by the non-degeneracy). In this case the cycle (2.1), and in fact every cycle of our permutation, has length $t = 1$, and the permutation is trivial. Therefore $\sigma = -1$ and $\alpha_{i_j}^{u} = \alpha_{i_{j+1}}^{-u} = \alpha_{i_{j+2}}^{u} = \cdots$, so that $\alpha_{i_j}/\alpha_{i_{j+2}}$ is a root of 1, and $i_j = i_{j+2}$, so that $t = 2$. In this case every cycle is of length 2. Therefore r is even and π is a product of $r/2$ cycles of length 2. After reordering, the cycles are $(1, 2), \ldots, (r-1, r)$ so that $\pi(i) = i + 1$, $\pi(i+1) = i$ for i odd in $1 \leq i < r$. Now $p'/q' = \sigma p/q = -p/q$ and $(\alpha_i \alpha_{i+1})^u = 1$ for i odd, and also $(\beta_i\beta_{i+1})^u = 1$, so that $\alpha_i\alpha_{i+1}$ and $\beta_i\beta_{i+1}$ are roots of 1. There cannot be a third permutation of $\beta_1, \dots, \beta_r$ with a property like (1.8), for if it belonged to p'', q'', then $p''/q'' = p/q$ or $p''/q'' = -p/q = p'/q'$, so that the permutation is either trivial or equal to π.

Conversely, if F, G are related satisfying (1.7), if r is even and $\alpha_i\alpha_{i+1}$ for each odd i is a root of 1, say $(\alpha_i\alpha_{i+1})^m = 1$, then $\alpha_i^{mp}\beta_{i+1}^{mq} = \alpha_i^{mp}\alpha_{i+1}^{mp} = 1$, also $\alpha_{i+1}^{mp}\beta_i^{mq} = \alpha_{i+1}^{mp}\alpha_i^{mp} = 1$, so that (1.9) holds with $p' = mp$, $q' = -mq$, and F, G are doubly related.

3. - Generalities

Consider equations

$$\sum_{i \in I} f_i(\mathbf{x})\, \underline{\alpha}_i^{\mathbf{x}} = 0 \tag{3.1}$$

where I is a finite set, $f_i(\mathbf{x}) = f_i(x_1, \dots, x_n)$ is a polynomial, and $\underline{\alpha}_i^{\mathbf{x}} = \alpha_{i1}^{x_1} \cdots \alpha_{in}^{x_n}$ with nonzero complex numbers α_{ij} $(i \in I,\ 1 \leq j \leq n)$. When $\mathcal{P}$ is a partition of I, and π a subset of I, write $\pi \in \mathcal{P}$ if π is among the subsets belonging to $\mathcal{P}$. Consider the system of equations

$$\sum_{i \in \pi} f_i(\mathbf{x})\, \underline{\alpha}_i^{\mathbf{x}} = 0 \qquad (\pi \in \mathcal{P}). \tag{3$\mathcal{P}$}$$

This system implies but is usually not implied by (3.1). Let $\Sigma(\mathcal{P})$ be the set of solutions $\mathbf{x} \in \mathbb{Z}^n$ of (3$\mathcal{P}$) which do not satisfy (3$\mathcal{Q}$) for any proper refinement $\mathcal{Q}$ of $\mathcal{P}$. Every solution $\mathbf{x} \in \mathbb{Z}^n$ of (3.1) belongs to some $\Sigma(\mathcal{P})$ (which need not be unique). Thus to solve (3.1) it suffices to know the sets $\Sigma(\mathcal{P})$.

Write $i \overset{\mathcal{P}}{\sim} j$ if $i, j \in I$ belong to the same subset of $\mathcal{P}$. Let $G(\mathcal{P}) \subseteq \mathbb{Z}^n$ consist of $\mathbf{x}$ having

$$\underline{\alpha}_i^{\mathbf{x}} = \underline{\alpha}_j^{\mathbf{x}} \qquad \text{for every } i, j \text{ with } i \overset{\mathcal{P}}{\sim} j.$$

Thus $G(\mathcal{P})$ is a subgroup of $\mathbb{Z}^n$.

M. Laurent [5] proved that *$\Sigma(\mathcal{P})$ is finite if $G(\mathcal{P}) = \{\mathbf{0}\}$*. A quantitative version, with explicit bounds for the cardinality $|\Sigma(\mathcal{P})|$, was given by the authors [7]. We like to call Laurent's assertion the "splitting theorem", since it allows to split (3.1) into systems (3$\mathcal{P}$) for partitions $\mathcal{P}$ with $G(\mathcal{P}) \neq \{\mathbf{0}\}$.

Suppose $G(\mathcal{P}) \neq \{\mathbf{0}\}$. Then also by Laurent [5], every solution $\mathbf{x} \in \Sigma(\mathcal{P})$ may be written as $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2$ with $\mathbf{x}_1 \in G(\mathcal{P})$ and

$$|\mathbf{x}_2| \leq c_0 \log^+ |\mathbf{x}_1|. \tag{3.2}$$

Here c_0 depends only on the equation (3.1), $\log^+ z = \log(\max(z, e))$, and $|\cdot|$ is the maximum norm.

4. - The Equation $F(x) = c$

It follows from the Skolem-Mahler-Lech Theorem that this equation has only finitely many solutions. We will give a proof to illustrate the present method in the simplest possible case. The equation is

$$(f_0(x)\alpha_0^x - c) + \sum_{i=1}^{r} f_i(x)\alpha_i^x = 0.$$

Say $\alpha_0^m = 1$, and restrict x to a given residue class modulo m. For such x, $f_0(x)\alpha_0^x - c$ is a polynomial $f_0^*(x)$, and the equation becomes

$$f_0^*(x) \cdot 1^x + \sum_{i=1}^{r} f_i(x)\alpha_i^x = 0.$$

There are $r+1$ summands here, which we indicate by the symbols $0, 1, \ldots, r$. We have to study $\Sigma(\mathcal{P})$ for every partition $\mathcal{P}$ of $\{0, 1, \ldots, r\}$ with $G(\mathcal{P}) \neq \{\mathbf{0}\}$. Let $p \neq 0$ be in $G(\mathcal{P})$. Then if $i \overset{\mathcal{P}}{\sim} j$ with $i < j$, we have $\alpha_i^p = \alpha_j^p$ if $i > 0$, and $1^p = \alpha_j^p$ if $i = 0$, therefore $p = 0$, which is impossible. Thus $\mathcal{P}$ consists of single element sets $\{i\}$, and $(3\mathcal{P})$ becomes

$$f_i(x) = 0 \qquad (0 \leq i \leq r).$$

There are only finitely many such x, since in §1 we made the hypothesis that $r \geq 1$ and that $f_i \neq 0$ for $1 \leq i \leq r$.

5. - On the Proof of Proposition 1

Equation (1.6) may be written as

$$\sum_{i=0}^{r} f_i(x)\alpha_i^x - \sum_{i=0}^{r'} g_i(y)\beta_i^y = 0. \tag{5.1}$$

The $r + r' + 2$ summands here will be symbolized by

$$0_x, 1_x, \ldots, r_x, \;\; 0_y, 1_y, \ldots, r'_y.$$

Given a partition $\mathcal{P}$ with $G(\mathcal{P}) \neq \{\mathbf{0}\}$ of this set of $r + r' + 2$ elements, we have to study solutions $(x, y) \in \Sigma(\mathcal{P})$. So let $(p, q) \neq (0, 0)$ lie in $G(\mathcal{P})$.

CASE A. *$\mathcal{P}$ contains a 1-element subset $\{i_x\}$ or $\{i_y\}$ with $i \geq 1$.* Say $\{i_y\}$; then $(3\mathcal{P})$ yields $g_i(y)\beta_i^y = 0$, which gives at most finitely many possibilities for y. When y is given, equation (1.6) becomes an equation in x of the type $F(x) = c$ with $c = G(y)$. This is the kind of equation dealt with in section 4, and it has only finitely many solutions.

CASE B. *$\mathcal{P}$ contains no 1-element subset $\{i_x\}$ or $\{i_y\}$ with $i \geq 1$.* Then Laurent [4, Lemme 2] has shown that $r = r'$ and there is a permutation $\{i_1, \ldots, i_r\}$ of $\{1, \ldots, r\}$ such that $\mathcal{P}$ contains the sets $\{1_x, (i_1)_y\}, \ldots, \{r_x, (i_r)_y\}$. Therefore after reordering, $\mathcal{P}$ contains the sets $\{1_x, 1_y\}, \ldots, \{r_x, r_y\}$, so that $i_x \sim i_y$ for $1 \leq i \leq r$, and (1.7) holds. In particular, F, G are related. Now (3$\mathcal{P}$) yields (1.11) for $1 \leq i \leq r$, therefore in fact for $0 \leq i \leq r$. (Note that $\mathcal{P}$ is *essentially* given, but it may either contain $\{0_x, 0_y\}$ or the singletons $\{0_x\}$, $\{0_y\}$).

It follows that $\Sigma(\mathcal{P})$ can be infinite only if F, G are related. Now suppose that $\mathcal{Q}$ is another partition with infinite $\Sigma(\mathcal{Q})$. Then $G(\mathcal{Q})$ contains some $(p', q') \neq (0,0)$, and we have again a $1-1$ correspondence between the sets $\{1_x, \ldots, r_x\}$ and $\{1_y, \ldots, r_y\}$. Say $i_x \overset{\mathcal{Q}}{\sim} \pi(i)_y$ where π is a permutation of $\{1, \ldots, r\}$, and we have (1.8). If $\mathcal{Q}$ is essentially different from $\mathcal{P}$, so that π is not trivial, then F, G are doubly related. After reordering, both (1.7), (1.9) hold, and $\mathcal{Q}$ contains the sets $\{1_x, 2_y\}$, $\{2_x, 1_y\}, \ldots, \{(r-1)_x, r_y\}$, $\{r_x, (r-1)_y\}$. Then (3$\mathcal{Q}$) leads to (1.12a), (1.12b).

6. - Proof of Theorem 1

We write (1.14) as

$$\sum_{i=1}^{r} (Af_i(x)\alpha_i^x + Bf_i(y)\alpha_i^y + Cf_i(z)\alpha_i^z) = 0.$$

We indicate the summands by the symbols

$$1_x, 1_y, 1_z, \ldots, r_x, r_y, r_z.$$

Given a partition $\mathcal{P}$ with $G(\mathcal{P}) \neq \{\mathbf{0}\}$ of this set of $3r$ elements, we have to study solutions $(x, y, z) \in \Sigma(\mathcal{P})$. So let $(p, q, s) \neq (0,0,0)$ lie in $G(\mathcal{P})$.

CASE A. *$\mathcal{P}$ contains a 1-element set.* Say the set $\{i_z\} \in \mathcal{P}$. Then (3$\mathcal{P}$) yields $f_i(z)\alpha_i^z = 0$, which gives only finitely many values for z. Given such z, write

$$F^*(x) = AF(x), \qquad G^*(y) = -BF(y) - CF(z),$$

and our equation becomes

$$F^*(x) = G^*(y). \tag{6.1}$$

By Proposition 1, all but finitely many pairs x, y with (6.1) have $f_0^*(x)\alpha_0^x = g_0^*(y)\beta_0^y$. But in our case, $f_0^*(x) = 0$, $g_0^*(y) = -CF(z)$, $\beta_0 = 1$ (recall our hypothesis

$f_0 = g_0 = 0$). Thus infinitely many solutions x, y are only possible if $F(z) = 0$. But this is not the case for proper solutions.

CASE B. *$\mathcal{P}$ contains no 1-element set. We have $i_x \sim j_x$ for some $i \neq j$* (where we write $\sim$ for $\overset{\mathcal{P}}{\sim}$). Then $\alpha_i^p = \alpha_j^p$, so that $p = 0$.

SUBCASE B.1 *$u_y \sim v_y$ for some $u \neq v$.* Then $q = 0$. Since $s \neq 0$, we cannot have $t_z \sim w_z$ for $t \neq w$. Therefore given t in $1 \leq t \leq r$, we have $t_z \sim w_x$ or $t_z \sim w_y$ for some w. Then $\alpha_t^s = \alpha_w^p = 1$ or $\alpha_t^s = \alpha_w^q = 1$, therefore $s = 0$, which is impossible. By symmetry, we are therefore led to the

SUBCASE B.2 *$u_y \not\sim v_y$, $u_z \not\sim v_z$ for $u \neq v$.* Now if $u_y \sim v_x$, then $q = 0$. Every t_z has $t_z \sim w_x$ or $t_z \sim w_y$ for some w, therefore $s = 0$, which is impossible. Therefore, by symmetry, there is no relation $u_y \sim v_x$ or $u_z \sim v_x$. Thus every set $\pi \in \mathcal{P}$ is contained in $\{1_x, \ldots, r_x\}$ or in $\{1_y, 1_z, \ldots, r_y, r_z\}$. But then $(3\mathcal{P})$ implies $F(x) = 0$, and the solution is improper.

By symmetry, we are reduced to:

CASE C. *$\mathcal{P}$ contains no 1-element set. We have $i_x \not\sim j_x$, $i_y \not\sim j_y$, $i_z \not\sim j_z$ for $i \neq j$.* Then given i, $1 \leq i \leq r$, we have $i_x \sim j_y$ or $i_x \sim j_z$ for some j. Consider the directed graph with vertices $1, \ldots, r$, and edges $i \to j$ if $i_x \sim j_y$ or $i_x \sim j_z$. From each vertex i, either one or two edges emanate (two edges $i \to j$, $i \to k$ if $i_x \sim j_y$ and $i_x \sim k_z$). The graph will contain some cycle

$$i_0 \to i_1 \to \cdots \to i_{t-1} \to i_0.$$

We have $(i_n)_x \sim (i_{n+1})_{u(n)}$ where $u(n) = y$ or z ($0 \leq n < t$, with the notation $i_t = i_0$). We further have

$$\alpha_{i_n}^p = \alpha_{i_{n+1}}^{v(n)}, \qquad (0 \leq n < t)$$

where $v(n) = q$ or s, depending on whether $u(n) = y$ or z. Then

$$\alpha_{i_0}^{p^t} = \alpha_{i_1}^{p^{t-1}v(0)} = \cdots = \alpha_{i_0}^{v(0)v(1)\cdots v(t-1)},$$

and $p^t = v(0)v(1)\cdots v(t-1)$. Say ν among the numbers $v(0), \ldots, v(t-1)$ equal q, and μ among them equal s. Then

$$p^t = q^\nu s^\mu. \tag{6.2}$$

Here $\nu + \mu = t$.

Now let P be a fixed prime number and set $\operatorname{ord} x = \xi$ if $P^\xi \mid x$ but $P^{\xi+1} \nmid x$. Then (6.2) yields

$$\operatorname{ord} p \leq \max(\operatorname{ord} q, \operatorname{ord} s).$$

By symmetry, $\operatorname{ord} q \leq \max(\operatorname{ord} p, \operatorname{ord} s)$, and $\operatorname{ord} s \leq \max(\operatorname{ord} p, \operatorname{ord} q)$. Therefore at least 2 among them are equal, say $\operatorname{ord} p \leq \operatorname{ord} q = \operatorname{ord} s$. Then by (6.2), in fact all three orders are equal. Since this is true for every prime P, we have $p = \pm q = \pm s$. Two among p, q, s must be equal, say $p = q$, and we have

$$p = q = s \tag{6.3}$$

or

$$p = q = -s. \tag{6.4}$$

Now if $i_x \sim j_y$, then $\alpha_i^p = \alpha_j^q = \alpha_j^p$, so that $i = j$. In the case (6.3), $i_x \sim k_z$ similarly yields $i = k$. Therefore i_x can only be associated with i_y, i_z, in fact i_x, i_y, i_z can only be associated with each other. Since $\mathcal{P}$ contains no singleton, we have $i_x \sim i_y \sim i_z$, and $\mathcal{P}$ consists of the triples $\{i_x, i_y, i_z\}$ $(1 \le i \le r)$. Then $(3\mathcal{P})$ becomes (1.15).

In the case (6.4), $j_z \sim i_x$ or $j_z \sim i_y$ yields $\alpha_j^s = \alpha_i^p = \alpha_i^{-s}$ or $\alpha_j^s = \alpha_i^q = \alpha_i^{-s}$, so that $(\alpha_i\alpha_j)^s = 1$, and $\alpha_i\alpha_j$ is a root of 1. Therefore $\alpha_1, \dots, \alpha_r$ occur in pairs, such that after reordering, $\alpha_i\alpha_{i+1}$ is a root of 1 for i odd, $1 \le i < r$. Then $F(x)$ is symmetric. For i odd, $i_x, i_y, (i+1)_z$ can only be associated with each other, and the same holds for $(i+1)_x, (i+1)_y, i_z$. Therefore $\mathcal{P}$ consists of precisely these triples, and $(3\mathcal{P})$ becomes (1.16z).

7. - The Equation

$$\sigma\delta^w = (t - \gamma)^e. \tag{7.1}$$

This is an equation in unknowns $t, w \in \mathbb{Z}$ with given complex $\delta \neq 0$, $\sigma \neq 0$, γ, and natural exponent e. We suppose that δ is not a root of 1.

LEMMA 2. *Equation* (7.1) *has only finitely many solutions, except possibly when* $\gamma \in \mathbb{Q}$, *and* $\delta^u \in \mathbb{Z}$ *for some* $u \in \mathbb{Z}$, $u \neq 0$. *If there are infinitely many solutions, they will make up a finite set plus one or two* 1*-parameter families* $t(s), w(s)$ *of the type*

$$t(s) = cR^s + \gamma, \quad w(s) = as + b \qquad (s \in \mathbb{N}), \tag{7.2}$$

with the possible second family being

$$t(s) = c'R^s + \gamma, \quad w(s) = as + b' \qquad (s \in \mathbb{N}). \tag{7.3}$$

Here $R > 1$, $R \in \mathbb{Z}$, *and is a rational power of* δ, *and* $a, b, b' \in \mathbb{Z}$ *with* $au > 0$. *Furthermore* $c, c' \in \mathbb{Q}^*$ *are such that* $t(s) \in \mathbb{Z}$ *for* $s \in \mathbb{N}$. *We can have two families only when* e *is even.*

PROOF. We have to solve equations

$$\sigma_1 \delta_1^w = t - \gamma,$$

where δ_1 is a fixed e-th root of δ, and σ_1 is any e-th root of σ. We apply an automorphism of $\mathbb{C}$ to this equation and write the new equation as $\sigma_2\delta_2^w = t - \gamma_2$. Then $\sigma_1\delta_1^w - \sigma_2\delta_2^w = \gamma_2 - \gamma$. But $G(w) = \sigma_1\delta_1^w - \sigma_2\delta_2^w + (\gamma - \gamma_2) = z_1 + z_2 + z_3$, say, can by Laurent's theorem have infinitely many solutions of $G(w) = 0$ only with $z_3 = 0$ (i.e., a partition of $\{1,2,3\}$ containing the singleton $\{3\}$). Therefore we can have infinitely many solutions only if $\gamma_2 = \gamma$. Since this holds for every automorphism of $\mathbb{C}$, we have $\gamma \in \mathbb{Q}$.

Now both sides of (7.1) lie in $\mathbb{Q}$, in fact in $\mathbb{Q}^e$. If t, w as well as t', w' are solutions, then $\delta^{w-w'} \in \mathbb{Q}^e$. So if we are to have infinitely many solutions, then $\delta^u \in \mathbb{Q}^e$ for some $u \in \mathbb{Z}\backslash\{0\}$. Let u be the least positive integer with this property, and choose $q \in \mathbb{Q}$ with $\delta^u = q^e$. Writing $w = uz + u_0$ with $z \in \mathbb{Z}$, $0 \le u_0 < u$, we have

$$\sigma\delta^{u_0} = \sigma\delta^w q^{-ez} = ((t-\gamma)q^{-z})^e \in \mathbb{Q}^e.$$

There can be at most one value of u_0, $0 \le u_0 < u$, with this property. If u_0 is such, set $\sigma\delta^{u_0} = r^e$. Then (7.1) becomes $(rq^z)^e = (t-\gamma)^e$, or

$$t = \gamma \pm rq^z \tag{7.4}$$

in unknowns $t, z \in \mathbb{Z}$. The $-$ sign can arise only when e is even.

Unless $q \in \mathbb{Z}$, z with (7.4) is bounded from above, and unless $1/q \in \mathbb{Z}$, z is bounded from below. Suppose $q \in \mathbb{Z}$; then $q > 1$ since δ was not a root of 1. Now z is bounded from below. The question now is: For what values of $z \ge 0$, $z \in \mathbb{Z}$, is $\gamma \pm rq^z \in \mathbb{Z}$? Write $\gamma = m/\ell$, $r = n/\ell$ with integers m, n, ℓ having g.c.d.$(m, n, \ell) = 1$. Then the condition becomes

$$m \pm nq^z \equiv 0 \pmod{\ell}.$$

Write $\ell = \ell_1\ell_2$ where ℓ_1 is made up of primes dividing q, and ℓ_2 is made up of other primes. Then $q^z \equiv 0 \pmod{\ell_1}$ when z is large, and infinitely many solutions are possible only if $m \equiv 0 \pmod{\ell_1}$. The condition on z then becomes

$$m \pm nq^z \equiv 0 \pmod{\ell_2}.$$

Let $v > 0$ be least with $q^v \equiv 1 \pmod{\ell_2}$. If $m + nq^z \equiv 0 \pmod{\ell}$ has a least solution $z_0 \ge 0$, then the other solutions are $z = z_0 + vs$ $(s \in \mathbb{N})$, so that

$$t = rq^{z_0}(q^v)^s + \gamma = cR^s + \gamma$$

with $R = q^v$, $c = rq^{z_0}$. Also

$$w = uz + u_0 = uvs + uz_0 + u_0 = as + b,$$

say, with $a = uv$. If $m - nq^z \equiv 0 \pmod{\ell}$ has a least solution $z_0' \geq 0$, then the other solutions are $z = z_0' + vs$, so that

$$t = c'R^s + \gamma$$

with $c' = -rq^{z_0'}$, and $w = as + b'$ with $a = uv$.

We had assumed that $\delta^u = q^e \in \mathbb{Z}$ with $u > 0$, and we had obtained $a = uv > 0$. If $\delta^u = q^e$ with $1/q \in \mathbb{Z}$, then the argument given above should be carried out with $-u$ in place of u, $1/q$ in place of q, and $a = -uv < 0$.

REMARK. There are $c_1, d_1 \in \mathbb{Z}$ with $c_1 \neq 0$ such that $t(s)$ as given by (7.2) may be written as

$$t(s) = c_1 \frac{R^s - 1}{R - 1} + d_1.$$

The same applies to (7.3), with integer coefficients $c_2 \neq 0$, d_2.

8. - The Equation

$$f(x)\alpha^x = g(y)\beta^y. \tag{8.1}$$

Here α, β are nonzero, and not roots of 1. Further f, g are nonzero polynomials. We seek solutions x, y in $\mathbb{Z}$. By Proposition 1, there are only finitely many solutions unless

$$\alpha^p = \beta^q \tag{8.2}$$

for certain nonzero integers $p > 0$, q. Set $\alpha^p = \beta^q = \delta$. Call the equation *exceptional of type* 1 if $\delta^u \in \mathbb{Z}$ for some $u \in \mathbb{Z}\backslash\{0\}$, and if g is constant but f has exactly one root γ (of arbitrary multiplicity), and this root is rational. Call it *exceptional of type* 2 if the roles of f, g are interchanged.

LEMMA 3. *Suppose* (8.1) *with* (8.2) *is not exceptional. Then in addition to finitely many solutions, the solutions of* (8.1) *make up a linear* 1*-parameter family*

$$\mathcal{F}: \qquad x = pt + a, \ \ y = qt + b \qquad (t \in \mathbb{Z}).$$

The family $\mathcal{F}$ occurs precisely if we have the following polynomial identity in t:

$$f(pt + a)\alpha^a = g(qt + b)\beta^b. \tag{8.3}$$

PROOF. It is trivial to check that every (x, y) in the family $\mathcal{F}$ is a solution if and only if the identity (8.3) holds.

Laurent [5, Lemme 6] shows that the solutions are as described in our Lemma, unless one of f, g is constant and the other has exactly one root γ. Say g is constant and f is a constant multiple of $(x-\gamma)^e$, so that (8.1) becomes $(x-\gamma)^e\alpha^x = \rho\beta^y$ with a coefficient $\rho \neq 0$. The integers x, y lie in arithmetic progressions

$$x = pt + a, \quad y = qs + b \qquad (t, s \in \mathbb{Z})$$

with $0 \leq a < p$, $0 \leq b < q$. For given a, b, we obtain an equation in t, s, namely $(pt + a - \gamma)^e\alpha^a\delta^t = \rho\beta^b\delta^s$, or

$$(t - \gamma')^e = \sigma\delta^{s-t}$$

with $\gamma' = (\gamma - a)/p$ and $\sigma = \rho\beta^b\alpha^{-a}p^{-e}$. Writing $s - t = w$, we obtain an equation of the type (7.1). By Lemma 2 we can have infinitely many solutions only if γ' and therefore γ is rational, and $\delta^u \in \mathbb{Z}$ for some $u \in \mathbb{Z}\backslash\{0\}$. But then the equation (8.1) is exceptional, against the hypothesis of the lemma.

LEMMA 4. *Suppose* (8.1) *with* (8.2) *is exceptional of type* 1. *Then in addition to finitely many solutions, the solutions of* (8.1) *comprise a finite number of exponential* 1*-parameter families* $\mathcal{G}_1, \ldots, \mathcal{G}_\ell$, *with*

$$\mathcal{G}_j\!: \qquad x = pc_jR^s + \gamma, \quad y = qc_jR^s + as + b_j \qquad (s \in \mathbb{N}).$$

Here $R \in \mathbb{Z}$, $R > 1$, *and* R *is a rational power of* α *and of* β. *Also* $a \in \mathbb{Z}\backslash\{0\}$, $c_j \in \mathbb{Q}^*$, $b_j \in \mathbb{Q}$ *such that* $x(s), y(s) \in \mathbb{Z}$ *for each* $s \in \mathbb{N}$. *We have* $au > 0$ *where* $u \in \mathbb{Z}\backslash\{0\}$ *with* $\beta^u \in \mathbb{Z}$.

PROOF. In view of (8.2) there is an ε such that $\alpha = \varepsilon^q$, $\beta = \varepsilon^p$, and (8.1) becomes $(x-\gamma)^e\varepsilon^{qx} = \tau\varepsilon^{py}$ with $\tau \neq 0$. Setting $qx = x'$, $py = y'$, $w = y' - x'$, the equation becomes

$$(x' - q\gamma)^e = \sigma\varepsilon^w$$

with $\sigma = \tau q^e$. By Lemma 2, the solutions in $x', w \in \mathbb{Z}$ are, with finitely many exceptions, in up to two families of the type

$$x'(t) = cR_0^t + q\gamma, \quad w(t) = a_0t + b_0 \qquad (t \in \mathbb{N}). \tag{8.6}$$

Then $y'(t) = x'(t) + w(t) = cR_0^t + a_0t + b_0$ $(s \in \mathbb{N})$. Since $x = x'/q$, $y = y'/p$, we need to check for what values of t do we have $x'(t) \equiv 0 \pmod{q}$, $y'(t) \equiv 0 \pmod{p}$, i.e.,

$$cR_0^t + q\gamma \equiv 0 \pmod{q}, \tag{8.7}$$

$$cR_0^t + a_0t + b_0 \equiv 0 \pmod{p}. \tag{8.8}$$

Since it is a little more complicated, let us deal with (8.8). Write $p = p_1 p_2$, where p_1 is a product of primes dividing R_0, and p_2 a product of other primes. Then (8.8) becomes

$$cR_0^t + a_0 t + b_0 \equiv 0 \pmod{p_i} \qquad (i = 1, 2). \tag{8.9}$$

We have $cR_0^t \equiv 0 \pmod{p_1}$ for $t > t_0$ (i.e., $cR_0^t \in \mathbb{Q}$ with numerator $\equiv 0 \pmod{p_1}$). Setting $t = t_0 + t'$, we have $cR_0^t + a_0 t + b_0 = c_1 R_0^{t'} + a_0 t' + b_1$ with $c_1 = cR_0^{t_0}$, $b_1 = b_0 + a_0 t_0$, and here $c_1 R_0^{t'} \equiv 0 \pmod{p_1}$ for $t' \in \mathbb{N}$. Since we may disregard finite sets of solutions, we may suppose after a change of notation that $cR_0^t \equiv 0 \pmod{p_1}$ for $t \in \mathbb{N}$. Then (8.9) with $i = 1$ becomes $a_0 t + b_0 \equiv 0 \pmod{p_1}$, and this is satisfied if t lies in certain residue classes modulo p_1.

Since g.c.d.$(R_0, p_2) = 1$, we have $R^{\varphi(p_2)} \equiv 1 \pmod{p_2}$, so that the residue class of $R^t \pmod{p_2}$ depends on the residue class of $t \pmod{\varphi(p_2)}$. The solutions of (8.9) with $i = 2$ then lie in certain residue classes modulo $p_2\varphi(p_2)$. Thus the solutions of (8.9) lie in certain residue classes to some modulus m, and both (8.7), (8.8) hold if t lies in certain residue classes to a modulus n, say $t = ns + n_j$ $(s \in \mathbb{N},\ 1 \le j \le \ell)$. Then

$$x(s) = (c/q)R_0^{ns+n_j} + \gamma = c_j p R^s + \gamma,$$

$$y(s) = (c/p)R_0^{ns+n_j} + (a_0/p)(ns + n_j) + b_0/p = c_j q R^s + as + b_j$$

with $R = R_0^n$, $a = a_0 n/p$, $c_j = (c/pq)R_0^{n_j}$, $b_j = (a_0 n_j + b_0)/p$. Notice that R, a do not depend on the family. In the possible second family of Lemma 2, only the parameters c, b are changed, and from this it is easily seen that the resulting families $\mathcal{G}_j$ may all be taken to have the same value of R and of a. If $\beta^u \in \mathbb{Z}$, then $\varepsilon^{pu} \in \mathbb{Z}$; and by Lemma 2 we have $a_0 pu > 0$ in (8.6), and therefore $au > 0$.

EXAMPLE. $(5x - 6)^4 \sqrt{2}^x = \sqrt{2}^y$.

Here $(5x - 6)^4 = \sqrt{2}^{y-x}$. Therefore $y - x = 8z$, $z \in \mathbb{Z}$, and $(5x - 6)^4 = 2^{4z}$, therefore $5x - 6 = \pm 2^z$. With the + sign, z needs to have $2^z \equiv -6 \equiv 4 \pmod 5$, so that $z = 2 + 4t$. Now $5x = 2^z + 6 = 4 \cdot 16^t + 6$, therefore

$$x(t) = \frac{4}{5}\ 16^t + \frac{6}{5}, \quad y(t) = x + 8z = \frac{4}{5}\ 16^t + 32t + \frac{86}{5}. \tag{8.10}$$

With the $-$ sign, z needs $2^z \equiv 6 \equiv 1 \pmod 5$, so that $z = 4t$. Now $5x = -2^z + 6 = -16^t + 6$, therefore

$$x(t) = -\frac{1}{5} \cdot 16^t + \frac{6}{5}, \quad y(t) = x + 8z = -\frac{1}{5} \cdot 16^t + 32t + \frac{6}{5}. \tag{8.11}$$

EXAMPLE. $(5x_1 - 6)^4 \sqrt{2}^{x_1} = (\sqrt{2})^{7y_1}$.

The solutions are $x_1 = x(t)$, $y_1 = y(t)/7$, with $x(t), y(t)$ as in (8.10) or (8.11), and t such that $y(t) \equiv 0 \pmod 7$. Depending on (8.10) or (8.11), we get the conditions

$$4 \cdot 2^t - t + 2 \equiv 0 \pmod 7,$$

$$2^t + t + 1 \equiv 0 \pmod 7.$$

The first is the same as $t \equiv 6$, or 10, or 11 (mod 21), and the second is the same as $t \equiv 2$, or 4, or 12 (mod 21). We obtain 6 families of solutions, the first being $x_1(s) = x(21s+6)$, $y_1(s) = \frac{1}{7}\, y(21s+6)$ with $x(t), y(t)$ as in (8.10), so that

$$x_1(s) = (2^{26}/5)2^{84s} + \frac{6}{5}, \quad y_1(s) = (2^{26}/35)2^{84s} + 96s + (1046/35).$$

9. - Proof of Propositions 2 and 3

We will suppose initially that $f_0 = g_0 = 0$, and we will study the system (1.11) with $1 \le i \le r$. Each of these equations (1.11) is of the type (8.1) with (8.2). We may suppose that each of the equations (1.11) has infinitely many solutions, since otherwise they have only finitely many common solutions.

CASE A. *Each equation is non-exceptional and has a linear solution family as in Lemma* 3. More precisely, let (p_i, q_i) be minimal with $\alpha_i^{p_i} = \beta_i^{q_i}$; then the i-th equation (1.11) has a solution family $\mathcal{F}_i : x = p_i t + a_i,\ y = q_i t + b_i\ (t \in \mathbb{Z})$. The minimal pair (p, q) with $\alpha_i^p = \beta_i^q$ $(i = 1, \ldots, r)$ is a common multiple of (p_i, q_i) for $i = 1, \ldots, r$; in fact it is the least common multiple. If $\mathcal{F}_1, \ldots, \mathcal{F}_r$ have a non-empty intersection, and if (a, b) lies in this intersection, then the intersection consists of pairs $(u + a, v + b)$ where (u, v) is a common multiple of the (p_i, q_i) $(i = 1, \ldots, r)$, hence is a multiple of (p, q). Therefore the intersection of $\mathcal{F}_1, \ldots, \mathcal{F}_r$ is the family $x = pt + a$, $y = qt + b$ $(t \in \mathbb{Z})$.

CASE B. *Some equations* (1.11) *are exceptional, some are not.* Then their solution families have finite intersection, since a family $\mathcal{F}$ has $qx - py$ constant, whereas a family $\mathcal{G}$ has $|qx - py|$ tending to infinity.

CASE C. *Some equations* (1.11) *are exceptional of type* 1, *some are exceptional of type* 2. We claim that their respective solution families have finite intersection. For the respective families have

$$x_1(s_1) = qc_1R_1^{s_1} + \gamma_1, \quad x_2(s_2) = qc_2R_2^{s_2} + as_2 + b$$

with $a \neq 0$. We need to solve $x_1(s_1) = x_2(s_2)$. By Proposition 1, all but finitely many solutions have $\gamma_1 = as_2 + b$, which shows that there are only finitely many solutions.

CASE D. *Each equation* (1.11) *is exceptional of type* 1. We have to study intersections of solution families $\mathcal{G}, \mathcal{G}'$ of the same type. Let two such families be

$$x_1(s_1) = qc_1R_1^{s_1} + \gamma_1, \quad y_1(s_1) = pc_1R_1^{s_1} + a_1s_1 + b_1, \tag{9.1}$$

$$x_2(s_2) = qc_2R_2^{s_2} + \gamma_2, \quad y_2(s_2) = pc_2R_2^{s_2} + a_2s_2 + b_2. \tag{9.2}$$

The equations $x_1(s_1) = x_2(s_2)$, $y_1(s_1) = y_2(s_2)$ are of the type studied in Proposition 1. With finitely many exceptions, they imply $\gamma_1 = \gamma_2$,

$$a_1s_1 + b_1 = a_2s_2 + b_2, \quad c_1R_1^{s_1} = c_2R_2^{s_2}. \tag{9.3}$$

Therefore, if there are infinitely many solutions $s_1, s_2 \in \mathbb{N}$, then $\operatorname{sgn} a_1 = \operatorname{sgn} a_2$ and $R_1^u = R_2^v$ with nonzero integers u, v. If (9.1), (9.2) belong respectively to the equation (1.11) with $i = 1$ and $i = 2$, say, then R_1, R_2 are rational powers of β_1, β_2, and it follows that there is a natural $N > 1$ which is an integral power of both β_1, β_2. Say $\beta_1^{u_1} = \beta_2^{u_2} = N$. Now $a_iu_i > 0$ $(i = 1, 2)$ by Lemma 4, so that $\operatorname{sgn} u_1 = \operatorname{sgn} u_2$, and either both $|\beta_1|, |\beta_2|$ are > 1, or both are < 1. In this way one sees that in Case D the system can have infinitely many solutions only if the ordered pair F, G is exceptional.

If there are infinitely many solutions s_1, s_2 to (9.3), they will make up a family $s_1 = u_0s + a$, $s_2 = v_0s + b$ $(s \in \mathbb{N})$. But then

$$x(s) = x_1(s_1) = x_1(u_0s + a) = qc_1R_1^aR_1^{u_0s} + \gamma_1,$$

$$y(s) = y_1(s_1) = y_1(u_0s + a) = pc_1R_1^aR_1^{u_0s} + a_1u_0s + a_1a + b_1$$

is again a family of exponential type.

Note that R_1 and a_1 (resp. R_2 and a_2), coming from (11.1) with $i = 1$ (resp. $i = 2$) are fixed. Further u_0, v_0 is the smallest pair (if such a pair exists) with $u_0 > 0$, $v_0 > 0$, $a_1u_0 = a_2v_0$, $R_1^{u_0} = R_2^{v_0}$. Therefore R^{u_0} and a_1u_0 depend only on the equations (1.11) with $i = 1, 2$, and are independent of the particular exponential families. The proof is now completed by induction on the number of equations (1.11).

CASE E. *Each equation* (1.11) *is exceptional of type* 2. Then infinitely many solutions are possible only if the ordered pair G, F is exceptional, and this is against the hypotheses of Propositions 2 and 3.

We now drop the assumption that f_0, g_0 be zero. We therefore also have to consider (1.11) with $i = 0$. The intersection of a linear family with this equation, i.e., with

$$f_0(x)\alpha_0^x = g_0(y)\beta_0^y, \tag{9.4}$$

yields $f_0(pt + a)\alpha_0^a = g_0(qt + b)\beta_0^b$. This either has only finitely many solutions t, or is satisfied identically, in which case the linear family satisfies (1.11) for

$0 \le i \le r$. Again there is only one solution family if p, q were chosen minimal with (1.7), (1.10). A comparison of magnitudes shows that the intersection of an exponential family with (9.4) is finite unless f_0, g_0 are constant. (It is essential that the coefficient a in Lemma 4 is nonzero). In the latter case, the equation $f_0 \alpha_0^{x(s)} = g_0 \beta_0^{y(s)}$ with $x(s), y(s)$ a family as in Lemma 4, is satisfied when s lies in the union of certain arithmetic progressions. But $x(s), y(s)$ with s in an arithmetic progression, runs again through an exponential family as in Lemma 4.

10. - The Equation

$$Af(x)\alpha^x + Bf(y)\alpha^y + Cf(z)\alpha^z = 0. \tag{10.1}$$

In this section, α, A, B, C are nonzero algebraic numbers, α is not a root of 1, and f is a nonzero polynomial with algebraic coefficients. We are interested in solutions $x, y, z \in \mathbb{Z}$ with

$$f(x)f(y)f(z) \neq 0. \tag{10.2}$$

LEMMA 5. *There is a finite set S such that the solutions x, y, z of* (10.1) *with* (10.2) *have $y - x \in S$ or $z - y \in S$ or $x - z \in S$.*

PROOF. We may suppose that f has leading coefficient 1. Let K be a number field containing α, A, B, C and the roots of f. We may consider K to be embedded in $\mathbb{C}$.

Since by (10.2) no summand in (10.1) vanishes, our solutions lie in $\Sigma(\mathcal{P})$, where $\mathcal{P}$ is the (trivial) partition of $\{1, 2, 3\}$ which consists of this set itself. Therefore $G(\mathcal{P})$ consists of the triples x, y, z with $\alpha^x = \alpha^y = \alpha^z$, i.e., with $x = y = z$. According to Laurent, solutions in $\Sigma(\mathcal{P})$ have

$$|y - x|, \ |z - x| \ll \log^+ |\mathbf{x}|, \tag{10.3}$$

where $\mathbf{x} = (x, y, z)$.

We begin with the case when $|\alpha| \neq 1$. Replacing α by α^{-1} and x, y, z by $-x, -y, -z$ if necessary, we may suppose that

$$|\alpha| > 1. \tag{10.4}$$

Without loss of generality, we consider solutions with

$$x \le y \le z. \tag{10.5}$$

When $|\mathbf{x}|$ is large, which we may suppose, then by (10.3), $|f(x)|, |f(y)|, |f(z)|$ will be of the same order of magnitude. In particular $|f(x)/f(z)| \leq 2$, $|f(y)/f(z)| \leq 2$, so that by (10.1), (10.4), (10.5),

$$|C||\alpha|^z \leq 2(|A| + |B|)|\alpha|^y.$$

Therefore $z - y$ will be bounded from above. Since it is nonnegative by (10.5), $z - y$ lies in a finite set.

It remains for us to deal with the case when

$$|\alpha| = 1. \tag{10.6}$$

We rewrite (10.1) as

$$f(x)(A\alpha^x + B\alpha^y + C\alpha^z) + B(f(y) - f(x))\alpha^y + C(f(z) - f(x))\alpha^z = 0.$$

When $|x|$ is large and $\deg f = \delta$, then $|f(x)| \gg |x|^\delta$, but

$$|f(y) - f(x)| \ll |y - x||x|^{\delta - 1} \ll |x|^{\delta - (1/2)}$$

by (10.3), and similarly for $|f(z) - f(x)|$. Therefore

$$|A\alpha^x + B\alpha^y + C\alpha^z| \ll |x|^{-1/2}. \tag{10.7}$$

More generally we will study solutions of

$$|A\alpha^x + B\alpha^y + C\beta^z| \ll |x|^{-1/2} \tag{10.8}$$

subject to (10.3), where α, β, A, B, C are algebraic, with $|\alpha| = |\beta| = 1$, $AB \neq 0$, and α not a root of 1. Setting $w_1 = \alpha^{y-x}$, $w_2 = \beta^z \alpha^{-x}$, we have

$$|Bw_1 + Cw_2 + A| \ll |x|^{-1/2},$$

or

$$-Cw_2 = Bw_1 + A + O(|x|^{-1/2}).$$

The same is true of the complex conjugates, so that

$$-\bar{C}/w_2 = \bar{B}/w_1 + \bar{A} + O(|x|^{-1/2}),$$

since $|w_1| = |w_2| = 1$. After multiplication,

$$|C|^2 = |A|^2 + |B|^2 + \bar{A}Bw_1 + A\bar{B}/w_1 + O(|x|^{-1/2}),$$

or

$$\bar{A}Bw_1^2 + (|A|^2 + |B|^2 - |C|^2)w_1 + A\bar{B} = O(|x|^{-1/2}).$$

The polynomial in w_1 on the left has algebraic roots ξ_1, ξ_2. Now w_1 is close to one of these roots, say to ξ_1. Then $|w_1 - \xi_1| \ll |x|^{-1/4}$ (in fact $\ll |x|^{-1/2}$ unless $\xi_1 = \xi_2$). We have $w_1 = \alpha^{y-x} = \alpha^t$ with $t = y - x$, therefore

$$(10.9) \qquad |\xi_1^{-1}\alpha^t - 1| \ll |x|^{-1/4}.$$

Here the left hand side vanishes for at most one value of t, say for $t = t_1$. When $t \neq t_1$, Baker's Theorem [1] says that the left-hand side is

$$> \exp(-c \log^+ t) = (\max(e, t))^{-c} = (\max(e, |y - x|)^{-c} \gg (\log |x|)^{-c}$$

by (10.3), with $c = c(\alpha, \xi_1)$. Comparison with (10.9) shows that x is bounded, therefore by (10.3) also y, therefore $x - y$, so that $y - x$ lies in a finite set. But when $y - x = t = t_1$, then $y - x$ is in fact fixed.

Call the equation (10.1) *exceptional* if $\alpha^u \in \mathbb{Z}$ for some $u \in \mathbb{Z}\backslash\{0\}$ and if f is of degree 1 with a rational root.

LEMMA 6. *Suppose* (10.1) *is not exceptional. Then with finitely many exceptions, the solutions of* (10.1), (10.2) *comprise a finite number* (*possibly zero*) *of linear* 1*-parameter families* $\mathcal{F}_1, \ldots, \mathcal{F}_\ell$ *with*

$$\mathcal{F}_j: \qquad x,\ y = x + h_j,\ z = x + k_j \qquad (x \in \mathbb{Z}).$$

The family $\mathcal{F}_j$ *occurs precisely if we have the following polynomial identity in* x:

$$(10.10) \qquad Af(x) + Bf(x + h_j)\alpha^{h_j} + Cf(x + k_j)\alpha^{k_j} = 0.$$

Such an identity cannot happen unless either $A + B + C = 0$ *and* $h_j = k_j = 0$, *or* $\deg f \leq 1$.

PROOF. By the preceding lemma and by symmetry, we may restrict ourselves to solutions with $z - y \in S$, where S is a finite set. We therefore may restrict ourselves to solutions with $z - y = m$, where m is fixed. Substitution of $z = y + m$ into (10.1) gives

$$(10.11) \qquad f(x)\alpha^x = g(y)\alpha^y$$

with

$$(10.12) \qquad g(y) = -(B/A)f(y) - (C/A)f(y + m)\alpha^m.$$

When $g = 0$, we get $f(x)\alpha^x = 0$, against (10.2). We may then suppose that $g \neq 0$, and we may apply the results of section 8.

We claim that equation (10.11) cannot be exceptional (in the sense of section 8), since $\deg g \leq \deg f$, so that exceptional would imply f to have a rational root γ, and g to be constant. But then by (10.12), f would be of degree ≤ 1, therefore of exact degree 1 with the root γ. Also if (10.11) were exceptional, then $\alpha^u \in \mathbb{Z}$ for some $u \in \mathbb{Z}\backslash 0$, so that (10.1) would be exceptional.

Since (10.11) is like (8.1) with $p = q = 1$, its solutions, with finitely many exceptions, are in families $y = x + h_j$ $(x \in \mathbb{Z})$, and then $x, y = x + h_j$, $z = x + k_j$ with $k_j = h_j + m$ $(x \in \mathbb{Z})$ are solutions of (10.1). We have $f(x) = g(x + h_j)\alpha^{h_j}$ identically in x, therefore (10.10).

As for the last assertion of the lemma, we note that (10.10) may be rewritten as

$$A'f(x) + B'f(x+h) + C'f(x+k) = 0 \tag{10.13}$$

with $h = h_j$, $k = k_j$, $A' = A$, $B' = B\alpha^h$, $C' = C\alpha^k$. If f is quadratic, say $f = (x+u)^2 + v$, the truth of (10.13) for f implies it for $f_1 = x^2 + v$. We obtain

$$A' + B' + C' = 0, \quad 2B'h + 2C'k = 0, \quad B'h^2 + C'k^2 = 0.$$

Since $B'C'(B' + C') = -A'B'C' \neq 0$, the last two of these 3 equations imply $h = k = 0$, and then the first gives $A+B+C = 0$. If $\deg f > 2$, a suitable derivative will be quadratic and again satisfy (10.13), so that again $h = k = A + B + C = 0$.

LEMMA 7. *Suppose* (10.1) *is exceptional. Then with finitely many exceptions, the solutions of* (10.1), (10.2) *may comprise a finite number of linear families* $\mathcal{F}_j$ *as in Lemma 6, as well as a finite number of exponential families*:

$$\mathcal{G}_j^x\colon \qquad x = c_j R^s + \gamma, \ \ y = x + as + b_j, \ \ z = x + as + b'_j \qquad (s \in \mathbb{N}),$$

as well as families $\mathcal{G}_j^y, \mathcal{G}_j^z$ *(obtained by a permutation in the roles of the variables). Here* $R \in \mathbb{Z}$, $R > 1$, R *is a rational power of* α, *and* $a \in \mathbb{Z}\backslash\{0\}$, $b_j, b'_j \in \mathbb{Z}$, $c_j \in \mathbb{Q}^*$ *such that* $x = x_j(s) \in \mathbb{Z}$ *for each* $s \in \mathbb{N}$.

PROOF. This time we cannot rule out that (10.11) is exceptional. (But this can happen for at most one value of m). Then it may have solution families $x = c_j R^s + \gamma$, $y = x + as + b_j$, giving rise to solution families $\mathcal{G}_j^x$ of (10.1).

The example below Supplement 3 to Theorem 2 illustrates our lemma.

11. - The System of Equations

$$Af_1(x)\alpha_1^x + Bf_1(y)\alpha_1^y + Cf_2(z)\alpha_2^z = 0, \tag{11.1}$$

$$Af_2(x)\alpha_2^x + Bf_2(y)\alpha_2^y + Cf_1(z)\alpha_1^z = 0. \tag{11.2}$$

Here α_1, α_2 are algebraic numbers such that $\alpha_1\alpha_2$ is a root of 1, but α_1, α_2 are not. Let $m > 0$ be given with $(\alpha_1\alpha_2)^m = 1$. Further A, B, C are nonzero algebraic numbers, and f_1, f_2 are nonzero polynomials with algebraic coefficients. We are interested in solutions where

$$\text{(11.3)} \qquad \text{no pair } f_1(x), f_2(x), \text{ or } f_1(y), f_2(y), \text{ or } f_1(z), f_2(z) \text{ is } 0, 0.$$

LEMMA 8. *With finitely many exceptions, the solutions of* (11.1), (11.2), (11.3) *comprise a finite number of linear* 1*-parameter families* $\mathcal{F}_1^z, \dots, \mathcal{F}_\ell^z$ *with*

$$\mathcal{F}_j^z: \qquad x = mt + a_j, \; y = mt + b_j, \; z = -mt + c_j \qquad (t \in \mathbb{Z}).$$

The family $\mathcal{F}_j^z$ *occurs precisely if we have the following two identities in* T:

$$\text{(11.4)} \qquad Af_1(T + a_j)\alpha_1^{a_j} + Bf_1(T + b_j)\alpha_1^{b_j} + Cf_2(-T + c_j)\alpha_2^{c_j} = 0,$$

$$\text{(11.5)} \qquad Af_2(T + a_j)\alpha_2^{a_j} + Bf_2(T + b_j)\alpha_2^{b_j} + Cf_1(-T + c_j)\alpha_1^{c_j} = 0.$$

PROOF. We will first prove that there is a finite set S such that all the solutions have

$$\text{(11.6)} \qquad x - y \in S.$$

For brevity write $\alpha_1 = \omega$, $\alpha_2^{-1} = \eta$, so that $\omega^m = \eta^m$, but ω and η are not roots of 1. Setting $z = -w$ we obtain

$$\text{(11.7)} \qquad Af_1(x)\omega^x + Bf_1(y)\omega^y + Cf_2(-w)\eta^w = 0,$$

$$\text{(11.8)} \qquad Af_2(x)\eta^{-x} + Bf_2(y)\eta^{-y} + Cf_1(-w)\omega^{-w} = 0.$$

These two equations involve 6 summands which we will symbolize by

$$\text{(11.9)} \qquad \begin{matrix} X & Y & W \\ X' & Y' & W'. \end{matrix}$$

In the notation of section 3, the system (11.7), (11.8) is a refinement of the equation

$$\text{(11.10)} \qquad X + Y + W + X' + Y' + W' = 0$$

of the type (11.10 $\mathcal{P}_0$), where $\mathcal{P}_0$ is the partition of the set (11.9) into the two sets $\{X, Y, W\}$, $\{X', Y', W'\}$. We have to study sets $\Sigma(\mathcal{P})$, where $\mathcal{P}$ is any refinement (not necessarily proper) of $\mathcal{P}_0$. We may discard partitions $\mathcal{P}$ containing the singletons $\{X\}, \{X'\}$, for then $(x, y, w) \in \Sigma(\mathcal{P})$ has $f_1(x) = f_2(x) = 0$, contrary to (11.3). Similarly we discard partitions containing both $\{Y\}, \{Y'\}$

or both $\{W\}, \{W'\}$. If $\mathcal{P}$ contains the three singletons $\{X\}, \{Y\}, \{W\}$, then $(x, y, z) \in \Sigma(\mathcal{P})$ has $f_1(x) = f_1(y) = f_2(-w) = 0$, so that $\Sigma(\mathcal{P})$ is finite. We may therefore suppose that $\mathcal{P}$ does not contain all 3 of $\{X\}, \{Y\}, \{W\}$, or of $\{X'\}, \{Y'\}, \{W'\}$. We claim that for the remaining partitions $\mathcal{P}$, the group $G(\mathcal{P})$ consists of triples p, q, s with

$$p = q = s. \tag{11.11}$$

If $\{X, Y, W\} \in \mathcal{P}$, then $(p, q, s) \in G(\mathcal{P})$ has $\omega^p = \omega^q = \eta^s$, which in view of $\omega^m = \eta^m$ gives (11.11). We may thus suppose that $\mathcal{P}$ splits $\{X, Y, W\}$ into a pair and a singleton, and the same supposition may be made of $\{X', Y', W'\}$. As an example, take the case when $\mathcal{P}$ consists of $\{X\}, \{Y, W\}, \{Y'\}, \{X', W'\}$. Then $(p, q, s) \in G(\mathcal{P})$ has $\omega^q = \eta^s$ and $\eta^{-p} = \omega^{-s}$, therefore (11.11).

By (3.2) and our characterisation of $G(\mathcal{P})$, elements of $\Sigma(\mathcal{P})$ have

$$|x - y|, |x - w| \ll \log^+ |\mathbf{x}|, \tag{11.12}$$

where $\mathbf{x} = (x, y, w)$. Therefore $|f_1(x)|$, $|f_1(y)|$, $|f_1(-w)|$ are of the same order of magnitude, and the same goes for $|f_2(x)|$, $|f_2(y)|$, $|f_2(-w)|$. We have $|\omega| = |\eta|$.

Now suppose that $|\omega| \neq 1$, say

$$|\omega| = |\eta| > 1. \tag{11.13}$$

Relations (11.7), (11.8) yield

$$\begin{aligned} C^2 f_1(-w) f_2(-w)(\eta/\omega)^w &= A^2 f_1(x) f_2(x)(\omega/\eta)^x + B^2 f_1(y) f_2(y)(\omega/\eta)^y \\ &+ AB f_1(x) f_2(y)\omega^x \eta^{-y} + AB f_1(y) f_2(x)\omega^y \eta^{-x}. \end{aligned} \tag{11.14}$$

The left-hand side and the first two summands of the right hand side are of the same order of magnitude. The third and fourth summands on the right have this magnitude, times extra factors $|\omega^x \eta^{-y}| = |\omega|^{x-y}$, $|\omega^y \eta^{-x}| = |\omega|^{y-x}$ respectively. Therefore these factors need to remain bounded, so that $|x - y|$ will be bounded, and $x - y$ will lie in a finite set $\mathcal{S}$.

Next, suppose that

$$|\omega| = |\eta| = 1. \tag{11.15}$$

Say $\deg f_2 \leq \deg f_1$. We may suppose that f_1 has leading coefficient 1, say $f_1(x) = x^\delta + \cdots$, and $f_2(-w) = \rho w^\delta + \cdots$, where possibly $\rho = 0$. (11.7) yields

$$f_1(x)(A\omega^x + B\omega^y + C\rho\eta^w) + B(f_1(y) - f_1(x))\omega^y + C(f_2(-w) - \rho f_1(x))\eta^w = 0.$$

Here $|f_1(x)| \gg |x|^\delta$, but by (11.12), $|f_1(y) - f_1(x)| \ll |x|^{\delta-(1/2)}$, and

$$|f_2(-w) - \rho f_1(x)| = |\rho w^\delta - \rho x^\delta + \text{terms of lower degree }| \ll |x|^{\delta-(1/2)}.$$

We obtain

$$|A\omega^x + B\omega^y + C\rho\eta^w| \ll |x|^{-1/2}.$$

This is a relation like (10.8), and (11.12) corresponds to (10.3). Therefore indeed $x - y$ lies in a finite set S, and (11.6) has been established.

It will therefore suffice to study solutions of (11.1), (11.2) with $y = x + h$ where h is fixed. Substitution yields

$$\begin{aligned} g_1(x)\alpha_1^x &= f_2(z)\alpha_2^z \\ g_2(x)\alpha_2^x &= f_1(z)\alpha_1^z \end{aligned} \tag{11.16}$$

with $g_i(x) = -(A/C)f_i(x) - (B/C)f_i(x+h)\alpha_i^h$ $(i = 1, 2)$. This is (except for notation) a system of equations such as (1.11), with $r = 2$ and $\beta_1 = \alpha_2$, $\beta_2 = \alpha_1$. Here $\alpha_i^m = \beta_i^{-m}$ $(i = 1, 2)$, so that (1.7) holds with $p = m$, $q = -m$. The system (11.16) is not exceptional (in the sense of Proposition 2), since $|\alpha_2| = 1/|\alpha_1|$, against condition (c). Therefore, according to Proposition 2, with finitely many exceptions, the solutions of (11.16) make up families of the type $x = mt + a$, $z = -mt + c$ $(t \in \mathbb{Z})$, and with $y = x + h$ this yields families $\mathcal{F}_j^z$ as in the Lemma. It is trivial to check that the family $\mathcal{F}_j^z$ occurs precisely when the identities (11.4), (11.5) hold.

12. - Proof of Theorem 2 and its Supplements

We have to solve systems (1.15) or (1.16x), (1.16y), or (1.16z). By Lemmas 6, 7, each equation (1.15) is solved by families $\mathcal{F}_j$ or $\mathcal{G}_j^x$ or $\mathcal{G}_j^y$ or $\mathcal{G}_j^z$. The arguments of section 9 show that families of distinct type have finite intersection, and families of like type have finite intersection, or an intersection which consists of a finite set, plus a number of families which are again of the same type. The system (1.16z) consists of pairs of equations as in section 11, so that its solutions make up families $\mathcal{F}_j^z$. The intersection of such families consists of a finite set plus a finite number of families of the type $\mathcal{F}^z$. This proves Theorem 2.

We have been a little careless. E.g., the first equation (1.15) may have solutions with $f_1(x) = 0$, which are not covered by Lemmas 6, 7. It is easily seen that there are only finitely many such solutions, except perhaps those with $f_1(x) = \cdots = f_r(x) = 0$, which are improper.

We turn to *Supplement* 1. It is trivial to check that $\mathcal{F}_j$ is a solution family if and only if the identities (1.17) hold. We know from Lemma 6 that this may happen only if either $A + B + C = 0$ and $h_j = k_j = 0$, or if each $\deg f_i \leq 1$. In the last case we claim that at most one f_i can be non-constant. For if there were two, say $f_i = x + u_i$ $(i = 1, 2)$, then (1.17) gives

$$A(x + u_i) + B(x + u_i + h)\alpha_i^h + C(x + u_i + k)\alpha_i^k = 0 \quad (i = 1, 2),$$

so that $A + B\alpha_i^h + C\alpha_i^k = 0$, $Bh\alpha_i^h + Ck\alpha_i^k = 0$. Unless $h = k = 0$ (whence $A + B + C = 0$) we get $\alpha_i^h = Ak/(B(h-k))$, so that $(\alpha_1/\alpha_2)^h = 1$, contradicting non-degeneracy.

As for *Supplement* 2, (1.16z) (or (1.16x), (1.16y)) can only arise when F is symmetric, so that the families $\mathcal{F}^x, \mathcal{F}^y, \mathcal{F}^z$ can only occur in this case. When $(\alpha_i\alpha_{i+1})^m = 1$, then each pair of equations in (1.16z) can be solved by families $\mathcal{F}^z$ as in Lemma 8, and hence so can the whole system (1.16z), with the same value of m.

Finally, consider *Supplement* 3. We need to look at intersections of families $\mathcal{G}_1^x, \dots, \mathcal{G}_r^x$, where $\mathcal{G}_i^x$ is a solution family of the i-th equation in (1.15). Then in $\mathcal{G}_i^x$, $x(s) = c_i R_i^s + \gamma_i$, and the arguments of section 9 show that these r families have an infinite intersection only if $\gamma_1 = \cdots = \gamma_r = \gamma$, say, and if the R_i are rational powers of each other. Therefore there is a natural N (a power of each of the R_i) which is an integral power of each α_i. Furthermore either each $|\alpha_i| > 1$, or each $|\alpha_i| < 1$. By Lemma 7, each f_i has to be linear with root γ. Therefore families $\mathcal{G}^x$ can occur in Theorem 2 only if F is exceptional. By definition, the families $\mathcal{G}^x$ of Lemma 7 were such that their values are in $\mathbb{Z}$ for each $s \in \mathbb{N}$, and the same is therefore true of the families $\mathcal{G}^x$ in Theorem 2.

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Universität Ulm, Germany
University of Colorado