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# Uniform Foliation Associated with the Hamiltonian System $\mathcal{H}_n$

HIRONOBU KIMURA

*Dedicated to Professor Tosihusa Kimura on his 60-th birthday*

## 0. - Introduction

In the last decade, the Painlevé equations are objects of many reserches. The Painlevé equations are six non-linear second order differential equations whose solutions are free from movable branch points (branch points whose positions change under the variation of integration constants). They are discovered by P. Painlevé [Pain] in the aim of defining new transcendental functions. We denote by $\mathrm{P_J}$ ($\mathrm{J} = \mathrm{I}, \cdots, \mathrm{VI}$) the equations of Painlevé. For example, $\mathrm{P_{VI}}$ is

$$\frac{d^2q}{dt^2} = \frac{1}{2}\left(\frac{1}{q} + \frac{1}{q-1} + \frac{1}{q-t}\right)\left(\frac{dq}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{q-t}\right)\left(\frac{dq}{dt}\right)$$
$$- \frac{q(q-1)(q-t)}{t^2(t-1)^2}\left[\alpha + \beta\frac{t}{q^2} + \gamma\frac{t-1}{(q-1)^2} + \delta\frac{t(t-1)}{(q-t)^2}\right],$$

where $\alpha$, $\beta$, $\gamma$ and $\delta$ are complex constants.

The adjective "new" implies that these transcendental functions are irreducible over the classical transcendental functions, that is, they cannot be expressed by using the exponential function, algebraic functions, functions which satisfy linear differential equations with algebraic coefficients, etc. (for the precise definition of irreducibility, see [Ume]).

Inspired by the work of Painlevé, certain mathematicians tried to generalize the results of Painlevé in two direction.

Firstly, in [Bur], [Chaz], [Gar.1], the authors have tried to classify all the algebraic ordinary differential equations of order greater than 2 free from movable branch point using the $\alpha$-method initiated by Painlevé (we refer the reader to [Inc] and other literatures for the detail).

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In other direction, many nonlinear differential equations and completely integrable Pfaffian systems are obtained using the theory of monodromy preserving deformations for linear ordinary differential equations. Non-existence of movable branch points are established for a large part of these equations by [Miw], [Malg]. This property is called the *Painlevé property*. Nevertheless, we have not yet arrived at the state where we can say that these equations define new special functions, because we know very little about the properties of their solutions. For the Painlevé equations $P_J$, a number of articles are devoted to the study of their general solutions.

In [Gar.2], [KimH.2], [Tak.2], [Shim.1], [YosS], they studied the local problem, i.e., the study of solutions at fixed singular points (a singular point is said to be *fixed* when it is not a movable one).

On the other hand, in [Okm.1], K. Okamoto has studied the equations $P_J$ from a geometric point of view. For each equation $P_J$, he has constructed explicitly a locally trivial complex analytic fiber space in which the equation $P_J$ defines a uniform foliation (see §1 for the definition).

The purpose of this paper is to realize the geometric interpretation, which is established for $P_J$ in [Okm.1], for a completely integrable Pfaffian system $\mathcal{H}_n$ (see below) enjoying the Painlevé property and generalizing $P_{VI}$.

The system $\mathcal{H}_n$ is a Hamiltonian system with independent variables $t = (t^1, \dots, t^n)$:

$$\mathcal{H}_n: \quad \begin{cases} \mathrm{d}q^i = \sum_j \{H_j, q^i\}\mathrm{d}t^j, \\ \qquad\qquad\qquad\qquad\qquad (1 \le i \le n), \\ \mathrm{d}p^i = \sum_j \{H_j, p^i\}\mathrm{d}t^j, \end{cases}$$

with Hamiltonians:

$$H_i := \frac{1}{e(t^i)}\left[\sum_{j,k} E_{ijk}(t,q)p^j p^k - \sum_j F_{ij}(t,q)p^j + \kappa q^i\right],$$

where

$$e(x) = x(x-1),$$

$\{\cdot,\cdot\}$ stands for the Poisson bracket

$$\{f,g\} = \sum_i \left(\frac{\partial f}{\partial p^i}\frac{\partial g}{\partial q^i} - \frac{\partial g}{\partial p^i}\frac{\partial f}{\partial q^i}\right),$$

and $E_{ijk}$, $F_{ij} \in \mathbb{C}(t)[q]$ are given by

$$E_{ijk} := \begin{cases} q^i q^j q^k, & \text{if } i,\ j,\ k \text{ distinct}, \\ q^i q^j (q^j - R_{ji}), & \text{if } i \neq j = k, \\ q^i q^k (q^k - R_{ki}), & \text{if } i = j \neq k, \\ q^i (q^i - 1)(q^i - t^i) - \sum_l^i S_{il} q^i q^l, & \text{if } i = j = k, \end{cases}$$

$$F_{ij} := \begin{cases} A q^i q^j - \theta_i R_{ij} q^j - \theta_j R_{ji} q^i, & \text{if } i \neq j, \\ (\theta_{n+1} - 1) q^i (q^i - 1) + \theta_{n+2} q^i (q^i - t^i) & \\ \qquad + \theta_i (q^i - 1)(q^i - t^i) & \\ \qquad + \sum_k^i \{\theta_k q^i (q^i - R_{ik}) - \theta_i S_{ik} q^k\}, & \text{if } i = j. \end{cases}$$

Here the symbol $\sum_k^i$ stands for the summation over $k = 1, \dots, i-1,\ i+1, \dots, n$ and

$$\begin{aligned} R_{ij} &:= \frac{t^i(t^j - 1)}{t^j - t^i}, \qquad S_{ij} := \frac{t^i(t^i - 1)}{t^i - t^j}, \\ A &:= \sum_{i=1}^{n+2} \theta_i - 1, \qquad \kappa = \frac{1}{4}(A^2 - \theta_\infty^2). \end{aligned} \tag{0.1}$$

Note that $E_{ijk}$, $F_{ij} \in \mathbb{C}(t)[q]$ are symmetric with respect to $i$, $j$, $k$:

$$E_{ijk} = E_{ikj} = E_{kji}, \qquad F_{ij} = F_{ji}, \tag{0.2}$$

and that $R_{ij}$ and $S_{ij}$ satisfy

$$R_{ij} + R_{ji} = 1, \qquad R_{ij} + S_{ji} = t^i. \tag{0.3}$$

In case $n = 1$, the Hamiltonian $H := H_1$ of the system $\mathcal{H}_1$ is

$$H = \frac{1}{e(t)}[q(q-1)(q-t)p^2 - \{(\theta_2 - 1)q(q-1) + \theta_3 q(q-t) + \theta_1 (q-1)(q-t)\}p + \kappa q]$$

and this system is equivalent to the sixth Painlevé equation in the sense that $P_{\mathrm{VI}}$ is obtained from $\mathcal{H}_1$ by eliminating $p$.

The independent variables $t$ of the system $\mathcal{H}_n$ is considered in

$$B := \{t \in \mathbb{C}^n \mid t^i \neq 0, 1, t^j \ (i \neq j)\}. \tag{0.4}$$

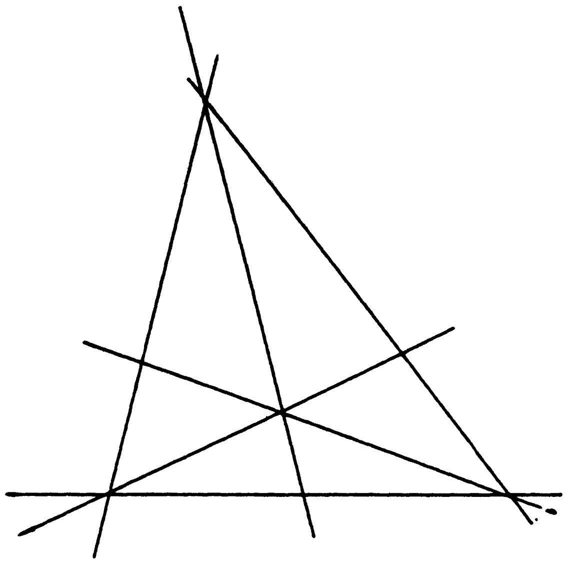

*Fig. 1*

In the case $n = 2$, the singularity $\Xi = \mathbb{P}^n \backslash B$ of $\mathcal{H}_n$ gives the line configuration drawed in Figure 1.

By virtue of the celebrated theorem of Miwa [Miw] and Malgrange [Malg] and the fact that the system $\mathcal{H}_n$ is equivalent to $2\times 2$ Schlesinger system ([YIKS] §3.6.4), we obtain the following result.

PROPOSITION 0.1. [KO.2] *The system $\mathcal{H}_n$ is completely integrable and it enjoys the Painlevé property. Moreover, any solution of $\mathcal{H}_n$ is continued meromorphically on the universal covering space $\tilde{B}$ of $B$.*

In order to formulate precisely a problem to be considered, we prepare several terminologies.

Let $\mathcal{P} := (E, \pi, B)$ be a locally trivial complex analytic fiber space consisting of complex manifolds $E$ of dimension $m+n$ and $B$ of dimension $n$, and of a surjective holomorphic mapping $\pi : E \to B$. The manifolds $E$ and $B$ are called the *total space* and the *base* of $\mathcal{P}$, respectively. Let $\mathcal{F}$ be a nonsingular complex analytic foliation of codimension $m$ defined in the total space $E$ of $\mathcal{P}$. Locally, the foliation $\mathcal{F}$ is defined by $m$ linearly independent holomorphic 1-forms.

DEFINITION 0.2. A leaf of a nonsingular foliation $\mathcal{F}$ in $\mathcal{P}$ is called *T-leaf* if it is transversal to the fibers of $\mathcal{P}$. A leaf of $\mathcal{F}$ is said to be *vertical* when it is entirely contained in some fiber of $\mathcal{P}$.

DEFINITION 0.3. A nonsingular complex analytic foliation $\mathcal{F}$ in $\mathcal{P} = (E, \pi, B)$ is said to be *uniform* for $\mathcal{P}$ if i) any leaf of $\mathcal{F}$ is $T$-leaf and ii) any path $\gamma$ in $B$ can be lifted in a leaf if a point is given in the intersection of the leaf and the fiber over the origin of $\gamma$.

Consider, in particular, a completely integrable Pfaffian system of the form

$$\text{(P):} \qquad \omega_i := \mathrm{d}x^i - \sum_{1 \le j \le n} G_{ij}(x,t)\mathrm{d}t^j = 0, \qquad (1 \le i \le m),$$

where $G_{ij} \in \mathbb{C}(t)[x]$. Let $\Xi \subset \mathbb{P}^n(t)$ be the set of fixed singular points of the system (P), i.e., $\Xi$ is a union of a projection image of pole divisors of $G_{ij}$'s to the $t$-space and of a hyperplane at infinity in $\mathbb{P}^n(t)$. Set $B := \mathbb{P}^n(t)\backslash\Xi$. Suppose that the system (P) enjoys the property stated in Proposition 0.1, and consider the system (P) in the total space of a trivial bundle $\mathcal{Q}^0 := (\mathbb{C}^m \times B, \pi, B)$. The system (P) defines a nonsingular complex analytic foliation $\mathcal{F}^0$ in $\mathbb{C}^m \times B$ whose leaves are all $T$-leaves. However, $\mathcal{F}^0$ is not uniform for $\mathcal{Q}^0$ in general because the second condition for the uniformity is not necessarily fulfilled.

DEFINITION 0.4. A fiber space $\mathcal{P} = (E, \pi, B)$ is called a P-*uniform* fibration (P-fibration) for the system (P) if

(i) $\mathcal{Q}^0 = (\mathbb{C}^m \times B, \pi, B)$ is a fiber subspace of $\mathcal{P}$ such that $E\backslash\mathbb{C}^m \times B$ is a proper analytic subset of $E$,

(ii) the foliation $\mathcal{F}^0$ prolongs to a uniform folation $\mathcal{F}$ for $E$,

(iii) any leaf of $\mathcal{F}$ in the total space $E$ intersects with the total space $\mathbb{C}^m \times B$ of $\mathcal{Q}^0$.

REMARK 0.5. Let $\mathcal{P} = (E, \pi, B)$ be a P-fibration for the system (P) and let $\varpi : \tilde{B} \to B$ be the universal covering space of $B$. Then the system (P) defines a uniform foliation $\tilde{\mathcal{F}}$ on the induced fiber space $(\varpi^* E, \pi, \tilde{B})$. Each leaf of the foliation $\tilde{\mathcal{F}}$ is mapped biholomorphically to the base $\tilde{B}$. Furthermore, the general solution of (P) defines an isomorphism between any two fibers $\pi^{-1}(t)$ and $\pi^{-1}(t')$ of the bundle $\mathcal{P}$.

DEFINITION 0.6. A fiber of the P-fibration for the system (P) is called a *space of initial conditions of* (P).

Heuristically speaking, in order to construct a P-fibration for the system (P), we study the behavior of its integral manifolds when they tend to "infinity" as we continue them analytically. To this end, we consider a locally trivial complex analytic fiber space $\mathcal{Q} = (F, \pi, B)$ whose fiber is an algebraic manifold and is a compactification of $\mathbb{C}^m$. Since the system (P) is algebraic, it extends naturally to the system on $F$ given by holomorphic 1-forms, and it defines, in general, a singular foliation (see [R]) on $F$ whose singular locus is the set of points where the 1-forms are linearly dependent. The system in $F$ thus obtained is called the *prolonged system* of (P).

Let $D$ be the singularity of the prolonged system in $F$ and let $\mathcal{F}$ be the foliation in $F \backslash D$ defined by the prolonged system of (P). Note that if a point $p \in D$ is contained in a closure of some $T$-leaf which pass through a point in a total space $\mathbb{C}^m \times B$ of $\mathcal{Q}^0$, then, by virtue of the Painlevé property assumed above for (P), a solution of the prolonged system which gives this leaf is continued meromorphically to this point. This observation leads to the following definition.

DEFINITION 0.7. A point in $p \in D$ is called an *accessible singular point* of the foliation $\mathcal{F}$ (or of the prolonged system) if it is contained in a closure of a $T$-leaf of $\mathcal{F}$.

Now we propose to study the following problem:

PROBLEM 0.8. *Construct a space of initial conditions for the Hamiltonian system* $\mathcal{H}_n$.

To present our problem more concretely, we consider, as an example, the Riccati equation:

$$\frac{dy}{dt} = a(t)y^2 + b(t)y + c(t). \tag{0.5}$$

where $a(t)$, $b(t)$ and $c(t)$ are holomorphic in a simply connected domain $B$ of $\mathbb{C}$.

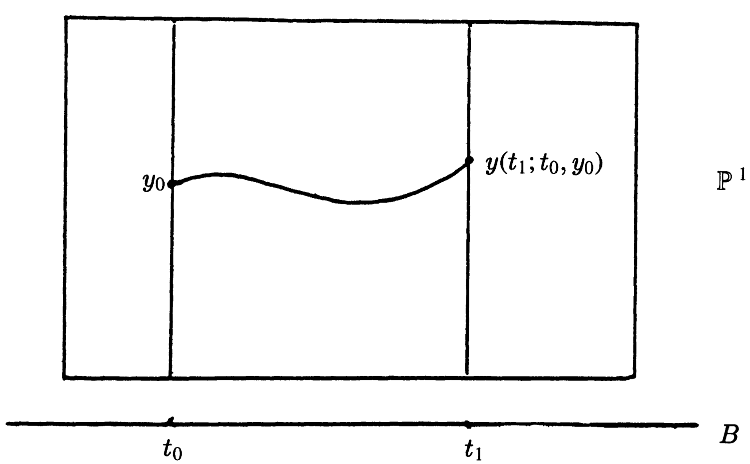


*Fig. 2*

If we consider a change of variable:

$$z := \frac{1}{y}, \tag{0.6}$$

then the equation of Riccati is transformed into

$$\frac{\mathrm{d}z}{\mathrm{d}t} = -\{a(t) + b(t)z + c(t)z^2\}. \tag{0.7}$$

We can interpret this procedure geometrically in the following manner. Given the Riccati equation (0.5) in a trivial bundle:

$$\mathcal{Q}^0 = (\mathbb{C} \times B, \pi, B),$$

then the fact that the equation (0.7) is obtained from (0.5) by the change of variable (0.6) implies that the equation (0.5) can be prolonged to a differential equation without singularity in the total space of fiber bundle: $\mathcal{P} = (\mathbb{P}^1 \times B, \pi, B)$ and this equation defines a foliation in the total space $\mathbb{P}^1 \times B$ of $\mathcal{P}$ whose leaves are transversal to the fibers.

Moreover, the Painlevé propery of the Riccati equations is deduced from the compactness of fibers by virtue of the theorem of Ehressmann. That is, if a point $(y_0, t_0) \in \mathbb{P}^1 \times B$ and a solution $y(t; t_0, y_0)$ of the equation (0.6) with the initial condition $y(t_0; t_0, y_0) = y_0$ are given, then this solution is continued meromorphically along any path in $B$ and gives, for $t_1 \in B$ arbitrary, the biholomorphic correspondence between two fibers $\pi^{-1}(t_0) \simeq \mathbb{P}^1$ and $\pi^{-1}(t_1) \simeq \mathbb{P}^1$:

$$\begin{array}{ccc} \pi^{-1}(t_0) & \rightarrow & \pi^{-1}(t_1) \\ y_0 & \mapsto & y(t_1; t_0, y_0). \end{array}$$

Hence we can say that $\mathbb{P}^1$ parametrizes all the solutions of the Riccati equation.

In order to study the same problem for the system $\mathcal{H}_n$, we find a manifold of dimension $2n$ which is a $2n$-dimensional analogue of the Hirzebruch surface $\Sigma_0$ (see Remark 1.4). We will see that this generalization is obtained in a natural way from the symmetries of the system $\mathcal{H}_n$.

In Section 1, we study the symmetries of the system $\mathcal{H}_n$. The structure of the group of symmetries suggests us to define a generalization $\Sigma_\alpha$ of Hirzebruch surface to $2n$ dimensional manifold and to prolong the system $\mathcal{H}_n$ to the total space of trivial bundle $\Sigma_\alpha \times B \to B$. After reviewing on a monoidal transformation we state the main theorem. In Section 2, we describe explicitly how the system $\mathcal{H}_n$, given in the total space of $(\mathbb{C}^{2n} \times B, \pi, B)$, is prolonged to a system $\mathcal{H}_n^{(0)}$ in $\Sigma_\alpha \times B$. Moreover, the set of singular points of $\mathcal{H}_n^{(0)}$ is studied. Section 3 is devoted to the determination of accessible singularity of the prolonged system $\mathcal{H}_n^{(0)}$. And finally, in Section 4, we complete the demonstration of the theorem.

## 1. - Symmetries of $\mathcal{H}_n$ and the main theorem

### *1.1. Symmetries of $\mathcal{H}_n$*

Consider a Pfaffian system:

$$\mathrm{P}: \qquad \mathrm{d}x^i = \sum_{1\le j\le n} G_{ij}(x,t,\theta)\mathrm{d}t^j, \qquad (1 \le i \le m),$$

where $G_{ij}(x,t,\theta)$ are rational functions in $(x,t) = (x^1,\cdots,x^m,t^1,\cdots,t^n)$ depending on parameters $\theta \in \mathbb{C}^N$. The system P with a parameter $\theta$ is denoted by $\mathrm{P}(\theta)$. For a birational transformation $S:(x,t)\to(x',t')$, we denote by $S\cdot\mathrm{P}(\theta)$ the system of differential equations in the variables $(x',t')$ obtained from $\mathrm{P}(\theta)$ by the transformation $S$.

DEFINITION 1.1. A *symmetry* for the system P is a pair $\sigma = (S,h)$ of a birational transformation $S:(x,t)\to(x',t')$ and an affine transformation $h:\mathbb{C}^N\to\mathbb{C}^N$ such that $S\cdot\mathrm{P}(\theta) = \mathrm{P}(h(\theta))$.

Let $\sigma = (S,h)$ and $\sigma' = (S',h')$ be symmetries for the system P. The product $\sigma\cdot\sigma'$ and the inverse $\sigma^{-1}$ to $\sigma$ defined by $\sigma\cdot\sigma' = (S\circ S', h\circ h')$ and $\sigma^{-1} = (S^{-1},h^{-1})$, are again symmetries of P.

Let $V = \{\theta = (\theta_1,\cdots,\theta_{n+2},\theta_\infty) \in \mathbb{C}^{n+3}\}$ be the space of parameters for the system $\mathcal{H}_n$, and let $\mathcal{H}_n(\theta)$ be the system $\mathcal{H}_n$ with fixed parameters $\theta$. Let us define the birational transformation $S_m:(q,p,t)\to(q',p',t')$ as follows:

$$S_m(m=1,\cdots,n): \qquad q'^i = \begin{cases} \dfrac{q^i}{R_{im}} & (i\neq m), \\[2ex] -\dfrac{t^m}{t^m-1}(g_t-1) & (i=m), \end{cases}$$

$$p'^i = \begin{cases} R_{im}\left(p^i - \dfrac{t^m}{t^i}p^m\right) & (i\neq m), \\[2ex] -(t^m-1)p^m & (i=m), \end{cases}$$

$$t'^i = \begin{cases} \dfrac{t^m-t^i}{t^m-1} & (i\neq m, n+1), \\[2ex] \dfrac{t^m}{t^m-1} & (i=m), \end{cases}$$

$$S_{n+1}: \qquad \begin{aligned} q'^i &= \frac{q^i}{t^i}, \\ p'^i &= t^i p^i, \\ t'^i &= \frac{1}{t^i}, \end{aligned}$$

$$S_{n+2}: \qquad \begin{aligned} q'^i &= \frac{q^i}{g_1 - 1}, \\ p'^i &= (g_1 - 1)\left(p^i + \alpha - \sum_a q^a p^a\right), \\ t'^i &= \frac{t^i}{t^i - 1}, \end{aligned}$$

where $R_{im}$ are given by (0.1) and

$$g_1 := \sum_a q^q, \qquad g_t := \sum_a \frac{q^a}{t^a}.$$

Let $h_m : V \to V$ be affine transformations defined by

$$\begin{aligned} h_m(m = 1, \cdots, n) &: (\theta_1, \cdots, \theta_m, \cdots, \theta_n, \theta_{n+1}, \theta_{n+2}, \theta_\infty) \\ &\mapsto (\theta_1, \cdots, \theta_{m-1}, \theta_{n+1}, \theta_{m+1}, \cdots, \theta_n, \theta_m, \theta_{n+2}, \theta_\infty), \\ h_{n+1} &: (\theta_1, \cdots, \theta_n, \theta_{n+1}, \theta_{n+2}, \theta_\infty) \mapsto (\theta_1, \cdots, \theta_n, \theta_{n+2}, \theta_{n+1}, \theta_\infty), \\ h_{n+2} &: (\theta_1, \cdots, \theta_n, \theta_{n+1}, \theta_{n+2}, \theta_\infty) \mapsto (\theta_1, \cdots, \theta_n, \theta_{n+1}, \theta_\infty, \theta_{n+2}). \end{aligned}$$

Let $L := \langle \sigma_1, \cdots, \sigma_{n+2} \rangle$ be the group generated by $\sigma_m = (S_m, h_m)$, $1 \leq m \leq n+2$. Then we have

PROPOSITION 1.2. [KimH.5], [YIKS] *The group $L$ is a group of symmetries for the system $\mathcal{H}_n$ and is isomorphic to the symmetric group $\mathrm{S}_{n+3}$ on $n+3$ letters.*

### *1.2. Compact manifold generalizing Hirzebruch's surface*

We introduce a compact complex manifold of dimension $2n$, which is a $\mathbb{P}^n$-bundle over $\mathbb{P}^n$. This manifold will serve as fibers of the fiber bundle over $B$ into which the system $\mathcal{H}_n$ is prolonged.

Let $z := (z^0, \cdots, z^n)$ be the homogeneous coordinates of $\mathbb{P}^n$, $(U_i, \varphi_i)$ be the $i$-th affine coordinate chart, i.e., $U_i := \{z \in \mathbb{P}^n \mid z^i \neq 0\}$ and $\varphi_i : U_i \to \mathbb{C}^n$ is defined by

$$\begin{aligned} \varphi_i(z) &= (z_i^0, \cdots, z_i^{i-1}, z_i^{i+1}, \cdots, z_i^n) \\ &:= \left(\frac{z^0}{z^i}, \cdots, \frac{z^{i-1}}{z^i}, \frac{z^{i+1}}{z^i}, \cdots, \frac{z^n}{z^i}\right). \end{aligned}$$

We make use of the following convention: the symbols $\mathbb{P}^n(t)$, $\mathbb{P}^n(z)$ and $\mathbb{P}^n(\xi)$ denote the complex projective space of dimension $n$ with homogeneous coordinates $t$, $z$ and $\xi$, respectively.

First we construct a non-compact complex manifold which is an affine bundle over $\mathbb{P}^n(z)$ with fibers isomorphic to $\mathbb{C}^n$ and will be compactified to give a generalization of the Hirzebruch's surface ([Hirz], [Kod]).

Set $W_{ii} = U_i \times \mathbb{C}^n$ $(1 \leq i \leq n)$. Then $(W_{ii}, \pi, U_i)$, with the projection $\pi$ to the first factor, is a trivial bundle over $U_i$. Let

$$(z_i, \varsigma_i) = (z_i, \varsigma_i^0, \cdots, \varsigma_i^{i-1}, \varsigma_i^{i+1}, \cdots, \varsigma_i^n)$$

be a system of coordinates in $W_{ii}$. In $\bigcup_{0 \leq i \leq n} W_{ii}$, we define an equivalence relation as follows: two points $(z_i, \varsigma_i) \in W_{ii}$ and $(z_j, \varsigma_j) \in W_{jj}$ are equivalent when

$$z_j^a = \begin{cases} z_i^a / z_i^j & (a \neq i, j) \\ 1/z_i^j & (a = i), \end{cases} \tag{1.1}$$

$$\varsigma_j^a = \begin{cases} z_i^j \varsigma_i^a & (a \neq i, j) \\ z_i^j \left(\alpha - \sum_b^i z_i^b \varsigma_i^b\right) & (a = i), \end{cases} \tag{1.2}$$

where $\alpha$ is a complex constant. Let $\Sigma_\alpha^{0n}$ be a complex manifold defined as a quotient space of $\bigcup_{0 \leq i \leq n} W_{ii}$ by the above equivalence relation.

Next we define a complex manifold obtained by compactifying fibers $\mathbb{C}^n$ of $\Sigma_\alpha^{0n}$ by $\mathbb{P}^n$. Consider a trivial $\mathbb{P}^n$-bundle $W_i := U_i \times \mathbb{P}^n(\xi_i)$ over $U_i$. Let $(z_i, \xi_i)$ be the coordinates in $W_i$, where $\xi_i = (\xi_i^0, \cdots, \xi_i^n)$ is the homogeneous coordinates of $\mathbb{P}^n(\xi_i)$.

In $\bigcup_{0 \leq i \leq n} W_i$, we define an equivalence relation as follows: points $(z_i, \xi_i) \in$

$W_i$ and $(z_j, \xi_j) \in W_j$ are identified when we have (1.1) and

$$\xi_j^a \xi_i^i = \begin{cases} z_i^j \xi_i^a \xi_j^j, & (a \neq i), \\ z_i^j \xi_j^j \left( \alpha \xi_i^i - \sum_b z_i^b \xi_i^b \right), & (a = i). \end{cases} \tag{1.3}$$

Let $\Sigma_\alpha^n$ be the quotient space of $\bigcup_{0 \leq i \leq n} W_i$ by the relation defined above. The set $\Sigma_\alpha^n$ has a natural structure of $\mathbb{P}^n$-bundle over $\mathbb{P}^n$ and will be called the *Hirzebruch manifold.* If there is no fear of confusion, we write $\Sigma_\alpha$ and $\Sigma_\alpha^0$ instead of $\Sigma_\alpha^n$ and $\Sigma_\alpha^{0n}$, respectively.

Set

$$W_{ik} = U_i \times \{\xi_i \in \mathbb{P}^n \mid \xi_i^k \neq 0\} \subset W_i \subset \Sigma_\alpha.$$

Note that $\{W_{ik}\}_{0 \leq i,k \leq n}$ forms an atlas consisting of affine charts of Hirzebruch manifold and $\Sigma_\alpha^0$ is an open submanifold of $\Sigma_\alpha$ with coordinate covering $\{W_{ii}\}_{0 \leq i \leq n}$.

REMARK 1.3. In each chart $W_{ii}$ of $\Sigma_\alpha^0$, there exists a natural symplectic structure $\Sigma_a^i dz_i^a \wedge d\varsigma_i^a$. The coordinate change $W_{ii} \to W_{jj}$ defined by (1.1) and (1.2) is a symplectic transformation.

REMARK 1.4. The manifold $\Sigma_0^1$ is a Hirzebruch surface (see [Kod]) and $\{\Sigma_\alpha^1\}_\alpha$ is its deformation of complex structure parametrized by a constant $\alpha$. This surface was used for the construction of a space of initial conditions for the Painlevé equations ([Okm.1]).

### *1.3. Monoidal transformation*

In this paragraph, we recall on monoidal transformations (blowing up) which we use in this paper.

Let $M$ be a complex manifold of dimension $m+n$ and let $C$ be a complex submanifold of $M$ of dimension $n$. Let $V_j$ $(j \in J)$ be open sets of $M$ such that $M = \bigcup_{j \in J} V_j$ and let $J' = \{j \in J;\ C \cap V_j \neq \emptyset\}$. We can suppose that we have the coordinates system on $V_j$ $(j \in J')$:

$$v_j = (v_j^0, \cdots, v_j^{m-1}, v_j^m, \cdots, v_j^{m+n-1})$$

such that

$$C \cap V_j = \{v_j \mid v_j^0 = \cdots = v_j^{m-1} = 0\}.$$

Let $\varsigma = (\varsigma^0, \cdots, \varsigma^{m-1})$ be the homogeneous coordinates of $\mathbb{P}^{m-1}$. The monoidal transformation of $M$ along $C$ is a pair $(\hat{M}, f)$ consisting of a complex manifold $\hat{M}$ such that $\dim \hat{M} = \dim M$ and a holomorphic mapping $f : \hat{M} \to M$ defined as follows.

Let $\hat{V}_j$ $(j \in J')$ be a subvariety of $V_j \times \mathbb{P}^{m-1}$ defined by

$$\hat{V}_j = \left\{(v_j, \varsigma) \in V_j \times \mathbb{P}^{m-1} \;\middle|\; v_j^a \varsigma^b - v_j^b \varsigma^a = 0,\ (0 \le a, b \le m-1)\right\}.$$

For $j \in J \backslash J'$, we set $\hat{V}_j = V_j$. Let $f_j : \hat{V}_j \to V_j$ be a holomorphic mapping induced from the projection to the first factor $p : V_j \times \mathbb{P}^{m-1} \to V_j$ for $j \in J'$ and let $f_j = \mathrm{id}.$ for $j \in J \backslash J'$.

$$\begin{array}{ccc} V_j \times \mathbb{P}^{m-1} & \hookleftarrow & \hat{V}_j \\ \downarrow & \swarrow f_j & \\ V_j & & \end{array}$$

We set $\hat{C}_j = f_j^{-1}(C \cap V_j)$. Then the complex structure of $M$ induces the equivalence relation in $\bigcup_{j \in J} \hat{V}_j$ and the mapping $f_j$ $(j \in J)$ are compatible with these relations. As a consequence, we have the manifolds:

$$\hat{M} = \bigcup_{j \in J} \hat{V}_j / \sim, \qquad \hat{C} = \bigcup_{j \in J} \hat{C}_j / \sim,$$

and a holomorphic mapping $f : \hat{M} \to M$ satisfying

(i) $f|_{\hat{V}_j} = f_j$,

(ii) $(\hat{C}, f, C)$ is a fiber bundle over $C$ with fibers isomorphic to $\mathbb{P}^{m-1}$,

(iii) $f : \hat{M} \backslash \hat{C} \to M \backslash C$ is biholomorphic.

In this paper we utilise the notation

$$\hat{M} = \mu_C^{-1}(M), \qquad f = \mu_C,$$

and the manifold $\hat{C}$ is called a *exceptional manifold*.

## *1.4. Main results*

In this part we explain our strategy and state the main result.

We want to consider the Hamiltonian system $\mathcal{H}_n$ in a trivial fiber bundle $\mathcal{Q}^{(0)} = (\Sigma_\alpha \times B, \pi, B)$ with base $B$ defined by (0.4), where $\Sigma_\alpha$ is a Hirzebruch manifold of dimension $2n$ with

$$\alpha := \frac{1}{2}\left(\sum_{i=1}^{n+2} \theta_i + \theta_\infty - 1\right).$$

To fix the idea, suppose the system $\mathcal{H}_n$ is defined in $(W_{00} \times B, \pi, B)$, where $W_{00} = U_0 \times \mathbb{C}^n$ is a coordinate chart of $\Sigma_\alpha^0$. Let $\mathcal{H}_n^{(0)}$ denote the prolonged system of $\mathcal{H}_n$ in $\Sigma_\alpha \times B$.

I. In Section 2, we shall show the following results:

(I-a) The set of singular points of $\mathcal{H}_n^{(0)}$ is a submanifold $D^0 := \left(\Sigma_\alpha \backslash \Sigma_\alpha^0\right) \times B$ of $\Sigma_\alpha \times B$ (Proposition 2.2). Let $\mathcal{F}^{(0)}$ be a nonsingular foliation in $\Sigma_\alpha^0 \times B$ defined by the system $\mathcal{H}_n^{(0)}$. The leaves of $\mathcal{F}^{(0)}$ are all transversal to the fibers of $\pi : \Sigma_\alpha^0 \times B \to B$;

(I-b) The set of accessible singular points of $\mathcal{H}_n^{(0)}$ is contained in $n+3$ mutually disjoint complex submanifolds $C_0^0, \dots, C_{n+2}^0$ of $D^0$ such that

(i) $\pi : C_m^0 \to B$ is a locally trivial fiber subspace of $(D^0, \pi, B)$,

(ii) $\dim C_m^0(t) = n - 1$. where $C_m^0(t) := C_m^0 \cap \pi^{-1}(t)$ (Proposition 2.4).

We set

$$C^0 := \bigcup_{0 \le m \le n+2} C_m^0, \qquad C^0(t) := C^0 \cap \pi^{-1}(t).$$

II. By the definition of accessible singularities, an infinite number of $T$-leaves of $\mathcal{F}^{(0)}$ may prolong holomorphically to a point of $C^0$. We try to separate all these $T$-leaves by performing a monoidal transformation in $\Sigma_\alpha \times B$ along $C^0$. In fact we will show that, for any $t \in B$, a family of $T$-leaves depending on $2n-1$ parameters pass through each component $C_m^0(t)$ of $C^0(t)$.

After performing a monoidal transformation $\mu_{C^0}$ in $\Sigma_\alpha \times B$ along $C^0$, we obtain a locally trivial fiber space $\mathcal{Q}^{(1)} := (E^{(1)}, \pi^{(1)}, B)$ whose fiber over $t \in B$ is $E^{(1)}(t) := \mu_{C^0(t)}(\Sigma_\alpha)$ and a Pfaffian system $\mathcal{H}_n^{(1)}$, which is a prolongation of the system $\mathcal{H}_n^{(0)}$ to $E^{(1)}$. Let us denote by $D_m^1$ the exceptional manifold $\mu_{C^0}^{-1}(C_0^m) \subset E^{(1)}$ $(0 \le m \le n+2)$ and denote by the same symbol $D^0$ the submanifold of $E^{(1)}$ which correspond to $D^0 \subset \Sigma_\alpha \times B$ by the mapping $\mu_{C^0}$. Note that $D^0$ and $D_m^1$ form locally trivial complex analytic fiber spaces:

$$(D^0, \pi^{(1)}, B), \qquad \mathcal{D}_m^1 := (D_m^1, \pi^{(1)}, B) \qquad (0 \le m \le n+2).$$

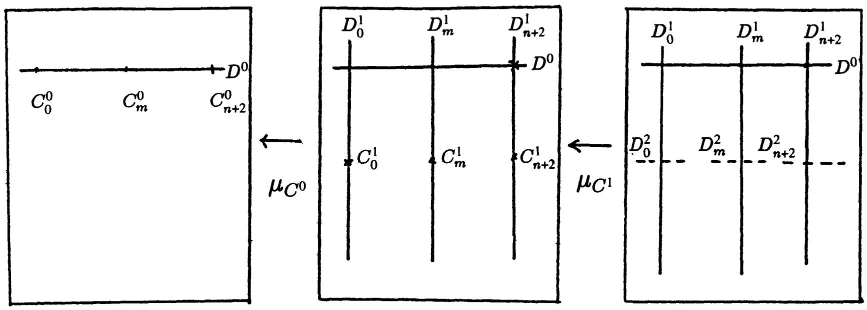


*Fig. 3*

Then we shall show in Section 4, the following results:

(II-a) The singularity of the system $\mathcal{H}_n^{(1)}$ consists of $n+4$ submanifolds $D^0$, $C_0^1, \ldots, C_{n+2}^1$ of $E^{(1)}$ such that

(i) $C_m^1$ is a submanifold of $D_m^1$ and $\pi^{(1)} : C_m^1 \to B$ forms a locally trivial fiber subspace of $D_m^1$ such that $\dim C_m^1(t) = 2n - 2$,

(ii) $C_i^1 \cap C_j^1 = \emptyset$ if $i \neq j$,

(iii) $D^0 \cap D_m^1$ is a subvariety of $D^0$ of dimension $3n - 3$ (Proposition 4.1).

Let $\mathcal{F}^{(1)}$ be a nonsingular complex analytic foliation in $E^{(1)} \backslash \left\{ D^0 \bigcup_{0 \le m \le n+2} C_m^1 \right\}$ defined by $\mathcal{H}_n^{(1)}$.

(II-b) The set of accessible singular points of the system $\mathcal{H}_n^{(1)}$ is contained in $C^1 := \bigcup_{0 \le m \le n+2} C_m^1$.

III. In order to separate $T$-leaves of $\mathcal{F}^{(1)}$ which pass a point of $C^1$, we perform once again a monoidal transformation $\mu_{C^1}$ in $E^{(1)}$ along $C^1$. Then we obtain a locally trivial complex analytic fiber space:

$$\mathcal{Q}^{(2)} = (E^{(2)}, \pi^{(2)}, B), \qquad E^{(2)} = \mu_{C^1}^{-1}(E^{(1)}),$$

and a prolonged system $\mathcal{H}_n^{(2)}$ in $E^{(2)}$ derived from $\mathcal{H}_n^{(1)}$. Let $D_m^2 := \mu_{C^1}^{-1}(C_m^1)$ be the exceptional manifolds in $E^{(2)}$ obtained from $C_m^1$ $(0 \leq m \leq n+2)$ by the monoidal transformation $\mu_{C^1}$. Let us denote by the same symbols $D^0$ and $D_m^1$ $(0 \leq m \leq n+2)$, as in II, the submanifold of $E^{(2)}$ which correspond to $D^0 \subset E^{(1)}$ and $D_m^1 \subset E^{(1)}$ by the mapping $\mu_{C^1}$, respectively. Then we shall prove in Section 4, the following assertions.

(III-a) The singularity of the system $\mathcal{H}_n^{(2)}$ is $D^0 \subset E^{(2)}$ (Proposition 4.3). Let $\mathcal{F}^{(2)}$ be the nonsingular foliation in $E^{(2)} \backslash D^0$ defined by $\mathcal{H}_n^{(2)}$.

(III-b) There is no accessible singularity for $\mathcal{H}_n^{(2)}$ (Propositión 4.4).

(III-c) The submanifolds $D_m^1$ $(0 \leq m \leq n+2)$ are filled with vertical leaves of $\mathcal{F}^{(2)}$ (Proposition 4.5).

(III-d) The leaves of $\mathcal{F}^{(2)}$ in $E^{(2)} \backslash \left\{ D^0 \bigcup_m D_m^1 \right\}$ are transversal to the fibers of $\mathcal{Q}^{(2)}$ (Proposition 4.6).

(III-e) There is no leaf in $E^{(2)} \backslash \left\{ D^0 \bigcup_m D_m^1 \right\}$ which is entirely contained in $\bigcup_m D_m^2$ (Proposition 4.7).

By virtue of the results (III-a),...,(III-d), we arrive at the situation where all leaves of $\mathcal{F}^{(2)}$ are separated into two groups; one consisting of leaves in $E^{(2)} \backslash \left\{ D^0 \bigcup_m D_m^1 \right\}$ and the other consisting of leaves in $\bigcup_m D_m^1$, the leaves of the former are transversal to the fibers and those of the latter are vertical ones. Moreover, by virtue of (III-e), we see that, for any $t \in B$, there is a $(2n-1)$-parameter family of solutions of $\mathcal{H}_n^{(0)}|_{\Sigma_\alpha^0 \times B}$ which pass through $C^0(t)$.

Let us denote by $\mathcal{P} := (M, \pi^{(2)}, B)$ the fibration over $B$ obtained from $\mathcal{Q}^{(2)}$ by removing the singularity of the system $\mathcal{H}_n^{(2)}$ and all vertical leaves of $\mathcal{F}^{(2)}$:

$$M = E^{(2)} \backslash \left\{ D^0 \bigcup_m D_m^1 \right\}$$

and let $\mathcal{F}$ be the foliation in $M$ obtained by restricting $\mathcal{F}^{(2)}$ to $M$.

Here is the principal result of this paper.

THEOREM 1.5. *The triplet $\mathcal{P} := (M, \pi^{(2)}, B)$ is a locally trivial fiber space and it gives a P-uniform fibration associated with the system $\mathcal{H}_n$. So the manifold $M(t) := M \cap (\pi^{(2)})^{-1}(t)$ is a space of initial conditions for $\mathcal{H}_n$.*

COROLLARY 1.6. *For any $t \in B$, there is a $(2n-1)$-parameter family of solutions of $\mathcal{H}_n^{(0)}|_{\Sigma_\alpha^0 \times B}$ which pass through $C^0(t)$.*

*1.5. Particular solutions of $\mathcal{H}_n$ and complex structure of $\Sigma_\alpha$*

Before terminate this section, we remark the relation between the analytic structure of $\Sigma_\alpha$ and particular solutions of $\mathcal{H}_n$.

In [KimH.4], we have shown that

$$\Sigma_\alpha \simeq \begin{cases} \mathbb{P}^n \times \mathbb{P}^n & \text{if } \alpha \neq 0, \\ PT^*\mathbb{P}^n & \text{if } \alpha = 0, \end{cases}$$

as complex manifolds, where $PT^*\mathbb{P}^n$ denotes a complex analytic fibers bundle on $\mathbb{P}^n$ obtained starting from the cotangent vector bundle on $\mathbb{P}^n$ by compactifying the fibers $\mathbb{C}^n$ by $\mathbb{P}^n$ in a natural way. Moreover the manifold $PT^*\mathbb{P}^n$ is not isomorphic to the trivial bundle $\mathbb{P}^n \times \mathbb{P}^n$. Recall that, for a construction of a space of initial conditions for $\mathcal{H}_n$, we used the manifold $\Sigma_\alpha$ where the constant $\alpha$ is given by

$$\alpha = \frac{1}{2}\left(\sum_{(i)} \theta_i + \theta_\infty - 1\right),$$

where $\sum_{(i)}$ stands for a summation over $i = 1, \ldots, n+2$.

PROPOSITION 1.7 [OK]. *Suppose that the parameters $\theta$ in the system $\mathcal{H}_n$ satisfies the condition $\alpha = 0$. Then the system $\mathcal{H}_n$ admits particular solutions $(q,p) = (q(t), 0)$ of the form*

$$q_i(t) = \frac{t_i(t_i - 1)}{\sum_{(i)} \theta_i - 1} \frac{\partial}{\partial t_i} \log\left((t_i - 1)^{\theta_i} u\left(1 - \theta_{n+2}, \theta_1, \ldots, \theta_n, \sum_{i=1}^{n+1} \theta_i; t\right)\right),$$

*$(i = 1, \ldots, n)$, where $u(\alpha, \beta_1, \ldots, \beta_n, \gamma; t)$ is any solution of the system of differential equations defining the hypergeometric functions $F_D(\alpha, \beta_1, \ldots, \beta_n, \gamma; t)$ of Lauricella:*

$$\left[(1 - t_i)\sum_j t_j D_i D_j + [\gamma - (\alpha+1)t_i]D_i - \beta_i \sum_j t_j D_j - \alpha\beta_i\right] u = 0,$$

$$(i = 1, \ldots, n).$$

The system of linear differential equations in the proposition is called a *hypergeometric equation $F_D$ of Lauricella.*

This proposition can be understood in the following manner. When $\alpha = 0$, the bundle $\mathcal{Q}^0 := \left(\Sigma_\alpha^0 \times B, \pi, B\right)$ admits a subbundle $(N, \pi, B)$ with fibers isomorphic to $\mathbb{P}^n$, i.e., $N = \{\text{zero section of } T^*\mathbb{P}^n\} \times B$. Since it can be seen that $N$ is preserved by the Hamiltonian flow of $\mathcal{H}_n^{(0)}$, we can restrict the system $\mathcal{H}_n^{(0)}$ on $N$. On the other hand, since $Aut(\mathbb{P}^n) \simeq PGL(n, \mathbb{C})$, the time evolution of any solution of $\mathcal{H}_n^{(0)}|_N$ can be expressed by using solutions of some linear differential equations of rank $n+1$. This explains why $\mathcal{H}_n$ admits a solution expressed in terms of solutions of $F_D$.

## 2. - System $\mathcal{H}_n^{(0)}$ and its singularity

### *2.1. Singularity of the system* $\mathcal{H}_n^{(0)}$

Suppose that the system $\mathcal{H}_n$ is defined in a total space of subbundle $(W_{00} \times B, \pi, B)$ of $\mathcal{Q}^0 = \left(\Sigma_\alpha^0 \times B, \pi, B\right)$, where $W_{00}$ is the coordinate chart of $\Sigma_\alpha^0$ given in the definition of $\Sigma_\alpha^0$. We take the constant $\alpha$ as

$$\alpha = \frac{1}{2}\left(\sum_{(i)} \theta_i + \theta_\infty - 1\right).$$

Then we have

PROPOSITION 2.1. *The Hamiltonian system $\mathcal{H}_n$ prolongs to a Hamiltonian system on $\Sigma_\alpha^0 \times B$ with holomorphic Hamiltonians on $\Sigma_\alpha^0 \times B$. That is in each subbundle $(W_{mm} \times B, \pi, B)$ $(0 \leq m \leq n)$, the prolonged system of $\mathcal{H}_n$ is written in the form of Hamiltonian system with polynomial Hamiltonians in the coordinates of $W_{mm}$ with holomorphic coefficients in $t \in B$.*

PROOF. Define $\tau_m = (T_m, \ell_m) \in L$ by

$$\tau_m := (\sigma_{n+2}\sigma_m)\sigma_{n+1}(\sigma_{n+2}\sigma_m)^{-1}, \qquad (1 \leq m \leq n).$$

Then $T_m : (q,p,t) \to (q',p',t')$ and $\ell_m : V \to V$ are given by

$$q'^i = \begin{cases} -\dfrac{q^i}{q^m} & (i \neq m), \\ \dfrac{1}{q^m} & (i = m), \end{cases}$$

$$p'^i = \begin{cases} -q^m p^i & (i \neq m), \\ -q^m \left(\alpha + \sum_a q^a p^a\right) & (i = m), \end{cases}$$

$$t'^i = \begin{cases} \dfrac{t^i}{t^m} & (i \neq m), \\ \dfrac{1}{t^m} & (i = m). \end{cases}$$

and

$$\ell_m : (\theta_1, \dots, \theta_{n+2}, \theta_\infty) \mapsto (\theta_1, \dots, \theta_{m-1}, \theta_\infty, \theta_{m+1}, \dots, \theta_{n+2}, \theta_m).$$

Let

$$\Phi_m : W_{00} \times B \ni (z_0, \varsigma_0, t) \mapsto (z_m, \varsigma_m, t) \in W_{mm} \times B$$

be the transition map of manifold $\Sigma_\alpha^0 \times B$ defined by (1.1) and (1.2). Define a birational map $\Psi_m$ by

$$\begin{aligned} \Psi_m : (q,p,t) &\mapsto (q', p', t_m) \\ &= \Big(-q^m, -q^2, \dots, -q^{m-1}, q^1, -q^{m+1}, \dots, q^n, \\ &\quad - p^m, -p^2, \dots, -p^{m-1}, p^1, -p^{m+1}, \dots, p^n, \\ &\quad \frac{t^1}{t^m}, \dots, \frac{t^{m-1}}{t^m}, \frac{1}{t^m}, \frac{t^{m+1}}{t^m}, \dots, \frac{t^n}{t^m}\Big). \end{aligned}$$

Moreover, for each $m = 1, \dots, n$, let $H_i(q,p,t)$ and $H_i^m(q,p,t)$ be the Hamiltonians of $\mathcal{H}_n(\theta)$ and $\mathcal{H}_n(\ell_m(\theta))$, respectively. If we define functions $K_i^m(q', p', t_m)$ by

$$\Psi_m^* \left(\sum K_i^m(q', p', t^m) dt_m^i\right) = \sum H_i^m(q,p,t) dt^i, \tag{2.1}$$

then the map $\Psi_m$ extends to a symplectic transformation

$$(q, p, t, H^m) \to (\Psi_m(q,p,t), K^m).$$

The composition $\Psi_m \circ T_m$ is shown to be equal to $\Phi_m$ and is extended to the symplectic transformation

$$(q,p,t,H) \to (\Phi_m(q,p,t), K^m).$$

Therefore the system $\mathcal{H}_n$ in $W_{00} \times B$ is transformed into the Hamiltonian system on $W_{mm} \times B$ with Hamiltonians $K^m = (K_j^m)_{j=1,\dots,n}$. Since $H^m = (H_j^m)_{j=1,\dots,n}$ are polynomials in $(q,p)$, so are $K^m = (K_j^m)_{j=1,\dots,n}$ by virtue of (2.1). ∎

Now we consider the prolongation of the system $\mathcal{H}_n$ to the total space $\Sigma_\alpha \times B$ of bundle $\mathcal{Q}^{(0)} = (\Sigma_\alpha \times B, \pi, B)$ and obtain the system $\mathcal{H}_n^{(0)}$.

PROPOSITION 2.2. *If $\theta_\infty \neq 0$, any solution of $\mathcal{H}_n^{(0)}|_{\Sigma_\alpha^0 \times B}$ passes through $W_{00} \times B$.*

PROOF. Let $\mathcal{F}^{(0)}$ be a foliation in $\Sigma_\alpha^0 \times B$ defined by $\mathcal{H}_n^{(0)}$. Suppose that there is a leaf of $\mathcal{F}^{(0)}$ contained entirely in $\left(\Sigma_\alpha^0 \backslash W_{00}\right) \times B$. Then, for some $1 \leq m \leq n$, this leaf is expressed by a solution $(z_m(t), \varsigma_m(t))$ of the Hamiltonian system $\mathcal{H}_n^{(0)}|_{W_{mm}\times B}$ satisfying $z_m^0(t) \equiv 0$. As it is seen in the proof of Proposition 2.1, the system $\mathcal{H}_n^{(0)}|_{W_{mm}\times B}$ is taken into $\mathcal{H}_n(\ell_m(\theta))$ by the transformation $\Psi_m^{-1}$. So there is a solution $(q(t),p(t))$ of $\mathcal{H}_n(\ell_m(\theta))$ satisfying $q^m(t) \equiv 0$. On the other hand, an equation for $q^m$ of $\mathcal{H}_n(\ell_m(\theta))$ has the form

$$\frac{\partial q^m}{\partial t^i} = 2\sum_j E_{ijm}(t,q)p^j - F_{im}(t,q), \tag{2.2}$$

where

$$\begin{aligned} &E_{ijm}(t,q)|_{q^m=0} = 0, \\ &F_{im}(t,q)|_{q^m=0} = \begin{cases} -\theta_\infty R_{mi} q^i, & (i \neq m), \\ \theta_\infty t^i - \theta_\infty \sum\limits_{k \neq m} S_{ik} q^k, & (i = m). \end{cases} \end{aligned} \tag{2.3}$$

The equalities (2.2) and (2.3) contradict with the existence of solution $(q(t),p(t))$ of $\mathcal{H}_n(\ell_m(\theta))$ such that $q^m(t) \equiv 0$. This proves the proposition. ∎

Let us give explicitly the set of singular points of the prolonged system $\mathcal{H}_n^{(0)}$.

PROPOSITION 2.3. *The singularity of the system* $\mathcal{H}_n^{(0)}$ *is* $D^0 := (\Sigma_\alpha \backslash \Sigma_\alpha^0) \times B$.

PROOF. We study the system $\mathcal{H}_n^{(0)}$ in $W_0 \times B \subset \Sigma_\alpha \times B$. Let $(q^1, \dots, q^n, \xi_0)$ be the coordinates of $W_0 = U_0 \times \mathbb{P}^n(\xi_0)$, and let $x_m = (x_m^1, \dots, x_m^n)$ be the affine coordinates in the $m$-th affine chart $\{\xi_0 \in \mathbb{P}^n(\xi_0) \mid \xi_0^m \neq 0\}$:

$$x_m^j = \begin{cases} \xi_0^j / \xi_0^m & (m = 0 \text{ or } j \neq m), \\ \xi_0^0 / \xi_0^m & (m \neq 0 \text{ and } j = m). \end{cases} \tag{2.4}$$

Then we have

$$x_m^j = \begin{cases} x_0^j / x_0^m & (j \neq m), \\ 1/x_0^m & (j = m). \end{cases} \tag{2.5}$$

The system $\mathcal{H}_n^{(0)}$ in $W_{00} \times B$ coincides with $\mathcal{H}_n$ by definition, hence it can be written as

$$\mathrm{d}q^i = \sum_j \{H_j, q^i\} \mathrm{d}t^j, \quad \mathrm{d}x_0^i = \sum_j \{H_j, x_0^i\} \mathrm{d}t^j, \quad (1 \leq i \leq n), \tag{2.6}$$

where $H_j$ is obtained by replacing $(p^1, \dots, p^n)$ by $(x_0^1, \dots, x_0^n)$ in the Hamiltonians of $\mathcal{H}_n$. In $W_{0m} \times B$ $(1 \leq m \leq n)$, the system $\mathcal{H}_n^{(0)}$ is written as

$$\begin{cases} x_m^m \mathrm{d}q^i = \displaystyle\sum_j A_{ij}^m (q, x_m, t) \mathrm{d}t^j, \\ x_m^m \mathrm{d}x_m^i = \displaystyle\sum_j B_{ij}^m (q, x_m, t) \mathrm{d}t^j, \quad (i \neq m) \\ \mathrm{d}x_m^m = \displaystyle\sum_j B_{mj}^m (q, x_m, t) \mathrm{d}t^j, \end{cases} \tag{2.7}$$

where $A_{ij}^m$, $B_{ij}^m \in \mathbb{C}(t)[q, x_m]$ are given by

$$e(t^i) A_{ij} = 2 \sum_a E_{ija} x^a \Big|_{x^m=1} - F_{ij} x^m,$$

$$e(t^i) B_{ij} = \sum_{a,b} E_{iab,j} x^a x^b \Big|_{x^m=1} - \sum_a^m F_{ia,j} x^a x^m$$
$$- F_{im,j} x^m + \kappa \delta_{ij} (x^m)^2 - x^j e(t^i) B_{im},$$

$$e(t^i) B_{im} = \sum_{a,b} E_{iab,m} x^a x^b \Big|_{x^m=1} - \sum_a^m F_{ia,m} x^a x^m$$
$$- F_{im,m} x^m + \kappa \delta_{im} (x^m)^2.$$

Here, for $f \in \mathbb{C}(q, x, t)$, a symbol $f_{,i}$ denotes the partial derivative of $f$ with respect to $q^i$.

We see that $A_{ij}^m$ and $B_{ij}^m$ are not divisible by $x_m^m$; it follows that the singularity of $\mathcal{H}_n^{(0)}$ in $W_{0m} \times B$ is $\{(q, x_m) \in W_{0m} \mid x_m^m = 0\} \times B$. Collecting these sets for $m = 1, \ldots, n$, the singularity of $\mathcal{H}_n^{(0)}$ in $W_0 \times B$ is

$$\{(q, \xi_0) \in W_0 \mid \xi_0^0 = 0\} \times B.$$

Taking the proof of Proposition 2.1 into consideration, we see that the set of singular points of $\mathcal{H}_n^{(0)}$ is given by

$$\bigcup_{0 \le i \le n} \{(z_i, \xi_i, t) \in W_i \times B \mid \xi_i^i = 0\},$$

which is written in the form $(\Sigma_\alpha \backslash \Sigma_\alpha^0) \times B$. This is what we want to prove. ■

## 2.2. *Accessible singularity of* $\mathcal{H}_n^{(0)}$

Now let us determine the set of accessible singular points of the system $\mathcal{H}_n^{(0)}$. Note that $\mathcal{H}_n^{(0)}|_{\Sigma_\alpha^0 \times B}$ has the Painlevé property. This fact implies that if a $T$-leaf in $\Sigma_\alpha^0 \times B$ contains a singular point of $\mathcal{H}_n^{(0)}$ in its closure, a solution representing this leaf can be continued holomorphically to this point. In order to fix the idea, we consider $\mathcal{H}_n^{(0)}$ in $W_0 \times B$. In an affine chart $W_{0m} \times B$, we have the system $\mathcal{H}_n^{(0)}$ in explicit form (2.7). We see in the proof of Lemma 3.1 that a necessary condition for a point $(q, x_m, t) \in W_{0m} \times B$ to belong to the set of accessible singular points is

$$\begin{aligned} & x_m^m = 0 \\ & A_{ij}^m(t, q, x_m) = 0 \quad (1 \le i, j \le n) \\ & B_{ij}^m(t, q, x_m) = 0 \quad (1 \le i, j \le n;\ i \ne m). \end{aligned} \tag{2.8}$$

In other open set $W_i \times B$ $(1 \le i \le n)$ of $\Sigma_\alpha \times B$, we will have the system of algebraic equations analogous to (2.8). By solving this system, we have

PROPOSITION 2.4. *The set of accessible singular points of the system $\mathcal{H}_n^{(0)}$ is given by mutually disjoint $n+3$ submanifolds $C_0^0, C_1^0, \ldots, C_{n+2}^0$ in $D^0$ of dimension $2n-1$:*

$$C_m^0 = \bigcup_{0\le j\le n} \{(z,\xi_j,t) \in W_j \times B \mid z^m = \xi_j^k = 0 \ (k \neq m)\} \quad (0 \le m \le n)$$

$$C_{n+1}^0 = \bigcup_{0\le j\le n} \{(z,\xi_j,t) \in W_j \times B \mid \overline{g}_1 = \xi_j^j = \xi_j^k + \xi_j^0 = 0 \ (k \neq 0, j)\},$$

$$C_{n+2}^0 = \bigcup_{0\le j\le n} \{(z,\xi_j,t) \in W_j \times B \mid \overline{g}_t = \xi_j^j = t^k \xi_j^k + \xi_j^0 = 0 \ (k \neq 0, j)\},$$

*where $z = (z^0, \ldots, z^n)$ is the homogeneous coordinates of $\mathbb{P}^n$ used in the definition of $\Sigma_\alpha$ such that $q^a = z^a/z^0$ $(1 \le a \le n)$ and*

$$\overline{g}_1 = z^0 - \sum_{a=1}^n z^a, \qquad \overline{g}_t = z^0 - \sum_{a=1}^n \frac{z^a}{t^a}.$$

This proposition will be proved in Section 3.

REMARK 2.5. $\pi : C_i^0 \to B$ is a locally trivial fiber space whose fibers $C_i^0(t)$, $t \in B$ are compact submanifolds of $D^0(t)$ of dimension $n-1$.

### 2.3. *Symmetry of the accessible singularity*

The generators $\sigma_i$ $(1 \le i \le n+2)$ of group $L$ of symmetries of $\mathcal{H}_n$, which are given in Section 1.1, can be extended in a natural manner to the bundle automorphism $\tilde{\sigma}_i$ $(1 \le i \le n+2)$ of bundle $\mathcal{Q}^{(0)}$. Denote by $\tilde{L} := \langle \tilde{\sigma}_1, \ldots, \tilde{\sigma}_{n+2} \rangle$ the group generated by the $\tilde{\sigma}_i$'s.

PROPOSITION 2.6. *The group $\tilde{L}$ acts transitively on the set of connected components of accessible singularity $\{C_0^0, \ldots, C_{n+2}^0\}$ of $\mathcal{H}_n^{(0)}$.*

Let $\tilde{\tau}_m$ $(1 \le m \le n)$ be the elements of $\tilde{L}$ defined by $\tilde{\tau}_m = (\tilde{\sigma}_m \circ \tilde{\sigma}_{n+2}) \circ \tilde{\sigma}_{n+1} \circ (\tilde{\sigma}_m \circ \tilde{\sigma}_{n+2})^{-1}$. Then the action of the group $\tilde{L}$ on $\{C_0^0, \ldots, C_{n+2}^0\}$ is given by the following table: where $\star$ denotes the manifold noted in the box at the first row and at the column where $\star$ is marked. For example, the table reads that the action of $\tilde{\sigma}_{n+1}$ is

$$\tilde{\sigma}_{n+1}(C_m^0) = C_m^0, \quad (0 \le m \le n)$$

$$\tilde{\sigma}_{n+1}(C_{n+1}^0) = C_{n+2}^0, \qquad \tilde{\sigma}_{n+1}(C_{n+2}^0) = C_{n+1}^0.$$

| | $C_0^0$ | $\cdots$ | $C_m^0$ | $\cdots$ | $C_n^0$ | $C_{n+1}^0$ | $C_{n+2}^0$ |
|---|---|---|---|---|---|---|---|
| $\tilde{\sigma}_m$ | $\star$ | $\star$ | $C_{n+2}^0$ | $\star$ | $\star$ | $C_m^0$ | $C_{n+1}^0$ |
| $\tilde{\sigma}_{n+1}$ | $\star$ | $\star$ | $\star$ | $\star$ | $\star$ | $C_{n+2}^0$ | $C_{n+1}^0$ |
| $\tilde{\sigma}_{n+2}$ | $C_{n+1}^0$ | $\star$ | $\star$ | $\star$ | $\star$ | $C_0^0$ | $\star$ |
| $\tilde{\tau}_m$ | $C_m^0$ | $\star$ | $C_0^0$ | $\star$ | $\star$ | $\star$ | $\star$ |

PROOF OF PROPOSITION 2.6. To show the proposition, it is sufficient to establish the above table. Let $(z, \xi)$ be the system of homogeneous coordinates in the chart $W_0$ of $\Sigma_\alpha$. If we restrict $\tilde{\sigma}_i$ to $\tilde{\sigma}_i^{-1}(W_0 \times B) \cap W_0 \times B$, we can write it in the form:

$$\begin{array}{rccc} \tilde{\sigma}_i : & W_0 \times B & \to & W_0 \times B \\ & (z, \xi, t) & \mapsto & (z', \xi', t') \\ & z' = A_i z, & & \xi' = B_i \xi, \end{array}$$

where $A_i$ and $B_i$ are the matrices with coefficients in $\mathbb{C}(t)$ and $\mathbb{C}(t)[z]$, respectively. In fact, for each $\tilde{\sigma}_i$, we obtain explicitly the matrices $A_i$ and $B_i$ and the transformation $t \to t'$ as follows.

$\tilde{\sigma}_m$ $(m = 1, \ldots, n)$ :

$$t'^j = \begin{cases} \dfrac{t^m - t^j}{t^m - 1} & (j \neq m), \\[2ex] \dfrac{t^m}{t^m - 1} & (j = m), \end{cases}$$

$$A_m = \operatorname{diag}\left(1, \frac{1}{R_{1m}}, \ldots, \frac{1}{R_{m-1,m}}, 0, \frac{1}{R_{m+1,m}}, \ldots, \frac{1}{R_{nm}}\right)$$

$$+ m) \begin{pmatrix} 0 \\ t'^m, -t^1 t'^m, \ldots, -t^n t'^m \\ 0 \end{pmatrix}$$

$$B_m = \operatorname{diag}(1, R_{1m}, \ldots, R_{m-1,m}, 0, R_{m+1,m}, \ldots, R_{nm})$$
$$- t^m \begin{pmatrix} 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \dfrac{R_{1m}}{t^1} & 0 & \cdots & 0 \\ & & & \vdots & & & \\ & & & \dfrac{R_{m-1,m}}{t^{m-1}} & & & \\ \vdots & & \vdots & 0 & \vdots & & \vdots \\ & & & \dfrac{R_{m+1,m}}{t^{m+1}} & & & \\ & & & \vdots & & & \\ 0 & \cdots & 0 & \dfrac{R_{nm}}{t^n} & 0 & \cdots & 0 \end{pmatrix}$$

$\tilde{\sigma}_{n+1}$:

$$t'^j = \frac{1}{t_j} \qquad (1 \leq j \leq n)$$
$$A_{n+1} = \operatorname{diag}\left(1, \frac{1}{t^1}, \cdots, \frac{1}{t^n}\right),$$
$$B_{n+1} = \operatorname{diag}(1, t^1, \ldots, t^n),$$

$\tilde{\sigma}_{n+2}$:

$$t'^j = \frac{t^j}{t^j - 1} \qquad (1 \leq j \leq n),$$
$$A_{n+2} = \begin{pmatrix} -1 & 1 & \ldots & 1 \\ 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \end{pmatrix},$$
$$B_{n+2} = \operatorname{diag}((z^0)^2, \alpha z^0, \cdots, \alpha z^0)$$
$$+ \overline{g}_1 \begin{pmatrix} 0 & 0 & \ldots & 0 \\ \alpha & z^1 & \ldots & z^n \\ \vdots & \vdots & & \vdots \\ 0 & z^1 & \ldots & z^n \end{pmatrix},$$

where $\operatorname{diag}(a_0, \ldots, a_n)$ denotes a diagonal $(n+1)$-matrix whose $(i, i)$-component is $a_i$.

Moreover, for $\tilde{\tau}_m$ $(1 \leq m \leq n)$, we obtain the following representation:

$\tilde{\tau}_m$:

$$t'^j = \begin{cases} \dfrac{t^j}{t^m} & (j \neq m), \\ \dfrac{1}{t^m} & (j = m), \end{cases}$$

$$A_m = \begin{array}{r} \\ \\ \\ \\ m) \\ \\ \\ \\ \end{array}\begin{pmatrix} 0 & 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ 0 & 1 & & & & & & \\ & & \ddots & & & & & \\ & & & 1 & & & & \\ 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{pmatrix}$$

$$B_m = \operatorname{diag}(-(z^0)^2, z^0 z^m, \dots, z^0 z^m, 0, z^0 z^m, \dots, z^0 z^m) + z^m \begin{pmatrix} 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \\ \alpha z^0 & z^1 & \dots & z^n \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

From the expressions of $\tilde{\sigma}_m$, we see for example that

$$\tilde{\sigma}_{n+1}(C_a^0 \cap (W_0 \times B)) = \begin{cases} C_a^0 \cap (W_0 \times B) & (0 \leq a \leq n), \\ C_{n+2}^0 \cap (W_0 \times B) & (a = n+1), \\ C_{n+1}^0 \cap (W_0 \times B) & (a = n+2). \end{cases}$$

Hence we obtain the results stated in the table for $\tilde{\sigma}_{n+1}$, since a closure of $C_a^0 \cap (W_0 \times B)$ in $\Sigma_\alpha \times B$ is $C_a^0$. We can prove in the same manner the results for $\tilde{\sigma}_i$'s and for $\tilde{\tau}_m$'s. ■

## 3. - Proof of Proposition 2.4

We determine the accessible singularity of the system $\mathcal{H}_n^{(0)}$ in an open set $W_0 \times B \simeq U_0 \times \mathbb{P}^n(\xi) \times B$ of $\Sigma_\alpha \times B$. Taking account of the proof of Proposition 2.1, the accessible singularities in the other $W_i \times B \subset \Sigma_\alpha \times B$ will be found from that in $W_0 \times B$.

Let $(q, \xi)$ be coordinates in $W_0 \simeq \mathbb{C}^n \times \mathbb{P}^n$ and $W_0 = \bigcup_{0 \le m \le n} W_{0m}$ be the covering by affine charts $W_{0m} = \{(q, \xi) \in W_0 \mid \xi^m \ne 0\}$. The affine coordinates $(q, x_m) := (q^1, \dots, q^n, x_m^1, \dots, x_m^n)$ in $W_{0m}$ are defined by

$$(x_m^1, \dots, x_m^n) = \left( \frac{\xi^1}{\xi^m}, \dots, \frac{\xi^{m-1}}{\xi^m}, \frac{\xi^0}{\xi^m}, \frac{\xi^{m+1}}{\xi^m}, \dots, \frac{\xi^n}{\xi^m} \right).$$

Note that the coordinates $(q, x_m)$ in $W_{0m}$ $(m \ne 0)$ are related with $(q, x_0)$ in $W_{00}$ by

$$x_m^a = \begin{cases} x_0^a / x_0^m & (a \ne m) \\ 1/x_0^m & (a = m). \end{cases} \tag{3.1}$$

In the following, we consider the system $\mathcal{H}_n^{(0)}$ in $W_{0m} \times B$ for some fixed $m \ge 1$. To avoid complexity of notations we write $x = (x^1, \dots, x^n)$ instead of $x_m = (x_m^1, \dots, x_m^n)$ to denote the affine coordinates in $W_{0m}$.

Recall the system $\mathcal{H}_n$ is given in $W_{00} \times B$. The system $\mathcal{H}_n^{(0)}$ restricted on $W_{0m} \times B$, which is obtained from $\mathcal{H}_n$ through the change of coordinates (3.2), is written as

$$\begin{cases} x^m \mathrm{d}q^i = \sum_j A_{ij}(q, x, t) \mathrm{d}t^j & (1 \le i \le n) \\ x^m \mathrm{d}x^i = \sum_j B_{ij}(q, x, t) \mathrm{d}t^j & (1 \le i \le n;\ i \ne m) \\ \mathrm{d}x^m = \sum_j B_{ij}(q, x, t) \mathrm{d}t^j \end{cases} \tag{3.2}$$

where $A_{ij}(q, x, t)$ $B_{ij}(q, x, t) \in \mathbb{C}(t)[q, x]$ $(1 \le i, j \le n)$, are given by

$$\begin{aligned} e(t^i) A_{ij} &= 2 \sum_a E_{ija}(t, q) x^a |_{x^m = 1} + \mathcal{O}(x^m), \\ e(t^i) B_{ij} &= \sum_{a,b} \{E_{iab,j}(t, q) - x^j E_{iab,m}(t, q)\} x^a x^b |_{x^m = 1} + \mathcal{O}(x^m), \\ e(t^i) B_{im} &= \sum_{a,b} E_{iab,m}(t, q) x^a x^b |_{x^m = 1} + \mathcal{O}(x^m), \end{aligned} \tag{3.3}$$

$\mathcal{O}(x^m)$ denoting polynomials in $x$ divisible by $x^m$ and, for $f \in \mathbb{C}(q, x, t)$, a symbol $f_j$ denoting the partial derivative of $f$ with respect to $q^j$.

Let $P_i(t,q)$ and $Q_{ij}(t,q)$ $(1 \le i,j \le n)$ be the symmetric matrices defined by

$$P_i = (E_{iab}(t,q))_{a,b=1,\dots,n}, \qquad Q_{ij} = (E_{iab,j}(t,q))_{a,b=1,\dots,n}$$

and let $X$ be a vector

$$X = {}^t(x^1,\dots,x^{m-1},1,x^{m+1},\dots,x^n).$$

Then

LEMMA 3.1. *If a point $(q,x,t) \in D^0 \cap (W_{0m} \times B)$ belongs to the set of accessible singular points of $\mathcal{H}_n^{(0)}$, it satisfies*

$$x^m = 0,$$

$$(P_i) \qquad P_iX = 0, \qquad (1 \le i \le n),$$

$$(Q_{ij}) \qquad Q_{ij}[X] - x^jQ_{im}[X] = 0, \qquad (1 \le i,j \le n;\ j \neq m),$$

*where $Q_{ij}[X] := {}^tXQ_{ij}X$.*

PROOF. Suppose that $(q_0,x_0,t_0) \in D^0 \cap (W_{0m} \times B)$ belongs to the set of accessible singular points of $\mathcal{H}_n^{(0)}$. By the definition of accessible singularity, there is a solution $(q(t),x(t))$ of the system (3.2) which is holomorphic at $t_0$ and satisfies $x^m(t_0) = 0$ and $x^m(t) \not\equiv 0$, since $(q_0,x_0,t_0) \in D^0$ implies $x_0^m = 0$. Put $(q(t),x(t))$ into the system (3.2). Since the terms $x^m(t)dq^i(t)$ and $x^m(t)dx^i(t)$ in (3.2) vanish at $t_0$, the point $(q_0,x_0,t_0)$ satisfies a system of algebraic equations:

$$\sum_a E_{iab}(t,q)x^b\Big|_{x^m=1} = 0,$$

$$\sum_a \{E_{iab,j}(t,q) - x^jE_{iab,m}(t,q)\}x^ax^b\Big|_{x^m=1} = 0, \quad (j \neq m) \quad (1 \le i,j \le n).$$

These equations can be expressed as $(P_i)$ and $(Q_{ij})$ by using matrices $P_i$ and $Q_{ij}$, respectively. ∎

First we seek for solutions of the equations: $x^m = 0$, $(P_i)$ and $(Q_{ij})$ under the condition:

$$\prod_{k=1}^n q^k \neq 0.$$

LEMMA 3.2. *Under the condition* $\prod_{k=1}^{n} q^k \neq 0$, *the solutions of the equation* $(P_m)$ *are given by*

$$\text{(a)} \qquad g_1 - 1 \neq 0, \quad g_t - 1 = 0, \quad x^a = \frac{t^m}{t^a},$$

$$\text{(b)} \qquad g_1 - 1 = 0, \quad g_t - 1 \neq 0, \quad x^a = 1,$$

$$\text{(c)} \qquad g_1 - 1 = 0, \quad g_t - 1 = 0, \quad R_{am}(x^a - 1) = R_{bm}(x^b - 1),$$

*where* $1 \leq a$, $b \leq n$ *and* $a$, $b \neq m$.

PROOF. In case $n = 1$, the result follows immediately. We may suppose $n \geq 2$ and $m \neq 1$ without loss of generality (see the proof below). By virtue of the hypothesis (3.3), the equation $(P_m)$ is equivalent to $P_m^{(1)} X = 0$, where the matrix $P_m^{(1)}$ is written as

$$P_m^{(1)} = \operatorname{diag}(-R_{1m}, \ldots, -R_{m-1,m}, 0, -R_{m+1,m}, \ldots, R_{nm}) + \begin{pmatrix} q^1 & \cdots & q^{m-1} & u^1 & q^{m+1} & \cdots & q^n \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ q^1 & \cdots & q^{m-1} & u^{m-1} & q^{m+1} & \cdots & q^n \\ q^1 u^1 & \cdots & q^{m-1}u^{m-1} & u^m & q^{m+1}u^{m+1} & \cdots & q^n u^n \\ q^1 & \cdots & q^{m-1} & u^{m+1} & q^{m+1} & \cdots & q^n \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ q^1 & \cdots & q^{m-1} & u^n & q^{m+1} & \cdots & q^n \end{pmatrix}$$

where

$$u^i = \begin{cases} q^m - R_{mi} & (i \neq m) \\ (q^m - 1)(q^m - t^m) - \sum_a^m S_{ma} q^a & (i = m). \end{cases}$$

In $P_m^{(1)}$, we subtract the first row from the $k$-th $(2 \leq k \leq n;\ k \neq m)$, and then we add $-(q^m - 1)\times$(first row) and $q^k \times$ ($k$-th row) $(2 \leq k \leq n;\ k \neq m)$ to the $m$-th row. The equation $P_m^{(1)} X = 0$ is taken into $P_m^{(2)} X = 0$ which is equivalent to $P_m^{(1)}X = 0$ and the matrix $P_m^{(2)}$ is given by

$$P_m^{(2)} = \operatorname{diag}(-R_{1m}, \dots, -R_{m-1,m}, 0, -R_{m+1,m}, \dots, R_{nm})$$
$$+ \begin{pmatrix} q^1 & \dots & q^{m-1} & q^m - R_{m1} & q^{m+1} & \dots & q^n \\ R_{1m} & & & R_{m1} - R_{m2} & & & \\ \vdots & & & \vdots & & & \\ R_{1m} & & & R_{m1} - R_{m,m-1} & & & \\ R_{1m}(g_1 - 1) & & & S_{m1}(1 - g_1) & & & \\ R_{1m} & & & R_{m1} - R_{m,m+1} & & & \\ \vdots & & & \vdots & & & \\ R_{1m} & & & R_{m1} - R_{mn} & & & \end{pmatrix}$$

Using the identities (0.3), we see that the equation $P_m^{(2)}X = 0$ can be written as

$$\begin{aligned} &Y - R_{1m}x^1 - R_{m1} = 0, \\ &R_{1m}(x^1 - 1) = R_{am}(x^a - 1), \qquad (a \neq m), \\ &R_{1m}(g_1 - 1)\left(x^1 - \frac{t^m}{t^1}\right) = 0, \end{aligned} \tag{3.4}$$

where $Y = (q^1, \dots, q^n)X = q^m + \sum_a^m q^a x^a$. If $g_1 - 1 \neq 0$, the third equation of (3.4) implies $x^1 = t^m/t^1$; putting it into the second, we have $x^a = t^m/t^a$ $(a \neq m)$. Then the first equation of (3.4), written as $t^m(g_t - 1) = 0$, leads to $g_t - 1 = 0$. This gives a solution (a). The solutions (b) and (c) can be obtained in a similar way. ∎

LEMMA 3.3. *Under the hypothesis* $\prod_k q^k \neq 0$, *solutions of* $(P_i)$ $(1 \leq i \leq n)$ *are given by* (a), (b) *or* (c).

PROOF. We seek for solutions of the equations $(P_i)$ $(i \neq m)$. If $n \leq 2$, the result follows immediately. We may suppose $n \geq 3$ and $i \neq 1$, $m$ without loss of generality. In a similar manner as in the proof of Lemma 3.2, the equation $(P_i)$ is taken into an equivalent equation $P_i^{(1)}X = 0$ with the matrix $P_i^{(1)}$ obtained from $P_m^{(2)}$ by exchanging suffices $m$ and $i$. Equation $P_i^{(1)}X = 0$ reads

$$\begin{cases} Y - R_{im}x^1 - R_{m1} = 0, \\ R_{am}(x^i - x^a) = R_{mi}(x^i - 1), \qquad (a \neq i, m), \\ (g_1 - 1)\left(x^1 - \dfrac{t^i}{t^1}x^i\right) = 0. \end{cases} \tag{3.5}$$

It is seen that the solutions of these equations are given by (a), (b) or (c) of Lemma 3.2. ∎

REMARK 3.4. The determinants of the matrices $P_i$ $(1 \leq i \leq n)$ are given by

$$\det P_i = \pm t^i (q^i)^n (g_1 - 1)(g_t - 1) \prod_k^i R_{ki} q^k.$$

Next let us find solutions of the equations $(P_i)$'s and $(Q_{ij})$'s. Using the explicit form of functions $E_{iab}$ given in the Introduction, we get

$$(3.6) \qquad Q_{ij}[X] = \begin{cases} q^i[2Yx^j - R_{ij}x^i x^j - R_{ji}(x^j)^2 - S_{ij}(x^i)^2] & (i \neq j), \\ Y^2 + 2q^i x^i Y - \{2(t^i+1)q^i - t^i\}(x^i)^2 & \\ \quad + \sum_a^i q^a\{R_{ai}(x^a)^2 + 2R_{ia}x^a x^i + S_{ia}(x^i)^2\} & (i = j). \end{cases}$$

LEMMA 3.5. *Under the condition* $\prod_k q^k \neq 0$, *solutions of the system of equations* $(P_i)$ *and* $(Q_{ij})$ $(1 \leq i, j \leq n;\ j \neq m)$ *are given by*

(i) $\qquad g_t - 1 = 0, \quad x^a = \dfrac{t^m}{t^a} \quad (1 \leq a \leq n;\ a \neq m);$

(ii) $\qquad g_1 - 1 = 0, \quad x^a = 1 \quad (1 \leq a \leq n;\ a \neq m).$

PROOF. 1) Firstly we consider the solution (a) of the equations $(P_i)$'s obtained in Lemma 3.2. Putting (a) into $Y = (q^1, \ldots, q^n)X$, we have

$$\begin{aligned} Y &= X^t(t^m/t^1, \ldots, t^m/t^{m-1}, 1, t^m/t^{m+1}, \ldots, t^m/t^n) \\ &= t^m g_t = t^m. \end{aligned}$$

Using the identities (0.3) and

$$(3.7) \qquad t^i R_{ji} - t^j S_{ij} = 0, \qquad (1 \leq i, j \leq n, i \neq j),$$

we obtain

$$\begin{aligned} Q_{ij}[X] &= q^i (t^m)^2 \left[\frac{2}{t^j} - \frac{2R_{ij}}{t^i t^j} - \frac{R_{ji}}{(t^j)^2} - \frac{S_{ij}}{(t^i)^2}\right] \\ &= \frac{(t^m)^2(t^i - 1)}{t^i t^j} q^i \qquad (i \neq j). \end{aligned}$$

$$\begin{aligned} Q_{ii}[X] &= (t^m)^2 \left[ 1 + 2\frac{q^i}{t^i} - \frac{2(t^i+1)q^i - t^i}{(t^i)^2} - \sum_a^i q^a \left\{ \frac{R_{ai}}{(t^a)^2} + \frac{2R_{ia}}{t^a t^i} + \frac{S_{ia}}{(t^i)^2} \right\} \right] \\ &= \frac{(t^m)^2(t^i-1)}{(t^i)^2} q^i. \end{aligned}$$

It follows that (i) is a solution of the equations $(Q_{ij})$.

2) Consider the solution (b) of the equations $(P_i)$'s in Lemma 3.2. Putting $X = {}^t(1, \dots, 1)$ into equalities (3.6), we have $Y = g_1 = 1$ and

$$Q_{ij}[X] = q^i[2 - 2R_{ij} - R_{ji} - S_{ij}] = (1 - t^i)q^i \qquad (i \neq j)$$

$$Q_{ii}[X] = 1 + 2q^i - \{2(t^i+1)q^i - t^i\} - \sum_a^i q^a\{R_{ai} + 2R_{ia} + S_{ia}\} = (1 - t^i)q^i$$

by virtue of the identities (0.3). It follows that (ii) in the lemma is a solution of $(Q_{ij})$.

3) What is left to be considered is a solution (c) of $(P_i)$'s:

$$g_1 - 1 = 0, \quad g_t - 1 = 0, \quad R_{am}(x^a - 1) = R_{bm}(x^b - 1),$$
$$(1 \le a, b \le n; \ a, b \neq m).$$

Set $C := R_{am}(x^a - 1)$, $(1 \le a \le n; \ a \neq m)$; we have

$$x^a = \frac{C}{R_{am}} + 1. \qquad (1 \le a \le n; \ a \neq m).$$

Putting these expressions for $x^a$'s into $Y = q^m + \sum_a^m q^a x^a$, we obtain

$$Y = \sum_a^m q^a \left( \frac{C}{R_{am}} + 1 \right) + q_m = C \sum_a^m \frac{q^a}{R_{am}} + g_1 = C + 1$$

by virtue of the identities $g_1 = g_t = 1$ and

$$\sum_{a}^{m} \frac{q^a}{R_{am}} = \frac{t^m g_t - g_1}{t^m - 1}. \tag{3.8}$$

Putting $Y = C+1$ into the equalities (3.6) and simplifying them by the help of (3.8), we have

$$\begin{aligned} Q_{mj}[X] = q^m \Bigg[ & 2(C+1)\left(\frac{C}{R_{mj}}+1\right) \\ & - 2R_{mj}\left(\frac{C}{R_{mj}}+1\right) - R_{jm}\left(\frac{C}{R_{mj}}+1\right)^2 - S_{mj}\Bigg] \qquad (m \neq j) \end{aligned}$$

$$\begin{aligned} Q_{mm}[X] = {} & (C+1)^2 + 2q^m(C+1) - 2(t^m+1)q^m + t^m \\ & - \sum_{a}^{m} q^a \left\{ R_{am}\left(\frac{C}{R_{am}}+1\right)^2 + 2R_{ma}\left(\frac{C}{R_{am}}+1\right) + S_{ma} \right\}. \end{aligned}$$

Using the identities (0.3), we deduce

$$Q_{mj}[X] - x^j Q_{mm}[X] = -R_{jm}^{-1} q^m C(C - t^m + 1).$$

Hence the equation $(Q_{mj})$ implies $C = 0$ or $C = t^m - 1$. If $C = 0$, we obtain solution (i), and if $C = t^m - 1$, we obtain solution (ii). ∎

Next we seek for solutions of the equations $(P_i)$ and $(Q_{ij})$ $(1 \leq i, j \leq n;\ j \neq m)$ under the hypothesis:

$$q_m = 0, \qquad \prod_{k \neq m} q_k \neq 0. \tag{3.9}$$

Note that the equation $(P_m)$ is trivial in this case.

LEMMA 3.6. *Under the hypothesis* (3.9), *solutions of the equation* $(P_i)$ $(i \neq m)$ *are given by*

(i) $\quad g_1 - 1 \neq 0, \quad g_t - 1 = 0, \quad t^1x^1 = \cdots \overset{m}{\vee} \cdots = t^nx^n \quad (\neq 0),$

(ii) $\quad g_1 - 1 = 0, \quad g_t - 1 \neq 0, \quad x^1 = \cdots \overset{m}{\vee} \cdots = x^n \quad (\neq 0),$

(iii) $\quad g_1 - 1 = 0, \quad g_t - 1 = 0, \quad R_{am}(x^a - x^i) = R_{bm}(x^b - x^i) \quad (\neq 0)$

(iv) $\quad q^m = 0, \quad x^1 = \cdots \overset{m}{\vee} \cdots = x^n = 0,$

*where* $(1 \leq a, b \leq n;\ a, b \neq i, m)$.

PROOF. In case $n = 2$, we have the lemma immediately. Suppose that $n \geq 3$. Moreover we can suppose $i,\ m \neq 1$ without loss of generality. Then by proceeding in the same way as in the proof of Lemma 3.2, we can show that the equations $(P_i)$ are equivalent to

$$\begin{aligned} &Y - R_{1i}x^1 - R_{i1}x^i = 0, \\ (3.10) \qquad &R_{1i}(x^1 - x^i) = R_{ai}(x^a - x^i), \qquad (a \neq m, i), \\ &(g_1 - 1)(t^1x^1 - t^ix^i) = 0. \end{aligned}$$

From these equations we conclude that

(i) if $g_1 - 1 \neq 0$, then $g_t - 1 = 0$, and $t^1x^1 = \cdots \overset{m}{\vee} \cdots = t^nx^n \neq 0$,

(ii) if $g_1 - 1 = 0$, $g_t - 1 \neq 0$, then $x^1 = \cdots \overset{m}{\vee} \cdots = x^n \neq 0$,

(iii) if $g_1 - 1 = 0$, $g_t - 1 = 0$, then $R_{am}(x^a - x^i) = R_{bm}(x^b - x^i)$,

where $1 \leq a, b \leq n$; $a, b \neq i, m$.

Moreover

(iv) $$q^m = 0, \quad x^1 = \cdots \overset{m}{\vee} \cdots = x^n = 0$$

is a solution of $(P_i)$. It is seen that the solutions with some non-zero $x^a$ are given by (i), (ii) or by (iii). ∎

LEMMA 3.7. *Under the condition* $q^m = 0$, $\prod_{k \neq m} q^k \neq 0$, *the solutions of the equations* $(P_i)$ *and* $(Q_{ij})$ $(1 \leq i, j \leq n,\ j \neq m)$ *are given by*

(i) $$g_t - 1 = 0, \quad x^a = \frac{t^m}{t^a}, \quad (1 \leq a \leq n;\ a \neq m),$$

(ii) $$g_t - 1 = 0, \quad x^a = 1, \quad (1 \leq a \leq n;\ a \neq m),$$

(iii) $$q^m = 0, \quad x^a = 0, \quad (1 \leq a \leq n;\ a \neq m).$$

PROOF. 1) First we consider solution (i) of $(P_i)$ $(i \neq m)$ obtained in Lemma 3.5. Set

$$C := t^a x^a \qquad (1 \leq a \leq n;\ a \neq m)$$

and determine $C$ so that $(Q_{ij})$ are also satisfied by (i). Because

$$Y = \sum_a^m q^a x^a = g_t C = C,$$

by putting $x^a = C/t^a$ into (3.6) and by using the identities (0.3), we can prove

$$\begin{aligned}
Q_{ij}[X] &= \frac{t^i-1}{t^it^j}\, q^iC^2 && (i,j,m \text{ distinct}),\\
Q_{ii}[X] &= \frac{t^i-1}{(t^i)^2}\, q^iC^2 && (i \neq m),\\
Q_{im}[X] &= q^i\left[2C - \frac{2CR_{im}}{t^i}R_{mi} - \frac{C^2S_{im}}{(t^i)^2}\right] && (i \neq m),\\
Q_{mj}[X] &= 0 && (j \neq m),\\
Q_{mm}[X] &= \left\{1 - \sum_a^m \frac{R_{am}q^a}{(t^a)^2}\right\}(C-t^m)^2.
\end{aligned}$$

Whence we obtain

$$\begin{aligned}
Q_{ij}[X] - \frac{C}{t^j}\, Q_{im}[X] &= \frac{S_{im}}{(t^i)^2t^j}\, q^iC(C-t^i)(C-t^m) && (i,j \neq m),\\
Q_{mj}[X] - \frac{C}{t^j}\, Q_{mm}[X] &= \frac{1}{t^j}\left\{\sum_a^m \frac{R_{am}}{(t^a)^2}\, q^a - 1\right\} C(C-t^m)^2 && (j \neq m).
\end{aligned}$$

Note that $C \neq 0$ in our case. Let us show $C = t^m$. Suppose the contrary: $C \neq t^m$. If $n \geq 3$, the equations $(Q_{ij})$ $(i, j \neq m)$ imply that there are at least two suffices $a, b \neq m$ such that $C = t^a = t^b$. But $t^a = t^b$ $(a \neq b)$ contradicts with the fact that we consider the system $\mathcal{H}_n^{(0)}$ in $\Sigma_\alpha \times B$ with $B = \{t \in \mathbb{C}^n \mid t^i \neq 0, 1, t^j\ (i \neq j)\}$. If $n = 2$, the equation $(Q_{mj})$ says that there is a suffix $j$ such that $j \neq m$ and that

$$q^j = \frac{(t^j)^2}{R_{jm}} = \frac{t^j(t^m - t^j)}{t^m - 1}.$$

On the other hand, we have $g_t - 1 = q^j/t^j - 1 = 0$. This is impossible by the same reason as above. Thus $C = t^m$.

2) Next let us consider the solution (ii) of the equations $(P_i)$'s:

$$g_1 - 1 = 0, \quad g_t - 1 \neq 0, \quad x^1 = \cdots {}^m \cdots = x^n \neq 0.$$

We follow the same line of argument as in the first part. Set $C := x^a$ $(1 \leq a \leq n;\ a \neq m)$ and let us show that the equations $(Q_{ij})$ are satisfied if and only if $C = 1$. Note that $Y = \sum_a^m q^ax^a = Cg_1 = C$. By putting $x^a = C$ into

(3.6), and by using identities (0.3), we obtain

$$
\begin{aligned}
&Q_{ij}[X] = (1-t^i)q^iC^2 && (i,j,m \text{ distinct}),\\
&Q_{ii}[X] = (1-t^i)q^iC^2 && (i \neq m),\\
&Q_{im}[X] = q^i[2C - 2CR_{im} - R_{mi} - C^2S_{im}] && (i \neq m),\\
&Q_{mj}[X] = 0, && (j \neq m),\\
&Q_{mm}[X] = \{1 - \sum_a^m R_{am}q^a\}(C-1)^2.
\end{aligned}
$$

Then

$$
\begin{aligned}
Q_{ij}[X] - \frac{C}{t^j}Q_{im}[X] &= S_{im}q^iC(C-1)\left(C - \frac{t^m}{t^i}\right) && (i,j \neq m),\\
Q_{mj}[X] - \frac{C}{t^j}Q_{mm}[X] &= \left\{\sum_a^m R_{am}q^a - 1\right\}C(C-1)^2 && (j \neq m).
\end{aligned}
$$

Suppose now that $C - 1 \neq 0$. If $n \geq 3$, the equations $(Q_{ij})$ $(i,j \neq m)$, we see that there are at least two suffices $a, b$ $(a \neq b;\ a, b \neq m)$ such that $C = t^m/t^a = t^m/t^b$, so $t^a = t^b$. This fact leads to the contradiction. If $n = 2$, from $(Q_{mj})$, we obtain $q^j = 1/R_{jm}$. On the other hand, we have $g_1 - 1 = q^j - 1 = 0$, which is impossible because of $R_{jm} \neq 1$. Consequently, we obtain $C - 1 = 0$.

3) Next we consider solution (iii) of $(P_\ell)$ for some $\ell$:

$$
g_1 - 1 = 0, \quad g_t - 1 = 0, \quad R_{am}(x^a - x^\ell) = R_{bm}(x^b - x^\ell)
$$

$(1 \leq a, b \leq n;\ a, b \neq \ell)$. Set as above

$$
C_\ell := R_{a\ell}(x^a - x^\ell) \qquad (a \neq \ell, m).
$$

Then

$$
\begin{aligned}
Y &= \sum_{a \neq m} q^a x^a = \sum_{a \neq m} q^a(x^a - x^m) + x^\ell \sum_{a \neq m} q^a\\
&= C_\ell \sum_{a \neq m,\ell} \frac{q^a}{R_{a\ell}} + x^\ell g_1 = C_\ell \frac{t^\ell g_t - g_1}{t^\ell - 1} + x^\ell g_1 = C_\ell + x^\ell
\end{aligned}
$$

by virtue of the equalities $g_1 = g_t = 1$ and

$$\sum_{a \neq \ell} \frac{q^a}{R_{a\ell}} = \frac{t^\ell g_t - g_1}{t^\ell - 1} = 1. \tag{3.11}$$

Let us examine the equation $(Q_{\ell\ell})$. Putting $Y = C + x^\ell$ into (3.6) and by using identities (0.3) and (3.11), we obtain

$$\begin{aligned} Q_{\ell\ell}[X] &= (C + x^\ell)^2 + 2q^\ell x^\ell (C + x^\ell) \\ &\quad - \sum_a^{m,\ell} \left\{ R_{a\ell} \left( \frac{C_\ell}{R_{a\ell}} + x^\ell \right)^2 + 2R_{\ell a} x^\ell \left( \frac{C}{R_{a\ell}} + x^\ell \right) + S_{\ell a}(x^\ell)^2 \right\} \\ &= q^\ell x^\ell (x^\ell - t^\ell x^\ell + 2C), \end{aligned}$$

$$Q_{\ell m}[X] = q^\ell [2(C + x^\ell) - 2R_{\ell m} x^\ell - R_{m\ell} - (x^\ell)^2 S_{im}] \qquad (\ell \neq m),$$

hence

$$Q_{\ell\ell}[X] - x^\ell Q_{\ell m}[X] = S_{\ell m} q^\ell x^\ell (x^\ell - 1) \left( x^\ell - \frac{t^m}{t^\ell} \right).$$

Since $S_{\ell m} q^\ell \neq 0$, we obtain the three cases to be considered:

$$\text{(A) } x^\ell = 0, \qquad \text{(B) } x^\ell = 1, \qquad \text{(C) } x^\ell = \frac{t^m}{t^\ell}.$$

We prove the followings:

(A′) If $x^\ell = 0$, then we have $x^a = 0$ $\quad (a \neq m)$.

(B′) If $x^\ell = 1$, then we have $x^a = 1$ $\quad (a \neq m)$.

(C′) If $x^\ell = t^m/t^\ell$, then we have $x^a = t^m/t^a$ $\quad (a \neq m)$.

(A′) In fact, we consider the equation $(Q_{i\ell})$ $(i \neq \ell, m)$. From identities (0.3) and $x^\ell = 0$, we obtain

$$Q_{i\ell}[X] = -S_{i\ell} q^i (x^i)^2,$$

hence

$$Q_{i\ell}[X] - x^\ell Q_{im}[X] = -S_{i\ell} q^i (x^i)^2.$$

Therefore the equations $(Q_{\ell i})$ leads to $x^i = 0$ $(i \neq \ell, m)$.

(B′) We proceed as in (A′). Using (0.3) and $x^\ell = 1$, we obtain

$$Q_{i\ell}[X] = q^i [2(C + 1) - 2R_{i\ell} x^i - R_{\ell i} - S_{i\ell}(x^i)^2],$$

$$Q_{im}[X] = q^i [2(C + 1) - 2R_{im} x^i - R_{mi} - S_{im}(x^i)^2]$$

for $i \neq \ell, m$, therefore

$$Q_{i\ell}[X] - Q_{im}[X] = (R_{i\ell} - R_{im})q^i(x^i)^2 \qquad (i \neq \ell, m).$$

Then the equation $(Q_{i\ell})$ implies $x^i = 1$ $(i \neq m, \ell)$.

(C′) Putting $x^\ell = t^m/t^\ell$ in the expressions (3.6) we have

$$Q_{i\ell}[X] = q^i \left[ 2\left(C + \frac{t^m}{t^\ell}\right)\frac{t^m}{t^\ell} - 2R_{im}x^i \frac{t^m}{t^\ell} - R_{\ell i}\left(\frac{t^m}{t^\ell}\right)^2 - S_{i\ell}(x^i)^2 \right],$$

$$Q_{im}[X] = q^i \left[ 2\left(C + \frac{t^m}{t^\ell}\right) - 2R_{im}x^i - R_{mi} - S_{im}(x^i)^2 \right].$$

We deduce

$$Q_{i\ell}[X] - \frac{t^m}{t^\ell} Q_{im}[X] = \left(\frac{t^m}{t^\ell} S_{im} - S_{i\ell}\right) q^i \left(x^i - \frac{t^m}{t^i}\right)^2 \qquad (i \neq \ell, m)$$

by virtue of identities (0.3). Then the equations $(Q_{i\ell})$ $(i \neq \ell, m)$ imply the assertion (3.13). Thus in cases (A), (B) and (C), we have obtained solutions (i), (ii) and (iii) of $(P_i)$'s and $(Q_{ij})$'s, respectively.

4) It is clear that solution $q^m = 0$, $x^a = 0$ $(1 \leq a \leq n;\ a \neq m)$ of $(P_i)$'s verifies the equations $(Q_{ij})$ $(1 \leq i, j \leq n;\ j \neq m)$. ∎

Finally by proceeding in the same way as in the proofs of the preceding lemmas, we can show the following proposition.

PROPOSITION 3.8. *All the solutions of the equations* $(P_i)$ *and* $(Q_{ij})$ $(1 \leq i, j \leq n;\ j \neq m)$ *are give by*

(i) $\quad g_t - 1 = 0, \quad x^a = \dfrac{t^m}{t^a}, \quad (1 \leq a \leq n;\ a \neq m),$

(ii) $\quad g_1 - 1 = 0, \quad x^a = 1, \quad (1 \leq a \leq n;\ a \neq m),$

(iii) $\quad q^m = 0, \quad x^a = 0, \quad (1 \leq a \leq n;\ a \neq m).$

## 4. - Demonstration of the theorem

### *4.1. The first monoidal transformation*

In Sections 2 and 3, we have seen the following:

(i) Accessible singular points of $\mathcal{H}_n^{(0)}$ are contained in $C^0 := \bigcup_m C_m^0 \subset \Sigma_\alpha \times B$, where $C_m^0$ are mutually disjoint submanifolds of $D^0 = (\Sigma_\alpha \backslash \Sigma_\alpha^0) \times B$ of dimension $2n-1$ given in Proposition 2.4.

(ii) A group $\tilde{L} \subset \mathrm{Aut}(\Sigma_\alpha \times B)$ acts transitively on $\{C_0^0, \dots, C_{n+2}^0\}$ as its permutation group.

Therefore, to study solutions of $\mathcal{H}_n^{(0)}$ at accessible singular points, it is sufficient to consider $\mathcal{H}_n^{(0)}$ at $C_m^0$ for some fixed $m \geq 1$. Moreover, taking into consideration the fact that $\tilde{\tau}_j : W_0 \times B \to W_j \times B$ is biholomorphic and that $\mathcal{H}_n^{(0)}|_{W_0 \times B}$ is taken into $\mathcal{H}_n^{(0)}|_{W_j \times B}$ by $\tilde{\tau}_j$, we can restrict our study of $\mathcal{H}_n^{(0)}$ at $C_m^0$ to its subset $C_m^0 \cap (W_0 \times B)$. Noting $C_m^0 \cap (W_0 \times B) \subset W_{0m} \times B$, we consider $\mathcal{H}_n^{(0)}$ in $W_{0m} \times B$.

Let $(q, x, t) := (q^1, \dots, q^n, x^1, \dots, x^n, t)$ be the affine coordinates of $W_{0m} \times B$ given in (2.1). The system $\mathcal{H}_n^{(0)}$ in $W_{0m} \times B$ can be written as

$$
(4.1) \qquad \begin{cases} x^m \mathrm{d}q^j = \sum_i A_{ij}(q, x, t)\mathrm{d}t^i & (1 \leq j \leq n) \\ x^m \mathrm{d}x^j = -\sum_i B_{ij}(q, x, t)\mathrm{d}t^i & (1 \leq j \leq n; j \neq m) \\ \mathrm{d}x^m = \sum_i B_{im}(q, x, t)\mathrm{d}t^i, \end{cases}
$$

where $A_{ij}$, $B_{ij} \in \mathbb{C}(t)[q, x]$ are given by

$$
(4.2) \qquad \begin{aligned} e(t^i)A_{ij} &= 2\sum_a E_{ija}x^a\Big|_{x^m=1} - F_{ij}x^m, \\ e(t^i)B_{ij} &= \sum_{a,b} E_{iab,j}x^a x^b\Big|_{x^m=1} - \sum_a^m F_{ia,j}x^a x^m \\ &\qquad - F_{im,j}x^m + \kappa\delta_{ij}(x^m)^2 - x^j e(t^i)B_{im}, \\ e(t^i)B_{im} &= \sum_{a,b} E_{iab,m}x^a x^b\Big|_{x^m=1} - \sum_a^m F_{ia,m}x^a x^m \\ &\qquad - F_{im,m}x^m + \kappa\delta_{ij}(x^m)^2. \end{aligned}
$$

Note that

$$
C_m^0 \cap (W_{0m} \times B) = \{(q, x, t) \in \mathbb{C}^{2n} \times B \mid q^m = x^1 = \cdots = x^n = 0\}.
$$

In the course of the demonstration of the theorem, we will see that $C_m^0$ is really a set of accessible singular points and an infinite number of $T$-leaves meet at each point of $C_m^0$. In order to make these leaves separated from each other, we perform a monoidal transformation in $\Sigma_\alpha \times B$ along $C_m^0$.

Since $C_j^0$'s are mutually disjoint, if we take an open set $C_m^0 \subset U \subset \Sigma_\alpha \times B$ containing no other $C_j^0$'s, the Pfaffian system in $(\mu_{C^0})^{-1}(U)$, obtained from $\mathcal{H}_n^{(0)}$ by a monoidal transformation $\mu_{C^0}$, is equivalent to the system in $(\mu_{C_m^0})^{-1}(U)$ obtained from $\mathcal{H}_n^{(0)}$ by the blowing up $\mu_{C_m^0}$. So, it suffices to consider $\mu_{C_m^0}$ in $W_{0m} \times B$.

For the sake of simplicity we write $W$ instead of $W_{0m}$ omitting its suffix. Let $\mu_{C_m^0} : W^{(1)} \times B \to W \times B$ be a blowing up and let $\mathcal{H}_n^{(1)}$ be a Pfaffian system on $W^{(1)} \times B$ obtained from $\mathcal{H}_n^{(0)}$ through $\mu_{C_m^0}$. In terms of the coordinate system, $\mathcal{H}_n^{(1)}$ is written in the following way.

Let $\eta = (\eta^0, \dots, \eta^n)$ be the homogeneous coordinates of $\mathbb{P}^n(\eta)$. Then

$$W^{(1)} \times B = \{(q, x, \eta, t) \in W \times \mathbb{P}^n(\eta) \times B$$
$$|x^a \eta^b - x^b \eta^a = 0,\ q^m \eta^b - x^b \eta^0 = 0\ (1 \le a, b \le n)\}.$$

Set

$$W^{(1)}(k) := \{(q, x, \eta) \in W^{(1)} \times \mathbb{P}^n(\eta) \mid \eta^k \neq 0\}.$$

By definition of $W^{(1)}$, we can take a coordinate system in $W^{(1)}(k) \times B$ as

$$\begin{aligned} &(q, y_0, t) && \text{in } W^{(1)}(0) \times B, \\ &(q', x^k, y_k, t) && \text{in } W^{(1)}(k) \times B \quad (1 \le k \le n), \end{aligned} \tag{4.3}$$

where $q' = (q^1, \dots, q^{m-1}, q^{m+1}, \dots, q^n)$ and

$$y_0 = (y_0^1, \dots, y_0^n) = \left(\frac{\eta^1}{\eta^0}, \dots, \frac{\eta^n}{\eta^0}\right) \text{ in } W^{(1)}(0),$$

$$y_k = (y_k^1, \dots, y_k^n) = \left(\frac{\eta^1}{\eta^k}, \dots, \frac{\eta^{k-1}}{\eta^k}, \frac{\eta^0}{\eta^k}, \frac{\eta^{k+1}}{\eta^k}, \dots, \frac{\eta^n}{\eta^k}\right) \text{ in } W^{(1)}(k).$$

We set $\phi_k := \mu_{C_m^0}|_{W^{(1)}(k) \times B}$. The mapping $\mu_{C_m^0} : W^{(1)} \times B \to W \times B$ is expressed as

$$(4.4)\qquad \begin{aligned} (q,x,t) &= \phi_0(q,y_0,t) = (q, q^m y_0, t) \\ (q,x,t) &= \phi_k(q', x^k, y_k, t) \\ &= (q^1,\dots,q^{m-1}, x^k y_k^k, q^{m+1},\dots,q^n, \\ &\qquad x^k y_k^1,\dots,x^k y_k^{k-1}, x^k, x^k y_k^{k+1},\dots,x^k y_k^n, t). \end{aligned}$$

Denote by $\mathcal{H}_n^{(1)}(k)$ the restriction of $\mathcal{H}_n^{(1)}$ to $W^{(1)}(k)\times B$. Then the systems $\mathcal{H}_n^{(1)}(k)$ are written as follows.

In $W^{(1)}(0)\times B$,

$$\mathcal{H}_n^{(1)}(0)\qquad \begin{cases} y_0^m \mathrm{d}q^i = \sum_i A_{ij}^0(q,y_0,t)\mathrm{d}t^i, \\ q^m y_0^m \mathrm{d}y_0^j = \sum_i B_{ij}^0(q,y_0,t)\mathrm{d}t^i, \quad (j\neq m) \\ q^m \mathrm{d}y_0^m = \sum_i B_{im}^0(q,y_0,t)\mathrm{d}t^i, \end{cases}$$

where the functions $A_{ij}^0$, $B_{ij}^0 \in \mathbb{C}(t)[q,y_0]$ are given by

$$\begin{aligned} A_{ij}^0 &= \frac{1}{q^m} A_{ij}\circ\phi_0, \\ B_{ij}^0 &= -\frac{1}{q^m}\{B_{ij}\circ\phi_0 + y_0^j A_{ij}\circ\phi_0\} \qquad (j\neq m), \\ B_{im}^0 &= B_{im}\circ\phi_0 - \frac{1}{q^m} A_{im}\circ\phi_0. \end{aligned}$$

Explicitly they are written as

$$(4.5)\qquad \begin{aligned} e(t^i)A_{ij}^0 &= \frac{2}{q^m} E_{ijm} - F_{im} y_0^m + h_{ij}^0, \\ e(t^i)B_{ij}^0 &= -\frac{1}{q^m} E_{imm,j} + F_{im,j} y_0^m \\ &\qquad - h_{im,j}^0 - q^m K_{ij}^0 - e(t^i) y_0^j B_{im}^0, \qquad (j\neq m) \\ e(t^i)B_{im}^0 &= E_{imm,m} - F_{im,m} q^m y_0^m \\ &\qquad + q^m h_{im,m}^0 + (q^m)^2 K_{im}^0 - e(t^i) A_{im}^0, \end{aligned}$$

where

$$h^0_{ij} = 2\sum_a^m E_{ija} y_0^a,$$

$$K^0_{ij} = \sum_{a,b}^m E_{iab,j} y_0^a y_0^b - y_0^m \sum_a^m F_{ia,j} y_0^a + \kappa\delta_{ij}(y_0^m)^2.$$

In $W^{(1)}(m) \times B$, we have

$$\mathcal{H}_n^{(1)}(m) \qquad \begin{cases} dq^j = \sum_i A^m_{ij}(q', x^m, y_m, t)dt^i & (j \neq m), \\ dx^m = \sum_i A^m_{im}(q', x^m, y_m, t)dt^i, \\ x^m dy^j_m = \sum_i B^m_{ij}(q', x^m, y_m, t)dt^i, \end{cases}$$

where the functions $A^m_{ij}$, $B^m_{ij} \in \mathbb{C}(t)[q', x^m, y_m]$ are determined from $A_{ij}$, $B_{ij}$ by

$$\begin{aligned} A^m_{ij} &= \frac{1}{x^m} A_{ij} \circ \phi_m, \\ A^m_{im} &= B_{im} \circ \phi_m, \\ B^m_{ij} &= -\frac{1}{x^m} B_{ij} \circ \phi_m - y^j_m B_{im} \circ \phi_m, \\ B^m_{im} &= \frac{1}{x^m} A_{im} \circ \phi_m - y^m_m B_{im} \circ \phi_m. \end{aligned}$$

Explicitly, we have

$$(4.6) \qquad \begin{aligned} e(t^i)A^m_{ij} &= \frac{2}{x^m} E_{ijm}(*) - F_{ij}(*) + h^m_{ij}, \\ e(t^i)A^m_{im} &= E_{imm,m}(*) - F_{im,m}(*)x^m + x^m h^m_{im,m} + (x^m)^2 K^m_{im}, \\ e(t^i)B^m_{ij} &= -\frac{1}{x^m} E_{imm,j}(*) + F_{im,j}(*) - h^m_{im,j} - x^m K^m_{ij}, \\ e(t^i)B^m_{im} &= \frac{2}{x^m} E_{imm}(*) - F_{im}(*) + h^m_{im} - y^m_m e(t^i)A^m_{im}, \end{aligned}$$

where $(*) = (t, q', x^m y^m_m)$ and

$$h^m_{ij} = 2\sum_a^m E_{ija}(*) y^a_m,$$

$$K^m_{ij} = \sum_{a,b}^m E_{iab,j}(*) y^a_m y^b_m - \sum_a^m F_{ia,j}(*) y^a_m + \kappa\delta_{ij}.$$

In $W^{(1)}(k)\times B$ $(k\neq 0,m)$, we have

$$\mathcal{H}_n^{(1)}(k)\qquad \begin{cases} y_k^m \mathrm{d}q^j = \sum_i A_{ij}^k(q',x^k,y_k,t)\mathrm{d}t^i & (j\neq m)\\ y_k^m \mathrm{d}x^k = \sum_i A_{im}^k(q',x^k,y_k,t)\mathrm{d}t^i & \\ x^k y_k^m \mathrm{d}y_k^j = \sum_i B_{ij}^k(q',x^k,y_k,t)\mathrm{d}t^i & (j\neq m)\\ x^k \mathrm{d}y_k^m = \sum_i B_{im}^k(q',x^k,y_k,t)\mathrm{d}t^i, & \end{cases}$$

where $A_{ij}^k$, $B_{ij}^k\in\mathbb{C}(t)[q',x^k,y_k]$ are determined from $A_{ij}$, $B_{ij}$ by

$$\begin{aligned} A_{ij}^k &= \frac{1}{x^k}\,A_{ij}\circ\phi_k,\\ A_{im}^k &= -\frac{1}{x^k}\,B_{ik}\circ\phi_k,\\ B_{ij}^k &= -\frac{1}{x^k}\{B_{ij}\circ\phi_k - y_k^j B_{ik}\circ\phi_k\},\\ B_{ik}^k &= \frac{1}{x^k}\{A_{im}\circ\phi_k + y_k^k B_{ik}\circ\phi_k\},\\ B_{im}^k &= B_{im}\circ\phi_k + \frac{1}{x^k}\,B_{ik}\circ\phi_k. \end{aligned}$$

We set

$$Z_{ij} := \frac{1}{x^k}\,E_{imm,j}(*) - F_{im,j}(*)y_k^m + h_{im,j}^k + x^k K_{ij}^k,$$

where

$$h_{ij}^k = 2E_{ikm,j}(*) + 2\sum_a^{k,m} E_{iam,j}(*)y_k^a,$$

$$\begin{aligned} K_{ij}^k = E_{ikk,j}(*) + 2\sum_a^{k,m} E_{ika,j}(*)y_k^a + \sum_{a,b}^{k,m} E_{iab,j}(*)y_k^a y_k^b \\ - F_{ik,j}(*)y_k^m - y_k^m\sum_a^{k,m} F_{ia,j}(*)y_k^a + \kappa\delta_{ij}(y_k^m)^2 \end{aligned}$$

and $(*) = (t, q', x^k y_k^k)$. Hence we obtain

$$
(4.7)\qquad
\begin{aligned}
e(t^i)A_{ij}^k &= \frac{2}{x^k} E_{ijm}(*) - F_{ij}(*)y_k^m + h_{ij}^k,\\
e(t^i)A_{im}^k &= -Z_{ik} + x^k Z_{im},\\
e(t^i)B_{ij}^k &= -Z_{ij} + y_k^j Z_{ik},\\
e(t^i)B_{ik}^k &= \frac{2}{x^k} E_{imm}(*) - F_{im}(*)y_k^m + h_{im}^k + y_k^k(Z_{ik} - x^k Z_{im}),\\
e(t^i)B_{im}^k &= Z_{ik}.
\end{aligned}
$$

We denote by $D_m^1 = (\mu_{C_m^0})^{-1}(C_m^0)$ the exceptional divisor and denote by the same symbol $D^0$ the proper transform $\overline{\mu_{C_m^0}^{-1}((D^0 \backslash C_m^0) \cap (W \times B))}$ ($\subset W^{(1)} \times B$) of the divisor $D^0 \cap (W \times B) \subset \Sigma_\alpha \times B$. In terms of the coordinates (4.3) in $W^{(1)}(k) \times B$, the sets $D_m^1$ and $D^0$ in $W^{(1)}(k) \times B$, the sets $D_m^1$ and $D^0$ in $W^{(1)} \times B$ are written as

$$
\begin{aligned}
D_m^1 = \{(q, y_0, t) \in W^{(1)}(0) \times B \mid q^m = 0\}&\\
\cup \bigcup_{k=1}^{n} \{(q', x^k, y_k, t) \in W^{(1)}(k) \times B \mid x^k = 0\},&\\
D^0 = \{(q, y_0, t) \in W^{(1)}(0) \times B \mid y_0^m = 0\}&\\
\cup \bigcup_{k \neq m} \{(q', x^k, y_k, t) \in W^{(1)}(k) \times B \mid y_k^m = 0\}.&
\end{aligned}
$$

Now let us determine the set of singular points Sing $\mathcal{H}_n^{(1)}$ of the system $\mathcal{H}_n^{(1)}$.

PROPOSITION 4.1.

$$\operatorname{Sing} \mathcal{H}_n^{(1)} = D^0 \cup C_m^1,$$

*where $C_m^1$ is a hypersurface of $D_m^1$ which is, by setting $C_{mi}^1 := C_m^1 \cap (W^{(1)}(i) \times B)$, given by*

$$
\begin{aligned}
C_{m0}^1 &= \{(q, y_0, t) \in W^{(1)}(0) \times B \mid q^m = 1 - \theta_m y_0^m = 0\}\\
C_{mm}^1 &= \{(q', x^m, y_m, t) \in W^{(1)}(m) \times B \mid x^m = y_m^m - \theta_m = 0\}\\
C_{mk}^1 &= \{(q', x^k, y_k, t) \in W^{(1)}(k) \times B \mid x^k = y_k^k - \theta_m y_k^m = 0\},
\end{aligned}
$$

*where $k \neq 0, m$.*

PROOF. We see immediately that the set $D^0$ belongs to the singularity of $\mathcal{H}_n^{(1)}(k)$. We shall show that, in $W^{(1)}(m) \times B$, the singularity of $\mathcal{H}_n^{(1)}(m)$ is

$$\{(q', x^m, y_m) \in W^{(1)}(m) \mid x^m = y_m^m - \theta_m = 0\} \times B.$$

Explicit form of $\mathcal{H}_n^{(1)}(m)$ says that the condition that a point $(q', x^m, y_m, t) \in W^{(1)}(m) \times B$ belongs to Sing $\mathcal{H}_n^{(1)}$ is

$$x^m = 0, \quad \det L_m = 0,$$

where $L_m = (e(t^i)B_{ij}^m)_{i,j=1,\cdots,n}$. On the other hand, by using the expressions of $E_{iab}$ and $F_{ia}$ in the Introduction, we have

$$\begin{aligned} e(t^i)B_{im}^m &= \frac{2}{x^m} E_{imm}(*) - F_{im}(*) - y_m^m E_{imm,m}(*) + O(x^m) \\ &= \begin{cases} R_{mi} q^i (\theta_m - y_m^m) + O(x^m) & (i \neq m), \\ \left(t^m - \sum_a^m S_{ma} q^a\right)(\theta_m - y_m^m) + O(x^m) & (i = m), \end{cases} \end{aligned} \tag{4.8}$$

$$\begin{aligned} e(t^i)B_{ij}^m &= \frac{1}{x^m} E_{imm,j}(*) - F_{im,j}(*) + O(x^m) \\ &= \begin{cases} \delta_{ij} R_{mi} q^i (\theta_m - y_m^m) + O(x^m) & (i \neq m), \\ S_{mj}(\theta_m - y_m^m) + O(x^m) & (i = m;\ j \neq m), \end{cases} \end{aligned} \tag{4.9}$$

where $(*) = (t, q', x^m y_m^m)$. It follows that

$$\det L_m \equiv (\theta_m - y_m^m)^n \det L_m' \qquad (\text{mod. } x^m) \tag{4.10}$$

where

$$L_m' = \begin{pmatrix} R_{m1} & & & R_{m1}q^1 & & & \\ & \ddots & & \vdots & & & \\ & & R_{m,m-1} & R_{m,m-1}q^{m-1} & & & \\ S_{m1} & \cdots & S_{m,m-1} & \Delta & S_{m,m+1} & \cdots & S_{mn} \\ & & & R_{m,m+1}q^{m+1} & R_{m,m+1} & & \\ & & & \vdots & & \ddots & \\ & & & R_{mn}q^n & & & R_{mn} \end{pmatrix},$$

$\Delta$ being given by

$$\Delta = \sum_a^m S_{ma} q^a - t^m. \tag{4.11}$$

Moreover

$$\begin{aligned} \det L_m' &= \det L_m'|_{q'=0} \\ &= (-1)^{n-m+1} t^m \prod_k^m R_{mk} \neq 0. \end{aligned}$$

Hence we have

$$x^m = \det L_m = 0 \Longleftrightarrow x^m = y^m_m - \theta_m = 0.$$

This proves the assertion for $\mathcal{H}_n^{(1)}(m)$. We can prove, in the same manner, that the singularities of $\mathcal{H}_m^{(1)}(k)$ $(k \neq m)$ are those given in the proposition. ∎

PROPOSITION 4.2. *The set of accessible singular points of the system* $\mathcal{H}_n^{(1)}$ *in* $W^{(1)} \times B$ *is contained in* $C^1_m$.

PROOF. From the definition of monoidal transformation, the mapping

$$\mu_{C^0_m} : (W^{(1)} \times B) \backslash D^1_m \to (W \times B) \backslash C^0_m$$

is biholomorphic. Since an accessible singular point of $\mathcal{H}_n^{(0)}$ in $W \times B$ belongs to $C^0_m$, those of $\mathcal{H}_n^{(1)}$ are in $D^1_m$. Noting $D^1_m \cap \operatorname{Sing} \mathcal{H}_n^{(1)} = C^1_m$ we have the proposition. ∎

## 4.2. *Second monoidal transformation along* $C^1_{mm}$

Perform a monoidal transformation $\mu_{C^1_m} : W^{(2)} \times B \to W^{(1)} \times B$ along $C^1_m$ and let $\mathcal{H}_n^{(2)}$ be the Pfaffian system on $W^{(2)} \times B$ which is obtained from $\mathcal{H}_n^{(1)}$ on $W^{(1)} \times B$. Let us denote by $D^0$ and $D^1_m$ the submanifolds of $W^{(2)} \times B$ given by $\overline{(\mu_{C^1_m})^{-1}(D^0 \backslash C^1_m)}$ and $\overline{(\mu_{C^1_m})^{-1}(D^1_m \backslash C^1_m)}$, respectively. And let $D^2_m := (\mu_{C^1_m})^{-1}(C^1_m)$ be the exceptional manifold.

In the rest of this section we will show

PROPOSITION 4.3. *The singularity of the system* $\mathcal{H}_n^{(2)}$ *in* $W^{(2)} \times B$ *is* $D^0$.

PROPOSITION 4.4. *The system* $\mathcal{H}_n^{(2)}$ *in* $W^{(2)} \times B$ *has no accessible singularity.*

PROPOSITION 4.5. *The manifold* $D^1_m \subset W^{(2)} \times B$ *is filled with vertical leaves.*

PROPOSITION 4.6. *The leaves in* $W^{(2)} \times B \backslash (D^0 \cup D^1_m)$ *are transversal to the fibers of* $\pi^{(2)} : W^{(2)} \times B \to B$.

PROPOSITION 4.7. *There is no leaf of the foliation defined by* $\mathcal{H}_n^{(2)}$ *which is entirely contained in* $D^2_m$.

Let us express the system $\mathcal{H}_n^{(2)}$ in terms of the coordinates in $W^{(2)} \times B$. Note that

$$W^{(1)} \times B = \bigcup_{0 \leq k \leq n} W^{(1)}(k) \times B,$$

where $W^{(1)}(k) \times B$ is the affine coordinate neighbourhood given in Subsection 4.1, and

$$C^1_{mk} = C^1_m \cap (W^{(1)}(k) \times B).$$

Set $W^{(2)}(k) \times B := (\mu_{C^1_m})^{-1}(W^{(1)}(k) \times B)$. In this subsection, we consider $\mathcal{H}_n^{(2)}$ on $W^{(2)}(m) \times B$. The system $\mathcal{H}_n^{(2)}$ on other $W^{(2)}(k) \times B$ will be considered in the next subsection.

Let $(q', x^m, y_m) \to (u, v)$ be an affine change of coordinates in $W^{(1)}(m)$ defined by

$$u = (q^1, \dots, q^{m-1}, x^m, q^{m+1}, \dots, q^n),$$
$$v = (y^1_m, \dots, y^{m-1}_m, y^m_m - \theta_m, y^{m+1}_m, \dots, y^n_m)$$

so that

$$C^1_{mm} = \{(u, v, t) \in W^{(1)}(m) \times B \mid u^m = v^m = 0\}.$$

Then the system $\mathcal{H}_n^{(1)}$ on $W^{(1)}(m) \times B$ is written as

$$\mathcal{H}_n^{(1)}(m) \qquad \begin{cases} \mathrm{d}u^j = \sum_i \tilde{A}_{ij}(u, v, t)\mathrm{d}t^i, \\ u^m \mathrm{d}v^j = \sum_i \tilde{B}_{ij}(u, v, t)\mathrm{d}t^i. \end{cases}$$

The functions $\tilde{A}_{ij}$, $\tilde{B}_{ij} \in \mathbb{C}(t)[u, v]$ are given by

$$(4.12) \qquad \begin{aligned} e(t^i)\tilde{A}_{ij} &= \frac{2}{u^m} E_{ijm}(*) - F_{ij}(*) + h_{ij}, \\ e(t^i)\tilde{A}_{im} &= E_{imm,m}(*) - F_{im,m}(*)u^m + u^m h_{im,m} + \mathcal{O}((u^m)^2), \\ e(t^i)\tilde{B}_{ij} &= -\frac{1}{u^m} E_{imm,j}(*) + F_{im,j}(*) + \mathcal{O}(u^m), \\ e(t^i)\tilde{B}_{im} &= \frac{2}{u^m} E_{imm}(*) - F_{im}(*) + h_{im} - (v^m + \theta_m)e(t^i)\tilde{A}_{im}, \end{aligned}$$

where $j \neq m$, $(*) = (t, u', (v^m + \theta_m)u^m)$ and

$$h_{ij} = 2\sum_a^m E_{ija}(*)v^a.$$

Noting that the defining equations of $C^1_{mm}$ is $u^m = v^m = 0$, we have

$$W^{(2)}(m) \times B = \{(u, v, \varsigma, t) \in W^{(1)}(m) \times \mathbb{P}^1(\varsigma) \times B \mid \varsigma^0 v^m - \varsigma^1 u^m = 0\}.$$

Moreover $W^{(2)}(m)$ is covered by two affine charts

$$W^{(2)}(m, \ell) = \{(u, v, \varsigma) \in W^{(2)}(m) \mid \varsigma^\ell \neq 0\} \qquad (\ell = 0, 1),$$

and, in each $W^{(2)}(m,\ell)$, we can take the coordinates

$$\left\{\begin{array}{ll}(u,v',w) & \text{in } W^{(2)}(m,0)\\ (u',z,v) & \text{in } W^{(2)}(m,1)\end{array}\right. \tag{4.13}$$

where

$$u' = (u^1,\dots,u^{m-1},u^{m+1},\dots,u^n),$$
$$v' = (v^1,\dots,v^{m-1},v^{m+1},\dots,v^n),$$

and $z$, $w$ are defined by

$$z = \varsigma^0/\varsigma^1, \qquad w = \varsigma^1/\varsigma^0.$$

In terms of coordinates (4.13), the map $\mu_{C^1_m} : W^{(2)}(m)\times B \to W^{(1)}(m)\times B$ is expressed as

$$(u,v',v^m,t) = \mu_{C^1_m}|_{W^{(2)}(m,0)\times B}(u,v',w,t) = (u,v',u^m w,t)$$
$$(u',v,u^m,t) = \mu_{C^1_m}|_{W^{(2)}(m,1)\times B}(u',v,z,t) = (u',v,zv^m,t).$$

Let $\mathcal{H}_n^{(2)}(m,\ell)$ be the restriction of $\mathcal{H}_n^{(2)}$ to $W^{(2)}(m,\ell)\times B$. Explicitly, in $W^{(2)}(m,0)\times B$, we have

$$\mathcal{H}_n^{(2)}(m,0) \qquad \left\{\begin{array}{ll} du^j = \sum\limits_i \tilde{A}^0_{ij}(u,v',w,t)dt^i & \\ u^m dv^j = \sum\limits_i \tilde{B}^0_{ij}(u,v',w,t)dt^i & (j\neq m)\\ u^m dw = \sum\limits_i \tilde{B}^0_{im}(u,v',w,t)dt^i & \end{array}\right.$$

where $\tilde{A}^0_{ij}$ and $\tilde{B}^0_{ij}$ are determined from $\tilde{A}_{ij}$ and $\tilde{B}_{ij}$ by

$$\begin{aligned}\tilde{A}^0_{ij} &= \tilde{A}_{ij}(u,v',u^m w,t),\\ \tilde{B}^0_{ij} &= \tilde{B}_{ij}(u,v',u^m w,t),\\ \tilde{B}^0_{im} &= \frac{1}{u^m}\tilde{B}_{im}(u,v',u^m w,t) - w\tilde{A}_{im}(u,v',u^m w,t).\end{aligned} \tag{4.14}$$

In $W^{(2)}(m,1)\times B$,

$$\mathcal{H}_n^{(1)}(m,1) \qquad \left\{\begin{array}{ll} du^j = \sum\limits_i \tilde{A}^1_{ij}(u',v,z,t)dt^i & (j\neq m)\\ v^m dz = \sum\limits_i \tilde{A}^1_{im}(u',v,z,t)dt^i & \\ z dv^j = \sum\limits_i \tilde{B}^1_{ij}(u',v,z,t)dt^i & \end{array}\right.$$

where $\tilde{A}^1_{ij}$ and $\tilde{B}^1_{ij}$ are determined from $\tilde{A}_{ij}$ and $\tilde{B}_{ij}$ by

$$
\begin{aligned}
&\tilde{A}^1_{ij} = \tilde{A}_{ij}(u', v, zv^m, t),\\
(4.15)\qquad &\tilde{A}^1_{im} = \tilde{A}_{im}(u', v, zv^m, t) - \frac{1}{v^m}\tilde{B}_{im}(u', v, zv^m, t),\\
&\tilde{B}^1_{ij} = \frac{1}{v^m}\tilde{B}_{ij}(u', v, zv^m, t).
\end{aligned}
$$

Note here that

$$
\begin{aligned}
D^0 \cap (W^{(2)} \times B) &= \emptyset,\\
D^1_m \cap (W^{(2)}(m,0) \times B) &= \emptyset,\\
D^1_m \cap (W^{(2)}(m,1) \times B) &= \{(u', v, z, t) \mid z = 0\},\\
D^2_m \cap (W^{(2)}(m,0) \times B) &= \{(u, v', w, t) \mid u^m = 0\},\\
D^2_m \cap (W^{(2)}(m,1) \times B) &= \{(u', v, z, t) \mid v^m = 0\}.
\end{aligned}
$$

LEMMA 4.8. *The system $\mathcal{H}^{(2)}_n(m,0)$ is free from singularities in $W^{(2)}(m,0)\times B$.*

LEMMA 4.9. *The system $\mathcal{H}^{(2)}_n(m,1)$ is free from singularities in $W^{(2)}(m,1)\times B$.*

SUBLEMMA 4.10. *For functions $f(x)$, $g(x)$ holomorphic at $x = 0$ and satisfying $f(0) = 0$, we have*

$$
\begin{aligned}
\frac{2}{X} f(XY) + g(XY) - (Y + Z)\{f'(XY) + Xg(XY)\}&\\
= f'(0)(Y - Z) + g(0) - \{Yf'(0) - g'(0)\}XZ + O(X^2)&
\end{aligned}
$$

PROOF OF LEMMA 4.8. Let us compute $\tilde{B}^0_{ij}$ modulo $(u^m)$. Use the notation: $(\sharp) = (t, u', (u^m w + \theta_m)u^m)$ and $(\sharp_0) = (t, u', 0)$. From the expressions (4.12) and (4.14) for $\tilde{A}^0_{ij}$ and $\tilde{B}^0_{ij}$, and using the equalities

$$
\theta_m E_{imm,m}(\sharp_0) - F_{im}(\sharp_0) = 0 \qquad (1 \le i \le n),
$$

we have, for $j \neq m$,

$$
\begin{aligned}
e(t^i)\tilde{B}^0_{ij} &= e(t^i)\tilde{B}_{ij}(u, v', u^m w, t)\\
&= -\frac{1}{u^m} E_{imm,j}(\sharp) + F_{im,j}(\sharp) + O(u^m)\\
&= -u^m w E_{imm,j,m}(\sharp_0) + O(u^m).
\end{aligned}
$$

Moreover, by applying Sublemma 4.10 to $f(x) = E_{imm}(t,u',x)$ and $g(x) = -F_{im}(t,u',x) + h_{im}(t,u',x,v')$ with $X = u^m$, $Y = u^m w + \theta_m$, $Z = u^m w$, we have

$$\begin{aligned}
e(t^i)u^m\tilde{B}^0_{im} &= e(t^i)\tilde{B}_{im}(u,v',u^m w,t) - u^m w\, e(t^i)\tilde{A}_{im}(u,v',u^m w,t) \\
&= \frac{2}{u^m} E_{imm}(\sharp) - F_{im}(\sharp) + h_{im}(\sharp,v') - (2u^m w + \theta_m)\{E_{imm,m}(\sharp) \\
&\qquad\qquad - u^m F_{im,m}(\sharp) + u^m h_{im,m}(\sharp,v') + O((u^m)^2)\}, \\
&= \theta_m E_{imm,m}(\sharp_0) - F_{im}(\sharp_0) + O((u^m)^2) \\
&= O((u^m)^2).
\end{aligned}$$

Hence

$$e(t^i)\tilde{B}^0_{ij} = O(u^m)$$

for $i, j = i, \ldots, n$. This proves the lemma. ∎

PROOF OF LEMMA 4.9. Set $(\sharp) = (t,u',(v^m+\theta_m)zv^m)$ and $(\sharp_0) = (t,u',0)$. We claim that $\tilde{A}^1_{ij}$, $\tilde{B}^1_{ij} \in \mathbb{C}(t)[u',v,z]$,

$$e(t^i)\tilde{A}^1_{im} = O(zv^m) \tag{4.16}$$

and

$$e(t^i)\tilde{B}^1_{ij} = \begin{cases} E_{imm,m,j}(\sharp_0) + O(z) & (j \neq m), \\ E_{imm,m}(\sharp_0) + O(z^2 v^m) & (j = m), \end{cases} \tag{4.17}$$

In fact, using the expressions (4.12) and (4.15) for $\tilde{A}^1_{ij}$ and $\tilde{B}^1_{ij}$, we have, for $j \neq m$,

$$\begin{aligned}
v^m e(t^i)\tilde{B}^1_{ij} &= e(t^i)\tilde{B}_{ij}(u',v,zv^m,t) \\
&= -\frac{1}{zv^m} E_{imm,j}(\sharp) + F_{im,j}(\sharp) + O(zv^m) \\
&= -E_{imm,m,j}(\sharp_0)(v^m+\theta_m) + F_{im,j}(\sharp_0) + O(zv^m) \\
&= -v^m E_{imm,m,j}(\sharp_0) + O(zv^m),
\end{aligned}$$

and applying Sublemma 4.10 to functions $f(x) = E_{imm}(t,u',x)$ and $g(x) = -F_{im}(t,u',x) + h_{im}(t,u',x,v')$ with $X = zv^m$, $Y = v^m + \theta_m$, $Z = 0$, we have

$$\begin{aligned}
v^m e(t^i)\tilde{B}^1_{im} &= E_{imm,m}(\sharp_0)(v^m+\theta_m) - F_{im}(\sharp_0) + O((zv^m)^2) \\
&= v^m E_{imm,m}(\sharp_0) + O((zv^m)^2).
\end{aligned}$$

This proves the assertion (4.17). Assertion (4.16) can be proved in a similar way. It follows from (4.16) and (4.17) that a point $p := (u', v, z, t) \in W^{(2)}(m, 1) \times B$ is a singular point of $\mathcal{H}_n^{(2)}(m, 1)$ if and only if

$$z = 0, \qquad \det L_m^1 = 0,$$

where $L_m^1 := (e(t^i)\tilde{B}_{ij}^1)_{1 \le i,j \le n}$. Let us show $\det L_m^1 \neq 0$ under the condition $z = 0$. Indeed, combining (4.17) with

$$E_{imm,m}(\sharp_0) = \begin{cases} -u^i R_{mi} & (i \neq m), \\ t^m - \sum_a^m S_{ma} u^a & (i = m), \end{cases} \tag{4.18}$$

$$E_{imm,j,m}(\sharp_0) = \begin{cases} 0 & (i \neq j, m) \\ -R_{mj} & (i \neq j), \\ -S_{mj} & (i = m), \end{cases}$$

we have, for $j \neq m$,

$$e(t^i)\tilde{B}_{ij}^1 \equiv \begin{cases} -\delta_{ij} R_{mi} & (i \neq m) \\ -S_{mj} & (i = m) \end{cases}$$

and

$$e(t^i)\tilde{B}_{im}^1 \equiv \begin{cases} R_{mi} u^i & (i \neq m) \\ \Delta & (i = m) \end{cases}$$

(mod. $z$), where $\Delta$ is that given by (4.11). Hence

$$\begin{aligned} &\det L_m^1 \\ &= \pm \det \begin{pmatrix} R_{m1} & & & R_{m1}u^1 & & & \\ & \ddots & & \vdots & & & \\ & & R_{m,m-1} & R_{m,m-1}u^{m-1} & & & \\ S_{m1} & \cdots & S_{m,m-1} & \Delta & S_{m,m+1} & \cdots & S_{mn} \\ & & & R_{m,m+1}u^{m+1} & R_{m,m+1} & & \\ & & & \vdots & & \ddots & \\ & & & R_{mn}u^n & & & R_{mn} \end{pmatrix} \\ &\equiv \pm t^m \prod_{i \neq m} R_{mi} \neq 0 \qquad (\text{mod. } z). \end{aligned}$$

This is what we want to prove. ■

PROOF OF PROPOSITION 4.5 FOR $W^{(2)}(m) \times B$. Note that

$$D_m^1 \cap (W^{(2)}(m) \times B) = D_m^1 \cap (W^{(2)}(m,1) \times B)$$
$$= \{(u', v, z, t) \mid z = 0\}.$$

As it is seen in the proof of Lemma 4.9, the matrix $L_m^1 := (e(t^i)\tilde{B}_{ij}^1)$ for the system $\mathcal{H}_n^{(2)}(m,1)$ is non-singular under the condition $z = 0$. It follows that $\mathcal{H}_n^{(2)}(m,1)$ implies $dt^1 = \cdots = dt^n = 0$ on $D_m^1 \cap (W^{(2)}(m,1) \times B)$. This shows that any leaf of $\mathcal{F}^{(2)}$ passing through $D_m^1 \cap (W^{(2)}(m,1) \times B)$ is contained in $D_m^1 \cap (W^{(2)}(m,1) \times B)$ and is a vertical one. ∎

PROOF OF PROPOSITION 4.6 FOR $W^{(2)}(m) \times B$. The result follows immediately from the form of the systems $\mathcal{H}_n^{(2)}(m,0)$ and $\mathcal{H}_n^{(2)}(m,1)$. ∎

PROOF OF PROPOSITION 4.7 FOR $W^{(2)}(m) \times B$. Note that

$$D_m^2 \cap (W^{(2)}(m,0) \times B) = \{(u, v', w, t) \mid u^m = 0\}.$$

Suppose that there is a leaf of $\mathcal{F}^{(2)}$ contained entirely in $D_m^2$, and let $(u(t), v'(t), w(t))$ be a solution of $\mathcal{H}_n^{(2)}(m,0)$ representing this leaf. We have $u^m(t) \equiv 0$. Putting this solution into $\mathcal{H}_n^{(2)}(m,0)$, we have

$$\tilde{A}_{im}^0(u'(t), 0, v'(t), 0, t) = E_{imm,m}(t, u'(t), 0) = 0.$$

But this is impossible because of (4.12), (4.14) and (4.18). The proof for $W^{(2)}(m,1) \times B$ is carried out in the same way. ∎

### 4.3. Second blowing up along $C_{mk}^1$

As in the second part of this section, we study the set of singularity of the system $\mathcal{H}_n^{(2)}$ on $W^{(2)}(k) \times B$ $(k \neq 0, m)$. First we make a change of coordinates $(q', x^k, y_k) \to (u, v)$ in $W^{(1)}(k)$ defined by

$$u = (q^1, \ldots, q^{m-1}, x^k, q^{m+1}, \ldots, q^n),$$
$$v = (y_k^1, \ldots, y_k^{k-1}, y_k^k - \theta_m y_k^m, y_k^{k+1}, \ldots, y_k^n),$$

so that

$$C_{mk}^1 = \{(u, v, t) \in W^{(1)}(k) \times B \mid u^m = v^k = 0\}. \tag{4.19}$$

Set $(*) = (t, u^1, \ldots, u^{m-1}, (v^k + \theta_m v^m)u^m, u^{m+1}, \ldots, u^n)$. Then the system

$\mathcal{H}_n^{(1)}(k)$ is rewritten in the form

$$\mathcal{H}_n^{(1)}(k) \qquad \begin{cases} v^m \mathrm{d}u^j = \sum_i \tilde{A}_{ij}(u,v,t)\mathrm{d}t^i \\ u^m v^m \mathrm{d}v^j = \sum_i \tilde{B}_{ij}(u,v,t)\mathrm{d}t^i \quad (j \neq m) \\ u^m \mathrm{d}v^m = \sum_i \tilde{B}_{im}(u,v,t)\mathrm{d}t^i \end{cases}$$

where $\tilde{A}_{ij},\ \tilde{B}_{ij} \in \mathbb{C}(t)[u,v]$ are

$$(4.20)\qquad \begin{aligned} e(t^i)\tilde{A}_{ij} &= \frac{2}{u^m} E_{ijm}(*) - F_{ij}(*)v^m + h_{ij}(*,v''), \\ e(t^i)\tilde{A}_{im} &= -Z_{ik} + u^m Z_{ik}, \\ e(t^i)\tilde{B}_{ij} &= -Z_{ij} + v^j Z_{ik}, \\ e(t^i)\tilde{B}_{ik} &= \frac{2}{u^m} E_{imm}(*) - F_{im}(*)v^m + h_{im}(*,v'') \\ &\qquad + v^k Z_{ik} - (v^k + \theta_m v^m)u^m Z_{im}, \\ e(t^i)\tilde{B}_{im} &= Z_{ik} \end{aligned}$$

and setting $v'' = (v^1, \cdots \overset{m}{\vee} \cdots \overset{k}{\vee} \cdots, v^n)$, $h_{ij}$ and $Z_{ij}$ $(1 \le i,j \le n)$ are given by

$$h_{ij} = 2E_{ijk}(*) + 2\sum_a^{k,m} E_{ija}(*)v^a,$$

$$Z_{ij} = \frac{1}{u^m} E_{imm,j}(*) - F_{im,j}(*)v^m + h_{im,j}(*,v'') + u^m K_{ij}$$

and

$$\begin{aligned} K_{ij} = E_{ikk,j}(*) + 2\sum_{a \neq k,m} E_{ika,j}(*)v^a + \sum_{a,b \neq k,m} E_{iab,j}(*)v^a v^b \\ - F_{ik,j}(*)v^m - v^m \sum_{a \neq k,m} F_{ia,j}(*)v^a + \kappa\delta_{ij}(v^m)^2. \end{aligned}$$

Set $W^{(2)}(k) \times B = (\mu_{C^1_m})^{-1}(W^{(1)}(k) \times B)$. Noting that $C^1_{mk}$ is given by (4.19), we have

$$W^{(2)}(k) \times B = \{(u,v,\varsigma,t) \in W^{(1)}(k) \times \mathbb{P}^1(\varsigma) \times B \mid \varsigma^0 v^k - \varsigma^1 u^m = 0\}.$$

Then $W^{(2)}(k) \times B$ is covered by two affine charts

$$W^{(2)}(k,\ell) \times B = \{(u,v,\varsigma,t) \in W^{(2)}(k) \times B \mid \varsigma^\ell \neq 0\}$$

with coordinates systems

$$
\text{(4.21)} \qquad \begin{cases} (u, v', w, t) \ \text{ in } W^{(2)}(k, 0) \\ (u', v, z, t) \ \text{ in } W^{(2)}(k, 1), \end{cases}
$$

where

$$
u' = (u^1, \dots, u^{m-1}, u^{m+1}, \dots, u^n),
$$
$$
v' = (v^1, \dots, v^{k-1}, v^{k+1}, \dots, v^n),
$$

and

$$
w = \varsigma^1/\varsigma^0, \qquad z = \varsigma^0/\varsigma^1.
$$

In terms of these coordinates the blowing up $\mu_{C_m^1} : W^{(2)}(k) \times B \to W^{(1)}(k) \times B$ is written as

$$
(u, v', v^k, t) = (\mu_{C_m^1}|_{W^{(2)}(k,0)\times B})(u, v', w, t) = (u, v', u^m w, t),
$$
$$
(u', v, u^m, t) = (\mu_{C_m^1}|_{W^{(2)}(k,1)\times B})(u', v, z, t) = (u', v, z v^k, t).
$$

Denote by $\mathcal{H}_n^{(2)}(k, \ell)$ the restriction of $\mathcal{H}_n^{(2)}(k)$ on a chart $W^{(2)}(k, \ell) \times B$.

In $W^{(2)}(k, 0) \times B$, we have

$$
\mathcal{H}_n^{(2)}(k,0) \qquad \begin{cases} v^m \mathrm{d}u^j = \sum_i \tilde{A}^0_{ij}(u, v', w, t)\mathrm{d}t^i, \\ v^m \mathrm{d}v^j = \sum_i \tilde{B}^0_{ij}(u, v', w, t)\mathrm{d}t^i \qquad (j \neq k, m), \\ v^m \mathrm{d}w = \sum_i \tilde{B}^0_{ik}(u, v', w, t)\mathrm{d}t^i, \\ \mathrm{d}v^m = \sum_i \tilde{B}^0_{im}(u, v', w, t)\mathrm{d}t^i, \end{cases}
$$

where $\tilde{A}^0_{ij}$ and $\tilde{B}^0_{ij}$ are obtained from $\tilde{A}_{ij}$ and $\tilde{B}_{ij}$ in (4.20) by

$$
\text{(4.22)} \qquad \begin{aligned} \tilde{A}^0_{ij} &= \tilde{A}_{ij}(u, v', u^m w, t), \\ \tilde{B}^0_{ij} &= \frac{1}{u^m}\tilde{B}_{ij}(u, v', u^m w, t), \\ \tilde{B}^0_{ik} &= \frac{1}{(u^m)^2}\{\tilde{B}_{ik}(u, v', u^m w, t) - u^m w \tilde{A}_{im}(u, v', u^m w, t)\}. \end{aligned}
$$

In $W^{(2)}(k,1)\times B$,

$$\mathcal{H}_n^{(2)}(k,1)\qquad \begin{cases} v^m \mathrm{d}u^j = \sum_i \tilde{A}^1_{ij}(u',v,z,t)\mathrm{d}t^i & (j\neq m),\\ v^m\mathrm{d}z = \sum_i \tilde{A}^1_{im}(u',v,z,t)\mathrm{d}t^i, & \\ zv^m\mathrm{d}v^j = \sum_i \tilde{B}^1_{ij}(u',v,z,t)\mathrm{d}t^i & (j\neq m),\\ z\mathrm{d}v^m = \sum_i \tilde{B}^1_{im}(u',v,z,t)\mathrm{d}t^i & \end{cases}$$

where

$$\begin{aligned} \tilde{A}^1_{ij} &= \tilde{A}_{ij}(u',v,zv^k,t),\\ \tilde{A}^1_{im} &= \frac{1}{v^k}\left(\tilde{A}_{im}(u',v,zv^k,t)-\frac{1}{v^k}\tilde{B}_{ik}(u',v,zv^k,t)\right),\\ \tilde{B}^1_{ij} &= \frac{1}{v^k}\tilde{B}_{ij}(u',v,zv^k,t). \end{aligned}\tag{4.23}$$

We shall show

LEMMA 4.11. *The singularity of the system* $\mathcal{H}_n^{(2)}(k,0)$ *is contained in* $D^0\cap(W^{(2)}(k,0)\times B)$.

LEMMA 4.12. *The set* $D^0\cap(W^{(2)}(k,0)\times B)$ *contains no accessible singular point.*

Note here that

$$\begin{aligned} D^0\cap(W^{(2)}(k,0)\times B) &= \{(u,v',w,t)\in W^{(2)}(k,0)\times B \mid v^m=0\},\\ D^0\cap(W^{(2)}(k,1)\times B) &= \{(u',v,z,t)\in W^{(2)}(k,1)\times B \mid v^m=0\},\\ D^1_m\cap(W^{(2)}(k,0)\times B) &= \emptyset,\\ D^1_m\cap(W^{(2)}(k,1)\times B) &= \{(u',v,z,t)\in W^{(2)}(k,1)\times B \mid z=0\},\\ D^2_m\cap(W^{(2)}(k,0)\times B) &= \{(u,v',w,t)\in W^{(2)}(k,0)\times B \mid u^m=0\},\\ D^2_m\cap(W^{(2)}(k,1)\times B) &= \{(u',v,z,t)\in W^{(2)}(k,1)\times B \mid v^k=0\}. \end{aligned}$$

PROOF OF LEMMA 4.11. We assert $\tilde{A}^0_{ij}$, $\tilde{B}^0_{ij}\in\mathbb{C}(t)[u,v',w]$. Set $(\sharp)=(t,u',(u^mw+\theta_mv^m)u^m)$, $(\sharp_0)=(t,u',0)$ and $\tilde{Z}_{ij}:=Z_{ij}(\sharp,v'')$. Note that

$$\tilde{Z}_{ij}=O(u^m),\qquad (j\neq m).$$

Using (4.20) and (4.22), we have

$$e(t^i)u^m\tilde{B}^0_{ij} = \tilde{B}_{ij}(u, v', u^m w, t) = -\tilde{Z}_{ij} + v^j\tilde{Z}_{ik} = \mathcal{O}(u^m) \qquad (j \neq k, m),$$
$$e(t^i)u^m\tilde{B}^0_{im} = -\tilde{Z}_{ik} = \mathcal{O}(u^m),$$

and applying Sublemma 4.10 to the functions $f(x) = E_{imm}(t, u', x)$ and $g(x) = -F_{im}(t, u', x)v^m + h_{im}(t, u', x, v'')$ with $X = u^m$, $Y = u^m w + \theta_m v^m$, $Z = u^m w$, we have

$$\begin{aligned} e(t^i)(u^m)^2\tilde{A}^0_{im} &= e(t^i)\tilde{B}_{ik} - u^m w\, e(t^i)\tilde{A}_{im} \\ &= \frac{2}{u^m} E_{imm}(\sharp) - F_{im}(\sharp)v^m + h_{im}(\sharp, v'') \\ &\qquad - (2u^m w + \theta_m v^m)u^m\tilde{Z}_{im} + \mathcal{O}((u^m)^2) \\ &= \mathcal{O}((u^m)^2). \end{aligned}$$

These equalities prove the lemma. ∎

PROOF OF LEMMA 4.12. If a point $p = (u^0, v^{0\prime}, w^0, t^0) \in D^0 \cap (W^{(2)}(k, 0) \times B)$ is an accessible singular point, it satisfies

$$u^{0m} = v^{0m} = 0, \quad \tilde{A}^0_{ij} = \tilde{B}^0_{i\ell} = 0, \qquad (1 \le i, j, \ell \le n;\ \ell \neq m).$$

On the other hand, under the condition $u^m = v^m = 0$, we have

$$e(t^i)\tilde{A}^0_{im} = E_{imm,m}(\sharp_0) + \mathcal{O}(u^m).$$

Hence we must have

$$E_{imm,m}(\sharp_0) = 0 \qquad (1 \le i \le n).$$

But it is impossible because of (4.18). ∎

Next we consider the system $\mathcal{H}^{(2)}_n(k, 1)$. Note that a singular point, if it exists, is contained in either of sets:

(i) $\{(u', v, z, t) \in W^{(2)}(k, 1) \times B \mid v^m = 0 \text{ and } z \neq 0\}$

(ii) $\{(u', v, z, t) \in W^{(2)}(k, 1) \times B \mid v^m \neq 0 \text{ and } z = \det L^1_k = 0\}$

(iii) $\{(u', v, z, t) \in W^{(2)}(k, 1) \times B \mid v^m = 0 \text{ and } z = 0\}$

where $L_k^1 = (e(t^i)\tilde{B}_{ij}^1)_{1\le i,j\le n}$. The first and the third set are $(D^0\backslash D_m^1)\cap(W^{(2)}(k,1)\times B)$ and $(D^0\cap D_m^1)\cap(W^{(2)}(k,1)\times B)$, respectively.

We show

LEMMA 4.13.

$$\{(u',v,z,t)\in W^{(2)}(k,1)\times B \mid v^m\neq 0 \text{ and } z=\det L_k^1=0\}=\emptyset.$$

LEMMA 4.14. *The set $(D^0\cap D_m^1)\cap(W^{(2)}(k,1)\times B)$ contains no accessible singular point.*

PROOF OF LEMMA 4.13. First we show $\tilde{A}_{ij}^1,\ \tilde{B}_{ij}^1\in\mathbb{C}(t)[u',v,z]$. Set $(\sharp)=(t,u',(v^k+\theta_m v^m)zv^k)$, $(\sharp_0)=(t,u',0)$ and $\tilde{Z}_{ij}:=Z_{ij}(\sharp,v'')$. Applying Sublemma 4.10 to $f(x)=E_{imm}(t,u',x)$, and $g(x)=F_{im}(t,u',x)v^m+h_{im}(t,u',x,v'')$ with $X=zv^k$, $Y=v^k+\theta_m v^m$, $Z=0$ or $v^k$, we have, for $j\neq k,m$,

$$\begin{aligned}
e(t^i)v^k\tilde{B}_{ij}^1 &= e(t^i)\tilde{B}_{ij}(u',v,zv^k,t)=-\tilde{Z}_{ij}+v^j\tilde{Z}_{ik}\\
&= -v^kE_{imm,m,j}(\sharp_0)+v^jv^kE_{imm,m,k}(\sharp_0)+O(zv^k),\\
e(t^i)v^k\tilde{B}_{im}^1 &= \tilde{Z}_{ik}=v^kE_{imm,m,k}(\sharp_0)+O(zv^j),\\
e(t^i)v^k\tilde{B}_{ik}^1 &= \frac{2}{zv^k}E_{imm}(\sharp)-F_{im}(\sharp)v^m\\
&\qquad +h_{im}(\sharp,v'')+(v^k+\theta_mv^m)(\tilde{Z}_{ik}-zv^k\tilde{Z}_{im})\\
&= v^kE_{imm,m}(\sharp_0)+O(zv^k)
\end{aligned}$$

and

$$\begin{aligned}
e(t^i)(v^k)^2\tilde{A}_{im}^1 &= v^ke(t^i)\tilde{A}_{im}-e(t^i)\tilde{B}_{ik}\\
&= -\frac{2}{zv^k}E_{imm}(\sharp)+F_{im}(\sharp)v^m-h_{im}(\sharp,v'')\\
&\qquad +(2v^k+\theta_mv^m)(zv^kZ_{im}-Z_{ik})\\
&= z(v^k)^2\{(\theta_m-v^m)F_{im,m}(\sharp_0)+h_{im,m}(\sharp_0,v'')\}+O((zv^k)^2).
\end{aligned}$$

Therefore, using (4.18), under the condition $z = 0$, we have

$$
\begin{aligned}
&\det L_k^1 \\
&= \pm \det \begin{pmatrix}
R_{m1} & & & R_{m1}u^1 & & & \\
& \ddots & & \vdots & & & \\
& & R_{m,m-1} & R_{m,m-1}u^{m-1} & & & \\
S_{m1} & \cdots & S_{m,m-1} & \Delta & S_{m,m+1} & \cdots & S_{mn} \\
& & & R_{m,m+1}u^{m+1} & R_{m,m+1} & & \\
& & & \vdots & & \ddots & \\
& & & R_{mn}u^n & & & R_{mn}
\end{pmatrix} \\
&= \pm t^m \prod_{i \neq m} R_{mi} \neq 0,
\end{aligned}
$$

where $\Delta = \sum_{a \neq m} S_{ma}u^a - t^m$. Then the lemma follows. ∎

PROOF OF LEMMA 4.14. Suppose that a point $p = (u^{0\prime}, v^0, z^0, t^0) \in D^0 \cap (W^{(2)}(k,1) \times B)$ such that $v^{0m} = z^0 = 0$ is an accessible singular point. By definition, there is a solution $(u'(t), v(t), z(t))$ of $\mathcal{H}_n^{(2)}(k,1)$ which is holomorphic at $t^0$ and satisfies

$$u'(t^0) = u^{0\prime}, \quad v(t^0) = v^0, \quad z(t^0) = z^0.$$

From the form of the system $\mathcal{H}_n^{(2)}(k,1)$, a point $p$ must satisfies

$$\tilde{A}_{ij}^1 = \tilde{B}_{ij}^1 = 0.$$

Taking the formulae in the proof of Lemma 4.12 into consideration, we deduce from them

$$E_{imm,m}(\natural_0) = 0 \qquad (1 \leq i \leq n).$$

But it is impossible because of (4.18). ∎

PROOF OF PROPOSITION 4.5 FOR $W^{(2)}(k) \times B$. Note that

$$D_m^1 \cap (W^{(2)}(k) \times B) = D_m^1 \cap (W^{(2)}(k,1) \times B) = \{(u', v, z, t) \mid z = 0\}.$$

As it is seen in the proof of Lemma 4.13, the matrix $L_k^1 = (e(t^i)\tilde{B}_{ij}^0)_{i,j=1,\dots,n}$ for the system $\mathcal{H}_n^{(2)}(k,1)$ is non-singular under the condition $z = 0$. This shows the proposition by using the same reasoning in the proof of Proposition 4.5 for $W^{(2)}(m) \times B$. ∎

The proofs of Propositions 4.6 and 4.7 for $W^{(2)}(k) \times B$ are carried out in a similar way as those for $W^{(2)}(m) \times B$.

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