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Approximation by Proper Holomorphic Maps into Convex Domains

AVNER DOR

Introduction

THEOREM 1. *Let $2 \leq N \leq M - 1$, and take $\Omega \subseteq \mathbb{C}^M$, an arbitrary convex domain, $f : \overline{B}^N \rightarrow \Omega$ continuous and holomorphic in B^N , and $\varepsilon > 0$. Then there exists a proper holomorphic map $F : B^N \rightarrow \Omega$ such that $|F - f| < \varepsilon$ on $(1 - \varepsilon)\overline{B}^N$.*

THEOREM 2. *Let $2 \leq N \leq M - 1$, and let $\Omega \subset\subset \mathbb{C}^M$ be a strongly convex domain with C^2 boundary, $f : \overline{B}^N \rightarrow \Omega$ continuous and holomorphic in B^N , and $\varepsilon > 0$. Then there exists a proper holomorphic map $F : B^N \rightarrow \Omega$ which is continuous on \overline{B}^N and $|F - f| < \varepsilon$ on $(1 - \varepsilon)\overline{B}^N$.*

In [D3] it was proved that there exists a continuous proper holomorphic map from the unit ball of \mathbb{C}^N to an arbitrary C^2 -smooth, bounded domain of higher dimension. Theorem 1 above shows that the proper holomorphic maps from the ball into a convex domain Ω of higher dimension are a dense subset of the holomorphic maps from the ball to the domain, in the topology of uniform convergence on compacta. Theorem 2 implies the same for C^2 -smooth, bounded strongly convex domains with continuous proper holomorphic maps. Here we have used the fact that any map on B^N is the limit (in the topology of uniform convergence on compacta) of maps which are continuous on the boundary. This fact is applied without further mention throughout the paper.

The construction in [D3] is local and is done near a small strongly convex point on the boundary of the target domain, while the considerations that take place here are global. The following questions are of interest: what domains admit a proper holomorphic map from a given lower dimensional ball (or, equivalently, from a strongly pseudoconvex domain with C^∞ -smooth boundary), and what domains admit proper holomorphic maps as a dense subset, in the sense described above. The answer to the first question tends to be local

while the second one depends on global properties of the target domain. In a recent paper F. Forstnerič and J. Globevnik [FG] constructed a domain D in \mathbb{C}^M ($2 \leq M$) which is bounded, smooth, and has a disconnected boundary, such that no proper holomorphic map from Δ (the unit disc of \mathbb{C}) to D goes through a prescribed point $x \in D$. Their proof can also be applied to show that the proper holomorphic maps from Δ to the same domain D are not dense in the above sense. This is true also when Δ is replaced by B^N for any $M \geq N \geq 1$. However, the result of [D3] mentioned above and Proposition 3 below (or direct observation of the domain D) imply that a proper holomorphic map from B^N to the domain D (of [FG]) exists for all $1 \leq N < M$. Thus, although proper holomorphic maps from a ball into a given higher dimensional, C^2 -smooth, bounded domain always exist, they may not form a dense subset of the holomorphic maps from the ball to the domain in the topology of uniform convergence on compacta.

The analysis in [D3] describes very weak local assumptions on a domain that are sufficient to prove that it admits a proper holomorphic map from a lower dimensional ball. However, it seems that for fixed $N, M, 1 \leq N < M$ the domains in \mathbb{C}^M that admit a dense subset of proper holomorphic maps from B^N , and provide for some additional assumptions (such as smoothness and connectedness of the boundary) must have a special suitable structure that has yet to be verified. We will now discuss some examples of “bad” domains that admit proper holomorphic maps. In these examples proper holomorphic maps avoid a set that is thin enough. The first result is a trivial consequence of Theorems 1, 2 and [D3], and the second one follows from an adaptation of the proofs of Theorems 1, 2. The next definition is a convenient convention for the following discussion.

DEFINITION. Let $j \geq 0, M, L \geq 1$ be integers. A subset S of \mathbb{C}^M (or \mathbb{R}^L) will be called a j -dimensional real curve, if S is a C^1 image of an open subset of \mathbb{R}^j (we define formally $\mathbb{R}^0 = \{0\}$).

For example, when S is a real affine subspace of real dimension $\leq j$ then S is a j -dimensional real curve.

COROLLARY 1. Take $N \geq 2, k \geq 0$ and $M \geq N + k + 2$. If $\Omega \subseteq \mathbb{C}^M$ is convex and $S_n \subset \mathbb{C}^M, 1 \leq n < \infty$, is a $(2k + 1)$ -dimensional real curve, then there exists a proper holomorphic map from B^N to Ω that does not intersect the set $E = \bigcup_{1 \leq n < \infty} S_n$. The same is true when $\Omega \subseteq \mathbb{C}^M$ is bounded (not necessarily convex) and C^2 -smooth. In this case the map can also be made continuous on the boundary. See proof after the proof of theorem 2.

The next theorem shows that even in the one co-dimensional case, apparently pathological domains can admit a dense set of proper holomorphic maps.

THEOREM 3. Let $N \geq 2, k \geq 0$, and $M \geq N + k + 1$. If $\Omega \subseteq \mathbb{C}^M$ is convex, and $S_n \subset \mathbb{C}^M, 1 \leq n < \infty$, are $(2k + 1)$ -dimensional real curves

where $E = (\bigcup_{1 \leq n < \infty} S_n) \cap \Omega$ is a relatively closed subset of Ω , then the proper holomorphic maps from B^N to Ω that avoid E are dense in the set of holomorphic maps from B^N to Ω , in the topology of uniform convergence on compacta. If Ω is C^2 -smooth, strongly convex and bounded, then this is true with continuous proper holomorphic maps.

Recall that a proper holomorphic map from B^N to Ω that avoids E is also a proper holomorphic map into the domain $\Omega \setminus E$. An interesting case is when S_n are real affine subspaces of real dimension $\leq 2k + 1$. The following corollary emphasizes the case $k = 0$.

COROLLARY 2. *When $N \geq 2$, $M \geq N + 1$, $\Omega \subseteq \mathbb{C}^M$ is convex, and $E \subset \Omega$ is a countable relatively closed subset (for example E can be a discrete subset of Ω) then the proper holomorphic maps from B^N into the domain $\Omega \setminus E$ are dense in the set of holomorphic maps from B^N to Ω in the topology of uniform convergence on compacta.*

The corollary can also be formulated when E is a countable union of one dimensional real curves which is closed in Ω .

The proof of Theorem 3 appears after the proof of Theorem 1. Using the same manipulations on the proof of Theorem 1 in [D3] we obtain:

THEOREM 4. *Let $N \geq 2$, $k \geq 0$ and $M \geq N + k + 1$. If $\Omega \subseteq \mathbb{C}^M$ is an arbitrary bounded and C^2 -smooth domain, and $S_n \subset \mathbb{C}^M$, $1 \leq n < \infty$, are $(2k + 1)$ -dimensional real curves where $E = (\bigcup_{1 \leq n < \infty} S_n) \cap \Omega$ is a relatively closed subset of Ω , then there exists a continuous proper holomorphic map from B^N to Ω which avoids E .*

The proof is analogous to the proof of Theorem 3 and we will not go through the details.

The following discussion demonstrates another type of “bad” domains that admit a dense subset of proper holomorphic maps. It includes domains that have a disconnected, real analytic boundaries. When we look at the proof of Theorem 1 in [D1] we get that:

PROPOSITION 1. *When $2 \leq N \leq M - 1$ and $f : B^N \rightarrow B^M$ is holomorphic and continuous on the boundary and $\varepsilon > 0$ then there exists $F : B^N \rightarrow B^M$ continuous proper holomorphic map such that $|F - f| < \varepsilon$ on $(1 - \varepsilon)\overline{B}^N$ and $|F| - |f| > -\varepsilon$ on \overline{B}^N .*

The following corollary is an immediate consequence:

COROLLARY 3. *Let $R > r > 0$, $2 \leq N \leq M - 1$. The continuous proper holomorphic maps from B^N to $\Omega = RB^M \setminus r\overline{B}^M$ are dense in the set of holomorphic maps from B^N to Ω in the topology of uniform convergence on compacta.*

This result (without the continuity of the maps on the boundary) can be proved also when $\Omega = D_2 \setminus \overline{D_1}$ where D_1, D_2 are convex domains and $D_1 \subset\subset D_2$. The proof is a simple adaptation of the proof of Theorem 1 with a suitable choice of the defining function but the details are numerous and will not be discussed here.

Note the contrast between these results (from Corollary 1 up to here) and the equidimensional case, which is demonstrated in the following elementary and well-known proposition.

PROPOSITION 2. (i) *Let $\Omega_1, \Omega_2 \subseteq \mathbb{C}^N (N \geq 1)$ be domains where Ω_1 is bounded and has connected boundary and let $K \subset \Omega_2, K \neq \emptyset$, be compact; then there is no proper holomorphic map from Ω_1 to $\Omega_2 \setminus K$.*

(ii) *Let $\Omega_1, \Omega_2 \subseteq \mathbb{C}^N (N \geq 1)$ be bounded domains where the boundary of Ω_1 is connected and the boundary of Ω_2 is disconnected, then there is no proper holomorphic map from Ω_1 to Ω_2 .*

(See proof at the end of this paper).

Theorem 1 of the paper [FG] provides a proper holomorphic map from Δ into an arbitrary C^k -smooth strongly pseudoconvex domain in $\mathbb{C}^N (N, k \geq 2)$ which is C^{k-0} -smooth on the boundary and goes through a prescribed point with a prescribed derivative at this point (up to a real scalar factor). It is not known however if there is a proper holomorphic map between balls of dimensions larger than 1 which is C^1 -smooth but not holomorphic on the boundary. In [F1] it was proved that a proper holomorphic map from B^N to $B^M (2 \leq N < M)$ which has a C^{M-N+1} extension to a larger domain is rational. In [D2] an example was given of a proper holomorphic map from Δ to B^2 which is C^∞ on the closure of Δ but does not have an holomorphic extension to any larger domain. For $M > N \geq 2$ there exists a proper holomorphic map from B^N to B^M which is continuous but not smooth on the boundary (see [D1]).

The existence of a proper holomorphic map from Δ to an arbitrary bounded smooth domain (without the assumption that it goes through a prescribed point) is a very simple consequence of the Riemann Mapping Theorem.

PROPOSITION 3. *Let $\Omega \subset \mathbb{C}^M (M \geq 2)$ be a bounded C^k -smooth domain ($k \geq 2$); then there exists a proper holomorphic embedding from Δ to Ω that is of class $C^{k-0}(\overline{\Delta})$. If Ω is real analytic then the map has an holomorphic extension to $\overline{\Delta}$.*

(See proof at the end of this paper).

Note that the easy proof of this proposition cannot help in any way to find such a map that goes through a prescribed point (in the case that the domain is strongly pseudoconvex). This is a deep result that requires an extremely difficult construction (see details in [FG]). However, in the case where the target domain is convex then the Riemann Mapping Theorem yields a map that goes through a prescribed point and has a prescribed derivative (up to proportion) at that point (see proof of Proposition 3 at the end of the paper).

Theorem 1 above generalizes a result in [D2] that provides a proper holomorphic map from a ball into a polydisc in co-dimension ≥ 1 . It is also proved in [D2] that such a map (and even the absolute value of its components) cannot have a continuous extension to an open subset of the boundary. Thus the map in Theorem 1 is not continuous on the closure in the general case. This stands in contrast to the situation in Theorem 2 where the domain is bounded, smooth and strongly convex.

In [D2] it is also proved that when $1 \leq N \leq M - 2$, then a proper holomorphic map from B^N to B^M extends to a proper holomorphic map from B^{N+1} to B^M . By combining techniques from this proof into the proof of Theorem 2 it can be proved that when $1 \leq N \leq M - 2$ and $\Omega \subset\subset \mathbb{C}^M$ is a smooth strongly convex domain then a proper holomorphic map from B^N to Ω can be extended to a proper holomorphic map from B^{N+1} to Ω . We will not go through the details of the proof here. By Theorem 2 of Globevnik [G], if $\Omega \subseteq \mathbb{C}^M$ is convex ($M \geq 2$) and $E \subset \Omega$ is a discrete subset then there exists a proper holomorphic map from Δ to Ω that goes through E . It follows then by induction that when $\Omega \subset\subset \mathbb{C}^M$ is a smooth strongly convex domain and $E \subset \Omega$ is discrete, there exists a proper holomorphic map from B^N to Ω that goes through E , for all $1 \leq N \leq M - 1$. We thus found that proper holomorphic maps from a ball to a bounded smooth strongly convex domain in co-dimension ≥ 1 can be made to go through a given discrete set or completely avoid a given discrete set.

Ideas and constructions from [D1], [D2], [D3] are applied here, which helps to keep the proofs relatively compact. Familiarity with constructions of proper holomorphic maps between domains in one co-dimension (such as [D1]) is required for reading the following proofs.

1. - Proof of Theorems 1 and 3

We start with Theorem 1.

We begin with a description of a convex exhaustion function on Ω . After this introduction the main Lemma is presented. It is designed to push the map toward the boundary of Ω or to infinity in an open part of the boundary and preserve the progress toward the boundary elsewhere. Comparing it with the main Lemma in the proof of Theorem 2, the reader will find that it is not possible to imitate the mechanism which causes the map in Theorem 2 to be continuous on the boundary.

1.1. There exists a C^∞ function $\rho : \Omega \rightarrow \mathbb{R}$ such that the following hold:

- (i) $\{w : \rho(w) < t\} \subset\subset \Omega$ for all $t > 0$ (thus ρ is an exhaustion function).

(ii) Define and for all $w \in \Omega$ $v \in \mathbb{C}^M$:

$$Q_w(v) \stackrel{\text{def}}{=} \text{Re} \left(\sum_{1 \leq j, k \leq M} D_j D_k \rho(w) v_j v_k + \sum_{1 \leq j, k \leq M} D_j \bar{D}_k \rho(w) v_j \bar{v}_k \right),$$

then there exists a constant $c > 0$ such that for $v \neq 0$: $Q_w(v) > c|v|^2$ (the notation D_j stands for $\partial/\partial z_j$ and \bar{D}_j stands for $\partial/\partial \bar{z}_j$).

The term $Q_w(v)$ is the second order Taylor term of ρ around the point w . For the proof of this standard lemma see the proof of Theorem 3.3.5 in [K, p. 117] or [G, 2.1]. Note that by replacing $\rho(w)$ with $\rho(w) + 2|w|^2$ we may assume henceforth that $c = 2$. Finally, let $\rho(w) = \infty$ for all $w \in \mathbb{C}^M \setminus \Omega$.

1.2. For a given domain $D \subset \mathbb{R}^n$ and a C^2 function $\psi : D \rightarrow \mathbb{C}$ define for all $x, y \in \mathbb{R}^n$, such that $x, x + y \in D$:

$$\begin{aligned} \alpha(x, y) = \psi(x + y) - & \left(\psi(x) + \sum_{1 \leq j \leq n} (\partial\psi/\partial x_j)(x) y_j \right. \\ & \left. + (1/2) \sum_{1 \leq j, k \leq n} (\partial^2\psi/\partial x_j \partial x_k)(x) y_j y_k \right). \end{aligned}$$

The following fact from calculus is frequently used in the proof:

When $\tau > 0$, and $K \subset D$ is compact there exist $\delta > 0$ such that for all $x \in K$ and $y \in \mathbb{R}^n$, $|y| < \delta$: $|\alpha(x, y)| < \tau|y|^2$.

1.3. The constant $\varepsilon_0 \stackrel{\text{def}}{=} 10^{-(10N)!!/\beta}$ will appear throughout the proof, where $\beta = \beta(N) > 0$ is defined in Lemma 3 of [D1].

1.4. Define now for all $w, v \in \mathbb{C}^M$ such that $w, w + v \in \Omega$:

$$\alpha(w, v) = \rho(w + v) - (\rho(w) + 2 \text{Re}((v, \bar{D}\rho(w)) + Q_w(v))).$$

When we view \mathbb{C}^M as \mathbb{R}^{2M} by identifying $(x_1 + iy_1, \dots, x_M + iy_M)$ with $(x_1, y_1, \dots, x_M, y_M)$ this definition coincides with the one in 1.2. Take now $\lambda > 0$ and define $K_\lambda = \{w \in \Omega : d(w, bD) \geq \lambda, |w| \leq 1/\lambda\}$; then $K_\lambda \subset \Omega$ is compact. By 1.2 there exists $1 > \delta(\lambda) > 0$ where $\lambda > \delta(\lambda)$ such that for all $w \in K_\lambda$, $v \in \mathbb{C}^M$, $|v| < \delta(\lambda)$: $|\alpha(w, v)| < 0.1 \cdot |v|^2$. The correspondence $\lambda \rightarrow \delta(\lambda)$ will be fixed from now on and we will assume (as we may) that δ is a decreasing function of λ . When $w \in \Omega$ we define $\lambda(w) = \min\{d(w, b\Omega), 1/|w|\}$. The facts that if $\lambda_1 < \lambda_2$ then $K_{\lambda_2} \subset K_{\lambda_1}$ and $\bigcup_{\lambda > 0} K_\lambda = \Omega$ will also be useful later.

This introduction is needed for the following Lemma which is the main step in the proof.

LEMMA 1. For a given $z_0 \in bB^N$ there exists W , an open neighbourhood of z_0 in the topology of $\overline{B^N}$, such that the following hold. When $f : \overline{B^N} \rightarrow \Omega$ is continuous and holomorphic in B^N , and $\varepsilon_0 > \varepsilon > 0$, there exists $g : \overline{B^N} \rightarrow \mathbb{C}^M$ continuous and holomorphic in B^N where:

- (a) for all $z \in \overline{B^N}$: $\rho(f(z)) - \varepsilon < \rho((f+g)(z)) < \infty$;
- (b) for all $z \in W \cap bB^N$: $\rho((f+g)(z)) > \rho(f(z)) + (\varepsilon_0)^2 \delta(\lambda(f(z)))^2$;
- (c) for all $z \in (1 - \varepsilon)\overline{B^N}$: $|g(z)| < \varepsilon$.

PROOF.

1.5. Let $M = \max\{|f(z)| : z \in \overline{B^N}\}$. We will assume that

$$0 < \varepsilon < (\varepsilon_0)^{10} \cdot \delta(\min\{d(f(\overline{B^N}), b\Omega), 1/M\})^2.$$

Using the proof of Lemma 1 in [D1] (or in [D3]) we can construct, on the same set W as in the proof of Lemma 1 in [D1], a continuous map $g : \overline{B^N} \rightarrow \mathbb{C}^M$, holomorphic in B^N , with the following properties:

1.6.

- (A) for all $z \in \overline{B^N}$: $|\overline{D}\rho(f(z)), g(z)| < \varepsilon^2$;
- (B) for all $z \in \overline{B^N}$: $|g(z)|^2 < (\varepsilon_0)^{1/2} \delta(\lambda(f(z)))^2$;
- (C) for all $z \in W \cap bB^N$: $|g(z)|^2 > (\varepsilon_0)^2 \delta(\lambda(f(z)))^2$;
- (D) for all $z \in (1 - \varepsilon)\overline{B^N}$: $|g(z)| < \varepsilon$.

We will show that g fulfills the requirements of Lemma 1.

1.7. Fix (until 1.9) $z \in \overline{B^N}$. Since by (B) $|g(z)| < \delta(\lambda(f(z))) < \lambda(f(z)) \leq d(f(z), b\Omega)$, then $f(z) + g(z) \in \Omega$.

When reading the proof of Theorem 2 note that if we had there $|g(z)| < d(f(z), b\Omega)$ we would not be able to verify continuity on the boundary.

1.8. Now,

$$\rho(f(z) + g(z)) = \rho(f(z)) + 2\operatorname{Re}((g(z), \overline{D}\rho(f(z)))) + Q_{f(z)}(g(z)) + \alpha(f(z), g(z)).$$

Since by (B) $|g(z)| < \delta(\lambda(f(z)))$ then by 1.4:

$$(1) \quad |\alpha(f(z), g(z))| < 0.1 \cdot |g(z)|^2$$

and from (A) it follows that:

$$(2) \quad |2\operatorname{Re}(g(z), \overline{D}\rho(f(z)))| < 2\varepsilon^2;$$

$$(3) \quad \text{finally 1.1 gives: } Q_{f(z)}(g(z)) > 2|g(z)|^2.$$

1.9. We conclude that: $\rho(f(z) + g(z)) > \rho(f(z)) + 1.5|g(z)|^2 - \varepsilon$. This and 1.7 imply (a), and with 1.5 and (C) it implies (b). Now (c) follows from (D). Lemma 1 is thus proved.

We shall proceed now with the proof of Theorem 1 by an inductive use of Lemma 1.

1.10. Fix W_1, \dots, W_m , open subsets of $\overline{B^N}$ where $\cup\{W_i : 1 \leq i \leq m\} \supseteq bB^N$, and W_i ($1 \leq i \leq m$) has the properties of W in Lemma 1.

1.11. For an integer n we define \bar{n} to be the only integer in $\{1, \dots, m\}$ so that $(n - \bar{n})/m$ is an integer.

1.12. *The induction hypothesis:* Define $f_1 = f$ (where f is from the statement of Theorem 1). Let $n \geq 1$, and assume inductively that the maps $g_1, \dots, g_{n-1}, f_1, \dots, f_n$ are defined, where for all $1 \leq i \leq n$ $f_i : \overline{B^N} \rightarrow \Omega$, is continuous and holomorphic in B^N , and for $1 \leq i \leq n - 1$ $g_i : \overline{B^N} \rightarrow \mathbb{C}^M$ is continuous and holomorphic in B^N , and $f_n = f_1 + g_1 + \dots + g_{n-1}$.

1.13. Let $\varepsilon_n = (\varepsilon_0)^{10} \cdot \varepsilon \cdot \delta(\min\{\lambda(f_i(z)) : z \in \overline{B^N}, 1 \leq i \leq n\})^2/2^n$, where $\varepsilon > 0$ is the one in the statement of Theorem 1 and we assume $\varepsilon < 1$.

1.14. By Lemma 1 there exists a continuous map $g_n : \overline{B^N} \rightarrow \mathbb{C}^M$, holomorphic in B^N , so that the following hold:

- (a) for all $z \in \overline{B^N}$: $\rho(f_n(z)) - \varepsilon_n < \rho((f_n + g_n)(z)) < \infty$;
- (b) for all $z \in W_{\bar{n}} \cap bB^N$: $\rho((f_n + g_n)(z)) > \rho(f_n(z)) + (\varepsilon_0)^2 \delta(\lambda(f_n(z)))^2$;
- (c) for all $z \in (1 - \varepsilon_n)\overline{B^N}$: $|g_n(z)| < \varepsilon_n$.

1.15. Define $f_{n+1} = f_n + g_n$. The induction hypothesis now holds for $n + 1$. It follows from (c) and 1.13 that $\sum_{1 \leq n} g_n$ converges uniformly on compact subsets of B^N . We will call its limit g . The map g is holomorphic in B^N and by (c) and 1.13 $|g| < \varepsilon$ on $(1 - \varepsilon)\overline{B^N}$. Define $F = f + g$; this map is also holomorphic in B^N and it is a uniform limit on compacta of $\{f_n\}_{1 \leq n < \infty}$. We will prove that F is a proper holomorphic map from B^N to Ω .

1.16. First, it will be shown that $F(B^N) \subset \Omega$. Take $z \in B^N$ and let n be large enough so that $\varepsilon_n < 1 - |z|$. Since $z \in (1 - \varepsilon_k)\overline{B^N}$ for all $k \geq n$ then by 1.14(c) and 1.13:

$$|F(z) - f_n(z)| \leq \sum_{\infty > k \geq n} |g_k(z)| < \sum_{\infty > k \geq n} \varepsilon_k \leq 2\varepsilon_n < d(f_n(z), b\Omega).$$

It follows that $F(z) \in \Omega$.

Now to show that F is a proper map from B^N to Ω we need to show that when $z \in bB^N$ and $\{z_n\}_{1 \leq n < \infty} \subset B^N$ converges to z then $\lim_{n \rightarrow \infty} \rho(F(z_n)) = \infty$. This is done in the following proposition. The main part is the proof of (3). Working, as we do, with the “utility” functions λ, δ makes this proof shorter and easier.

1.17. Proposition:

- (1) Let $z \in \overline{B^N}$ and $1 \leq n < k < \infty$ be integers, then $\rho(f_k(z)) > \rho(f_n(z)) - 2\varepsilon_n$.
- (2) Let $z \in B^N$ and $1 \leq n < \infty$ be an integer, then $\rho(F(z)) \geq \rho(f_n(z)) - 2\varepsilon_n$.
- (3) Let $z \in bB^N$, then $\lim_{n \rightarrow \infty} \rho(f_n(z)) = \infty$.
- (4) Let $z \in bB^N$, $\{z_n\}_{1 \leq n < \infty} \subset B^N$, and $\lim_{n \rightarrow \infty} z_n = z$ then $\lim_{n \rightarrow \infty} \rho(F(z_n)) = \infty$.

PROOF. (1) By 1.14(a) and 1.13 we have:

$$\begin{aligned} \rho(f_k(z)) - \rho(f_n(z)) &= \sum_{n \leq j \leq k-1} \rho(f_{j+1}(z)) - \rho(f_j(z)) \\ &> \sum_{n \leq j \leq k-1} -\varepsilon_j > \sum_{n \leq j < \infty} -\varepsilon_j \geq -2\varepsilon_n. \end{aligned}$$

(2) Since ρ is continuous in Ω and $F(z) = \lim_{k \rightarrow \infty} f_k(z)$ it follows from (1) that: $\rho(F(z)) - \rho(f_n(z)) = \lim_{k \rightarrow \infty} \rho(f_k(z)) - \rho(f_n(z)) \geq -2\varepsilon_n$.

(3) Assume, to get a contradiction, that there exists $A > 0$ such that $A \geq \rho(f_n(z))$ for all $\infty > n \geq 1$, then $\{f_n(z) : 1 \leq n < \infty\}$ is contained in the compact subset of Ω , $K = \{w \in \Omega : \rho(w) \leq A\}$. Take $\lambda_0 > 0$ such that $K \subset K_{\lambda_0}$ (see definition 1.4) then $\lambda_0 \leq \lambda(f_n(z))$ for all $\infty > n \geq 1$ and therefore $\delta(\lambda_0) \leq \delta(\lambda(f_n(z)))$ for all $n \geq 1$. Now let $m \geq \ell \geq 1$ be such that $z \in W_\ell$ then by 1.14(a) and 1.14(b) for all $\infty > n \geq 1$:

$$\begin{aligned} \rho(f_n(z)) - \rho(f_1(z)) &= \sum_{1 \leq j \leq n-1} \rho(f_{j+1}(z)) - \rho(f_j(z)) \\ &> \sum_{1 \leq j \leq n-1, \bar{j}=\ell} (\varepsilon_0)^2 \delta(\lambda(f_j(z)))^2 - \sum_{1 \leq j \leq n-1} \varepsilon_j \\ &> \sum_{1 \leq j \leq n-1, \bar{j}=\ell} (\varepsilon_0)^2 (\delta(\lambda_0))^2 - \sum_{1 \leq j \leq n-1} \varepsilon_j > ((n/m) - 1)(\varepsilon_0)^2 (\delta(\lambda_0))^2 - 1. \end{aligned}$$

As we let $n \rightarrow \infty$ we obtain that $\rho(f_n(z)) \rightarrow \infty$, which is a contradiction to our assumption. We conclude that $\{\rho(f_n(z)) : 1 \leq n < \infty\}$ is not bounded. Take $A > 0$; there exists n_0 such that $\rho(f_{n_0}(z)) > A + 1$, if $n \geq n_0$; then by (1) $\rho(f_n(z)) > \rho(f_{n_0}(z)) - 2\varepsilon_n > A$. It follows that $\lim_{n \rightarrow \infty} \rho(f_n(z)) = \infty$.

(4) Let $z \in bB^N$, and take $\{z_n\}_{1 \leq n < \infty} \subset B^N$, such that $\lim_{n \rightarrow \infty} z_n = z$. Take $A > 0$. By (3) there exists n_0 such that $\rho(f_{n_0}(z)) > A + 2$. By the continuity of f_{n_0} on $\overline{B^N}$ there exists $n_1 \geq 1$ such that for all $n > n_1 : \rho(f_{n_0}(z_n)) > A + 1$. Now (2) implies that for all $n > n_1 : \rho(F(z_n)) > A$.

Theorem 1 is now proved.

PROOF OF THEOREM 3. We will show how to modify the proof of Theorem 1 to obtain this result. This is done with the aid of the next sublemma which is common knowledge.

Sublemma 1: Take integers m, n, k where $n \geq 1, k \geq 0$, and $m > n + k$. Let $D \subset \mathbb{R}^n$ be open, and $\varepsilon > 0$. If $f : D \rightarrow \mathbb{R}^m$ is C^1 and $S_\ell \subset \mathbb{R}^m, 1 \leq \ell < \infty$, are k -dimensional real curves, then there exists $y \in \mathbb{R}^m$ where $|y| < \varepsilon$ such that $(y + f(D)) \cap (\bigcup_{1 \leq \ell < \infty} S_\ell) = \emptyset$.

(Note that $f(D)$ is an n -dimensional real curve in \mathbb{R}^m).

PROOF. For each $1 \leq \ell < \infty$ take U_ℓ , an open subset of \mathbb{R}^k , and $g_\ell : U_\ell \rightarrow \mathbb{R}^m$, a C^1 map, where $g_\ell(U_\ell) = S_\ell$ and define for all $(x_1, x_2) \in U_\ell \times D : F_\ell(x_1, x_2) = g_\ell(x_1)f(x_2)$; then $F_\ell : U_\ell \times D \rightarrow \mathbb{R}^m$ is a C^1 map. Since $m > n + k$ then $F_\ell(U_\ell \times D)$ has a zero Lebesgue measure in \mathbb{R}^m for all $1 \leq \ell < \infty$ and therefore the same is true for $\bigcup_{1 \leq \ell < \infty} F_\ell(U_\ell \times D)$. It follows that there exists $y \in \mathbb{R}^m$ where $|y| < \varepsilon$ such that $y \notin \bigcup_{1 \leq \ell < \infty} F_\ell(U_\ell \times D)$. Now if $(y + f(D)) \cap S_\ell \neq \emptyset$ for some $1 \leq \ell < \infty$ then there exist $x_1 \in U_\ell$ and $x_2 \in D$ such that: $y + f(x_2) = g_\ell(x_1)$, therefore $y = F_\ell(x_1, x_2)$ and a contradiction.

Coming back to our proof, an additional perturbation is inserted into the induction step in the proof of Theorem 1, with the use of sublemma 1 to move the map away from E . Therefore at any step n , the distance between the compact set $f_n((1 - \varepsilon_{n-1})\overline{B^N})$ and E is positive. Thus we need to ensure that the perturbations that will come in all the next steps will add up to less than this distance. The choice of $\varepsilon_1, \varepsilon_2, \dots$ will thus be done accordingly.

(1) Let $f : B^N \rightarrow \Omega$ be the map we wish to approximate. The map f is holomorphic and continuous on $\overline{B^N}$. Let also $1 > \varepsilon > 0$. Until the induction step in 1.12, our proof goes exactly the same as the proof of Theorem 1 and will not be repeated. We therefore continue with the induction assumption.

The induction hypothesis:

(2) Define $f_1 = f$. Let $n \geq 1$, and assume inductively that the maps f_1, \dots, f_n are defined, where, for all $2 \leq k \leq n, f_k : B^N \rightarrow \Omega \setminus E$ is holomorphic and continuous on $\overline{B^N}$.

(3) Assume also that the positive numbers $\varepsilon_1, \dots, \varepsilon_{n-1}$ are defined. Let δ_k (for $2 \leq k \leq n$) be the distance between $f_k((1 - \varepsilon_{k-1})\overline{B^N})$ and E . Since E is closed in Ω then $\delta_k > 0$.

The induction step. Take $\varepsilon_n > 0$, such that (for $n \geq 2$) $\varepsilon_n <$

$\min\{\varepsilon_{n-1}, \delta_n\}/10$ and $\varepsilon_n < (\varepsilon_0)^{10} \cdot \varepsilon \cdot \delta(\min\{\lambda(f_i(z)) : z \in \overline{B^N}, 1 \leq i \leq n\})/10^n$.

(4) By Lemma 1 above there exists a continuous map $g_n : \overline{B^N} \rightarrow \mathbb{C}^M$, holomorphic in B^N , so that the following holds:

- (a) for all $z \in \overline{B^N}$: $\rho(f_n(z)) - \varepsilon_n/2 < \rho((f_n + g_n)(z)) < \infty$;
- (b) for all $z \in W_{\overline{n}} \cap bB^N$: $\rho((f_n + g_n)(z)) > \rho(f_n(z)) + (\varepsilon_0)^2 \delta(\lambda(f_n(z)))^2$;
- (c) for all $z \in (1 - \varepsilon_n)\overline{B^N}$: $|g_n(z)| < \varepsilon_n/2$.

(5) Define $h_n = f_n + g_n$. then $h_n : \overline{B^N} \rightarrow \Omega$ is continuous and holomorphic in B^N . Until now the construction is essentially the same as in proof of Theorem 1, but the image of the map h_n might intersect E . Let $K_n = h_n(\overline{B^N})$; then K_n is a compact subset of Ω . Let $r_n > 0$ be small enough that if $w \in \mathbb{C}^M$, $|w| < r_n$ and $v \in K_n$ then $|\rho(v+w) - \rho(v)| < \varepsilon_n/2$. By Sublemma 1 there exists $w_n \in \mathbb{C}^M$, where $|w_n| < \min\{r_n, \varepsilon_n\}/2$, such that $(h_n(B^N) + w_n) \cap E = \emptyset$. Define $f_{n+1} = h_n + w_n$. Now by (4) and the size of ε_n in (3) we obtain:

- (6) (a) for all $z \in \overline{B^N}$: $\rho(f_n(z)) - \varepsilon_n < \rho((f_{n+1})(z)) < \infty$;
- (b) for all $z \in W_{\overline{n}} \cap bB^N$: $\rho((f_{n+1})(z)) > \rho(f_n(z)) + (1/2)(\varepsilon_0)^2 \delta(\lambda(f_n(z)))^2$
- (c) for all $z \in (1 - \varepsilon_n)\overline{B^N}$: $|f_{n+1}(z) - f_n(z)| < \varepsilon_n$.

Note now that the induction hypothesis holds for $n + 1$. We can proceed from here in the same way as in the proof of Theorem 1 to show that $\{f_n\}_{1 \leq n < \infty}$ converges uniformly on compacta to a map F , and F is a proper holomorphic map from B^N to Ω . Furthermore, it is clear from (c) that $|F - f| < \varepsilon$ on $(1 - \varepsilon)\overline{B^N}$. It remains to prove that $F(B^N) \cap E = \emptyset$.

To do this we will prove that $d(F((1 - \varepsilon_{n-1})\overline{B^N}), E) \geq \delta_n/2$ for all $n \geq 2$. Fix $n \geq 2$, it follows from 6(c) that for all $m \geq n$ and $z \in (1 - \varepsilon_{n-1})\overline{B^N}$: $|f_{m+1}(z) - f_m(z)| < \varepsilon_n$. Therefore for all $z \in (1 - \varepsilon_{n-1})\overline{B^N}$

$$|F(z) - f_n(z)| \leq \sum_{\infty > m \geq n} |f_{m+1}(z) - f_m(z)| < \sum_{\infty > m \geq n} \varepsilon_m < 2\varepsilon_n.$$

In the last inequality assumption (3) was used. Since by (3) $\varepsilon_n < \delta_n/10$ and $\delta_n = d(f_n((1 - \varepsilon_{n-1})\overline{B^N}), E)$ we therefore have:

$$d(F(z), E) \geq \delta_n - 2\varepsilon_n \geq \delta_n/2.$$

The proof is complete. Inserting this method of perturbation into the proof of Theorem 2 gives the second half of the theorem.

2. - Proof of Theorem 2

As in the beginning of the previous proof we first need a summary of the properties of the defining function.

2.1. There exists a real C^2 function ρ from a neighborhood $\Omega' \subset \mathbb{C}^M$ of $\overline{\Omega}$ to \mathbb{R} such that:

- (i) $\Omega = \{z \in \Omega' : \rho(z) < 0\}$, and $\rho(z) > -1$ for all $z \in \Omega$;
- (ii) $\overline{D}(\rho)(z) \neq 0$ for all $z \in b\Omega$;
- (iii) there exists a constant $c > 0$ such that for all $w \in \overline{\Omega}$, $v \in \mathbb{C}^M$, $v \neq 0 : |v|^2 > Q_w(v) > c|v|^2$ (where $Q_w(v)$ is defined in 1.1), note that $1 > c$. For a proof use Lemma 3.1.6 in [K, p. 101] with exercise E.2.8 in [Ra, p. 66].

2.2. Let $\alpha(w, v)$ be as in 1.4; then by 1.2 there exists a constant $1 > \lambda > 0$ such that for all $w \in \overline{\Omega}$, $v \in \mathbb{C}^M$, $|v| < \lambda : |\alpha(w, v)| < 0.1 \cdot c \cdot |v|^2$.

2.3. The definition of ε_0 will take into account the constant λ and the constant c which, this time, has an upper bound. Let $\varepsilon_0 \stackrel{\text{def}}{=} (c\lambda)^{10} \cdot 10^{-(10N)!/\beta}$.

The following lemma yields Theorem 2.

LEMMA 2. For a given $z_0 \in bB^N$ there exists W , an open neighbourhood of z_0 in the topology of \overline{B}^N , where the following holds:

For every continuous map $f : \overline{B}^N \rightarrow \Omega$, which is holomorphic in B^N , and $\varepsilon_0 > \varepsilon > 0$, there exists $g : \overline{B}^N \rightarrow \mathbb{C}^M$ continuous and holomorphic in B^N such that:

- (a) for all $z \in \overline{B}^N$: $\rho((f))(z) - \varepsilon < \rho((f + g)(z)) < 0$;
- (b) for all $z \in W \cap bB^N$: $|\rho((f + g)(z))| < |\rho((f)(z))|(1 - (\varepsilon_0)^3)$;
- (c) for all $z \in \overline{B}^N$: $|g(z)|^2 < (\varepsilon_0)^{1/2} |\rho((f)(z))|$;
- (d) for all $z \in (1 - \varepsilon)\overline{B}^N$: $|g(z)| < \varepsilon$.

PROOF 2.4. We will assume that $\varepsilon < (\varepsilon_0)^{10} \cdot \min\{|\rho(f(z))| : z \in \overline{B}^N\}$. Using the proof of Lemma 1 of [D1], or the proof of Lemma 1 of [D3], as before, we can construct, on same set W as in the proof of Lemma 1 in [D1], a continuous map $g : \overline{B}^N \rightarrow \mathbb{C}^M$, holomorphic in B^N , where:

2.5.

- (A) For all $z \in \overline{B}^N$: $|(\overline{D}\rho(f(z)), g(z))| < \varepsilon^2$.
- (B) For all $z \in \overline{B}^N$: $|g(z)|^2 < (\varepsilon_0)^{1/2} |\rho(f(z))|$.
- (C) For all $z \in W \cap bB^N$: $|g(z)|^2 > (\varepsilon_0)^2 |\rho(f(z))|$.
- (D) For all $z \in (1 - \varepsilon)\overline{B}^N$: $|g(z)| < \varepsilon$.

We will show that g is the function we are looking for.

2.6. Since by (B), 2.1(i), 2.3 and 2.4: $|g(z)| < \lambda$, for all $z \in \overline{B}^N$, then by 2.1(iii) and 2.2, $0.1 \cdot Q_{f(z)}(g(z)) \geq |\alpha(f(z), g(z))|$ for all $z \in \overline{B}^N$.

2.7. We then conclude from the equality:

$$\rho(f(z) + g(z)) = \rho(f(z)) + 2 \operatorname{Re}((g(z), \overline{D}\rho(f(z)))) + Q_{f(z)}(g(z)) + \alpha(f(z), g(z)),$$

(for all $z \in \overline{B}^N$) and from (A) that:

$$\begin{aligned} 2|g(z)|^2 + 2\varepsilon^2 &> \rho(f(z) + g(z)) - \rho(f(z)) \\ &> (1/2)Q_{f(z)}(g(z)) - 2\varepsilon^2 > (c/2)|g(z)|^2 - 2\varepsilon^2 \end{aligned}$$

for all $z \in \overline{B}^N$.

2.8. Now, (c) and (a) follow from this, (B) and 2.4. If we look also at (C) then (b) follows. Finally (d) is the same as (D). The lemma is proved.

2.9. We proceed at first as in the proof of Theorem 1. Take W_1, \dots, W_m open subsets of \overline{B}^N where $\cup\{W_i : 1 \leq i \leq m\} \supseteq bB^N$, and $W_i (1 \leq i \leq m)$ has the properties of W in Lemma 2. Assume that $m \geq 100$. The integer m and the sets W_1, \dots, W_m will be fixed.

2.10. For an integer n let \bar{n} to be the only integer in $\{1, \dots, m\}$ such that $(n - \bar{n})/m$ is an integer.

2.11. Define $f_1 = f$, $g_0 \equiv 0$. Let $n \geq 1$, and assume inductively that the maps g_0, \dots, g_{n-1} , f_1, \dots, f_n are defined, where $f_i : \overline{B}^N \rightarrow \Omega$, $g_j : \overline{B}^N \rightarrow \mathbb{C}^M (1 \leq i \leq n, 0 \leq j \leq n-1)$, are continuous and holomorphic in B^N . Assume also that $f_n = f_1 + g_1 + \dots + g_{n-1}$.

2.12. Define $\varepsilon_n = \varepsilon \cdot (m^{-1}\varepsilon_0)^{10} \cdot \min\{|\rho(f_i(z))| : z \in \overline{B}^N, 0 \leq i \leq n\}/2^n$, where $\varepsilon > 0$ is the one in the statement of Theorem 2 which is assumed to be smaller than 1.

2.13. By Lemma 2 there exists a continuous map $g_n : \overline{B}^N \rightarrow \mathbb{C}^M$ holomorphic in B^N , such that the following (a)-(d) hold:

- (a) for all $z \in \overline{B}^N$: $\rho((f_n)(z)) - \varepsilon_n < \rho((f_n + g_n)(z)) < 0$;
- (b) for all $z \in W_{\bar{n}} \cap bB^N$: $|\rho((f_n + g_n)(z))| < |\rho((f_n)(z))|(1 - (\varepsilon_0)^3)$;
- (c) for all $z \in \overline{B}^N$: $|g_n(z)|^2 < (\varepsilon_0)^{1/2} |\rho((f_n)(z))|$;
- (d) for all $z \in (1 - \varepsilon_n)\overline{B}^N$: $|g_n(z)| < \varepsilon_n$.

Define $f_{n+1} = f_n + g_n$. The inductive assumption holds for $n + 1$.

2.14. When one looks at the definition of $\varepsilon_1, \varepsilon_2, \dots$ in 2.12 it easily follows from (b) and (c) that for all $n \geq 1$ and $z \in bB^N$:

$$|\rho((f_n)(z))|(1 - (\varepsilon_0)^4) > |\rho((f_{n+m})(z))|.$$

2.15. We obtain from this and 2.1(i) that for all $n > 1$ and $z \in bB^N$:
 $(1 - (\varepsilon_0)^4)^{n/m-1} > |\rho((f_n)(z))|$.

It follows then from (c) that $\sum_{0 \leq n < \infty} g_n$ converges uniformly on $\overline{B^N}$; call its limit g . Then g is continuous on $\overline{B^N}$ and holomorphic on B^N .

Define $F = f + g$; then F is a uniform limit of $\{f_n\}$ in $\overline{B^N}$, and F is continuous on $\overline{B^N}$ and holomorphic on B^N . By 2.15 and the continuity of ρ , $\rho(F(z)) = 0$ whenever $z \in bB^N$. Thus F is a continuous proper map from B^N to Ω . Now 2.13(d) and 2.12 imply that $|g(z)| < \varepsilon$ whenever $z \in (1 - \varepsilon)\overline{B^N}$. The proof is concluded.

PROOF OF COROLLARY 1. Take $V \subset \mathbb{C}^M$, a complex affine subspace of (complex) dimension $N+1$ that does not intersect any of the sets S_n , and which goes through Ω . To obtain V we can take any complex affine subspace of \mathbb{C}^M of complex dimension $N+1$ that goes through Ω , and translate it slightly in a suitable manner (see sublemma 1 which appears at the beginning of the proof of Theorem 3). By Theorem 1 there is a proper holomorphic map into $V \cap \Omega$. This map is also a proper holomorphic map into Ω . In the case that Ω is smooth and bounded the proof of Proposition 3 below provides such V where $V \cap \Omega$ is convex, by Theorem 2 there exists a continuous proper holomorphic map from B^N to $V \cap \Omega$.

PROOF OF PROPOSITION 2. The proof is a standard argument and it is presented here for completeness' sake.

(i) Since proper holomorphic maps between domains in the same dimension must be onto (see [Ru, p. 301]), the proof is concluded if we show that a proper holomorphic map from Ω_1 to $\Omega_2 \setminus K$ must be a proper holomorphic map into Ω_2 as well. Let $f : \Omega_1 \rightarrow \Omega_2 \setminus K$ be a proper holomorphic map. Define for $r > 0$

$$\Omega_1(r) = \{z \in \Omega_1 : d(z, b\Omega_1) < r\},$$

$$\Omega_2(r) = \{z \in \Omega_2 : d(z, b\Omega_2) < r, \text{ or } d(z, K) > r + 1/r\},$$

$$K(r) = \{z \in \Omega_2 : d(z, K) < r\}.$$

Let $d = d(K, b\Omega_2)$; then $d > 0$. For each $r > 0$ there exists $\varepsilon_r > 0$ small enough such that $f(\Omega_1(\varepsilon_r)) \subset \Omega_2(r) \cup K(r)$. When $0 < r < d/2$ then $\Omega_2(r)$ and $K(r)$ are disjoint open sets. Therefore, since $\Omega_1(\varepsilon)$ is connected for all $\varepsilon > 0$, only one of the following two possibilities holds:

(a) $f(\Omega_1(\varepsilon_r)) \subset \Omega_2(r)$ for all $0 < r < d/2$

(b) $f(\Omega_1(\varepsilon_r)) \subset K(r)$ for all $0 < r < d/2$.

If (a) holds then f is a proper holomorphic map from Ω_1 to Ω_2 and the proof is concluded. Otherwise take $0 < r < d/2$ and let V be a real affine hyperplane in \mathbb{C}^N such that $K(r)$ is in one side of V (call this half space V^+) and the other side of V (call it V^-) has a nonempty intersection with $\Omega_2 \setminus K$. Since $f(\Omega_1(\varepsilon_r)) \subset K(r) \subset V^+$ we get from the maximum principle that

$f(\Omega_1) \subset V^+$ but this is a contradiction since $f(\Omega_1) = \Omega_2 \setminus K$ and V^- has a nonempty intersection with $\Omega_2 \setminus K$.

(ii) It can be deduced from (i) or proved directly in the following manner. Since $b\Omega_2$ is a disconnected compact set there exist two nonempty compact sets K_1 and K_2 such that $K_1 \cup K_2 = b\Omega_2$ and $d = d(K_1, K_2) > 0$. Define as above $\Omega_1(r) = \{z \in \Omega_1 : d(z, b\Omega_1) < r\}$ and, for $i = 1, 2, K_i(r) = \{z \in \Omega_2 : d(z, K_i) < r\}$; then for all $r > 0$ there exists $\varepsilon_r > 0$ such that $f(\Omega_1(\varepsilon_r)) \subset K_1(r) \cup K_2(r)$. Fix $r > 0$ where $d/2 > r$. Since $K_1(r)$ and $K_2(r)$ are disjoint there is $i \in \{1, 2\}$ such that $f(\Omega_1(\varepsilon_r)) \subset K_i(r)$; assume that $i = 1$. Let $A = \Omega_1 \setminus \Omega_1(\varepsilon_r)$; then A is a compact subset of Ω_1 , and therefore, for some small $\delta > 0$, $f(A) \cap K_2(\delta) = \emptyset$. Take such a δ where $\delta < r$, then since $f(\Omega_1(\varepsilon_r)) \cap K_2(\delta) = \emptyset$ it follows that $f(\Omega_1) \cap K_2(\delta) = \emptyset$ and this is a contradiction since $f(\Omega_1) = \Omega_2$.

PROOF OF PROPOSITION 3. Let $z_0 \in b\Omega$ be such that $|z_0| = \max\{|z| : z \in \overline{\Omega}\}$ then z_0 is a point of strong convexity (since the ball $B(0, |z_0|)$ contains Ω and its boundary is tangent to $b\Omega$ at z_0). Since Ω is C^k -smooth, where $k \geq 2$, there is a neighborhood G of z_0 in $b\Omega$ such that every point in G is a point of strong convexity. We can take $\varepsilon > 0$ small enough such that the affine hyperplane $W = (1 - \varepsilon)z_0 + (z_0)^\perp$ intersects $b\Omega$ in G (i.e. $W \cap b\Omega \subset G$). Let V be a one-dimensional complex affine subspace of W that goes through $(1 - \varepsilon)z_0$; then $V \cap \Omega$ is a nonempty strongly convex one-dimensional domain with C^k boundary. By the Riemann Mapping Theorem there exists a biholomorphic map from Δ to $V \cap \Omega$ and by another classical theorem (see [BK, 8]) this map extends C^{k-0} to the boundary. In the case that Ω is real analytic, the Schwarz Reflection Theorem implies that the map is holomorphic on $\overline{\Delta}$. Clearly this map is a proper holomorphic embedding into Ω .

Note that if $\Omega \subset \mathbb{C}^M$ is convex and bounded (not necessarily smooth), $z_0 \in \Omega$ and $v \in \mathbb{C}^M \setminus \{0\}$ then by taking $V = z_0 + \mathbb{C} \cdot v$ and applying the Riemann mapping Theorem we obtain a proper holomorphic embedding from Δ to Ω that goes through z_0 whose derivative is proportional to v . This map is C^{k-0} on the boundary when Ω is C^k -smooth and when Ω is real analytic it is holomorphic on the boundary. This is a special case of Theorem 1 in [FG].

REFERENCES

- [BK] S. BELL - S. KRANTZ, *Smoothness to the boundary of conformal maps*, Rocky Mountain J. Math., **17**, No. 1, (1987), 23-40.
- [D1] A. DOR, *Proper holomorphic maps between balls in one co-dimension*, Ark. Mat., **28**, No. 1, (1990), 49-100.
- [D2] A. DOR, *Lifting of proper holomorphic maps*, preprint.
- [D3] A. DOR, *Continuous proper holomorphic maps into bounded domains*, preprint.

- [F1] F. FORSTNERIČ, *Extending proper holomorphic mappings of positive co-dimension*, Invent. Math., **95**, No. 1, (1989), 31-61.
- [F2] F. FORSTNERIČ, *Embedding strictly pseudoconvex domains into balls*, American Math. Soc., **295**, (1986), 347-368.
- [FG] F. FORSTNERIČ - J. GLOBEVNIK, *Discs in pseudoconvex domains*, preprint (1991).
- [G] J. GLOBEVNIK, *Relative embeddings of discs into convex domains*, Invent. Math., **98**, No. 2, (1989), 331-350.
- [H] M. HAKIM, *Applications holomorphes propres continues de domaines strictement pseudoconvexes de \mathbb{C}^n dans la boule unité de \mathbb{C}^{n+1}* , Duke Math. J., **60**, No. 1, (1990), 115-133.
- [HS] M. HAKIM - N. SIBONY, *Fonction holomorphes sur la boule unité de \mathbb{C}^n* , Invent. Math., **67**, (1982), 213-222.
- [K] S. KRANTZ, *Function theory of several complex variables*, John Wiley and Sons, 1982.
- [L] E. LØW, *A construction of inner functions on the unit ball in \mathbb{C}^p* , Invent. Math., **67**, (1982), 223-229.
- [N] A. NOELL - B. STENSØNES, *Proper holomorphic maps from weakly pseudoconvex domains*, preprint.
- [Ra] M. RANGE, *Holomorphic functions and integral representations in several complex variables*, Springer-Verlag, New York, 1986.
- [Ru] W. RUDIN, *Function theory in the unit ball of \mathbb{C}^n* , Springer-Verlag, New York, 1980.
- [S] B. STENSØNES, *Proper holomorphic mappings from strongly pseudoconvex domains in \mathbb{C}^2 to the unit polydisc in \mathbb{C}^3* , Math. Scand., **65**, (1989), 129-139.

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