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Introduction

Let $n \geq 2$ and let Ω be an exterior domain in $\mathbb{R}^n$, i.e., a domain having a compact complement $\mathbb{R}^n/\Omega$, and assume that the boundary $\partial\Omega$ is of class $C^{2+\mu}$ with $0 < \mu < 1$. Consider the following boundary value problem for the Stokes equations in Ω:

$$(S) \qquad -\Delta u + \nabla p = f, \quad \text{div } u = g \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

where $u = (u^1(x), \cdots, u^n(x))$ and $p = p(x)$ denote the unknown velocity and pressure, respectively; $f = (f^1(x), \cdots, f^n(x))$ and $g = g(x)$ denote the given external force and the scalar divergence, respectively.

The purpose of the present paper is to extend the well-known concept of generalized solutions u of (S) having a finite Dirichlet integral

$$(D) \qquad \int_\Omega |\nabla u(x)|^2 \, dx < \infty$$

(see, *e.g.*, Chang-Finn [11], Finn [14], Fujita [16], Heywood [21]). We consider here a much larger class of the generalized solutions u of (S) satisfying

$$(CL_q) \qquad \int_\Omega |\nabla u(x) - A|^q \, dx < \infty \text{ with some matrix } A,$$

where $1 < q < \infty$. In particular, setting $A \equiv 0$, we treat the class

$$(D_q) \qquad \int_\Omega |\nabla u(x)|^q \, dx < \infty$$

which generalizes the Dirichlet integral to L^q-spaces.

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In the class (CL_q), we can investigate the motion of the fluid past an obstacle rotating around its axis. Such a fluid motion is governed by (S) with the boundary condition at infinity

$$(B.C.)_\infty \qquad u(x) \to Ax + a \quad \text{as } x \to \infty,$$

where A denotes a skew-symmetric matrix and the vector a is a constant. Another physical phenomenon described in the class (CL_q) is the flow due to an obstacle embedded in a pure straining tensor: far from the obstacle the fluid is in a pure stretching specified by the rate-of-strain tensor A, with $\mathrm{Tr}\ A = 0$. Then the velocity u can be written as

$$u(x) = Ax + u_0(x),$$

where u_0 represents the changes due to the presence of the obstacle, with $u_0(x)$ small for large $|x|$. Such a solution describes a *suspension*, *i.e.*, the motion of a small particle in the fluid, by which one can calculate an effective viscosity being different from that of the original fluid and determine the radius of particles. Einstein [12] calculated their quantities when the obstacle is a sphere (see Batchelor [3] and Landau-Lifschitz [23]).

In 1850, G.G. Stokes showed that, in general, in two-dimensional exterior domains, there is no solution u of (S) tending to a prescribed *non-zero constant vector at infinity*. We shall first generalize the "Stokes paradox" to higher dimensions and determine the exact class of solutions in which the paradox holds. Indeed, we shall treat the simpler class (D_q) and show that $u \equiv 0$ is the only solution of (S) with $f \equiv g \equiv 0$, if $1 < q \leq n/(n-1)$. In the two-dimensional case, Finn [14] and Heywood [20] obtained similar results. Secondly, we shall give a concrete characterization of the *null-space* for the solutions of (S) in the class (CL_q). Here we shall see that a non-trivial null-space appears when q varies and that the case $q = n/(n-1)$ is critical. Finally, based on these results of the null-space, we shall give a theorem on the existence and uniqueness for the solutions of (S) in the class (CL_q). This theorem holds if one can solve (S) with the boundary condition $(B.C.)_\infty$ at infinity.

Our basic tool consists of the two fundamental facts, a *regularity theory* (Theorem 3.1) and an *a priori estimate* (Theorem 3.3) in L^q-spaces for the gradient of solutions of (S). The regularity theorem is useful to show the Stokes paradox in higer dimensions and enables us to see why the critical value $q = n/(n-1)$ appears in solvability of (S). The a priori estimate plays a basic role in characterization of the null-space and range of solutions. Such an estimate has been got by several authors for $n \geq 3$ (Kozono-Sohr [22], Borchers-Miyakawa [7]). Recently, Galdi-Simader [18] obtained a similar result to ours by using the hydrodynamic potentials. Our method is however different from Galdi-Simader's [18]: we are based on the cut-off procedure. Making use of a simple embedding argument about a certain functional space, we shall show the same a priori estimate holding for all $n \geq 2$.

Concerning characterization of the null-space in the class (CL_q), Maslennikova-Timoshin [25] solved (S) explicitly in an exterior domain of the unit sphere in $\mathbb{R}^3$ and announced a similar result to ours. They have used the special functions (the Legendre functions) for representation of the solution. We shall give a more systematic treatement for generalized solutions of (S). Our approach is so different from [25] that we can apply it to all dimensions $n \geq 2$. For another investigation such as strong solutions, see, *e.g.*, Sohr-Varnhorn [30].

1. - Main Results

1.1. Before stating our results we introduce some notations. For $1 < q < \infty$ $(q' = q/(q-1))$, $\|\cdot\|_q$ and $(\cdot,\cdot)$ denote the usual norm of $L^q(\Omega)$ and the inner product between $L^q(\Omega)$ and $L^{q'}(\Omega)$, respectively. In general we shall denote by (f, ϕ) the value of the distribution f at $\phi \in C_0^\infty(\Omega)$. $\hat{H}_0^{1,q}(\Omega)$ is the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|\nabla u\|_q$. Since Ω is an exterior domain, $\hat{H}_0^{1,q}(\Omega)$ is larger than $H_0^{1,q}(\Omega)$. Having introduced $\hat{H}_0^{1,q}(\Omega)$, it is also useful to define $\hat{H}^{-1,q}(\Omega) := \hat{H}_0^{1,q'}(\Omega)^*$ (X^*; dual space of X), and $\|\cdot\|_{-1,q}$ denotes the norm of $\hat{H}^{-1,q}(\Omega)$ defined by $\|f\|_{-1,q} := \sup\{|(f,\phi)|/\|\nabla\phi\|_{q'};\ \phi \in C_0^\infty(\Omega),\ \phi \neq 0\}$. We shall denote by $C_0^\infty(\Omega)^n$, $L^q(\Omega)^n, \cdots$, and $C_0^\infty(\Omega)^{n^2}$, $L^q(\Omega)^{n^2}, \cdots$ the corresponding spaces for the vector-valued and the matrix-valued functions, respectively. In such spaces, we shall also use the same notations $\|\cdot\|_q$ and $(\cdot,\cdot)$.

Let $f \in \hat{H}^{-1,q}(\Omega)^n$ and $g \in L^q_{\mathrm{loc}}(\overline{\Omega})$, where $g \in L^q_{\mathrm{loc}}(\overline{\Omega})$ means that $\int_{\Omega\cap B} |g(x)|^q\,dx < \infty$ for all open balls B in $\mathbb{R}^n$ with $\Omega \cap B \neq \phi$. A pair $\{u,p\} \in H^{1,q}_{\mathrm{loc}}(\overline{\Omega})^n \times L^q(\Omega)$ with $u\big|_{\partial\Omega} = 0$ (in the trace sense) is called a *generalized solution* of (S) if

$$(\nabla u, \nabla \Phi) - (p, \operatorname{div}\Phi) = (f, \Phi),\quad -(u, \nabla\phi) = (g, \phi)$$

for all $\Phi \in C_0^\infty(\Omega)^n$ and all $\phi \in C_0^\infty(\Omega)$, respectively.

1.2. Our result on the generalized Stokes paradox now reads:

THEOREM A. (Stokes paradox). *Let $n \geq 2$ and $1 < q \leq n'$ $(n' \equiv n/(n-1))$. Suppose that $\{u,p\} \in H^{1,q}_{\mathrm{loc}}(\overline{\Omega})^n \times L^q(\Omega)$ is a generalized solution of (S) with $f \equiv 0$, $g \equiv 0$ satisfying $\nabla u \in L^q(\Omega)^{n^2}$. Then it follows that $u \equiv 0$, $p \equiv 0$ in Ω.*

By Bogovskii's result [6], the pressure p is determined by u, and hence we can restate the above theorem without p.

THEOREM A$'$. *Let $n \geq 2$ and $1 < q \leq n'$. Suppose that $u \in H^{1,q}_{\mathrm{loc}}(\overline{\Omega})^n$ satisfies* $\operatorname{div} u = 0$ *in Ω, $u\big|_{\partial\Omega} = 0$, and $(\nabla u, \nabla\Phi) = 0$ for all $\Phi \in C_0^\infty(\Omega)^n$ with* $\operatorname{div}\Phi = 0$. *If, in addition, $\nabla u \in L^q(\Omega)^{n^2}$, then we have $u \equiv 0$ in Ω.*

REMARKS. 1. In the above theorem, we do not assume any integrability condition on u itself. It follows that there is no solution u of (S) with $f \equiv 0$, $g \equiv 0$ in the class (D_q) for $1 < q \leq n'$ such that $u(x) \to a$ as $x \to \infty$, where a is a *non-zero* constant vector in $\mathbb{R}^n$.

2. Heywood [20] showed the same result in the special case $n = q = 2$. Chang-Finn [11] gave a similar result for $n = 2$ in the class $u(x) = o(\log |x|)$ as $x \to \infty$.

1.3. We next proceed to the characterization of the null-space for (S) in the class (CL_q).

Let us denote by $\mathbb{N}_q$ the set of all generalized solutions $\{u, p\} \in H^{1,q}_{\text{loc}}(\overline{\Omega})^n \times L^q(\Omega)$ of (S) with $f \equiv 0$, $g \equiv 0$ satisfying $\nabla u - A \in L^q(\Omega)^{n^2}$ for some matrix $A \in \mathbb{R}^{n^2}$ with $\operatorname{Tr} A = 0$. $\mathbb{N}_q^0$ is the subspace of $\mathbb{N}_q$ defined by $\mathbb{N}_q^0 \equiv \left\{\{u, p\} \in \mathbb{N}_q;\ \nabla u \in L^q(\Omega)^{n^2}\right\}$.

Our second result now reads:

THEOREM B. (Characterization of the null-space). (i) *Let* $1 < q \leq n'$ *for* $n \geq 3$ *and* $1 < q < 2$ *for* $n = 2$. *Then* $\dim \mathbb{N}_q = n^2 - 1$ *and* $\dim \mathbb{N}_q^0 = 0$. *For every* $A \in \mathbb{R}^{n^2}$ *with* $\operatorname{Tr} A = 0$ *and* $a \in \mathbb{R}^n$ *satisfying the condition*

$$\int_{\partial\Omega} \left\{(Ax + a) \cdot \frac{\partial v}{\partial \nu} - \chi(Ax + a) \cdot \nu\right\} \, \mathrm{d}S = 0 \tag{1.1}$$

for all $\{v, \chi\} \in \mathbb{N}_{q'}^0$, *there exists a unique* $\{u, p\} \in \mathbb{N}_q$ *such that*

$$\nabla u - A \in L^q(\Omega)^{n^2}, \tag{1.2}$$

$$u \in C^0(\overline{\Omega})^n, \quad \lim_{x \to \infty} |u(x) - (Ax + a)| = 0, \tag{1.3}$$

where ν *denotes the unit outer normal to* $\partial\Omega$ *and* $\mathrm{d}S$ *is the surface element of* $\partial\Omega$. *Conversely, for every* $\{u, p\} \in \mathbb{N}_q$, *there are unique* $A \in \mathbb{R}^{n^2}$ *with* $\operatorname{Tr} A = 0$ *and* $a \in \mathbb{R}^n$ *such that* (1.2) *and* (1.3) *hold.*

(ii) *Let* $n' < q < \infty$, $n \geq 2$. *Then* $\dim \mathbb{N}_q = n^2 + n - 1$ *and* $\dim \mathbb{N}_q^0 = n$. *For every* $A \in \mathbb{R}^{n^2}$ *with* $\operatorname{Tr} A = 0$ *and* $a \in \mathbb{R}^n$, *there exists a unique* $\{u, p\} \in \mathbb{N}_q$ *such that* (1.2) *and* (1.3) *hold if* $n \geq 3$, *and such that* (1.2) *and*

$$\int_{\Omega} |\nabla[u(x) - Ax - E(x)a]|^2 \, \mathrm{d}x < \infty \tag{1.3'}$$

hold if $n = 2$, *where* $E = (E_{ij}(x))_{i,j=1,2}$ *denotes the fundamental tensor of the Stokes equations in* $\mathbb{R}^2$: $E_{ij}(x) = (4\pi)^{-1}\left[\log\left(\frac{\delta_{ij}}{|x|}\right) + \frac{x_i x_j}{|x|^2}\right]$. *Conversely, for every* $\{u, p\} \in \mathbb{N}_q$, *there are unique* $A \in \mathbb{R}^{n^2}$ *with* $\operatorname{Tr} A = 0$ *and* $a \in \mathbb{R}^n$ *such that* (1.2)-(1.3) *hold if* $n \geq 3$, *and such that* (1.2)-(1.3′) *hold if* $n = 2$.

(iii) *Let* $n = q = 2$. *Then* $\dim \mathbb{N}_2 = n^2 - 1 = 3$ *and* $\dim \mathbb{N}_2^0 = 0$. *For every* $A \in \mathbb{R}^{2^2}$ *with* $\mathrm{Tr}\ A = 0$, *there is a unique* $\{u, p\} \in \mathbb{N}_2$ *such that* (1.2) *holds with* $q = 2$. *Conversely, for every* $\{u, p\} \in \mathbb{N}_2$, *there is a unique* $A \in \mathbb{R}^{2^2}$ *with* $\mathrm{Tr}\ A = 0$ *such that* (1.2) *holds with* $q = 2$.

REMARK. For $\{v, \chi\} \in \mathbb{N}_{q'}^0$, we have $\nabla v - \chi I \in L^{q'}(\Omega)^{n^2}$ and $\mathrm{div}\,(\nabla v - \chi I) = 0$ in the sense of distributions in Ω, where I is the identity matrix in $\mathbb{R}^{n^2}$. Using the trace theorem as in Miyakawa [26, Proposition 1.2] and Simader-Sohr [29], we see that $\dfrac{\partial v}{\partial \nu} - \chi I \cdot \nu \in \left(H^{1/q',q}(\partial\Omega)^n\right)^*$ and hence (1.1) should be understood in such a generalized sense as the duality between $\dfrac{\partial v}{\partial \nu} - \chi I \cdot \nu \in \left(H^{1/q',q}(\partial\Omega)^n\right)^*$ and $Ax + a \in H^{1/q',q}(\partial\Omega)^n$. However, from the regularity theorem in bounded domains (as Cattabriga [10] showes), we get $v \in H^{2,q'}_{\mathrm{loc}}(\overline{\Omega})^n$, $\chi \in H^{2,q'}_{\mathrm{loc}}(\overline{\Omega})$; therefore (1.1) may be also regarded in the usual sense.

1.4. We are next concerned with the necessary and sufficient condition for the solvability of (S) in the class (CL_q).

THEOREM C. (Inhomogeneous case). (i) *Let* $1 < q \le n'$ *for* $n \ge 3$ *and* $1 < q < 2$ *for* $n = 2$. *Then for every* $f \in \hat{H}^{-1,q}(\Omega)^n$, $g \in L^q_{\mathrm{loc}}(\overline{\Omega})$, $A \in \mathbb{R}^{n^2}$ *with* $g - \mathrm{Tr}\ A \in L^q(\Omega)$ *and* $a \in \mathbb{R}^n$, *there exists a generalized solution* $\{u, p\} \in H^{1,q}_{\mathrm{loc}}(\overline{\Omega})^n \times L^q(\Omega)$ *of* (S) *satisfying* (1.2) *and*

$$(1.4) \qquad \int_\Omega |u(x) - (Ax + a)|^{nq/(n-q)} dx < \infty,$$

if and only if the compatibility condition

$$(1.5) \qquad (f, v) - (g - \mathrm{Tr}\ A, \chi) + \int_{\partial\Omega} \left\{ (Ax + a) \cdot \frac{\partial v}{\partial \nu} - \chi (Ax + a) \cdot \nu \right\} dS = 0$$

holds for all $\{v, \chi\} \in \mathbb{N}_{q'}^0$. *Such* $\{u, p\}$ *is unique and subject to the inequality*

$$(1.6) \qquad \|\nabla u - A\|_q + \|p\|_q \le C \left(\|f\|_{-1,q} + \|g - \mathrm{Tr}\ A\|_q + |A| + |a| \right)$$

with $C = C(\Omega, n, q) > 0$ *independent of* u *and* p, *where* $|A|$ *and* $|a|$ *denote the standard Euclidian norms in* $\mathbb{R}^{n^2}$ *and* $\mathbb{R}^n$, *respectively.*

(ii) *Let* $n' < q < n$, $n \ge 3$. *Then for every* $f \in \hat{H}^{-1,q}(\Omega)^n$, $g \in L^q_{\mathrm{loc}}(\overline{\Omega})$, $A \in \mathbb{R}^{n^2}$ *with* $g - \mathrm{Tr}\ A \in L^q(\Omega)$ *and* $a \in \mathbb{R}^n$, *there exists a unique generalized solution* $\{u, p\} \in H^{1,q}_{\mathrm{loc}}(\overline{\Omega})^n \times L^q(\Omega)$ *of* (S) *such that* (1.2) *and* (1.4) *hold. Such* $\{u, p\}$ *is subject to the inequality* (1.6). *If in addition* $f \in \hat{H}^{-1,r}(\Omega)^n$, $g - \mathrm{Tr}\ A \in L^r(\Omega)$ *for some* $r > n$, *we have also* $\nabla u - A \in L^r(\Omega)^{n^2}$, $p \in L^r(\Omega)$ *and* (1.3).

(iii) *Let $n \leq q < \infty$ for $n \geq 3$ and $2 < q < \infty$ for $n = 2$. Then for every $f \in \hat{H}^{-1,q}(\Omega)^n$, $g \in L^q_{\mathrm{loc}}(\overline{\Omega})$ and $A \in \mathbb{R}^{n^2}$ with $g - \mathrm{Tr}\ A \in L^q(\Omega)$, there exists at least one generalized solution $\{u,p\} \in H^{1,q}_{\mathrm{loc}}(\overline{\Omega})^n \times L^q(\Omega)$ of (S) satisfying* (1.2). *Such $\{u,p\}$ is unique modulo $\mathbb{N}^0_q$ and subject to the inequality*

$$\begin{aligned}(1.7)\qquad &\inf\left\{\|\nabla u - A - \nabla v\|_q + \|p - \chi\|_q;\ \{v,\chi\} \in \mathbb{N}^0_q\right\} \\ &\leq C\left(\|f\|_{-1,q} + \|g - \mathrm{Tr}\ A\|_q + |A|\right),\end{aligned}$$

where $C = C(\Omega, n, q) > 0$.

(iv) *Let $n = q = 2$. Then for every $f \in \hat{H}^{-1,2}(\Omega)^2$, $g \in L^2_{\mathrm{loc}}(\overline{\Omega})$ and $A \in \mathbb{R}^{2^2}$ with $g - \mathrm{Tr}\ A \in L^2(\Omega)$, there exists a unique generalized solution $\{u,p\} \in H^{1,2}_{\mathrm{loc}}(\overline{\Omega})^2 \times L^2(\Omega)$ of (S) satisfying* (1.2) *with $q = 2$. Such $\{u,p\}$ is subject to the inequality*

$$(1.8)\qquad \|\nabla u - A\|_2 + \|p\|_2 \leq C\left(\|f\|_{-1,2} + \|g - \mathrm{Tr}\ A\|_2 + |A|\right),$$

where $C = C(\Omega) > 0$.

REMARKS. 1. In case (i), the compatibility condition (1.5) is *necessary and sufficient* for the solvability of (S).

2. In case (ii), the additional condition $f \in \hat{H}^{-1,r}(\Omega)^n$, $g - \mathrm{Tr}\ A \in L^r(\Omega)$ $(r > n)$ enables us to get the smoothness of u and its asymptotic behaviour (1.3). In case (iii) we cannot prescribe $a \in \mathbb{R}^n$ so that the uniqueness follows. However, if we assume in addition that $f \in \hat{H}^{-1,\gamma}(\Omega)^n$, $g - \mathrm{Tr}\ A \in L^\gamma(\Omega)$ for some $n' < \gamma < n$, then we can prescribe $a \in \mathbb{R}^n$ so as to get the unique solvability under the condition (1.3).

2. - Preliminaries

2.1. *Homogeneous Sobolev space $\hat{H}^{1,q}_0(\Omega)$.*

In this subsection we shall give a concrete characterization of $\hat{H}^{1,q}_0(\Omega)$ and some elementary lemmas for the proof of the main results.

Let D be a domain in $\mathbb{R}^n$ $(n \geq 2)$. We denote by $\|\cdot\|_{q,D}$ and $(\cdot,\cdot)_D$ the norm of $L^q(D)$ and the inner product between $L^q(D)$ and $L^{q'}(D)$, respectively. $\hat{H}^{1,q}_0(D)$ is the completion of $C_0^\infty(D)$ with respect to the norm $\|\nabla u\|_{q,D}$. If D is bounded, the Poincaré inequality states that $\hat{H}^{1,q}_0(D) = H^{1,q}_0(D)$, but, in general, $\hat{H}^{1,q}_0(D)$ is larger than $H^{1,q}_0(D)$. $\hat{H}^{-1,q}(D)$ is the dual space of $\hat{H}^{1,q'}_0(D)$ $(1/q + 1/q' = 1)$ whose norm is denoted by $\|\cdot\|_{-1,q,D}$. In case $D = \Omega$, we shall call these norms $\|\cdot\|_q$, $(\cdot,\cdot)$, and $\|\cdot\|_{-1,q}$. In what follows C denotes a constant which may change from line to line. In particular, $C = C(*,\cdots,*)$ denotes a constant depending only on the quantities appearing in the parentheses.

The following inequality is simple but very useful for the forthcoming arguments (see also Simader-Sohr [29]).

Variational inequality in L^q. Let $n \geq 2$ and $1 < q < \infty$. Suppose that $u \in L^q_{\text{loc}}(\mathbb{R}^n)$ with $\nabla u \in L^q(\mathbb{R}^n)^n$. Then we have

$$
\begin{aligned}
&\|\nabla u\|_{q,\mathbb{R}^n} \\
(2.1) \qquad &\leq C \sup\left\{ \frac{|(\nabla u, \nabla \phi)_{\mathbb{R}^n}|}{\|\nabla \phi\|_{q',\mathbb{R}^n}};\ \phi \in C_0^\infty(\mathbb{R}^n),\ \phi \neq 0 \right\}
\end{aligned}
$$

with $C = C(n, q)$ independent of u.

Indeed, the Calderon-Zygmund inequality gives

$$
\|\nabla\nabla\psi\|_{q',\mathbb{R}^n} \leq C\|\Delta\psi\|_{q',\mathbb{R}^n} \ (\psi \in C_0^\infty(\mathbb{R}^n)),
$$

Then, since the space $H \equiv \{\Delta\psi;\ \psi \in C_0^\infty(\mathbb{R}^n)\}$ is dense in $L^{q'}(\mathbb{R}^n)$, we have for each $i = 1, \ldots, n$

$$
\begin{aligned}
&\sup\left\{ \frac{|(\nabla u, \nabla \phi)_{\mathbb{R}^n}|}{\|\nabla \phi\|_{q',\mathbb{R}^n}};\ \phi \in C_0^\infty(\mathbb{R}^n),\ \phi \neq 0 \right\} \\
&\geq \sup\left\{ \frac{|(\nabla u, \nabla(\partial_i\psi))_{\mathbb{R}^n}|}{\|\nabla(\partial_i\psi)\|_{q',\mathbb{R}^n}};\ \psi \in C_0^\infty(\mathbb{R}^n),\ \psi \neq 0 \right\} \\
&\geq C \sup\left\{ \frac{|(\partial_i u, \Delta\psi)_{\mathbb{R}^n}|}{\|\Delta\psi\|_{q',\mathbb{R}^n}};\ \psi \in C_0^\infty(\mathbb{R}^n),\ \psi \neq 0 \right\} \\
&= C \sup\left\{ \frac{|(\partial_i u, g)_{\mathbb{R}^n}|}{\|g\|_{q',\mathbb{R}^n}};\ g \in L^{q'}(\mathbb{R}^n),\ g \neq 0 \right\} \\
&= C\|\partial_i u\|_{q,\mathbb{R}^n}
\end{aligned}
$$

with $C = C(n, q)$, and (2.1) follows.

Based on the above variational inequality, we get the following approximation lemma.

LEMMA 2.1. *Let $n \geq 2$ and $1 < q < \infty$. Then for every $u \in L^q_{\text{loc}}(\overline{\Omega})$ with $\nabla u \in L^q(\Omega)^n$, there is a sequence $\{u_j\}_{j=1}^\infty$ in $C_0^\infty(\overline{\Omega})$ such that $\nabla u_j \to \nabla u$ in $L^q(\Omega)^n$, where $C_0^\infty(\overline{\Omega})$ is the set of all C^∞-functions ϕ with compact support in $\overline{\Omega}$ (ϕ may not vanish on $\partial\Omega$). The same assertion is true with Ω replaced by $\mathbb{R}^n$.*

PROOF. By the extension theorem (Adams [1]), for each $u \in L^q_{\text{loc}}(\overline{\Omega})$ with $\nabla u \in L^q(\Omega)^n$, there is a function $\tilde{u} \in L^q_{\text{loc}}(\mathbb{R}^n)$ with $\nabla\tilde{u} \in L^q(\mathbb{R}^n)^n$ such that $\tilde{u} = u$ in Ω, so we may only prove the assertion on $\mathbb{R}^n$. Let $L^{1,q} = \{u \in L^q_{\text{loc}}(\mathbb{R}^n);\ \nabla u \in L^q(\mathbb{R}^n)^n\}$. We denote by $[u]$ the set of all $v \in L^{1,q}$ such that $u - v$ is a constant function on $\mathbb{R}^n$, and set $L^{1,q}/\mathbb{R} \equiv \{[u];\ u \in L^{1,q}\}$ and $G_q \equiv \{\nabla u \in L^q(\mathbb{R}^n)^n;\ [u] \in L^{1,q}/\mathbb{R}\}$. We may regard G_q as a closed subspace of $L^q(\mathbb{R}^n)^n$; equipped with the norm $\|[u]\|_{L^{1,q}/\mathbb{R}} := \|\nabla u\|_{q,\mathbb{R}^n}$, $L^{1,q}/\mathbb{R}$ is a Banach space isometric to G_q. Hence it suffices to prove that the space $W \equiv \{\nabla\phi;\ \phi \in C_0^\infty(\mathbb{R}^n)\}$ is dense in G_q. To this end, let us consider a map $A_q : \nabla u \in G_q \to A_q(\nabla u) \in G_{q'}^*$ defined by $\langle A_q(\nabla u), \nabla v\rangle = (\nabla u, \nabla v)_{\mathbb{R}^n}$ for $\nabla v \in G_{q'}$, where $\langle\cdot,\cdot\rangle$ denotes the duality between $G_{q'}^*$ and $G_{q'}$. Then by (2.1) we see that A_q is injective and that its range is closed in $G_{q'}^*$. Since A_q^* coincides with $A_{q'}$ (T^*; adjoint operator of T), it follows from the closed range theorem that A_q is also surjective and hence bijective. Now, suppose that $F \in G_q^*$ satisfies $\langle F, \nabla\phi\rangle = 0$ for all $\phi \in C_0^\infty(\mathbb{R}^n)$. Since $A_{q'}$ is also bijective, there is a unique $\nabla u \in G_{q'}$ such that $\langle F, \nabla u\rangle = \langle A_{q'}(\nabla u), \nabla v\rangle = (\nabla u, \nabla v)_{\mathbb{R}^n}$ holds for all $\nabla v \in G_q$. Then by the assumption and (2.1) we get $\nabla u = 0$ and hence $F = 0$, which implies that W is dense in G_q. □

REMARK. Simader [28] gave another proof of this lemma by using the Poincaré inequality on annular domains and a scaling argument.

The following concrete characterization of the space $\hat{H}_0^{1,q}(\Omega)$ is essentially due to Galdi-Simader [18, Theorem 1.1]. Based on Lemma 2.1, we give here another proof.

LEMMA 2.2 (Galdi-Simader). (i) *For* $1 < q < n$, *we have*

$$\hat{H}_0^{1,q}(\Omega) = \left\{u \in L^{nq/(n-q)}(\Omega);\ \nabla u \in L^q(\Omega)^n,\ u\big|_{\partial\Omega} = 0\right\}.$$

(ii) *For* $n \le q < \infty$, *we have*

$$\hat{H}_0^{1,q}(\Omega) = \left\{u \in L^q_{\mathrm{loc}}(\overline{\Omega});\ \nabla u \in L^q(\Omega)^n,\ u\big|_{\partial\Omega} = 0\right\}.$$

If $n < q$, *the function* $u \in \hat{H}_0^{1,q}(\Omega)$ *is continuous on* $\overline{\Omega}$ *(after redefinition on a set of measure zero of* Ω*) and satisfies*

$$u(x) = O\left(|x|^{1-n/q}\right) \quad \text{as } x \to \infty.$$

PROOF. Let H_q be the space defined by the right-hand side of (i) and (ii). By the Sobolev inequality, it is easy to see that $\hat{H}_0^{1,q}(\Omega) \subset H_q$ and so we may only prove the converse inclusion. To this end, we introduce an extension operator Γ. Take $R > 0$ so that $\partial\Omega \subset B_R \equiv \{x \in \mathbb{R}^n;\ |x| < R\}$ and consider a continuous extension operator $\Gamma : H^{1-1/q,q}(\partial\Omega) \to H^{1,q}(\Omega)$ satisfying $\operatorname{supp} \Gamma\phi \subset B_R$ for all $\phi \in H^{1-1/q,q}(\partial\Omega)$.

(i) Case $1 < q < n$. Let $u \in H_q$. Then by Lemma 2.1, there is a sequence $\{u_j\}_{j=1}^\infty$ in $C_0^\infty(\overline{\Omega})$ such that $\nabla u_j \to \nabla u$ in $L^q(\Omega)^n$. Since $u \in L^{nq/(n-q)}(\Omega)$, it follows from the Sobolev inequality that $u_j \to u$ in $L^{nq/(n-q)}(\Omega)$. Then by the trace theorem, we get $u_j\big|_{\partial\Omega} \to 0$ in $H^{1-1/q,q}(\partial\Omega)$. Setting $w_j = u_j - \Gamma(u_j\big|_{\partial\Omega})$, we get $w_j \in H_0^{1,q}(\Omega)$ and it follows from the continuity of Γ that

$$\|\nabla w_j - \nabla u\|_q \le \|\nabla u_j - \nabla u\|_q + C\left\|u_j\big|_{\partial\Omega}\right\|_{H^{1-1/q,q}(\partial\Omega)}$$

with C independent of j. Hence $\nabla w_j \to \nabla u$ in $L^q(\Omega)^n$, and since $C_0^\infty(\Omega)$ is dense in $H_0^{1,q}(\Omega)$, we obtain $u \in \hat{H}_0^{1,q}(\Omega)$.

(ii) Case $n \le q < \infty$. Let $u \in H_q$. Then it follows from Lemma 2.1 and a standard argument that there are sequences $\{u_j\}_{j=1}^\infty$ in $C_0^\infty(\overline{\Omega})$ and $\{c_j\}_{j=1}^\infty$ in $\mathbb{R}$ such that $\nabla u_j \to \nabla u$ in $L^q(\Omega)^n$ and $u_j + c_j \to u$ in $L^q_{\mathrm{loc}}(\overline{\Omega})$. We shall next approximate the sequence $\{c_j\}_{j=1}^\infty$ in terms of a sequence of functions in $C_0^\infty(\Omega)$ with respect to the norm $\|\nabla \cdot\|_q$. Take $\varsigma \in C_0^\infty(\mathbb{R}^n)$ satisfying $0 \le \varsigma \le 1$, $\varsigma(x) = 1$ for $|x| \le 1$ and $\varsigma(x) = 0$ for $|x| \ge 2$ and set $\varsigma_k(x) = \varsigma(x/k)$ $(k = 1, 2, \cdots)$. The sequence $\{\varsigma_k\}_{k=1}^\infty$ will be called a sequence of n-dimensional cut-off functions. Then we have $\varsigma_k(x) = 1$ for $|x| \le k$ and $\|\nabla\varsigma_k\|_{q,\mathbb{R}^n} \le Ck^{-1+n/q}$ $(k = 1, 2, \cdots)$ with C independent of k. Since $n \le q$, by Mazur's theorem ([33, p. 120 Theorem 2), we can choose a sequence $\{\overline{\varsigma}_k\}_{k=1}^\infty$ of convex combinations of ς_k's so that

$$\nabla\overline{\varsigma}_k \to 0 \text{ in } L^q(\mathbb{R}^n)^n, \ \overline{\varsigma}_k \to 1 \text{ locally uniformly in } \mathbb{R}^n.$$

Hence there is a subsequence $\{\overline{\varsigma}_{k(j)}\}_{j=1}^\infty$ of $\{\overline{\varsigma}_k\}_{k=1}^\infty$ such that $c_j\|\nabla\overline{\varsigma}_{k(j)}\|_{q,\mathbb{R}^n} \to 0$ as $j \to \infty$. Defining $\overline{u}_j = u_j + c_j\overline{\varsigma}_{k(j)}$ $(j = 1, 2, \cdots)$, we have $\overline{u}_j \in C_0^\infty(\overline{\Omega})$ and $\nabla\overline{u}_j \to \nabla u$ in $L^q(\Omega)^n$, $\overline{u}_j \to u$ in $L^q_{\mathrm{loc}}(\overline{\Omega})$. Now, making use of a sequence $w_j = \overline{u}_j - \Gamma(\overline{u}_j|_{\partial\Omega})$ $(j = 1, 2, \cdots)$ as in the case of (i), we can prove similarly as above that $u \in \hat{H}_0^{1,q}(\Omega)$.

Finally, the asymptotic behaviour $u(x) = O(|x|^{1-n/q})$, $x \to \infty$ for $u \in \hat{H}_0^{1,q}(\Omega)$ with $q > n$ follows from Friedman [15, p. 23 Theorem 9.2]. □

We shall next consider the complex interpolation space $[X, Y]_\theta$ $(0 \le \theta \le 1)$. For all $1 < q$, $r < \infty$, the norms $\|\nabla u\|_q$ and $\|\nabla u\|_r$ are consistent on $C_0^\infty(\Omega)$, so the pair $\{\hat{H}_0^{1,q}(\Omega), \ \hat{H}_0^{1,r}(\Omega)\}$ is interpolation couple. See Reed-Simon [27, p. 35]. Using the Riesz-Thorin theorem [32, 1.18.7], we obtain from Lemma 2.2 the following result:

If $1 < q < n$, $1 < r < n$ and if $n \le q < \infty$, $n \le r < \infty$, then

$$\left[\hat{H}_0^{1,q}(\Omega), \ \hat{H}_0^{1,r}(\Omega)\right]_\theta = \hat{H}_0^{1,s}(\Omega), \tag{2.2}$$

where $1/s = (1-\theta)/q + \theta/r$, $0 \le \theta \le 1$.

In the whole space $\mathbb{R}^n$, we shall prove the corresponding result without restriction on q and r.

LEMMA 2.3. *Let $n \ge 2$ and $1 < q < \infty$, $1 < r < \infty$. Then we have*

$$\left[\hat{H}_0^{1,q}(\mathbb{R}^n), \ \hat{H}_0^{1,r}(\mathbb{R}^n)\right]_\theta = \hat{H}_0^{1,s}(\mathbb{R}^n),$$

where $1/s = (1-\theta)/q + \theta/r$, $0 \le \theta \le 1$.

PROOF. Let $E_q \equiv \{\nabla u \in L^q(\mathbb{R}^n)^n; \ u \in \hat{H}_0^{1,q}(\mathbb{R}^n)\}$. Then we may regard E_q as a *closed* subspace of $L^q(\mathbb{R}^n)^n$. Hence E_q is a Banach space with the norm

$\|\nabla u\|_{E_q} := \|\nabla u\|_{q,\mathbb{R}^n}$ for $\nabla u \in E_q$, and isometric to $\hat{H}_0^{1,q}(\mathbb{R}^n)$. Now it suffices to show that

$$[E_q, E_r]_\theta = E_s \text{ for } q,\ r,\ s \text{ and } \theta \text{ as above.} \tag{2.3}$$

To this end, we need to solve the equation $\Delta\chi = \operatorname{div}\, u$ in $\mathbb{R}^n$ in the following weak sense:

For every $u \in L^q(\mathbb{R}^n)^n$, there is a unique $\chi \in \hat{H}_0^{1,q}(\mathbb{R}^n)$ such that

$$(\nabla\chi, \nabla\phi)_{\mathbb{R}^n} = (u, \nabla\phi)_{\mathbb{R}^n} \text{ for all } \phi \in \hat{H}_0^{1,q'}(\mathbb{R}^n). \tag{2.4}$$

Based on (2.1), we see as in the proof of Lemma 2.1 that the map $B_q : \nabla u \in E_q \to B_q(\nabla u) \in E_{q'}^*$ defined by $\langle B_q(\nabla u), \nabla v\rangle := (\nabla u, \nabla v)_{\mathbb{R}^n}$ for $\nabla v \in E_{q'}$ is a bijective operator. Here $\langle\cdot,\cdot\rangle$ denotes the duality between $E_{q'}^*$ and $E_{q'}$. Since the map $\nabla\phi \in E_{q'} \to (u, \nabla\phi)_{\mathbb{R}^n} \in \mathbb{R}$ is a continuous functional on $E_{q'}$, we can solve (2.4) uniquely for every given $u \in L^q(\mathbb{R}^n)^n$.

Now, it is easy to see that the map $Q : u \to \nabla\chi$ defined by the relation (2.4) is a projection operator from $L^q(\mathbb{R}^n)^n$ onto E_q. Then (2.3) follows from Bergh-Löfström [4, Theorem 6.4.2]. □

We need further the following two lemmas.

LEMMA 2.4. *Let* $1 < q < \infty$ *and* $h \in L^q(\mathbb{R}^n)$. *If*

$$\sup\left\{\frac{|(h, \Delta\phi)_{\mathbb{R}^n}|}{\|\Delta\phi\|_{r',\mathbb{R}^n}};\ \phi \in C_0^\infty(\mathbb{R}^n),\ \ \phi \neq 0\right\} < \infty$$

for some $1 < r < \infty$, *then we have also* $h \in L^r(\mathbb{R}^n)$.

PROOF. Here we follow Simader-Sohr [29]. Since the space $H \equiv \{\Delta\phi;\ \phi \in C^\infty(\mathbb{R}^n)\}$ is a dense subspace in $L^{r'}(\mathbb{R}^n)$, we see by the assumption that the map $\Delta\phi \in H \to (h, \Delta\phi)_{\mathbb{R}^n} \in \mathbb{R}$ is uniquely extended as a continuous functional on $L^{r'}(\mathbb{R}^n)$. Hence there is a unique $\eta \in L^r(\mathbb{R}^n)$ such that $(\eta, \Delta\phi)_{\mathbb{R}^n} = (h, \Delta\phi)_{\mathbb{R}^n}$ holds for all $\phi \in C_0^\infty(\mathbb{R}^n)$. Since $w := h - \eta \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, Weyl's lemma states that the function w is of class C^∞ and harmonic in $\mathbb{R}^n$ in the classical sense. Applying the mean value property to w on the ball $B_{|x|}(x)$ centered at $x \neq 0$ with radius $|x|$, and then using the Hölder inequality, we obtain the estimate

$$|w(x)| \le C\left(\|h\|_{q,\mathbb{R}^n}|x|^{-n/q} + \|\eta\|_{r,\mathbb{R}^n}|x|^{-n/r}\right),$$

where $C = C(n, q, r)$. Then it follows from the Liouville theorem that $w \equiv 0$ in $\mathbb{R}^n$ and hence $h \in L^r(\mathbb{R}^n)$. □

LEMMA 2.5 (Embedding argument). *Let* Ω_0 *be a subdomain of* Ω *with closure* $\overline{\Omega}_0$ *contained in* Ω. *Then for each* $1 < q < \infty$, *there is a constant* $C = C(\Omega, \Omega_0, n, q)$ *such that*

$$\|f\|_{-1,q,\mathbb{R}^n} \le C\|f\|_{-1,q,\Omega}$$

holds for all $f \in \hat{H}^{-1,q}(\mathbb{R}^n)$ *with* supp $f \subset \overline{\Omega}_0$.

PROOF. (i) Case $1 < q \leq n'$. Then we have $n \leq q'$. Let us take a subdomain Ω_1 of Ω so that $\overline{\Omega}_0 \subset \Omega_1$ and so that $D \equiv \Omega/\overline{\Omega}_1$ is a bounded domain in $\mathbb{R}^n$. We show first that the space

$$S_D \equiv \left\{ \phi \in C_0^\infty(\mathbb{R}^n); \; \int_D \phi(x)\, dx = 0 \right\}$$

is *dense* in $\hat{H}_0^{1,q'}(\mathbb{R}^n)$. Indeed, taking the sequence $\{\varsigma_k\}_{k=1}^\infty$ of n-dimensional cut-off functions as in the proof of Lemma 2.2, we see $\varsigma_k(x) = 1$ for $|x| \leq k$ and $\|\nabla \varsigma_k\|_{q',\mathbb{R}^n} \leq Ck^{-1+n/q'}$ with C independent of k. Letting $\phi \in C_0^\infty(\mathbb{R}^n)$, we set $\phi_k(x) = \phi(x) - (\text{vol } D)^{-1} \left(\int_D \phi(y)\, dy \right) \cdot \varsigma_k(x)$, $(k = 1, 2, \cdots)$. For large k, we have $\phi_k \in S_D$, so we may assume that $\phi_k \in S_D$ for all $k \geq 1$. Since $\|\nabla \phi_k - \nabla \phi\|_{q',\mathbb{R}^n} \leq C\|\phi\|_{L^\infty(\mathbb{R}^n)} \cdot k^{-1+n/q'}$ and since $C_0^\infty(\mathbb{R}^n)$ is dense in $\hat{H}_0^{1,q'}(\mathbb{R}^n)$, we see that S_D is dense in $\hat{H}_0^{1,q'}(\mathbb{R}^n)$ if $q' > n$ *i.e.*, if $1 < q < n'$. In case $q' = n$ *i.e.*, in case $q = n'$, again by Mazur's theorem, we can choose a sequence $\{\overline{\phi}_k\}_{k=1}^\infty$ of convex combinations of ϕ_k's so that $\nabla\overline{\phi}_k \to \nabla \phi$ in $L^n(\mathbb{R}^n)^n$ as $k \to \infty$, and we see that S_D is also dense in $\hat{H}_0^{1,n}(\mathbb{R}^n)$.

Let $f \in \hat{H}^{-1,q}(\mathbb{R}^n)$ with supp $f \subset \overline{\Omega}_0$. Taking a function $\eta \in C^\infty(\mathbb{R}^n)$ satisfying $0 \leq \eta \leq 1$, $\eta(x) = 1$ for $x \in \overline{\Omega}_1$ and $\eta(x) = 0$ for $x \in \mathbb{R}^n/\Omega$, we have

$$|(f, \phi)_{\mathbb{R}^n}| = |(f, \eta\phi)_\Omega| \leq C\|f\|_{-1,q} \left(\|(\nabla\eta)\phi\|_{q'} + \|\nabla\phi\|_{q',\mathbb{R}^n} \right) \tag{2.5}$$

for all $\phi \in S_D$ with C independent of ϕ. Since supp $\nabla\eta \subset D$ and since $\int_D \phi(x)\, dx = 0$, we have by the Poincaré inequality on D that $\|(\nabla\eta)\phi\|_{q'} \leq C\|\nabla\phi\|_{q',D}$. Hence from (2.5) it holds

$$|(f, \phi)_{\mathbb{R}^n}| \leq C\|f\|_{-1,q}\|\nabla\phi\|_{q',\mathbb{R}^n} \text{ for all } \phi \in S_D.$$

Since S_D is dense in $\hat{H}_0^{1,q'}(\mathbb{R}^n)$, the above inequality holds for all $\phi \in \hat{H}_0^{1,q'}(\mathbb{R}^n)$, from which we get the desired result in case $1 < q \leq n'$.

(ii) Case $n' < q < \infty$. Since $1 < q' < n$, we can take $r \in (q', \infty)$ so that $1/r = 1/q' - 1/n$. Then it follows from the Sobolev inequality in $\mathbb{R}^n$ that

$$\|(\nabla\eta)\phi\|_{q'} \leq C\|\phi\|_{q',D} \leq C\|\phi\|_{r,\mathbb{R}^n} \leq C\|\nabla\phi\|_{q',\mathbb{R}^n}$$

for all $\phi \in C_0^\infty(\mathbb{R}^n)$. Now we get the desired result by making use of (2.5) with $\phi \in S_D$ replaced by $\phi \in C_0^\infty(\mathbb{R}^n)$. □

2.2. *Stokes equations in bounded domains.*

In this subsection, we recall the L^q-theory for the Stokes equations in bounded domains due to Cattabriga [10].

THEOREM 2.6 (Cattabriga). *Let $n \geq 2$ and $G \subset \mathbb{R}^n$ be a bounded domain with boundary ∂G of class $C^{2+\mu}$ $(0 < \mu < 1)$. Let $1 < q < \infty$. Then for every $f \in \hat{H}^{-1,q}(G)^n$ and $g \in L^q(G)$ with $\int_G g(x)\,dx = 0$, there is a unique pair $\{u,p\} \in H_0^{1,q}(G)^n \times L^q(G)$ with $\int_G p(x)\,dx = 0$ such that*

$$-\Delta u + \nabla p = f, \ \operatorname{div} u = g \ \text{in } G \tag{2.6}$$

in the sense of distributions. Such $\{u,p\}$ is subject to the inequality

$$\|\nabla u\|_{q,G} + \|p\|_{q,G} \leq C\left(\|f\|_{-1,q,G} + \|g\|_{q,G}\right), \tag{2.7}$$

where $C = C(G,n,q)$.

REMARK. Since G is bounded, we have $\hat{H}^{-1,q}(G) = H_0^{1,q'}(G)^*$. Cattabriga [10] gave the above result for $n = 3$ under the weaker assumption that ∂G is of class C^2. Galdi-Simader [18] extended Catabriga's result for $n \geq 2$. Another proof was given by Kozono-Sohr [22] (see also Borchers-Miyakawa [7]).

The following corollary is an immediate consequence of Theorem 2.6.

COROLLARY 2.7 (Regularity in bounded domains). *Under the same assumption on G, q, f and g as in Theorem 2.6, suppose that $\{u,p\} \in H_0^{1,q}(G)^n \times L^q(G)$ satisfies (2.6) in the sense of distributions. If, in addition, $f \in \hat{H}^{-1,r}(G)^n$ and $g \in L^r(G)$ for some $1 < r < \infty$, then we have also $u \in H_0^{1,r}(G)^n$ and $p \in L^r(G)$.*

2.3. *Stokes equations in $\mathbb{R}^n$.*

In this subsection, we shall give a result on $\mathbb{R}^n$ corresponding to that of the preceding subsection.

LEMMA 2.8 (Regularity theory in $\mathbb{R}^n$). *Let $n \geq 2$, $1 < q < \infty$ and let $f \in \hat{H}^{-1,q}(\mathbb{R}^n)^n$, $g \in L^q(\mathbb{R}^n)$. Suppose that $\{u,p\} \in L^q_{loc}(\mathbb{R}^n)^n \times L^q(\mathbb{R}^n)$ with $\nabla u \in L^q(\mathbb{R}^n)^{n^2}$ satisfy*

$$-\Delta u + \nabla p = f, \ \operatorname{div} u = g \ \text{in } \mathbb{R}^n \tag{2.8}$$

in the sense of distributions. If, in addition, $f \in \hat{H}^{-1,r}(\mathbb{R}^n)^n$ and $g \in L^r(\mathbb{R}^n)$ for some $1 < r < \infty$, then we have also $\nabla u \in L^r(\mathbb{R}^n)^{n^2}$ and $p \in L^r(\mathbb{R}^n)$.

PROOF. By Lemma 2.1, there are sequences $\{u_j\}_{j=1}^\infty$ in $C_0^\infty(\mathbb{R}^n)^n$ and $\{p_j\}_{j=1}^\infty$ in $C_0^\infty(\mathbb{R}^n)$ such that

$$\nabla u_j \to \nabla u \ \text{in } L^q(\mathbb{R}^n)^{n^2}, \ p_j \to p \ \text{in } L^q(\mathbb{R}^n). \tag{2.9}$$

Set $f_j := -\Delta u_j + \nabla p_j$ and $g_j := \operatorname{div} u_j$ ($j = 1, 2, \cdots$). Then we have by (2.8) and (2.9)

$$(2.10) \qquad (f_j, \Phi)_{\mathbb{R}^n} \to (f, \Phi)_{\mathbb{R}^n}, \ (g_j, \phi)_{\mathbb{R}^n} \to (g, \phi)$$

for all $\Phi \in C_0^\infty(\mathbb{R}^n)^n$ and all $\phi \in C_0^\infty(\mathbb{R}^n)$, respectively. On the other hand, using the fundamental solution F_n of $-\Delta$ in $\mathbb{R}^n$, we can represent u_j and p_j as

$$u_j = F_n * f_j - F_n * \nabla p_j, \ p_j = -\operatorname{div} F_n * (f_j + \nabla g_j),$$

where $*$ denotes the convolution. Then it follows that

$$(p_j, \Delta\phi)_{\mathbb{R}^n} = -(f_j, \nabla\phi)_{\mathbb{R}^n} + (g_j, \Delta\phi)_{\mathbb{R}^n} \text{ for all } \phi \in C_0^\infty(\mathbb{R}^n).$$

Letting $j \to \infty$ and then using the Calderon-Zygmund inequality, we have by (2.9)-(2.10) that

$$|(p, \Delta\phi)_{\mathbb{R}^n}| \le C \left(\|f\|_{-1,r,\mathbb{R}^n} + \|g\|_{r,\mathbb{R}^n} \right) \|\Delta\phi\|_{r',\mathbb{R}^n}$$

for all $\phi \in C_0^\infty(\mathbb{R}^n)$. Hence Lemma 2.4 states that $p \in L^r(\mathbb{R}^n)$. Concerning the regularity of ∇u, we have similarly

$$(\partial_k u_j, \Delta\Phi)_{\mathbb{R}^n} = (f_j, \partial_k\Phi)_{\mathbb{R}^n} + (p_j, \ \operatorname{div} (\partial_k\Phi))_{\mathbb{R}^n}, \ (k = 1, \cdots, n),$$

for all $\Phi \in C_0^\infty(\mathbb{R}^n)^n$. Since $p \in L^r(\mathbb{R}^n)$, the same argument as above yields

$$|(\partial_k u, \Delta\Phi)_{\mathbb{R}^n}| \le C \left(\|f\|_{-1,r,\mathbb{R}^n} + \|p\|_{r,\mathbb{R}^n} \right) \|\Delta\Phi\|_{r',\mathbb{R}^n}, \ (k = 1, \cdots, n),$$

for all $\Phi \in C_0^\infty(\mathbb{R}^n)^n$. Again from Lemma 2.4, we get $\partial_k u \in L^r(\mathbb{R}^n)^n$, ($k = 1, \cdots, n$). □

Concerning the existence and uniqueness of solutions in the class $\hat{H}_0^{1,q}(\mathbb{R}^n)$, we have

LEMMA 2.9 (A priori estimate in $\mathbb{R}^n$). *Let $n \ge 2$ and $1 < q < \infty$. Then for every $f \in \hat{H}^{-1,q}(\mathbb{R}^n)^n$ and $g \in L^q(\mathbb{R}^n)$, there is a unique $\{u, p\} \in \hat{H}_0^{1,q}(\mathbb{R}^n)^n \times L^q(\mathbb{R}^n)$ such that* (2.8) *holds in the sense of distributions. Such $\{u, p\}$ is subject to the inequality*

$$\|\nabla u\|_{q,\mathbb{R}^n} + \|p\|_{q,\mathbb{R}^n} \le C \left(\|f\|_{-1,q,\mathbb{R}^n} + \|g\|_{q,\mathbb{R}^n} \right),$$

where $C = C(n, q)$.

PROOF. By the definition of the space $\hat{H}_0^{1,q'}(\mathbb{R}^n)$, we see that the operator $-\nabla : \hat{H}_0^{1,q'}(\mathbb{R}^n) \to L^{q'}(\mathbb{R}^n)^n$ is injective and has a closed range. Hence by the closed range theorem, the adjoint operator $\operatorname{div} = (-\nabla)^* : L^q(\mathbb{R}^n)^n \to \hat{H}^{-1,q}(\mathbb{R}^n)$ is surjective. Since the null space Ker(div) of div is a closed subspace in $L^q(\mathbb{R}^n)^n$, for each $h \in \hat{H}^{-1,q}(\mathbb{R}^n)$, there is at least one $u \in L^q(\mathbb{R}^n)^n$ such that $-(u, \nabla\phi)_{\mathbb{R}^n} = (h, \phi)_{\mathbb{R}^n}$ holds for all $\phi \in \hat{H}_0^{1,q'}(\mathbb{R}^n)$ and that $\|u\|_{q,\mathbb{R}^n} \le C\|h\|_{-1,q,\mathbb{R}^n}$ with C independent of h. Let us now use the properties of the space E_q and the bijective operator $B_q : E_q \to E_{q'}^*$ in the proof of Lemma 2.3. Since $u \in L^q(\mathbb{R}^n)^n$, the map $\nabla\phi \in E_{q'} \to -(u, \nabla\phi)_{\mathbb{R}^n} \in \mathbb{R}$ is an element in $E_{q'}^*$, so we can choose $\pi \in \hat{H}_0^{1,q}(\mathbb{R}^n)$ so that

$$(2.11) \qquad (\nabla\pi, \nabla\phi)_{\mathbb{R}^n} = -(u, \nabla\phi)_{\mathbb{R}^n} = (h, \phi)_{\mathbb{R}^n} \text{ for all } \phi \in \hat{H}_0^{1,q'}(\mathbb{R}^n).$$

By (2.1) such a π is uniquely determined by h and so (2.11) defines a bounded linear operator $J_q : h \in \hat{H}^{-1,q}(\mathbb{R}^n) \to \pi \in \hat{H}_0^{1,q}(\mathbb{R}^n)$. A direct calculation shows that

$$(\text{div } J_q(\nabla\psi) + \psi,\ \Delta\phi)_{\mathbb{R}^n} = 0 \text{ for all } \psi \in L^q(\mathbb{R}^n),\ \phi \in C_0^\infty(\mathbb{R}^n).$$

Since the space $H = \{\Delta\phi;\ \phi \in C_0^\infty(\mathbb{R}^n)\}$ is dense in $L^{q'}(\mathbb{R}^n)$, the above identity yields that div $J_q(\nabla\psi) = -\psi$ for all $\psi \in L^q(\mathbb{R}^n)$. Then we see that the pair $\{u, p\}$ defined by

$$u = J_q f + J_q(\nabla \text{ div } J_q(f + \nabla g)),\ \ p = -\text{div } J_q(f + \nabla g)$$

has the desired property.

Now it remains to show the uniqueness. Let $\{u', p'\} \in \hat{H}_0^{1,q}(\mathbb{R}^n)^n \times L^q(\mathbb{R}^n)$ satisfy (2.8) in the sense of distributions. Then $\overline{u} = u - u'$, $\overline{p} = p - p'$ satisfies (2.8) with $f = 0$, $g = 0$. Applying the operator div to both sides of the first equation, we get $\Delta\overline{p} = 0$ in the sense of distributions in $\mathbb{R}^n$. Since $\overline{p} \in L^q(\mathbb{R}^n)$, it follows from the Liouville theorem that $\overline{p} \equiv 0$ in $\mathbb{R}^n$. Therefore $\Delta\overline{u} \equiv 0$ in $\mathbb{R}^n$. Since $\overline{u} \in \hat{H}_0^{1,q}(\mathbb{R}^n)^n$, we have by (2.1) that $\overline{u} \equiv 0$ in $\mathbb{R}^n$. □

REMARK. There have been several results related to Lemma 2.9 (Kozono-Sohr [22, Proposition 2.9], Borchers-Miyakawa [7, Proposition 3.7], Galdi-Simader [18, Theorem 3.1]). Our proof seems to be rather simple: we used only the variational inequality (2.1).

3. - Stokes equations in the class (D_q)

In this section we shall give some results in $\hat{H}_0^{1,q}(\Omega)$ analogous to those of subsection 2.3. In exterior domains, because of the boundary condition, we have restrictions on q and r.

THEOREM 3.1 (Regularity theory in Ω). *Let $n \ge 2$, $1 < q < \infty$ and $r > n'(= n/(n-1))$ and let $f \in \hat{H}^{-1,q}(\Omega)^n \cap \hat{H}^{-1,r}(\Omega)^n$ and $g \in L^q(\Omega) \cap L^r(\Omega)$. Suppose that $\{u, p\} \in \hat{H}_0^{1,q}(\Omega)^n \times L^q(\Omega)$ is a generalized solution of (S). Then* *we have* $\nabla u \in L^r(\Omega)^{n^2}$ *and* $p \in L^r(\Omega)$. *In case* $r \geq n$ $(n \geq 3)$ *and* $r > 2$ $(n = 2)$, *we have* $u \in \hat{H}_0^{1,r}(\Omega)^n$, *and in case* $1 < q < n$, *we have also* $u \in \hat{H}_0^{1,r}(\Omega)^n$.

PROOF. We use the cut-off procedure. Take a ball $B_R \equiv \{x \in \mathbb{R}^n;\ |x| < R\}$ so that $\partial\Omega \subset B_R$ and take a function $\psi_1 \in C_0^\infty(\mathbb{R}^n)$ satisfying $0 \leq \psi_1 \leq 1$, $\psi_1(x) = 1$ for $x \in \mathbb{R}^n/\Omega$, $\psi_1(x) = 0$ for $|x| \geq R$, and set $\psi_2 = 1 - \psi_1$. Then from (S) it follows

$$(S_i) \qquad -\Delta(\psi_i u) + \nabla(\psi_i p) = f_i,\ \text{div}\ (\psi_i u) = g_i (i = 1, 2),$$

where $f_i = \psi_i f - 2\nabla\psi_i\nabla u - (\Delta\psi_i)u + (\nabla\psi_i)p$, $g_i = \psi_i g + (\nabla\psi_i)u$ $(i = 1, 2)$. We may regard (S_1) and (S_2) as equations in $\Omega_R \equiv \Omega \cap B_R$ and in $\mathbb{R}^n$, respectively. Set $\Omega_1 \equiv \Omega_R$ and $\Omega_2 \equiv \mathbb{R}^n$.

Let us first assume that $1/q - 1/n \leq 1/r < 1/n' = 1 - 1/n$. Taking $s \in (1, \infty)$ so that $1/s = 1/r + 1/n$, we have $s \leq q$ and $1/s' = 1/r' - 1/n$. Since $q' \leq s'$, it follows from the Sobolev embedding $H_0^{1,r'}(\Omega_i) \subset L^{s'}(\Omega_i)$ that

$$\|\phi_i\|_{q',\Omega_R} \leq C\|\phi_i\|_{s',\Omega_R} \leq C\|\nabla\phi_i\|_{r',\Omega_i} \text{ for all } \phi_i \in C_0^\infty(\Omega_i)\ (i = 1, 2).$$

Since supp $\nabla\psi_i$ and supp $\Delta\psi_i$ are contained in Ω_R, we have by assumption and the above inequality that $f_i \in \hat{H}^{-1,r}(\Omega_i)^n$ $(i = 1, 2)$. By the Sobolev embedding $H^{1,q}(\Omega_R) \subset L^r(\Omega_R)$, we get easily $g_i \in L^r(\Omega_i)$ $(i = 1, 2)$, and also

$$\int_{\Omega_1} g_1 \ dx = \int_{\Omega_R} \text{div}\ (\psi_1 u)\ dx = \int_{\partial\Omega} u \cdot \nu \ dS = 0.$$

Now applying Corollary 2.7 and Lemma 2.8 to $\{\psi_1 u, \psi_1 p\}$ and $\{\psi_2 u, \psi_2 p\}$, respectively, we obtain

$$(3.1) \qquad \nabla(\psi_i u) \in L^r(\Omega_i)^{n^2},\ \psi_i p \in L^r(\Omega_i)\ (i = 1, 2).$$

We next consider the case $1/q - 2/n \leq 1/r < 1/q - 1/n$. Taking $\bar{q} = (1/q - 1/n)^{-1}$, we have by (3.1) that $\nabla u \in L^{\bar{q}}(\Omega)^{n^2}$ and $p \in L^{\bar{q}}(\Omega)$. Now, taking $\bar{q}$ instead of q in the above, we get (3.1) also for $r > n'$ with $1/r \geq 1/q - 2/n$. Proceeding in the same way to the case $1/r < 1/q - 2/n$, by the bootstrap argument, we get (3.1) for all $r > n'$ and hence $\nabla u \in L^r(\Omega)^{n^2}$, $p \in L^r(\Omega)$ for all $r > n'$.

It remains to show that $u \in \hat{H}_0^{1,r}(\Omega)^n$ in case $r \geq n$ $(n \geq 3)$, $r > 2$ $(n = 2)$, and in case $1 < q < n$. To this end, we may show $\psi_2 u \in \hat{H}_0^{1,r}(\Omega)^n$ in (3.1). Consider first the case when $r \geq n$ $(n \geq 3)$ and $r > 2$ $(n = 2)$. Since $\nabla(\psi_2 u) \in L^r(\mathbb{R}^n)^{n^2}$ and since $\psi_2 u$ vanishes in a neighbourhood of $\partial\Omega$, we get by Lemma 2.2(ii) that $\psi_2 u \in \hat{H}_0^{1,r}(\Omega)^n$. We next consider the case when $1 < q < n$, $n' < r < n$ $(n \geq 3)$. Since $f_2 \in \hat{H}^{-1,r}(\mathbb{R}^n)^n$ and $g_2 \in L^r(\mathbb{R}^n)$, it follows from Lemma 2.9 that there is a unique pair $\{v, \chi\} \in \hat{H}_0^{1,r}(\mathbb{R}^n)^n \times L^r(\mathbb{R}^n)$ such that $-\Delta v + \nabla\chi = f_2$, div $v = g_2$ in the sense of distributions in $\mathbb{R}^n$. Taking

$w = v - \psi_2 u$ and $\eta = \chi - \psi_2 p$, we see that $\{w, \eta\}$ satisfies (2.8) with $f = 0$ and $g = 0$; applying div to both sides of the first equation, we obtain that η is harmonic in $\mathbb{R}^n$. Since $\eta \in L^r(\mathbb{R}^n)$, the Liouville theorem yields that $\eta \equiv 0$ in $\mathbb{R}^n$; hence w is also harmonic in $\mathbb{R}^n$. Moreover, by the Sobolev embedding theorem, we obtain $w \in L^{\overline{q}}(\mathbb{R}^n) + L^{\overline{r}}(\mathbb{R}^n)$, where $1/\overline{q} = 1/q - 1/n$ and $1/\overline{r} = 1/r - 1/n$. Using the same argument as in the proof of Lemma 2.4, we get $w \equiv 0$ in $\mathbb{R}^n$, from which $\psi_2 u \in \hat{H}_0^{1,r}(\mathbb{R}^n)^n$ follows. Now, again by Lemma 2.2(i), we have $\psi_2 u \in \hat{H}_0^{1,r}(\Omega)^n$. □

REMARK. The restriction $n' < r$ is a critical condition; we *cannot* take $1 < r \leq n'$ in Theorem 3.1. Indeed, let us assume the main results in Section 1. Taking some $n < q < \infty$ and $A = 0$, $a \neq 0$ in Theorem B(ii), we get such $\{u, p\} \in \mathbb{N}_q^0$ as $\lim_{x \to \infty} u(x) = a$ in case $n \geq 3$ and as $\int_\Omega |\nabla u(x) - \nabla E(x) a|^2 dx < \infty$ in case $n = 2$, and by Lemma 2.2(ii), we have $u \in \hat{H}_0^{1,q}(\Omega)^n$. Suppose now that Theorem 3.1 is true for $1 < r \leq n'$. Then it follows that $\nabla u \in L^r(\Omega)^{n^2}$ for some $1 < r \leq n'$. Thus by Theorem A, we get $u \equiv 0$ in Ω, which contradicts $a \neq 0$. Note that $\int_\Omega |\nabla E(x)|^2 \, dx = \infty$ in case $n = 2$.

We shall next give an a priori estimate in the class (D_q). For this purpose we need:

LEMMA 3.2. *Let $n \geq 2$ and $1 < q < \infty$ and let $\{f, g\} \in \hat{H}^{-1,q}(\Omega)^n \times L^q(\Omega)$. Suppose that $\{u, p\} \in \hat{H}_0^{1,q}(\Omega)^n \times L^q(\Omega)$ is a generalized solution of (S). Then we have*

$$
\begin{aligned}
&\|\nabla u\|_q + \|p\|_q \\
&\leq C \left(\|f\|_{-1,q} + \|g\|_q + \|u\|_{q,\Omega_R} + \|p\|_{-1,q,\Omega_R} + \left| \int_{\Omega_R} \psi_1(x) p(x) \, dx \right| \right),
\end{aligned}
\tag{3.2}
$$

where $\Omega_R = \Omega \cap B_R$ and ψ_1 are the same as in the proof of Theorem 3.1 *and where C is a constant independent of u and p.*

PROOF. We use again the cut-off method. Recalling the equations (S_i) ($i = 1, 2$) in the proof of Theorem 3.1, we first consider (S_1) in Ω_R. Since supp $\nabla \psi_1$, supp $\Delta \psi_1 \subset \Omega_R$, we obtain

$$
\begin{aligned}
\|f_1\|_{-1,q,\Omega_R} &\leq C \left(\|f\|_{-1,q} + \|u\|_{q,\Omega_R} + \|p\|_{-1,q,\Omega_R} \right), \\
\|g_1\|_{q,\Omega_R} &\leq C \left(\|g\|_q + \|u\|_{q,\Omega_R} \right).
\end{aligned}
$$

Applying Theorem 2.6 to $\{\psi_1 u, \psi_1 p\}$ in (S_1) and then using the above inequalities, we get

$$\begin{aligned} &\|\nabla(\psi_1 u)\|_{q,\Omega_R} + \|\psi_1 p\|_{q,\Omega_R} \\ &\leq C\left(\|f\|_{-1,q} + \|g\|_q + \|u\|_{q,\Omega_R} + \|p\|_{-1,q,\Omega_R} + \left|\int_{\Omega_R} \psi_1(x)p(x)\ \mathrm{d}x\right|\right) \end{aligned} \tag{3.3}$$

with C independent of u and p.

We next consider (S_2) in $\mathbb{R}^n$. Since supp $f_2 \subset$ supp ψ_2, it follows from Lemma 2.5 that

$$\|f_2\|_{-1,q,\mathbb{R}^n} \leq C\|f_2\|_{-1,q} \text{ with } C = C(n,q). \tag{3.4}$$

Since the inequality $\|\phi\|_{q',\Omega_R} \leq C\|\nabla\phi\|_{q'}$ holds for all $\phi \in C_0^\infty(\Omega)$ and since supp $\nabla\phi_2$, supp $\Delta\phi_2 \subset \Omega_R$, we obtain

$$\begin{aligned} &\|f_2\|_{-1,q} \leq C\left(\|f\|_{-1,q} + \|u\|_{q,\Omega_R} + \|p\|_{-1,q,\Omega_R}\right), \\ &\|g_2\|_q \leq C\left(\|g\|_q + \|u\|_{q,\Omega_R}\right). \end{aligned} \tag{3.5}$$

Now applying Lemma 2.9 to $\{\psi_2 u, \psi_2 p\}$ in (S_2) and then using (3.4)-(3.5), we obtain

$$\begin{aligned} &\|\nabla(\psi_2 u)\|_{q,\mathbb{R}^n} + \|\psi_2 p\|_{q,\mathbb{R}^n} \\ &\leq C\left(\|f\|_{-1,q} + \|g\|_q + \|u\|_{q,\Omega_R} + \|p\|_{-1,q,\Omega_R}\right). \end{aligned} \tag{3.6}$$

Then the desired result follows from (3.3) and (3.6). □

Now we introduce the weak Stokes operator S_q. Let $X_q \equiv \hat{H}_0^{1,q}(\Omega)^n \times L^q(\Omega)$ and $Y_q \equiv \hat{H}^{-1,q}(\Omega)^n \times L^q(\Omega)$. We define two bounded linear operators S_q and T_q by

$$S_q : \{u,p\} \in X_q \to \{-\Delta u + \nabla p,\ \text{div } u\} \in Y_q;$$

$$T_q : \{v,\chi\} \in X_q \to \{-\Delta v - \nabla\chi,\ -\text{div } v\} \in Y_q,$$

respectively. It is easy to see that

$$S_q^* \text{ (adjoint operator of } S_q) = T_{q'} \text{ for all } 1 < q < \infty. \tag{3.7}$$

Then Lemma 3.2 enables us to apply such a standard argument as Lions-Magenes [24, p. 153, Lemma 5.1], so we see that

(3.8) Ker S_q *(the kernel of* S_q*) is of finite dimension and*
$R(S_q)$ *(the range of* S_q*) closed in* Y_q.

More precisely we have

THEOREM 3.3 (A priori estimate). *Let $n \geq 2$, $1 < q < n$ for $n \geq 3$ and $1 < q \leq 2$ for $n = 2$, and let $\{f, g\} \in \hat{H}^{-1,q}(\Omega)^n \times L^q(\Omega)$. Suppose that $\{u, p\} \in \hat{H}_0^{1,q}(\Omega)^n \times L^q(\Omega)$ is a generalized solution of (S). Then it holds*

$$\|\nabla u\|_q + \|p\|_q \leq C \left(\|f\|_{-1,q} + \|g\|_q\right) \tag{3.9}$$

where $C = C(\Omega, q, n)$.

PROOF. We show first that Ker $S_q = \{0, 0\}$ for such q as in the theorem. Let $\{u, p\} \in$ Ker S_q. Then it is enough to show that $\{u, p\} \in \hat{H}_0^{1,2}(\Omega)^n \times L^2(\Omega)$, because we can insert $\Phi = u$ as a test function in the definition of the generalized solution and hence $\|\nabla u\|_2^2 = 0$, $\nabla p = 0$ follows. Then we get $u \equiv 0$, $p \equiv 0$. If $n \geq 3$, then we can take $r = 2 > n'$ in Theorem 3.1 and get $\{u, p\} \in \hat{H}_0^{1,2}(\Omega)^n \times L^2(\Omega)$. If $n = 2$, then we get by Theorem 3.1 and the interpolation property that $\nabla u \in L^r(\Omega)^{n^2}$, $p \in L^r(\Omega)$ for all finite $r \geq q$. Since $n = 2$, it follows from Lemma 2.2(ii) that $u \in \hat{H}_0^{1,2}(\Omega)^2$.

Now we prove (3.9) by contradiction. Suppose the contrary. Then there is a sequence $\{u_k, p_k\}_{k=1}^{\infty}$ in $\hat{H}_0^{1,q}(\Omega)^n \times L^q(\Omega)$ such that $\|\nabla u_k\|_q + \|p_k\|_q = 1$ and that $-\Delta u_k + \nabla p_k \to 0$ in $\hat{H}^{-1,q}(\Omega)^n$, div $u_k \to 0$ in $L^q(\Omega)$ as $k \to \infty$. A well known compactness argument yields that there is a subsequence, which we denote by $\{u_k, p_k\}_{k=1}^{\infty}$ for simplicity, such that $\{u_k\}_{k=1}^{\infty}$, $\{p_k\}_{k=1}^{\infty}$ and $\left\{\int_{\Omega_R} \psi_1(x) p_k(x)\, dx\right\}_{k=1}^{\infty}$ converge strongly in $L^q(\Omega_R)^n$, $\hat{H}^{-1,q}(\Omega_R)$ and $\mathbb{R}$, respectively. Then, applying Lemma 3.2 to $\{u_k - u_{k'},\ p_k - p_{k'}\}_{k,k'=1}^{\infty}$, we see that $\{u_k\}_{k=1}^{\infty}$ and $\{p_k\}_{k=1}^{\infty}$ are Cauchy sequences in $\hat{H}_0^{1,q}(\Omega)^n$ and in $L^q(\Omega)$, respectively. Thus, there is a pair $\{u, p\} \in \hat{H}_0^{1,q}(\Omega)^n \times L^q(\Omega)$ such that $u_k \to u$ in $\hat{H}_0^{1,q}(\Omega)^n$ and $p_k \to p$ in $L^q(\Omega)$. Moreover, we have $\{u, p\} \in$ Ker S_q and that $\|\nabla u\|_q + \|p\|_q = 1$, but this contradicts Ker $S_q = \{0, 0\}$. □

Using (3.7)-(3.8) and a closed range theorem, we have by Theorem 3.3 the following corollary.

COROLLARY 3.4. (i) *Let $1 < q \leq n'$ for $n \geq 3$ and $1 < q < 2$ for $n = 2$. Then we have* Ker $S_q = \{0, 0\}$ *and* $R(S_q) = (\text{Ker } T_{q'})^{\perp}$.

(ii) *Let $n' < q < n$ for $n \geq 3$ and $q = 2$ for $n = 2$. Then we have* Ker $S_q = \{0, 0\}$ *and* $R(S_q) = Y_q$.

(iii) *Let $n \leq q < \infty$ for $n \geq 3$ and $2 < q < \infty$ for $n = 2$. Then we have* Ker $S_q = R(T_{q'})^{\perp}$ *and* $R(S_q) = Y_q$.

Here $W^{\perp}$ denotes the annihilator of the subspace W.

REMARK. Theorem 3.3 was first proved by Kozono-Sohr [22] in case $n \geq 3$ and $n' < q < n$. Borchers-Miyakawa [7] extended the result to the case when $n \geq 3$ and $1 < q \leq n'$. Recently Galdi-Simader [18] gave a similar result for $n \geq 2$, but with a different method from ours.

4. - Proof of the main results

4.1. *Stokes paradox*; *Proof of Theorems* A *and* A′.

Let us first give the following auxiliary lemma due to Bogovskii [5, 6].

LEMMA 4.1 (Bogovskii). (i) *Let* $1 < q < \infty$. *Suppose that* $w \in \hat{H}^{-1,q}(\Omega)^n$ *satisfies* $(w, \Phi) = 0$ *for all* $\Phi \in C_0^\infty(\Omega)^n$ *with* div $\Phi = 0$. *Then there is a unique* $p \in L^q(\Omega)$ *such that* $w = \nabla p$, *i.e.*, $(w, \Psi) = -(p, \text{ div } \Psi)$ *for all* $\Psi \in \hat{H}_0^{1,q'}(\Omega)^n$.

(ii) *Let* $1 < r < n$ *and let* $u \in L^1_{\mathrm{loc}}(\overline{\Omega})$ *with* $\nabla u \in L^r(\Omega)^n$. *Then there is a constant* $C = C(u, n, r)$ *such that* $u + C \in L^q(\Omega)$ *with* $1/q = 1/r - 1/n$.

For the proof, see also Giga-Sohr [19, Corollary 2.2] and Borchers-Sohr [9, Lemma 4.1].

In the forthcoming argument, we use the linear extension operator $\Gamma : C^2(\partial\Omega)^n \to C_0^2(B_R)^n$ satisfying

$$\Gamma\phi = \phi \text{ on } \partial\Omega, \ \ \|\Gamma\phi\|_{H^{m,q}(B_R)} \leq C\|\phi\|_{H^{m-1/q,q}(\partial\Omega)}, \ \ (m = 1, 2)$$

for all $\phi \in C^2(\partial\Omega)^n$ with $C = C(\partial\Omega, R, n, m, q)$. Here $B_R = \{x \in \mathbb{R}^n; \ |x| < R\}$ is a ball containing $\partial\Omega$.

PROOF OF THEOREM A. If $n = n' = q = 2$, then the desired result follows from Corollary 3.4(ii) and Lemma 2.2(ii), so we may prove only the case $1 < q \leq n'$ for $n \geq 3$ and $1 < q < 2$ for $n = 2$. Since $\nabla u \in L^q(\Omega)^{n^2}$, it follows from Lemma 4.1(ii) that there is a constant vector $a = a(u, q) \in \mathbb{R}^n$ such that $u - a \in L^{nq/(n-q)}(\Omega)^n$. Set $w = \Gamma a \in C_0^2(B_R)$ and define $\hat{u} = u - a + w$. Then we see by Lemma 2.2(i) that $\hat{u} \in \hat{H}_0^{1,q}(\Omega)^n$, and by assumption we get

$$-\Delta\hat{u} + \nabla p = -\Delta w, \ \text{div } \hat{u} = \text{div } w \text{ in } \Omega \tag{4.1}$$

in the sense of distributions. Since $\{-\Delta w, \text{ div } w\} \in \hat{H}^{-1,\gamma}(\Omega)^n \times L^\gamma(\Omega)$ for all $\gamma > 1$, it follows from Theorem 3.1 that $\{\hat{u}, p\} \in \hat{H}_0^{1,r}(\Omega)^n \times L^r(\Omega)$ for all $r > n'$. Moreover, since $q' \geq n$ for $n \geq 3$ and $q' > 2$ for $n = 2$, we obtain from Lemma 2.2(ii) that $\{u, -p\} \in \text{Ker } T_{q'}$. Therefore it follows from (4.1) and Corollary 3.4(i) that

$$\begin{aligned} 0 &= (-\Delta w, u) + (\text{div } w, -p) \\ &= -\int_\Omega |\nabla u(x)|^2 \, dx + (\nabla u, \nabla\hat{u}) - (p, \text{ div } \hat{u}). \end{aligned}$$

Since $\{\hat{u}, p\} \in \hat{H}_0^{1,q'}(\Omega)^n \times L^{q'}(\Omega)$ and since $C_0^\infty(\Omega)^n \times C_0^\infty(\Omega)$ is dense in $\hat{H}_0^{1,q'}(\Omega)^n \times L^{q'}(\Omega)$, it follows from the assumption on $\{u, p\}$ that $(\nabla u, \nabla\hat{u}) - (p, \text{ div } \hat{u}) = 0$. So we get $u \equiv 0$ in Ω, then $\nabla p = 0$ in Ω. Since $p \in L^q(\Omega)$, we have also $p \equiv 0$ in Ω. □

PROOF OF THEOREM A$'$. Since $\nabla u \in L^q(\Omega)^{n^2}$, we have $-\Delta u \in \hat{H}^{-1,q}(\Omega)^n$ and, by assumption, $(-\Delta u, \Phi) = 0$ for all $\Phi \in C_0^\infty(\Omega)^n$ with div $\Phi = 0$. Then it follows from Lemma 4.1(i) that there is a scalar function $p \in L^q(\Omega)$ such that

$$(\nabla u, \nabla \Psi) - (p, \ \mathrm{div}\ \Psi) = 0 \text{ for all } \Psi \in C_0^\infty(\Omega)^n.$$

Now Theorem A yields that $u \equiv 0$ in Ω. □

4.2. *Characterization of the null space; Proof of Theorem* B.

We shall first consider the cases (i) and (ii), *i.e.*, $n \geq 2$, $1 < q < \infty$ except for $n = q = 2$. The proof will be done by three lemmas. Let us define the vector spaces V and $\hat{V}_q$ for $1 < q \leq n'$ $(n \geq 3)$ and $1 < q < 2$ $(n = 2)$ as follows:

$$V = \{\{A, a\} \in \mathbb{R}^{n^2} \times \mathbb{R}^n;\ \mathrm{Tr}\ A = 0\},$$

$$\hat{V}_q = \left\{ \{A, a\} \in V;\ \int_{\partial\Omega} \left[(Ax + a) \cdot \frac{\partial v}{\partial \nu} - \chi(Ax + a) \cdot \nu \right] \mathrm{d}S = 0 \right.$$
$$\left. \text{for all } \{v, \chi\} \in \mathbb{N}_{q'}^0 \right\}.$$

Then the existence of a generalized solution with (1.2) and (1.3-3$'$) in Theorem B(i)-(ii) is guaranteed by the following lemma.

LEMMA 4.2 (Existence). (i) *Let* $1 < q \leq n'$ *for* $n \geq 3$ *and* $1 < q < 2$ *for* $n = 2$. *Then there is a linear operator* $K_q : \{A, a\} \to \{u, p\}$ *from* $\hat{V}_q$ *to* $\bigcap_{r \geq q} \mathbb{N}_r$ *such that* $\nabla u - A \in L^r(\Omega)^{n^2}$ *for all* $r \geq q$ *and such that* (1.3) *holds.*

(ii) *There is a linear operator* $L : \{A, a\} \to \{u, p\}$ *from* V *to* $\bigcap_{r > n'} \mathbb{N}_r$ *such that:* $\nabla u - A \in L^r(\Omega)^{n^2}$ *for all* $r > n'$, (1.3) *holds if* $n \geq 3$, *and*

$$\int_\Omega |\nabla[u(x) - Ax - E(x)a]|^s \,\mathrm{d}x < \infty \tag{4.2}$$

holds for all $s \geq 2$ *if* $n = 2$.

PROOF. (i) Since $q' \geq n$ for $n \geq 3$ and $q' > 2$ for $n = 2$, we have by Lemma 2.2(ii) that $\{v, -\chi\} \in \mathrm{Ker}\ T_{q'}$ for $\{v, \chi\} \in \mathbb{N}_{q'}^0$. Taking $w = \Gamma(A \cdot + a)$ for $\{A, a\} \in \hat{V}_q$, where Γ is the extension operator defined above and where $A \cdot + a$ is a function on $\partial\Omega$ defined by $x \in \partial\Omega \to Ax + a \in \mathbb{R}^n$, a direct calculation shows that

$$(-\Delta w, v) + (\mathrm{div}\ w, -\chi)$$
$$= \int_{\partial\Omega} \left\{ (Ax + a) \cdot \frac{\partial v}{\partial \nu} - \chi(Ax + a) \cdot \nu \right\} \mathrm{d}S = 0$$

for all $\{v,\chi\} \in \mathbb{N}^0_{q'}$. Hence it follows from Corollary 3.4(i) that there is a unique generalized solution $\{\hat{u},p\} \in \hat{H}^{1,q}_0(\Omega)^n \times L^q(\Omega)$ of (4.1). Moreover, by Theorem 3.1 and the interpolation inequality, $\nabla\hat{u} \in L^r(\Omega)^{n^2}$ and $p \in L^r(\Omega)$ for all $r \geq q$. If $n \geq 3$, we have again by Theorem 3.1 that $\hat{u} \in \hat{H}^{1,\gamma}_0(\Omega)^n$ for all $\gamma > n'$. For $\gamma \in (n/2,n)$, $\hat{H}^{1,\gamma}_0(\Omega)$ is continuously embedded into $L^{n\gamma/(n-\gamma)}(\Omega)$ (see Lemma 2.2(i)). For such γ, we have $n\gamma/(n-\gamma) > n$ and hence, in particular, $\hat{u} \in H^{1,s}(\Omega)^n$ for $s > n$. By the Sobolev embedding theorem, $\hat{u} \in C^0(\overline{\Omega})^n$ and $\lim_{x\to\infty} |\hat{u}(x)| = 0$. If $n = 2$, we have by Lemma 2.2(i) that $\hat{u} \in H^{1,2q/(2-q)}_0(\Omega)^2$. Since $2q/(2-q) > 2$, $\hat{u}$ has the same properties as above. Now, setting $u = \hat{u}+Ax+a-w$ and then defining $K_q\{A,a\} = \{u,p\}$, we obtain the operator K_q.

(ii) We first consider the case $n \geq 3$. Let $\{A,a\} \in V$. We set $w = \Gamma(A\cdot{+a})$. Then it follows from Corollary 3.4(ii) and Theorem 3.1 that there is a unique generalized solution $\{\hat{u},p\}$ of (4.1) such that $\{\hat{u},p\} \in \hat{H}^{1,r}_0(\Omega)^n \times L^r(\Omega)$ for all $r > n'$. Then in the same way as above, we can show that $\hat{u} \in C^0(\overline{\Omega})^n$, $\lim_{x\to\infty} |\hat{u}(x)| = 0$ and that the map $L : \{A,a\} \to \{u,p\}$ with $u = \hat{u} + Ax + a - w$ satisfies the required conditions.

We next construct L for $n = 2$. Without loss of generality, we may assume that $0 \in \mathbb{R}^n/\overline{\Omega}$. Set $w = \Gamma(A\cdot{+Ea})$, where $A\cdot{+Ea}$ is the function on $\partial\Omega$ defined by $x \in \partial\Omega \to Ax + E(x)a \in \mathbb{R}^2$. Then it follows from Corollary 3.4(ii) and Theorem 3.1 that there is a unique generalized solution $\{\hat{u},p\}$ of (4.1) belonging to $\hat{H}^{1,r}_0(\Omega)^2 \times L^r(\Omega)$ for all $r \geq 2$. Setting $u = \hat{u} + Ax + Ea - w$, we see that the map $L : \{A,a\} \to \{u,p\}$ enjoys the desired properties. □

We next show the uniqueness of generalized solutions.

LEMMA 4.3 (Uniqueness). (i) *Let* $1 < q \leq n'$ *for* $n \geq 3$ *and* $1 < q < 2$ *for* $n = 2$. *Then for every* $\{A,a\} \in \hat{V}_q$, *there is a unique* $\{u,p\} \in \mathbb{N}_q$ *with properties* (1.2) *and* (1.3).

(ii) *Let* $n' < q < \infty$, $n \geq 2$. *Then for every* $\{A,a\} \in V$, *there exists a unique* $\{u,p\} \in \mathbb{N}_q$ *with properties* (1.2)-(1.3) *if* $n \geq 3$, *and* (1.2)-(1.3′) *if* $n = 2$.

PROOF. The proof of existence is contained in Lemma 4.2 so we may only prove uniqueness.

(i) Suppose that $\{\tilde{u},\tilde{p}\} \in \mathbb{N}_q$ satisfies (1.2) and (1.3) with u replaced by $\tilde{u}$. Set $\overline{u} = u\tilde{u}$ and $\overline{p} = p - \tilde{p}$ (note that $p, \tilde{p}$ and $\overline{p}$ do *not* denote integral exponents but functions of the pressure). Then we have $\nabla\overline{u} \in L^q(\Omega)^{n^2}$, $\overline{u} \in C^0(\overline{\Omega})^n$ and from Theorem 3.1 that there is a unique generalized solution $\lim_{x\to\infty} |\overline{u}(x)| = 0$. On the other hand, by Lemma 4.1(ii), there is a constant vector $C \in \mathbb{R}^n$ such that $\overline{u} + C \in L^{nq/(n-q)}(\Omega)^n$. Since $\lim_{x\to\infty} |\overline{u}(x)| = 0$, we have $C = 0$ and hence $\overline{u} \in L^{nq/(n-q)}(\Omega)^n$. Then from Lemma 2.2(i) we obtain $\overline{u} \in \hat{H}^{1,q}_0(\Omega)^n$ and so $\{\overline{u},\overline{p}\} \in \operatorname{Ker} S_q$. Now applying Corollary 3.4(i), we get $\overline{u} \equiv 0$, $\overline{p} \equiv 0$ and the assertion on uniqueness follows.

(ii) Let us first assume $n \geq 3$. Let $\{\tilde{u},\tilde{p}\} \in \mathbb{N}_q$ and $\{\overline{u},\overline{p}\}$ as above. If $n' < q < n$, we can argue in the same way as above and get $\{\overline{u},\overline{p}\} \in \operatorname{Ker} S_q$.

Then it follows from Corollary 3.4(ii) that $\overline{u} \equiv 0$, $\overline{p} \equiv 0$. If $n \le q < \infty$, we see by Lemma 2.2(ii) that $\overline{u} \in \hat{H}_0^{1,q}(\Omega)^n$. From Theorem 3.1, we obtain $\nabla \overline{u} \in L^r(\Omega)^{n^2}$, $\overline{p} \in L^r(\Omega)$ for all $r > n'$. In the same way as in (i), we get $\{\overline{u}, \overline{p}\} \in \mathrm{Ker}\ S_\gamma$ for some γ with $n' < \gamma < n$ and Corollary 3.4(ii) yields $\overline{u} \equiv 0$, $\overline{p} \equiv 0$. In case $n = 2$, we see by Lemma 2.2(ii) and (1.3′) that $\{\overline{u}, \overline{p}\} \in \mathrm{Ker}\ S_2 = \{0, 0\}$. □

Now it remains to give the dimensions of $\mathbb{N}_q$ and $\mathbb{N}_q^0$. To this end, we shall make use of the operators K_q and L constructed in Lemma 4.2.

LEMMA 4.4. (i) *For each* q *with* $1 < q \le n'$ $(n \ge 3)$ *and with* $1 < q < 2$ $(n = 2)$, K_q *defines a bijection from* $\hat{V}_q$ *onto* $\mathbb{N}_q$.

(ii) *For each* q *with* $n' < q < \infty$ $(n \ge 2)$, L *defines a bijection from* V *onto* $\mathbb{N}_q$.

PROOF. (i) *Injectivity*. Let $K_q\{A, a\} = \{0, 0\}$ for $\{A, a\} \in \hat{V}_q$. Then by (1.3), $|Ax + a| \to 0$ as $x \to \infty$; hence we get $A = 0$ and $a = 0$.

Surjectivity. Suppose that $\{u, p\} \in \mathbb{N}_q$. Then $\nabla u - A \in L^q(\Omega)^{n^2}$ for some $A \in \mathbb{R}^{n^2}$ with $\mathrm{Tr}\ A = 0$. By Lemma 4.1(ii), there is a constant vector $a \in \mathbb{R}^n$ such that $u - Ax - a \in L^{nq/(n-q)}(\Omega)^n$. Introducing $w = \Gamma(A \cdot +a)$ as in the proof of Lemma 4.2 and then defining $\hat{u} = u - Ax - a + w$, we see by Lemma 2.2(i) that $\{\hat{u}, p\} \in \hat{H}_0^{1,q}(\Omega)^n \times L^q(\Omega)$ and that $\{\hat{u}, p\}$ satisfies (4.1) in the sense of distributions. Moreover it follows from Theorem 3.1 that $\hat{u} \in \hat{H}_0^{1,r}(\Omega)^n$ for all $r > n'$. Now using the same argument as in the proof of Lemma 4.2, we get $\hat{u} \in C^0(\overline{\Omega})^n$, $\lim_{x \to \infty} |\hat{u}(x)| = 0$ and hence u satisfies (1.3). Then the uniqueness stated in Lemma 4.3(i) yields that $\{u, p\} = K_q\{A, a\}$.

(ii) *Injectivity*. Let $L\{A, a\} = \{0, 0\}$ for $\{A, a\} \in V$. If $n \ge 3$, we get in the same way as above $A = 0$, $a = 0$. If $n = 2$, we obtain by (4.2) that $A - \nabla E a \in L^s(\Omega)^{2^2}$ for all $s \ge 2$. The explicit expression of E shows that ∇E is *not* in $L^2(\Omega)^{2^3}$, but in $L^r(\Omega)^{2^3}$ for all $r > 2$. Hence $A = 0$, $a = 0$.

Surjectivity. Let us first assume that $n \ge 3$. The proof for q with $n' < q < n$ is parallel to that of case (i), so we may only show it for $n \le q < \infty$. Suppose that $\{u, p\} \in \mathbb{N}_q$ $(q \ge n)$. Then $\nabla u - A \in L^q(\Omega)^{n^2}$ for some $A \in \mathbb{R}^{n^2}$ with $\mathrm{Tr}\ A = 0$, and taking $w = \Gamma(A\cdot)$ and $\hat{u} = u - Ax + w$, we see by Lemma 2.2(ii) that $\{\hat{u}, p\} \in \hat{H}_0^{1,q}(\Omega)^n \times L^q(\Omega)$ and that $\{\hat{u}, p\}$ is a generalized solution of (4.1). Moreover, by Theorem 3.1, $\nabla \hat{u} \in L^r(\Omega)^{n^2}$ for all $r > n'$, and in particular, we have $\nabla \hat{u} \in L^\gamma(\Omega)^{n^2}$ for γ with $n/2 < \gamma < n$. By Lemma 4.1(ii), there is a constant vector $a \in \mathbb{R}^n$ such that $\hat{u} - a \in L^\sigma(\Omega)^n$ with $1/\sigma = 1/\gamma - 1/n$. Since $\sigma > n$, we have $\hat{u} - a \in H^{1,\sigma}(\Omega)^n$ and hence by the Sobolev embedding theorem $\hat{u} - a \in C^0(\overline{\Omega})^n$ and $\lim_{x \to \infty} |\hat{u}(x) - a| = 0$, from which (1.3) follows. Now the uniqueness result of Lemma 4.3(ii) yields that $\{u, p\} = L\{A, a\}$.

We next consider the case $n = 2$ and $q > 2$. Let $\{u, p\} \in \mathbb{N}_q$. Then $\nabla u - A \in L^q(\Omega)^{2^2}$ for some $A \in \mathbb{R}^{2^2}$ with $\mathrm{Tr}\ A = 0$. Since $u - Ax - \Gamma(A\cdot) \in \hat{H}_0^{1,q}(\Omega)^2$, we obtain by Lemma 2.2(ii) that $u(x) - Ax = O(|x|^{1-2/q})$ as $|x| \to \infty$.

Applying the regularity theorem of Finn-Smith [13, Theorem 5.11-12], we see $u - Ax \in C^2(\Omega) \cap C^1(\overline{\Omega})$ and $p \in C^1(\Omega) \cap C^0(\overline{\Omega})$. Then it follows from the representation formula of Chang-Finn [11, Theorem 1] that

$$u(x) - Ax - E(x)a = u^\infty - \int_{\partial\Omega} A\xi \cdot TE(x-\xi)\nu_\xi \mathrm{d}S_\xi$$

$$- \int_{\partial\Omega} (E(x-\xi) - E(x))\, T\,(u(\xi)A\xi)\,\nu_\xi \mathrm{d}S_\xi,$$

where $a = -\int_{\partial\Omega} T\,(u(\xi) - A\xi)\,\nu_\xi \mathrm{d}S_\xi$, $u^\infty \in \mathbb{R}^2$ and T denotes the stress tensor. Using the explicit expression of E, we see that

$$\sup_{\xi\in\partial\Omega} |\nabla_x TE(x-\xi)| = \mathrm{O}(|x|^{-2}), \quad \sup_{\xi\in\partial\Omega} |\nabla_x E(x-\xi) - \nabla_x E(x)| = \mathrm{O}(|x|^{-2})$$

as $x \to \infty$. Therefore $\nabla(u - Ax - Ea) \in L^2(\Omega)^{2^2}$ and it follows from the uniqueness proved in Lemma 4.2(ii) that $\{u,p\} = L\{A,a\}$. □

Properties of dim $\mathbb{N}_q$ *and* dim $\mathbb{N}_q^0$.

Let us first consider the case $n' < q < \infty$ $(n \geq 2)$. Then by Lemma 4.4(ii) and the definition of V, we obtain

$$\text{(4.3)} \qquad \dim\ \mathbb{N}_q = \dim\ V = n^2 + n - 1,\ \dim\ \mathbb{N}_q^0 = n,\ n' < q < \infty\ (n \geq 2).$$

Hence Theorem B(ii) follows from (4.3) and Lemmas 4.2-3(ii).

We next consider the case $1 < q \leq n'$ for $n \geq 3$ and $1 < q < 2$ for $n = 2$. By Theorem A and the definition of $\mathbb{N}_q^0$, we have

$$\text{(4.4)} \qquad \dim\ \mathbb{N}_q^0 = 0,\ 1 < q \leq n'\ (n \geq 3),\ 1 < q < 2\ (n = 2).$$

Moreover, it follows from Lemma 4.4(i) that $\mathbb{N}_q^0$ is isometric to the subspace W_q of $\hat{V}_q$:

$$W_q \equiv \left\{ a \in \mathbb{R}^n;\ \int_{\partial\Omega} \left\{ a \cdot \frac{\partial v}{\partial \nu} - \chi a \cdot \nu \right\} \mathrm{d}S = 0 \text{ for all } \{v,\chi\} \in \mathbb{N}_{q'}^0 \right\}.$$

Hence $W_q = \{0\}$. On the other hand, by (4.3), we see dim $\mathbb{N}_{q'}^0 = n$. Therefore it follows that dim $\hat{V}_q = n^2 + n - 1 - \dim\ \mathbb{N}_{q'}^0 = n^2 - 1$. Now, Lemma 4.4(i) yields

$$\text{(4.5)} \qquad \dim\ \mathbb{N}_q = \dim\ \hat{V}_q = n^2 - 1,\ 1 < q \leq n'\ (n \geq 3),\ 1 < q < 2\ (n = 2).$$

Hence Theorem B(i) follows from (4.4-5) and Lemmas 4.2-3(i).

(iii) *Case* $n = q = 2$. In the same way as in Lemmas 4.2-4.3, we can construct a bijective operator $L' : A \to \{u,p\}$ from $V' \equiv \{A \in \mathbb{R}^{2^2};\ \mathrm{Tr}\ A = 0\}$ onto $\mathbb{N}_2$ such that u satisfies (4.2) with $a = 0$. Hence we get dim $\mathbb{N}_2 = 3$. By Lemma 2.2(ii) and Corollary 3.4(ii), we have $\mathbb{N}_2^0 = \mathrm{Ker}\ S_2 = \{0,0\}$. And therefore existence and uniqueness derive from the same argument as before, so we may omit the details. □

4.3. *Inhomogeneous equations*; *Proof of Theorem* C.

Recall the function $A \cdot + a : x \in \partial\Omega \to Ax + a \in \mathbb{R}^n$ and set $w = \Gamma(A \cdot + a)$. Taking $\hat{u} = u - Ax - a + w$, we get from (S)

$$(4.6)\qquad \begin{aligned} &-\Delta\hat{u} + \nabla p = f - \Delta w,\ \ \mathrm{div}\ \hat{u} = g\mathrm{Tr}\ A + \mathrm{div}\ w\ \text{ in } \Omega,\\ &\hat{u} = 0 \text{ on } \partial\Omega. \end{aligned}$$

In order to solve (S), we shall make use of (4.6).

Proof of Theorem C. (i) *Case* $1 < q \le n'$ *for* $n \ge 3$ *and* $1 < q < 2$ *for* $n = 2$. As we have seen in the proof of Theorem B, (1.5) is equivalent to the identity

$$(f - \Delta w, v) + (g - \mathrm{Tr}\ A + \mathrm{div}\ w, -\chi) = 0 \text{ for all } \{v,\chi\} \in \mathbb{N}_{q'}^0.$$

This implies that $\{f - \Delta w,\ g - \mathrm{Tr}\ A + \mathrm{div}\ w\} \in (\mathrm{Ker}\ T_{q'})^\perp$. Hence by Corollary 3.4(i), there is a unique generalized solution $\{\hat{u},p\} \in \hat{H}_0^{1,q}(\Omega)^n \times L^q(\Omega)$ of (4.6). By Lemma 2.2(i) we have also $\hat{u} \in L^{nq/(n-q)}(\Omega)^n$. Moreover, from Theorem 3.3 and the continuity of the extension operator Γ, we obtain

$$\begin{aligned} \|\nabla\hat{u}\|_q + \|p\|_q &\le C\left(\|f - \Delta w\|_{-1,q} + \|g - \mathrm{Tr}\ A + \mathrm{div}\ w\|_q\right)\\ &\le C\left(\|f\|_{-1,q} + \|g - \mathrm{Tr}\ A\|_q + |A| + |a|\right), \end{aligned}$$

where $C = C(\Omega,n,q)$. Taking $u = \hat{u} + Ax + a - w$, we see that $\{u,p\}$ is the desired generalized solution of (S). The uniqueness follows from the fact that $\mathbb{N}_q^0 = \{0,0\}$. Conversely, suppose that $\{u,p\} \in H_{\mathrm{loc}}^{1,q}(\overline{\Omega})^n \times L^q(\Omega)$ is a generalized solution of (S) satisfying (1.2) and (1.4) for some $A \in \mathbb{R}^{n^2}$ with $\mathrm{Tr}\ A - g \in L^q(\Omega)$ and $a \in \mathbb{R}^n$. Taking $w = \Gamma(A \cdot + a)$, we see by Lemma 2.2(i) that $\hat{u} = u - Ax - a + w \in \hat{H}_0^{1,q}(\Omega)^n$ and that $\{\hat{u},p\}$ is a generalized solution of (4.6). Hence it follows from Corollary 3.4(i) that

$$\{f - \Delta w,\ g - \mathrm{Tr}\ A + \mathrm{div}\ w\} \in R(S_q) = (\mathrm{Ker}\ T_{q'})^\perp,$$

from which we get (1.5).

(ii) *Case* $n' < q < n$ *for* $n \ge 3$. Take $w = \Gamma(A \cdot + a)$ and consider (4.6) for $\{\hat{u},p\}$. Then we have by Corollary 3.4(ii) that (4.6) is uniquely solvable in $\hat{H}_0^{1,q}(\Omega)^n \times L^q(\Omega)$ for all f, g, A and a as given in the assumptions. Then the proof of existence and uniqueness is quite the same as in the case (i) above. Suppose in addition that $f \in \hat{H}^{-1,r}(\Omega)^n$ and $g - \mathrm{Tr}\ A \in L^r(\Omega)$ for some $r > n$. Since $1 < r' < q' < n$, it follows from (2.2) and an interpolation argument (see, e.g., Triebel [32, 1.11.2]) that $f \in \hat{H}^{-1,\gamma}(\Omega)^n$ and $g - \mathrm{Tr}\ A \in L^\gamma(\Omega)$ for all $q \le \gamma \le r$. Hence we have by Theorem 3.1 that $\{\hat{u}, p\} \in \hat{H}_0^{1,\gamma}(\Omega)^n \times L^\gamma(\Omega)$ for all $q \le \gamma \le r$ and that, in particular, $\hat{u} \in H^{1,s}(\Omega)^n$ for some $s > n$. By the Sobolev embedding theorem, we obtain $\hat{u} \in C^0(\overline{\Omega})^n$ and $\lim_{x \to \infty} |\hat{u}(x)| = 0$. Now it is easy to see that $u = \hat{u} + Ax + a - w$ satisfies (1.3).

(iii) *Case* $n \le q < \infty$ *for* $n \ge 3$ *and* $2 < q < \infty$ *for* $n = 2$. Taking $w = \Gamma(A\cdot)$ in (4.6), we have by Corollary 3.4(iii) that there is at least one generalized solution $\{\hat{u}, p\} \in \hat{H}_0^{1,q}(\Omega)^n \times L^q(\Omega)$ of (4.6). On the other hand, we have by Lemma 2.2(ii) that $\mathbb{N}_q^0 = \mathrm{Ker}\ S_q$ and that $R(S_q)$ is isometric to the quotient space $X_q/\mathbb{N}_q^0$. Therefore it follows that

$$\begin{aligned}
&\inf\left\{ \|\nabla\hat{u} - \nabla v\|_q + \|p - \chi\|_q;\ \{v, \chi\} \in \mathbb{N}_q^0 \right\} \\
&\le C\left(\|f - \Delta w\|_{-1,q} + \|g - \mathrm{Tr}\ A + \mathrm{div}\ w\|_q \right) \\
&\le C\left(\|f\|_{-1,q} + \|g - \mathrm{Tr}\ A\|_q + |A| \right),
\end{aligned}$$

where $C = C(\Omega, n, q)$. Taking $u = \hat{u} + Ax - w$, we see that $\{u, p\} \in H_{\mathrm{loc}}^{1,q}(\overline{\Omega})^n \times L^q(\Omega)$ has the desired property.

Suppose that $\{\tilde{u}, \tilde{p}\} \in H_{\mathrm{loc}}^{1,q}(\overline{\Omega})^n \times L^q(\Omega)$ is another generalized solution of (S) with (1.2). Set $u' = u - \tilde{u}$ and $p' = p - \tilde{p}$. Then we get $\{u', p'\} \in \mathbb{N}_q^0$. Hence

$$\begin{aligned}
&\inf\left\{ \|\nabla\tilde{u} - A - \nabla v\|_q + \|\tilde{p} - \chi\|_q;\ \{v, \chi\} \in \mathbb{N}_q^0 \right\} \\
&= \inf\left\{ \|\nabla u - A - \nabla v\|_q + \|p - \chi\|_q;\ \{v, \chi\} \in \mathbb{N}_q^0 \right\}
\end{aligned}$$

so uniqueness and (1.7) follow.

(iv) *Case* $n = q = 2$. By Theorem B(iii), we see that $\mathbb{N}_2^0 = \{0, 0\}$; so the proof is quite similar to that of the case (ii) above. □

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