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The Stefan Problem with Kinetic Condition at the Free Boundary

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1. - Formulation of the problem

Consider the free boundary problem: find a function $u(x,y,t)$ and a curve

$$\Gamma : y = g(x,t) \qquad (g(x,t) > 0) \tag{1.1}$$

such that u satisfies the differential equation

$$\Delta u \equiv u_{xx} + u_{yy} = 0 \text{ in } \Omega_t \equiv \big\{(x,y);\ -\infty < x < \infty,\ 0 < y < g(x,t)\big\} \tag{1.2}$$

and the boundary conditions

$$u(x,0,t) = b(x,t) \qquad (-\infty < x < \infty), \tag{1.3}$$

$$\frac{\partial u}{\partial n} + u = 0 \text{ on } \Gamma, \tag{1.4}$$

$$u = V_n \text{ on } \Gamma, \tag{1.5}$$

and g satisfies the initial condition

$$g(x,0) = g_0(x),\ \ g_0(x) > 0 \qquad (-\infty < x < \infty). \tag{1.6}$$

Here n is the outward normal to Γ,

$$n = \frac{(-g_x, 1)}{\sqrt{1+g_x^2}},$$

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and V_n is the velocity of the free boundary,

$$V_n = \frac{g_t}{\sqrt{1+g_x^2}}.$$

In view of (1.5), equation (1.4) can also be written in the form

$$-\frac{\partial u}{\partial n} = V_n \text{ on } \Gamma. \tag{1.4'}$$

The relations (1.4′) and $u = 0$ on Γ constitute the standard free boundary conditions for the Stefan problem. The modified condition $u = V_n$ on Γ represents kinetic heating. For one space dimension with independent variable y and with Δu replaced by $u_t = u_{yy}$, this problem was studied by Dewynne, Howison, Ockendon and Xie [1] and by Xie [5].

Our interest in the present problem arises from the modeling of titanium silicide film growth. The problem actually involves three free boundaries; see [3; Chap. 8]. Here however we restrict ourselves to a subproblem whereby the lower part of Ω_t is a fixed curve, which for simplicity is taken to be the x-axis. The function u represents the concentration of titanium silicide. Relation (1.4′) is the conservation of mass, whereas (1.4) models the rate of conversion of titanium to titanium silicide.

Consider first the special one-dimension problem where $g_0(x)$ and $b(x,t)$ are independent of x. Given

$$g_0 = \overline{s}_0, \quad b = b(t), \quad \big(\overline{s} > 0, \ b(t) > 0\big), \tag{1.7}$$

one easily finds a unique solution $u_0(y,t)$, with free boundary $y = s_0(t)$:

$$u_0(y,t) = -\frac{b(t)y}{1+s_0(t)} + b(t), \quad 0 < y < s_0(t), \tag{1.8}$$

$$s_0'(t) = \frac{b(t)}{1+s_0(t)}, \quad s_0(0) = \overline{s}_0. \tag{1.9}$$

From the last equation we get

$$\frac{1}{2}s_0^2(t) + s_0(t) = \frac{1}{2}\,\overline{s}_0^2 + \overline{s}_0 + \int_0^t b(\tau)\mathrm{d}\tau. \tag{1.10}$$

Note that, if $b(t) \sim C$ as $t \to \infty$ $(C > 0)$, then $s_0(t) \sim \sqrt{2Ct}$ as $t \to \infty$.

In this paper we shall prove the existence of a local classical solution of (1.1)–(1.6). We shall also prove that a global solution exists if the data g, b are "close" to the data (1.7):

$$g_0(x) = \overline{s}_0 + \varepsilon g_1(x), \tag{1.11}$$

$$b(x,t) = b(t) + \varepsilon b_1(x,t) \tag{1.12}$$

where ε is a sufficiently small positive constant. The global solution will have the form

$$u(x,t) = -\frac{b(t)y}{1+s_0(t)} + b(t) + \varepsilon u_1(x,y,t), \tag{1.13}$$

$$g(x,t) = s_0(t) + \varepsilon g_1(x,t) \tag{1.14}$$

with suitable functions u_1, g_1 (which depend on ε).

In the modeling of titanium silicide film growth the free boundary is actually nearly flat, so that the assumptions (1.11), (1.12) possibly include practical cases.

In Section 2 we formulate the problem for u_1, g_1. In Sections 3–6 we derive a priori estimates on g_1 and its $C_x^{2,\alpha}$ norm. In Section 7 we establish local existence (for general data). Then, by combining local existence with the a priori estimates on g_1, global existence for the data (1.11), (1.12) immediately follows. Finally, uniqueness is proved in Section 8.

2. - The reduced problem

From (1.14) we get

$$n = \frac{(-\varepsilon g_{1x}, 1)}{\sqrt{1+\varepsilon^2 g_{1x}^2}}.$$

Condition (1.4) written in terms of u_1 is

$$\frac{\partial u_1}{\partial n} + u_1 - \frac{b(t)}{1+s_0(t)} g_1(x,t) + \frac{\varepsilon b(t)}{1+s_0(t)} F(\varepsilon g_{1x}) g_{1x}^2 = 0 \tag{2.1}$$

where

$$F(\lambda) = \frac{1}{\sqrt{1+\lambda^2}\left(1+\sqrt{1+\lambda^2}\right)}.$$

Condition (1.5) can be reduced to

$$V_n = \frac{s_0'(t) + \varepsilon g_{1t}}{\sqrt{1+\varepsilon^2 g_{1x}^2}} = \frac{-b(t)\left(s_0(t)+\varepsilon g_1\right)}{1+s_0(t)} + b(t) + \varepsilon u_1$$

or, upon recalling (1.9),

$$\begin{aligned} g_{1t} + \frac{b(t)}{1+s_0(t)} g_1 \sqrt{1+\varepsilon^2 g_{1x}^2} - \frac{\varepsilon b(t)}{1+s_0(t)} G(\varepsilon g_{1x}) g_{1x}^2 \\ - \sqrt{1+\varepsilon^2 g_{1x}^2}\, u_1\left(x, s_0(t)+\varepsilon g_1, t\right) = 0 \end{aligned} \tag{2.2}$$

where

$$G(\lambda) = \frac{1}{1+\sqrt{1+\lambda^2}}.$$

We also have

$$\Delta u_1 = 0 \text{ if } x \in \mathbb{R},\ 0 < y < s_0(t) + \varepsilon g_1(x), \tag{2.3}$$

and

$$u_1(x,0,t) = b_1(x,t), \tag{2.4}$$

$$g_1(x_0) = g_1(x). \tag{2.5}$$

We shall henceforth assume:

$$b(t) \text{ is continuous, } 0 < \bar{b} \le b(t) \le \bar{\bar{b}} < \infty, \tag{2.6}$$

$$b_1(x,t) \text{ is continuous,} \tag{2.7}$$

$$\left|b_1(x,t)\right| \le B(t),\ B(t) \le \frac{1}{1+t},\ \int_0^\infty B(t)\mathrm{d}t < \infty. \tag{2.8}$$

We also assume that $g_1(x)$ satisfies:

$$\|g_1\|_{L^\infty},\ \|g_{1x}\|_{L^\infty},\ \|g_{1xx}\|_{L^\infty},\ [g_{1xx}]_{C^\alpha_x} \le 1 \tag{2.9}$$

where the norms are taken in $\mathbb{R}$.

We seek a solution u, g of the form (1.13), (1.14) such that $g_1(x,t)$ satisfies:

$$\begin{gathered}\left\|g_1(\cdot,t)\right\|_{L^\infty} + \left\|g_{1x}(\cdot,t)\right\|_{L^\infty} + \left\|g_{1xx}(\cdot,t)\right\|_{L^\infty} \le \frac{K}{\sqrt{1+t}},\\ \left[g_{1xx}(\cdot,t)\right]_{C^\alpha_x} \le K \qquad (K > 1).\end{gathered} \tag{2.10}$$

In Sections 3–6 we assume that a classical solution exists for all $0 \le t \le T$, for some $T > 0$ (all the derivatives g_{1t}, g_{1x}, g_{1xx} are continuous and g_{1xx} is Hölder continuous in x), and that (2.10) holds for some K and all $0 \le t \le T$. We shall then prove that for all $0 \le t \le T$

$$\left\|g_1(\cdot,t)\right\|_{L^\infty} \le \frac{C}{\sqrt{1+t}} \qquad \text{(in Section 3)}, \tag{2.11}$$

$$\left\|u_1(\cdot,\cdot,t)\right\|_{C^{2,\alpha}_{x,y}(D_t)} \le \frac{C}{1+t} + \frac{C}{\sqrt{1+t}}\left\|g_1(\cdot,t)\right\|_{C^{1,\alpha}_x} \quad \text{(in Section 4)}, \tag{2.12}$$

where D_t is defined in (4.1).

$$\left\| g_{1xx}(\cdot, t) \right\|_{L^\infty} \leq \frac{C}{\sqrt{1+t}} \qquad \text{(in Section 5)}, \tag{2.13}$$

and

$$\left[g_{1xx}(\cdot, t) \right]_{C_x^\alpha} \leq C \qquad \text{(in Section 6)}, \tag{2.14}$$

where all norms are taken in $\mathbb{R}$. The crucial facts here are that ε is assumed to be sufficiently small depending on K but not on T, and that C is a constant independent of K, T, ε.

In Section 7 we prove the existence of a classical solution of (1.1)–(1.6) for general data $b(x,t)$, $g_0(x)$, for $0 \leq t \leq \bar{t}$ where $\bar{t}$ is sufficiently small. By combining this result with the estimates derived in Sections 3–6, we easily construct a global classical solution of (1.1)–(1.6) for the data (1.11), (1.12) provided ε is sufficiently small. In Section 8 we prove uniqueness of the solution.

3. - Proof of (2.11)

In this section we prove:

LEMMA 3.1. *If u, g is a solution satisfying* (2.10), *for* $0 \leq t \leq T$, *then* (2.11) *holds for* $0 \leq t \leq T$ *provided* $0 < \varepsilon < 1/K$; *C is a constant independent of K, T, ε.*

We shall need the following fact:

$$\text{if } h(x) \geq 0 \ (x \in \mathbb{R}) \text{ and } \|h''\|_{L^\infty} \leq A \qquad (0 < A < \infty), \text{ then}$$

$$h(x) \geq \frac{\left(h'(x)\right)^2}{2A} \text{ for all } x \in \mathbb{R}. \tag{3.1}$$

To prove it we use Taylor's expansion and the assumption $h(y) \geq 0$ to deduce that

$$h(x) \geq (x-y)h'(x) - \frac{1}{2}(x-y)^2 \|h''\|_{L^\infty} \qquad \forall y \in \mathbb{R}.$$

Choosing y such that $(x-y)h'(x) \geq 0$ and

$$(x-y)h'(x) = (x-y)^2 A,$$

(3.1) follows.

Denote by $w_1(y,t)$, $s_1(t)$ the solution corresponding to

$$\overline{s}_1 = \overline{s}_0 - 2\varepsilon,$$
$$b_1(t) = b(t) - \varepsilon B(t),$$

i.e. (see Section 1),

$$w_1(y,t) = -\frac{b_1(t)y}{1+s_1(t)} + b_1(t), \tag{3.2}$$
$$\frac{1}{2}s_1^2(t) + s_1(t) = \frac{1}{2}\overline{s}_1^2 + \overline{s}_1 + \int_0^t b_1(\tau)d\tau.$$

LEMMA 3.2. *If* $\varepsilon < 1/K$ *then*

$$g(x,t) > s_1(t), \tag{3.3}$$
$$u(x,y,t) \geq w_1(y,t)$$

for $x \in \mathbb{R}$, $0 \leq y \leq g(x,t)$, $0 \leq t \leq T$.

PROOF. We first show that

(3.4)
if $g(x,t) > s_1(t)$ for $x \in \mathbb{R}$, $0 \leq t \leq T$,
then $u(x,y,t) \geq w_1(y,t)$ for $x \in \mathbb{R}$, $0 \leq y \leq g(x,t)$, $0 \leq t \leq T$.

Indeed,

$$u(x,0,t) \geq b_1(t) = w_1(0,t).$$

Also, on $\Gamma : y = g(x,t) = s_0(t) + \varepsilon g_1(x,t)$,

$$\frac{\partial w_1}{\partial n} + w_1 = -\frac{b_1(t)}{1+s_1(t)}n_y \frac{b_1(t)}{1+s_1(t)}g(x,t) + b_1(t) \tag{3.5}$$
$$= -\frac{b_1(t)}{1+s_1(t)}\left[n_y + g(x,t) - 1 - s_1(t)\right].$$

Since $g(x,t) \geq s_1(t)$, we have by (3.1),

$$g(x,t) - s_1(t) = \frac{g_x^2}{2K\varepsilon} = \frac{1}{2K\varepsilon}\varepsilon^2 g_{1x}^2.$$

Also

$$n_y - 1 = \frac{1}{\sqrt{1+\varepsilon^2 g_{1x}^2}} - 1 \geq -\frac{1}{2}\varepsilon^2 g_{1x}^2.$$

It follows that the right-hand side of (3.5) is ≤ 0 if $\varepsilon < 1/K$. Applying the maximum principle to $u - w_1$ in the region $\{0 \le y \le g(x,t)\}$, we conclude that $u \ge w_1$.

Next we establish that

$$\text{(3.6)} \qquad \begin{aligned} &\text{if } u(x, g(x,t), t) \ge w_1(g(x,t), t) \text{ for } x \in \mathbb{R},\ 0 \le t \le T, \\ &\text{then } g(x,t) \ge s_1(t) + \delta_T \text{ for } x \in \mathbb{R},\ 0 \le t \le T, \end{aligned}$$

where δ_T is some positive constant.

Indeed, by (1.5),

$$\begin{aligned} g_t &= \sqrt{1 + \varepsilon^2 g_{1x}^2}\, u\big(x, g(x,t), t\big) \\ &\ge w_1\big(g(x,t), t\big) = -\frac{b_1(t)}{1 + s_1(t)} g + b_1(t) \end{aligned}$$

and therefore

$$\text{(3.7)} \qquad g(x,t) \ge \tilde{s}(t),$$

where $\tilde{s}(t)$ is defined by

$$\begin{aligned} \tilde{s}'(t) &= -\frac{b_1(t)}{1 + s_1(t)} \tilde{s}(t) + b_1(t), \\ \tilde{s}(0) &= \overline{s}_0 - \varepsilon \end{aligned}$$

(recall that $|g_1(x,0)| \le 1$). Since $s_1(t)$ satisfies the same differential equation as $\tilde{s}(t)$ with $s_1(0) = \overline{s}_0 - 2\varepsilon < \tilde{s}(0)$, we have, for some $\delta_T > 0$,

$$\tilde{s}(t) \ge s_1(t) + \delta_T \qquad (0 \le t \le T).$$

This together with (3.7) complete the proof of (3.6).

Finally, by combining (3.4) with (3.6), the assertion of Lemma 3.2 follows.

We shall next compare u, g with the solution $w_2(y,t)$, $s_2(t)$ corresponding to

$$\begin{aligned} \overline{s}_2 &= \overline{s}_0 + 2\varepsilon, \\ b_2(t) &= b(t) + \varepsilon B(t). \end{aligned}$$

Clearly,

$$\text{(3.8)} \qquad \begin{aligned} w_2(y,t) &= -\frac{b_2(t) y}{1 + s_2(t)} + b_2(t), \\ \frac{1}{2} s_2^2(t) + s_2(t) &= \frac{1}{2} \overline{s}_2^2 + \overline{s}_2 + \int_0^t b_2(\tau) \mathrm{d}\tau. \end{aligned}$$

LEMMA 3.3. *There holds:*

$$
\begin{aligned}
&g(x,t) < s_2(t), \\
&u(x,y,t) \le w_2(y,t),
\end{aligned}
\tag{3.9}
$$

for $x \in \mathbb{R}$, $0 \le y \le g(x,t)$, $0 \le t \le T$.

PROOF. We first prove that

$$
\begin{aligned}
&\text{if } g(x,t) \le s_2(t) \text{ for } x \in \mathbb{R},\ 0 \le t \le T, \\
&\text{then } u(x,y,t) < w_1(y,t) \text{ for } x \in \mathbb{R},\ 0 \le y \le g(x,t), \qquad 0 \le t \le T.
\end{aligned}
\tag{3.10}
$$

Indeed, the proof is similar to the proof of (3.4); it does not require that $\varepsilon < 1/K$, since

$$n_y + g(x,t) - 1 - s_2(t) \le n_y - 1 \le 0.$$

Next we prove:

$$
\begin{aligned}
&\text{if } u(x,g(x,t),t) \le w_2(g(x,t),t) \text{ for } x \in \mathbb{R},\ 0 \le t \le T, \\
&\text{then } g(x,t) \le s_2(t) - \delta_T \text{ for } x \in \mathbb{R},\ 0 \le t \le T,
\end{aligned}
\tag{3.11}
$$

where δ_T is some positive constant. To prove it we note that

$$g_t = \sqrt{1+g_x^2}u(x,g,t) \le \sqrt{1+g_x^2}\ w_2(g,t) \equiv H(t,g,g_x).$$

Let $\tilde{s}(t)$ be the solution to

$$
\begin{aligned}
&\tilde{s}'(t) = w_2\big(\tilde{s}(t),t\big), \\
&\tilde{s}(0) = \overline{s}_0 + \varepsilon.
\end{aligned}
$$

Then by comparison

$$\tilde{s}(t) \le s_2(t) - \delta_T \qquad (0 \le t \le T) \tag{3.12}$$

for some $\delta_T > 0$. Also

$$\tilde{s}' = H(t,\tilde{s},0). \tag{3.13}$$

Taking the difference of the inequality $g_t \le H(t,g,g_x)$ and (3.13) and noting that $H(t,v,v_x)$ is Lipschitz in the variables v, v_x, we get, for $z = g - \tilde{s}$,

$$z_t \le a z_x + bz$$

where a, b are bounded functions. Since also $z(x,0) \le 0$, it follows that $z(x,t) \le 0$, i.e., $g(x,t) \le \tilde{s}(t)$. Upon recalling (3.12), the assertion (3.11) follows.

Finally, Lemma 3.3 follows by combining (3.10) and (3.11).

PROOF OF LEMMA 3.1. We shall estimate the function $s_2(t) - s_1(t)$. We have $s_1(t) < s_2(t)$ and, from (3.2), (3.8) we easily find that

$$\big(s_2(t) - s_1(t)\big)\left(\frac{s_1(t)+s_2(t)}{2}+1\right) \le \left(C + 2\int_0^\infty B(s)\mathrm{d}s\right)\varepsilon.$$

From (2.6), (2.8) and (3.2), (3.8) we also have

$$\frac{1}{C}\sqrt{1+t} \le s_1(t) \le s_0(t) \le s_2(t) \le C\sqrt{1+t} \tag{3.14}$$

and therefore

$$0 \le s_2(t) - s_1(t) \le \frac{C\varepsilon}{\sqrt{1+t}}. \tag{3.15}$$

Recalling Lemmas 3.2, 3.3, (1.14) and using (3.14), (3.15), the assertion (2.11) follows.

4. - Proof of (2.12)

Set

$$D_t = \left\{x \in \mathbb{R},\ g(x,t) - \frac{1}{2}\bar{s}_0 \le y \le g(x,t)\right\}. \tag{4.1}$$

In this section we prove:

LEMMA 4.1. *If u, g is a solution satisfying* (2.10) *for $0 \le t \le T$, then* (2.12) *holds for $0 \le t \le T$ provided $0 < \varepsilon < \varepsilon_K$; ε_K is a constant independent of T and C is a constant independent of K, T, ε.*

PROOF. From (1.13), Lemmas 3.2, 3.3, (3.15) and the assumption $B(t) \le 1/(1+t)$ we find that

$$\big|u_1(x,y,t)\big| \le \frac{C}{1+t}. \tag{4.2}$$

Next, by (2.10),

$$\|g_x\|_{C_x^{1,\alpha}} = \|\varepsilon g_{1x}\|_{C_x^{1,\alpha}} \le 1 \text{ if } \varepsilon K < 1, \tag{4.3}$$

so that $\Gamma : y = g(x,t)$ is uniformly in $C^{2,\alpha}$ (independently of K and t).

Consider the expression

$$\tilde{F} = F(\varepsilon g_{1x})g_{1x}^2$$

which appears in (2.1). Clearly

$$\tilde{F}_x = \varepsilon F'(\varepsilon g_{1x}) g_{1x}^2 g_{1xx} + F(\varepsilon g_{1x}) 2 g_{1x} g_{1xx}.$$

Using (2.10) we easily find that

$$\|\tilde{F}\|_{L^\infty} \le \frac{CK^2}{1+t},$$

$$\|\tilde{F}_x\|_{L^\infty} \le \frac{CK^3}{1+t},$$

and

$$\|\tilde{F}_x\|_{C_x^\alpha} \le \frac{C_K}{\sqrt{1+t}}.$$

It follows that

$$\|\varepsilon\tilde{F}\|_{C_x^{1,\alpha}} \le \frac{\varepsilon C_K}{\sqrt{1+t}} < \frac{1}{\sqrt{1+t}}$$

if $\varepsilon \le \varepsilon_K$, and consequently, by (2.1),

$$\left\| \left[\frac{\partial u_1}{\partial n} + u_1 \right]_{y=g(x,t)} \right\|_{C_x^{1,\alpha}} \le \frac{C}{1+t} + \frac{C}{\sqrt{1+t}} \|g_1\|_{C_x^{1,\alpha}}. \tag{4.4}$$

Using (4.2), (4.3) and (4.4), we can now apply the interior-boundary Schauder estimates to u_1 to obtain the assertion (2.12).

5. - Proof of (2.13)

In this section we prove:

LEMMA 5.1. *If u, g is a solution satisfying* (2.10) *for $0 \le t \le T$, then* (2.13) *holds for $0 \le t \le T$ provided $0 < \varepsilon < \varepsilon_K$; ε_K is a constant independent of T, and C is a constant independent of K, T, ε.*

PROOF. We first assume that

$$g_{xxx}, g_{xxt} \text{ are continuous.} \tag{5.1}$$

We wish to differentiate (2.2) twice with respect to x so as to obtain an equation of the form

$$(g_{1xx})_t + A(g_{1xx})_x = B,$$

and then integrate along characteristics to derive (2.13). We begin with

$$\frac{\partial}{\partial x}\sqrt{1+\varepsilon^2 g_{1x}^2} = \frac{\varepsilon^2 g_{1x} g_{1xx}}{\sqrt{1+\varepsilon^2 g_{1x}^2}} \equiv J_1.$$

By (2.10)

$$\|J_1\|_{C_x^\alpha} \le \frac{\varepsilon}{\sqrt{1+t}} \text{ if } \varepsilon < \varepsilon_K.$$

Next

$$\frac{\partial^2}{\partial x^2}\sqrt{1+\varepsilon^2 g_{1x}^2} = \frac{\varepsilon^2 g_{1x}}{\sqrt{1+\varepsilon^2 g_{1x}^2}} g_{1xxx} + J_2 \tag{5.2}$$

where

$$J_2 = \varepsilon^2\left(\frac{g_{1xx}}{\sqrt{1+\varepsilon^2 g_{1x}^2}} - g_{1x}\frac{\varepsilon^2 g_{1x} g_{1xx}}{(1+\varepsilon^2 g_{1x}^2)^{3/2}}\right) g_{1xx}.$$

Again using (2.10), we get

$$\|J_2\|_{C_x^\alpha} \le \frac{\varepsilon}{\sqrt{1+t}} \text{ if } \varepsilon \le \varepsilon_K.$$

We now turn to the expression $\varepsilon G g_{1x}^2$ in (2.2). Clearly

$$\frac{\partial}{\partial x}\left(\varepsilon G(\varepsilon g_{1x}) g_{1x}^2\right) = \left[\varepsilon^2 G'(\varepsilon g_{1x}) g_{1x}^2 + 2\varepsilon G(\varepsilon g_{1x}) g_{1x}\right] g_{1xx}$$

so that

$$\begin{aligned} \frac{\partial^2}{\partial x^2}\left(\varepsilon G(\varepsilon g_{1x}) g_{1x}^2\right) &= \left[\varepsilon^2 G'(\varepsilon g_{1x}) g_{1x}^2 + 2\varepsilon G(\varepsilon g_{1x}) g_{1x}\right] g_{1xxx} + J_4 \\ &\equiv J_3 g_{1xxx} + J_4, \end{aligned} \tag{5.3}$$

where

$$J_4 = \frac{\partial J_3}{\partial x} g_{1xx} = \varepsilon Q(g_{1x}) g_{1xx}^2$$

and $Q(s)$ is a smooth function. Using (2.10) we find that

$$\|J_3\|_{W_x^{1,\infty}} \le \frac{\varepsilon^{2/3}}{\sqrt{1+t}} \text{ if } \varepsilon \le \varepsilon_K,$$

and

$$\begin{aligned} \|J_4\|_{C_x^\alpha} &\le \varepsilon \|Q(g_{1x})\|_{C_x^\alpha}\left\{2\|g_{1xx}\|_{L^\infty}\|g_{1xx}\|_{C_x^\alpha}\right\} \\ &\le \frac{1}{\sqrt{1+t}} \text{ if } \varepsilon \le \varepsilon_K. \end{aligned}$$

We now differentiate (2.2) twice in x to obtain

$$\begin{aligned} &(g_{1xx})_t + \frac{b(t)}{1+s_0(t)}\sqrt{1+\varepsilon^2 g_{1x}^2}\, g_{1xx} + \frac{2b(t)}{1+s_0(t)} J_1 g_{1x} \\ &+ \frac{b(t)}{1+s_0(t)} g_1\left(\frac{\varepsilon^2 g_{1x}}{\sqrt{1+\varepsilon^2 g_{1x}^2}} g_{1xxx} + J_2\right) \\ &- \frac{b(t)}{1+s_0(t)}(J_3 g_{1xxx} + J_4) - \frac{\varepsilon^2 g_{1x} u_1}{\sqrt{1+\varepsilon^2 g_{1x}^2}} g_{1xxx} - J_2 u_1 - J_5 = 0, \end{aligned} \tag{5.4}$$

where J_5 involves the first and second derivatives of u_1. By (2.10), (2.12) and the estimates on J_1, J_2 we find that

$$\|J_5\|_{C_x^\alpha} \le \frac{C}{1+t} + \frac{C}{\sqrt{1+t}}\|g_1(\cdot,t)\|_{C_x^{1,\alpha}}$$

provided $\varepsilon \le \varepsilon_K$. We can rewrite equation (5.4) in the form

$$(g_{1xx})_t + H(x,t)(g_{1xx})_x + \frac{b(t)}{1+s_0(t)}\sqrt{1+\varepsilon^2 g_{1x}^2}\, g_{1xx} = J_6 \tag{5.5}$$

where, by (2.10) and the estimates on the J_i $(i<6)$,

$$\|J_6\|_{C_x^\alpha} \le \frac{C}{1+t} + \frac{C}{\sqrt{1+t}}\|g_1(\cdot,t)\|_{C_x^{1,\alpha}} \tag{5.6}$$

and

$$\|H(\cdot,t)\|_{W_x^{1,\infty}} \le \frac{\sqrt{\varepsilon}}{1+t} \tag{5.7}$$

if $\varepsilon \le \varepsilon_K$. Introduce the characteristics $\xi = \xi(x,t)$ by

$$\frac{\mathrm{d}\xi}{\mathrm{d}t} = H(\xi,t), \quad \xi(x,t) = x; \tag{5.8}$$

in view of (5.7), these are uniquely defined and Lipschitz continuous in t. We can rewrite (5.5) as

$$\frac{\mathrm{d}}{\mathrm{d}t} g_{1xx}\big(\xi(x,t),t\big) + \frac{b(t)}{1+s_0(t)}\sqrt{1+\varepsilon^2 g_{1x}} g_{1xx}\big(\xi(x,t),t\big) = J_6\big(\xi(x,t),t\big). \tag{5.9}$$

By integration,

$$\begin{aligned} &g_{1xx}\big(\xi(x,t),t\big) - g_{1xx}(x,0)\exp\left(-\int_0^t \frac{b(\tau)}{1+s_0(\tau)}\sqrt{1+\varepsilon^2 g_{1x}^2}\,\mathrm{d}\tau\right) \\ &= \int_0^t \exp\left(-\int_s^t \frac{b(\tau)}{1+s_0(\tau)}\sqrt{1+\varepsilon^2 g_{1x}^2}\,\mathrm{d}\tau\right) J_6\big(\xi(x,s),s\big)\mathrm{d}s, \end{aligned} \tag{5.10}$$

so that

$$\begin{aligned} &\|g_{1xx}\|_{L^\infty}(t) \\ &\le \|g_{1xx}\|_{L^\infty}(0)e^{-c_0\sqrt{1+t}} + \int_0^t e^{-c_0\left(\sqrt{1+t}-\sqrt{1+s}\right)}\|J_6\|_{L^\infty}(s)\mathrm{d}s \end{aligned} \tag{5.11}$$

where c_0 is a positive constant such that

$$\frac{b(\tau)}{1+s_0(\tau)} \geq \frac{c_0}{2\sqrt{1+\tau}}.$$

Let

$$U_2(t) = \|g_{1xx}\|_{L^\infty}(t), \ \ U_{1,\alpha}(t) = \|g_1\|_{C_x^{1,\alpha}}(t).$$

Using (5.6), we obtain from (5.11)

$$U_2(t) \leq U_2(0)e^{-c_0\sqrt{1+t}} + Ce^{-c_0\sqrt{1+t}} \int_0^t e^{c_0\sqrt{1+s}} \left(\frac{1}{1+s} + \frac{U_{1,\alpha}(s)}{\sqrt{1+s}}\right) \mathrm{d}s.$$

By interpolation and Lemma 3.1,

$$U_{1,\alpha}(s) \leq \mu U_2(s) + \tilde{C}_\mu \|g_1\|_{L^\infty}(s) \leq \mu U_2 + \frac{C_\mu}{\sqrt{1+s}}$$

where μ is any positive number and $\tilde{C}_\mu$, C_μ depend only on μ (but not on K, T). It follows that

$$U_2(t) \leq U_2(0)e^{-c_0\sqrt{1+t}} + Ce^{-c_0\sqrt{1+t}} \int_0^t e^{c_0\sqrt{1+s}} \left(\frac{C_\mu^*}{1+s} + \frac{\mu U_2(s)}{\sqrt{1+s}}\right) \mathrm{d}s \tag{5.12}$$

where $C_\mu^* = C_\mu + 1$.

We shall need the following inequality:

$$\int_0^t \frac{e^{\lambda\sqrt{1+s}}}{1+s} ds \leq \frac{\overline{C}_\lambda}{\sqrt{1+t}} e^{\lambda\sqrt{1+t}}, \tag{5.13}$$

which holds for any $\lambda > 0$ and suitable constant C_λ. To prove it denotes the difference of the right-hand side from the left-hand side by $h(t)$. If $0 \leq t \leq t_0$, where $\sqrt{1+t_0} = 2/\lambda$, then $h(t) < 0$ provided $\overline{C}_\lambda$ is sufficiently large. On the other hand, if $t > t_0$

$$h'(t) = \frac{e^{\lambda\sqrt{1+t}}}{1+t}\left[1 - \frac{\overline{C}_\lambda}{2}\left(\lambda - \frac{1}{\sqrt{1+t}}\right)\right] < 0 \text{ if } \lambda\overline{C}_\lambda > 4,$$

and (5.13) follows.

We shall apply to (5.12) Gronwall's inequality which states:

$$\begin{aligned} &\text{if } \varphi(t) \leq \psi(t) + \int_0^t \chi(s)\varphi(s) \ \ (\varphi > 0, \ \psi > 0, \ \chi > 0) \\ &\text{then } \varphi(t) \leq \psi(t) + \int_0^t \chi(s)\psi(s) \exp\left(\int_s^t \chi(\tau)\mathrm{d}\tau\right) \mathrm{d}s. \end{aligned} \tag{5.14}$$

We take

$$\varphi(t) = e^{c_0\sqrt{1+t}}U_2(t),$$

$$\psi(t) = U_2(0) + CC_\mu^* \int_0^t \frac{e^{c_0\sqrt{1+s}}}{1+s},$$

$$\chi(t) = \frac{C\mu}{\sqrt{1+t}}.$$

By (5.13)

$$\psi(t) \leq \frac{\tilde{C}_\mu}{\sqrt{1+t}} e^{c_0\sqrt{1+t}}.$$

So

$$\int_0^t \chi(s)\psi(s)\exp\left(\int_s^t \chi(\tau)\mathrm{d}\tau\right)\mathrm{d}s \leq \int_0^t \frac{C\mu}{\sqrt{1+s}} \frac{\tilde{C}_\mu}{\sqrt{1+s}} e^{c_0\sqrt{1+s}} e^{2C\mu\left(\sqrt{1+t}-\sqrt{1+s}\right)}\mathrm{d}s.$$

Choosing μ such that $2C\mu < c_0$ and using (5.13), we find that the right-hand side is bounded by

$$e^{2C\mu\sqrt{1+t}} C\mu\hat{C}_\mu \int_0^t \frac{e^{(c_0-2C\mu)\sqrt{1+s}}}{1+s}\mathrm{d}s \leq \overline{\overline{C}}_\mu \frac{e^{c_0\sqrt{1+t}}}{\sqrt{1+t}}.$$

We now use Gronwall's inequality (5.14) to immediately deduce from (5.12) that

$$e^{c_0\sqrt{1+t}}U_2(t) \leq \frac{Ce^{c_0\sqrt{1+t}}}{\sqrt{1+t}}.$$

Thus Lemma 5.1 follows, provided (5.1) holds. The assumptions (5.1) can actually be avoided. Since we need only the integral equation (5.10) along characteristics, we may proceed as follows: we first differentiate (2.2) in x once, write the integral equation along characteristics, and then differentiate it once in x. In this way we avoid using g_{xxx}, g_{xxt}.

6. - Proof of (2.14)

LEMMA 6.1. *If* u, g *is a solution satisfying* (2.10) *for* $0 \leq t \leq T$, *then* (2.14) *holds for* $0 \leq t \leq T$ *provided* $0 < \varepsilon < \varepsilon_K$; ε_K *is a constant independent of* T, *and* C *is a constant independent of* K, T, ε.

PROOF. For simplicity we again make the assumption (5.1). Then (5.7) is satisfied and, by (5.6) and Lemma 5.1,

$$\|J_6\|_{C_x^\alpha} \leq \frac{C}{1+t}. \tag{6.1}$$

For fixed x, $\overline{x}$, consider the function

$$w(t) = g_{1xx}\big(\xi(x,t),t\big) - g_{1xx}\big(\xi(\overline{x},t),t\big).$$

Using (2.10) and (6.1) we find, from (5.9), that

$$\frac{\mathrm{d}}{\mathrm{d}t}w + \frac{b(t)}{1+s_0(t)}\sqrt{1+\varepsilon^2 g_{1x}^2}\,w = R(x,t), \tag{6.2}$$

where

$$\big\|R(\,\cdot\,,t)\big\|_{L^\infty} \le \frac{C}{1+t}\big|\xi(x,t)-\xi(\overline{x},t)\big|^\alpha \tag{6.3}$$

and as beforre (cf. (5.11))

$$\big|w(t)\big| \le \big|w(0)\big|e^{-c_0\sqrt{1+t}} + e^{-c_0\sqrt{1+t}}\int_0^t e^{c_0\sqrt{1+s}}\frac{C}{1+s}\big|\xi(x,s)-\xi(\overline{x},s)\big|^\alpha \mathrm{d}s. \tag{6.4}$$

From (5.8), (5.7) we have

$$\frac{\mathrm{d}\xi_x(x,t)}{\mathrm{d}t} = H_x(\xi,t)\xi_x, \quad \xi_x(x,0)=1$$

with

$$\big|H_x(x,t)\big| \le \frac{\sqrt{\varepsilon}}{1+t}$$

and then, by comparison,

$$\frac{1}{(1+t)^{\sqrt{\varepsilon}}} \le \xi_x \le (1+t)^{\sqrt{\varepsilon}}. \tag{6.5}$$

This implies that the inverse function $x = x(\xi,t)$ exists and satisfies:

$$\frac{1}{(1+t)^{\sqrt{\varepsilon}}} \le x_\xi \le (1+t)^{\sqrt{\varepsilon}}. \tag{6.6}$$

From (6.5) it follows that

$$\big|\xi(x,s)-\xi(\overline{x},s)\big|^\alpha \le C(1+s)^{\frac{1}{4}}|x-\overline{x}|^\alpha$$

if $\sqrt{\varepsilon}\alpha < \frac{1}{4}$. Using this in (6.4), we get

$$\begin{aligned}|w(t)| &\le |x-\overline{x}|^\alpha e^{-c_0\sqrt{1+t}} + e^{-c_0\sqrt{1+t}}\int_0^t \frac{C}{(1+s)^{3/4}}e^{c_0\sqrt{1+s}}|x-\overline{x}|^\alpha \mathrm{d}s \\ &\le \frac{C}{(1+t)^{1/4}}|x-\overline{x}|^\alpha,\end{aligned} \tag{6.7}$$

since

$$\int_0^t \frac{e^{\lambda\sqrt{1+s}}}{(1+s)^{3/4}}ds \le \frac{\overline{C}_\lambda}{(1+t)^{1/4}}e^{\lambda\sqrt{1+s}}$$

(the proof is the same as for (5.13)).

For any x_1, x_2, t, choose x and $\overline{x}$ such that

$$x_1 = \xi(x,t), \quad x_2 = \xi(\overline{x},t).$$

Then

$$|x_1 - x_2| = \left|\xi_x(\hat{x})\right| |x - \overline{x}| \ge \frac{1}{(1+t)^{\sqrt{\varepsilon}}}|x - \overline{x}|$$

by (6.5). Using this in (6.7) we get

$$\begin{aligned} \left|g_{1xx}(x_1,t) - g_{1xx}(x_2,t)\right| &\le \frac{C}{(1+t)^{1/4}}(1+t)^{\sqrt{\varepsilon}\alpha}|x_1 - x_2|^\alpha \\ &\le C|x_1 - x_2|^\alpha \text{ if } \sqrt{\varepsilon}\alpha < \frac{1}{4}. \end{aligned}$$

This completes the proof of the lemma.

We summarize the results of Sections 3–6:

THEOREM 6.2. *Consider the problem* (1.1)–(1.6), (1.11), (1.12) *under the assumptions* (2.6)–(2.9). *If* u, g *is a solution for* $0 \le t \le T$ $(0 < T < \infty)$ *satisfying* (2.10), *then it also satisfies* (2.11)–(2.14), *provided* $0 < \varepsilon < \varepsilon_K$; ε_K *is a positive constant independent of* T, *and* C *is a positive constant independent of* K, T, ε.

REMARK 6.1. From (2.2) it follows that also

$$\left\|g_{1t}(\cdot,t)\right\|_{L^\infty} \le \frac{C}{\sqrt{1+t}}, \quad \left\|g_{1xt}(\cdot,t)\right\|_{L^\infty} \le \frac{C}{\sqrt{1+t}}. \tag{6.8}$$

REMARK 6.2. The proof of Theorem 6.2 breaks down if we relax the growth conditions on $B(t)$ in (2.8). Indeed, suppose (instead of $\int_0^\infty B(t)dt < \infty$) that

$$\left|B(t)\right| \le \frac{C}{(1+t)^\kappa} \qquad \left(\frac{1}{2} \le \kappa < 1\right).$$

The comparison results of Section 2 are still valid, yielding the estimate

$$|g_1| \le \frac{C}{(1+t)^{\kappa-1/2}}.$$

This suggests the extension of Theorem 6.2 with $K/\sqrt{1+t}$ in (2.10) replaced by $K/(1+t)^{\kappa-1/2}$; a term $C/(1+t)^{\kappa-1/2}$ should then be added to the right-hand sides of (2.11), (2.13), and $C/(1+t)$ in (2.12) should be replaced by $C/(1+t)^\kappa$. Next, in (5.12),

$$\frac{C^*_\mu}{1+s} \text{ is replaced by } \frac{C^*_\mu}{(1+s)^\kappa},$$

and, analogously to (5.13),

$$\int_0^t \frac{e^{\lambda\sqrt{1+s}}}{(1+s)^\kappa} ds \le \frac{\overline{C}_\mu}{(1+t)^{\kappa-1/2}} e^{\lambda\sqrt{1+t}}.$$

With these changes we can now proceed as before to derive the estimate

$$\left\| g_{1xx}(\,\cdot\,,t) \right\|_{L^\infty} \le \frac{C}{(1+t)^{\kappa-1/2}}$$

which replaces (2.13). This estimate, however, is too weak for establishing the appropriate bound on the Hölder coefficient of g_{1xx}. Indeed, instead of (6.5) we only get

$$\exp\left\{ -\sqrt{\varepsilon}\frac{(1+t)^{1-\kappa}}{1-\kappa} \right\} \le \xi_x \le \exp\left\{ \sqrt{\varepsilon}\frac{(1+t)^{1-\kappa}}{1-\kappa} \right\},$$

which is insufficient for the proof of (2.14).

7. - Existence theorems

Consider (1.1)–(1.6) with

$$g_0(x) \ge c_0 > 0, \quad \|g_0\|_{C^{2,\alpha}_x} \le K < \infty, \tag{7.1}$$

$$b(x,t) \text{ continuous and } |b(x,t)| \le C < \infty \text{ for } x \in \mathbb{R},\ t \ge 0. \tag{7.2}$$

We shall prove that for some small $T > 0$ there exists a solution (u, g) with g in the class

$$\begin{aligned} B_{K,M} \equiv \Big\{ & g(x,t),\ 0 \le t \le T;\ g(x,t) \ge \frac{1}{2}c_0,\ \left\| g(\,\cdot\,,t) \right\|_{L^\infty} \le 2K, \\ & \left\| g_x(\,\cdot\,,t) \right\|_{L^\infty} \le 2K,\ \left\| g_{xx}(\,\cdot\,,t) \right\|_{L^\infty} \le 2K,\ \left[g_{xx}(\,\cdot\,,t) \right]_{C^\alpha_x} \le 2K, \\ & \left\| g_t(\,\cdot\,,t) \right\|_{L^\infty} \le M, \text{ and } g(x,0) = g_0(x) \Big\}, \end{aligned}$$

where M is a positive constant to be determined.

For any $g \in B_{K,M}$ define $\Gamma : y = g(x,t)$ and let u be the solution of (1.2)–(1.4). By the maximum principle

$$|u| \le C^* \qquad (C^* \text{ independent of } K, M, T). \tag{7.3}$$

and by the Schauder estimates

$$\big\|u(\cdot\,,\,\cdot\,,t)\big\|_{C^{2,\alpha}_{x,y}(D_t)} \le C_K \qquad (C_K \text{ independent of } M, T), \tag{7.4}$$

where D_t is defined as in (4.1). Let

$$v(x,t) = u\big(x, g(x,t), t\big)$$

and let φ_δ be mollifiers in x, and set

$$v_\delta(x,t) = \big(\varphi_\delta * v(\cdot\,,t)\big)(x).$$

Then

$$\|v_\delta(\cdot\,,t)\|_{C^{2,\alpha}_x} \le C_K, \ \ |v_\delta| \le C^*, \tag{7.5}$$

We also introduce

$$g_\delta(x) = \varphi_\delta * g_0(x);$$

clearly

$$\|g_\delta\|_{C^{2,\alpha}} \le K. \tag{7.6}$$

For any small $\varepsilon > 0$, let $\tilde{g}(x,t)$ be the solution of

$$\tilde{g}_t = \sqrt{1+\tilde{g}_x^2}\,v_\delta(x,t) + \varepsilon\tilde{g}_{xx}, \tag{7.7}$$

$$\tilde{g}(x,0) = g_\delta(x). \tag{7.8}$$

By comparison [2; p. 52]

$$\begin{aligned} &\tilde{g}(x,t) \le C^* t + \max g_\delta(x), \\ &\tilde{g}(x,t) \ge -C^* t + \inf g_\delta(x), \end{aligned} \tag{7.9}$$

so that

$$\tilde{g}(x,t) > \frac{c_0}{2}, \ \ \|\tilde{g}\|_{L^\infty} \le 2K \tag{7.10}$$

provided T is small (depending only on C^*, K). Next differentiate (7.7) in x to obtain

$$(7.11)\qquad \mathcal{L}\tilde{g}_x \equiv \frac{\partial}{\partial t}\tilde{g}_x - \frac{v_\delta(x,t)}{\sqrt{1+\tilde{g}_x^2}}\tilde{g}_x\frac{\partial}{\partial x}\tilde{g}_x - v_{\delta,x}(x,t)\sqrt{1+\tilde{g}_x^2} - \varepsilon\frac{\partial^2}{\partial x^2}\tilde{g}_x = 0.$$

The function $w = K + Ct$ satisfies

$$\mathcal{L}w = C - v_{\delta,x}(x,t)\sqrt{1+w^2} > 0$$

if $C \geq C_K$ and T is small depending only on K. It follows, by comparison with $\tilde{g}_x$, that

$$(7.12)\qquad \tilde{g}_x \leq w \leq 2K$$

if T is small enough, and similarly

$$(7.13)\qquad \tilde{g}_x \geq -2K.$$

Differentiating (7.11) once more in x and using (7.10), (7.12) (7.13), we obtain by comparison, as before,

$$(7.14)\qquad |\tilde{g}_{xx}| \leq Ct + K \leq 2K$$

where $C = C_K$, and T is small enough, depending only on K.

Finally, from (7.7),

$$(7.15)\qquad |\tilde{g}_t| \leq M$$

where M depends only on K, provided T is small enough (depending only on K).

Observe that u is continuous in t (by compactness and uniqueness). Therefore also

$$(7.16)\qquad v_\delta \text{ is continuous in } (x,t).$$

We next observe that the problem

$$(7.17)\qquad g_t = \sqrt{1+g_x^2}\,v_\delta(x,t),$$

$$(7.18)\qquad g(x,0) = g_\delta(x)$$

has at most one solution. Indeed, this follows by estimating the difference of two solutions, making use of the Lipschitz continuity of $v_\delta(x,t)$ in x and its continuity in t (by (7.16)).

From the above observations and the estimates (7.10), (7.12), (7.13), (7.14), (7.15), it follows that the family $\tilde{g} \equiv \tilde{g}_\varepsilon$ converges to a (unique) solution g^* of (7.17), (7.18) as $\varepsilon \to 0$.

By differentiating (7.17) formally twice in x we get

$$\begin{aligned}\frac{\partial}{\partial t} g^*_{xx} - \frac{g^*_x}{\sqrt{1+g^{*2}_x}} g^*_{xxx} &= \frac{1}{(1+g^{*2}_x)^{3/2}} (g^*_{xx})^2 v_\delta(x,t) \\ &+ \frac{2g^*_x}{\sqrt{1+g^{*2}_x}} g^*_{xx} v_{\delta,x}(x,t) + \sqrt{1+g^{*2}_x}\, v_{\delta,xx}(x,t).\end{aligned} \tag{7.19}$$

To justify this differentiation note that by differentiating (7.11) successively in x and comparing with functions of the form $Ct + C_1$ we can estimate the derivatives $\tilde{g}_{xxx}$, $\tilde{g}_{xxxx}$, etc. as we have done in (7.14). The constants depend on δ but not on ε. Hence differentiating (7.7) twice in x and then letting $\varepsilon \to 0$, equation (7.19) follows.

Next we introduce the characteristics

$$\frac{\mathrm{d}\xi}{\mathrm{d}t} = -\left(\frac{g^*_x}{\sqrt{1+g^{*2}_x}} v_\delta\right)(\xi,t), \quad \xi(x,0) = x \tag{7.20}$$

and note that

$$\frac{1}{2} \le \frac{\mathrm{d}\xi}{\mathrm{d}x} \le 2$$

if T is small. Writing (7.19) in integrated form along characteristics, we can derive the inequality

$$\begin{aligned}\left| g^*_{xx}\big(\xi(x_1,t),t\big) - g^*_{xx}\big(\xi(x_2,t),t\big) \right| &\le \left| g_{0,xx}(x_1) - g_{0,xx}(x_2) \right| \\ &+ \int_0^t \left| A_1 \cdot \left[g^*_{xx}\big(\xi(x_1,s),s\big) - g^*_{xx}\big(\xi(x_2,s),s\big) \right] + A_2 \right| \mathrm{d}s\end{aligned}$$

where $|A_j| \le C_K$, C_K independent of δ. It easily follows that

$$[g^*_{xx}]_{C^\alpha_x} \le 2K$$

if T is small.

Consider the mapping W defined by $g \to Wg = g^*$. We have proved that W maps $B_{K,M}$ into itself provided T is sufficiently small (depending on K, but not on δ).

If we provide $B_{K,M}$ with the uniform topology, then $B_{K,M}$ is compact. From the uniqueness of solution to (7.17), (7.18) and compactness, it follows that W is continuous. Hence, by the Schauder fixed-point theorem, W has a fixed point g^*_δ. Letting $\delta \to 0$ through an appropriate subsequence, we obtain a limiting function g, which together with the corresponding u, provide a solution to (1.1)–(1.6).

We summarize:

THEOREM 7.1. *If* (7.1), (7.2) *hold, then there exists a solution* (u, g) *of* (1.1)–(1.6) *with* g *in the set* $B_{K,M}$, *provided* T *is sufficiently small.*

The proof shows that T depends only on $\|g_0\|_{C_x^{2,\alpha}}$ (and that all the norms in (2.10) are continuous in t). Hence the solution can be extended step-by-step for all times as long as one can establish a priori estimate on

$$\|g(\cdot\,,t)\|_{C_x^{2,\alpha}}$$

independently of t. Such an estimate has already been derived in Theorem 6.2. We may therefore state:

THEOREM 7.2. *Consider the problem* (1.1)–(1.6), (1.11), (1.12) *under the assumptions* (2.6)–(2.9). *If* ε *is sufficiently small (depending on* $\|g_0\|_{C^{2,\alpha}}$*), then there exists a global solution.*

The solution satisfies (2.10) and (6.8), g_{xx} is continuous in t, and the norms in (2.10) are continuous in t.

REMARK 7.1. The reason for introducing the mollifiers φ_δ in the proof of Theorem 7.1 is to justify the calculations which involve third derivations of g. The diffusion term εg_{xx} was introduced in (7.7) so that we can use a parabolic comparison theorem.

8. - Uniqueness

In Section 7 we proved the existence of solutions (u, g) such that

(8.1) all the norms in (2.10) and g, g_x, g_{xx} are continuous in t.

We shall now establish uniqueness of such solutions for general data.

THEOREM 8.1. *Assume that, for some* $T > 0$, (u, g), $\tilde{u}, \tilde{g})$ *are two solutions of* (1.1)–(1.6) *satisfying* (8.1). *Then* $u \equiv \tilde{u}$ *and* $g \equiv \tilde{g}$.

PROOF. By assumption

$$\|g(\cdot\,,t)\|_{C_x^{2,\alpha}},\ \|\tilde{g}(\cdot\,,t)\|_{C_x^{2,\alpha}} \le C$$

and therefore by Schauder's estimates, for any $\delta > 0$,

$$\|u(\cdot\,,\cdot\,,t)\|_{C^{2,\alpha}(\Omega_t^\delta)},\ \|\tilde{u}(\cdot\,,\cdot\,,t)\|_{C^{2,\alpha}(\tilde{\Omega}_t^\delta)} \le C_\delta \tag{8.2}$$

where $\Omega_t^\delta = \{(x,y);\ \delta < y < g(x,t)\}$ and $\tilde{\Omega}_t^\delta = \{(x,y);\ \delta < y < \tilde{g}(x,t)\}$. Set

$$V(t) = \sup_x \left| g(x,t) - \tilde{g}(x,t) \right| \tag{8.3}$$

and introduce the domain

$$G_t = \{(x,y);\ 0 < y < g(x,t) - V(t)\}.$$

Then $\partial G_t = \{y = 0\} \cup S_t$, where

$$S_t = \{y = g(x,t) - V(t)\}$$

is uniformly in $C_x^{2,\alpha}$. The outward normal along S_t is

$$n = \frac{(-g_x, 1)}{\sqrt{1 + g_x^2}}.$$

Set

$$J_1 = \left(\frac{\partial u}{\partial n} + u \right)\Big|_{y=g(x,t)-V(t)} - \left(\frac{\partial u}{\partial n} + u \right)\Big|_{y=g(x,t)},$$

$$J_2 = \left(\frac{\partial \tilde{u}}{\partial n} + \tilde{u} \right)\Big|_{y=g(x,t)-V(t)} - \left(\frac{\partial \tilde{u}}{\partial n} + \tilde{u} \right)\Big|_{y=\tilde{g}(x,t)}.$$

Then, by (8.2),

$$\|J_1\|_{C_x^\alpha} \le CV(t) \tag{8.4}$$

and

$$\|J_2\|_{C_x^\alpha} \le CV(t) + C\left\|g(\cdot,t) - \tilde{g}(\cdot,t)\right\|_{C_x^\alpha}. \tag{8.5}$$

Introducing the normal to $\{y = \tilde{g}(x,t)\}$,

$$\tilde{n} = \frac{(-\tilde{g}_x, 1)}{\sqrt{1 + \tilde{g}_x^2}},$$

we also have

$$\left\| \left(\frac{\partial \tilde{u}}{\partial n} - \frac{\partial \tilde{u}}{\partial \tilde{n}} \right)_{y=\tilde{g}(x,t)} \right\|_{C_x^\alpha} \le C\|g_x - \tilde{g}_x\|_{C_x^\alpha}(t). \tag{8.6}$$

Using the free boundary condition (1.4) for both u and $\tilde{u}$ and the estimates (8.4)–(8.6), we easily obtain

$$\left\| \left(\frac{\partial (u - \tilde{u})}{\partial n} + (u - \tilde{u}) \right)_{y=g(x,t)-V(t)} \right\|_{C_x^\alpha} \le C\|g - \tilde{g}\|_{C_x^{1,\alpha}}(t). \tag{8.7}$$

By (8.7) and the maximum principle

$$\big\|(u-\tilde{u})(\cdot\,,t)\big\|_{L^\infty(G_t)} \le C\|g_x-\tilde{g}_x\|_{C_x^\alpha}(t). \tag{8.8}$$

Denote by w the harmonic conjugate of $u-\tilde{u}$. Then, by (8.7),

$$\left\|\frac{\partial w}{\partial s}\right\|_{C_x^\alpha} \le C\|u-\tilde{u}\|_{C_x^\alpha} + C\|g-\tilde{g}\|_{C_x^{1,\alpha}},$$

where w and $u-\tilde{u}$ are evaluated on S_t. Applying to w elliptic $C^{1,\alpha}$ estimates [4, Theorem 2.4], we then easily get

$$\|w\|_{C_{x,y}^{1,\alpha}(G_t)} \le C\|u-\tilde{u}\|_{C_x^\alpha(S_t)} + C\|g-\tilde{g}\|_{C_x^{1,\alpha}}.$$

Since, for any $\delta>0$,

$$\|u-\tilde{u}\|_{C_x^\alpha} \le \delta\|u-\tilde{u}\|_{C_x^{1,\alpha}} + C_\delta\|u-\tilde{u}\|_{L^\infty} \text{ on } S_t$$

and

$$\|u-\tilde{u}\|_{C_x^{1,\alpha}(S_t)} \le C\|w\|_{C_{x,y}^{1,\alpha}(G_t)},$$

it follows (by choosing δ small enough) that

$$\|(u-\tilde{u})\|_{C_{x,y}^{1,\alpha}(G_t)} \le C\|g-\tilde{g}\|_{C_x^{1,\alpha}}(t). \tag{8.9}$$

Next, differentiating in x the free boundary condition

$$g_t = \sqrt{1+g_x^2}\,u(x,g,t),$$

we get

$$g_{xt} = H(x,t)g_{xx} + K(x,t), \tag{8.10}$$

where

$$H(x,t) = \frac{g_x}{\sqrt{1+g_x^2}}u\big(x,g(x,t),t\big),$$

$$K(x,t) = \sqrt{1+g_x^2}\Big(u_x\big(x,g(x,t),t\big) + u_y\big(x,g(x,t),t\big)g_x\Big).$$

A similar formula holds for $\tilde{g}$. Using (8.2) and (8.9) we can estimate

$$\|K-\tilde{K}\|_{C_x^\alpha} + \big\|(H-\tilde{H})\tilde{g}_{xx}\big\|_{C_x^\alpha} \le C\big\|g(\cdot\,,t)-\tilde{g}(\cdot\,,t)\big\|_{C_x^{1,\alpha}}.$$

Consequently, the function $g^* = g-\tilde{g}$ satisfies

$$\big\|g_{xt}^* - H(x,t)g_{xx}^*\big\|_{C_x^\alpha} \le C\|g^*\|_{C_x^{1,\alpha}}(t); \tag{8.11}$$

also,

$$\|H(\cdot,t)\|_{C_x^{1,\alpha}} \le C. \tag{8.12}$$

Introduce the characteristics

$$\frac{d\xi}{dt} = -H(\xi,t), \quad \xi(x,0) = 0. \tag{8.13}$$

By (8.12),

$$\frac{1}{2} \le \frac{d\xi}{dx} \le 2, \quad \text{for } 0 < t < t_0, \tag{8.14}$$

if t_0 is small enough. Integrating (8.11) along characteristics, we obtain

$$\|g_x^*\|_{L_x^\infty}(t) \le Ct \max_{0\le\tau\le t} \|g^*(\cdot,\tau)\|_{C_x^{1,\alpha}}. \tag{8.15}$$

Using (8.14) and proceeding as in Section 6, we can also get (much more simply) the estimate

$$[g_x^*]_{C_x^\alpha}(t) \le Ct \max_{0\le\tau\le t} \|g^*(\cdot,\tau)\|_{C_x^{1,\alpha}}. \tag{8.16}$$

From (8.15), (8.16), it follows that

$$g(x,t) \equiv \tilde{g}(x,t) \text{ for } 0 \le t \le \tau,$$

if τ is small enough, and then also $u(x,y,t) \equiv \tilde{u}(x,y,t)$ for $0 \le t < \tau$.

We can now proceed step-by-step to prove that $\tilde{g} = g$, $u = \tilde{u}$, for all $0 \le t \le T$.

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