

ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

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Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4^e série, tome 19,
n° 1 (1992), p. 69-86

http://www.numdam.org/item?id=ASNSP_1992_4_19_1_69_0

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Holomorphic Symplectic Normalization of a Real Function

S.M. WEBSTER*

Introduction

The invariant theory of a real-valued function r defined on an open set in \mathbb{C}^n or other complex manifold is generally simpler than that of an equation $r = 0$, or real hypersurface. If f is a local biholomorphic mapping defined near a point z , the transformation $r \rightarrow r \circ f$ clearly preserves the value $r(z)$, the complex differential $\partial r(z)$, the complex Hessian (or Levi form) $\partial\bar{\partial}r(z)$, and its null space and signature. If the Levi form is non-degenerate, it defines a Kähler metric, and higher order invariants may be systematically derived from the curvature tensor via covariant differentiation. This is entirely analogous but much simpler than the invariant theory of a non-degenerate real hypersurface (see Chern-Moser [3]). Degenerate, in particular weakly pseudoconvex, real hypersurfaces play an important role in function theory and have been much studied. However, there is as yet nothing like a systematic invariant theory for them. As a step toward such a theory we consider the invariants of a function r near a point where the Levi form degenerates, but under the larger pseudogroup of local holomorphic symplectic transformations. Enlarging the group naturally tends to reduce the number of invariants, and of the above mentioned ones we shall see that only the null space of the Levi form remains.

To describe the transformation procedure, we let $T^*(\mathbb{C}^n)$ denote the holomorphic $(1,0)$ -cotangent bundle and $\pi : T^*(\mathbb{C}^n) \rightarrow \mathbb{C}^n$ the natural projection. On $T^*(\mathbb{C}^n)$ we have the canonical one- and two-forms

$$(0.1) \quad \theta = \sum_{\alpha=1}^n p_{\alpha} dz^{\alpha}, \quad \omega = d\theta = \sum_{\alpha=1}^n dp_{\alpha} \wedge dz^{\alpha},$$

where the p_{α} are holomorphic fiber coordinates relative to the holomorphic coordinate system z^{α} on \mathbb{C}^n . If we identify $T^*(\mathbb{C}^n)$ with the real cotangent bundle of $\mathbb{C}^n \cong \mathbb{R}^{2n}$ in the usual way, then $\text{Re}\theta$ and $\text{Re}\omega$ correspond to the

* Partially supported by N.S.F. Grant DMS-8913354.

Pervenuto alla Redazione il 14 Maggio 1991.

canonical forms of real symplectic geometry. As is well known, the graph of dr , or in our notation the image of ∂r in $T^*(\mathbb{C}^n)$, which we write as

$$(0.2) \quad M : R_\alpha \equiv p_\alpha - \partial_\alpha r = 0, \quad \partial_\alpha = \frac{\partial}{\partial z^\alpha}, \quad 1 \leq \alpha \leq n,$$

is a maximal submanifold on which $\operatorname{Re} \omega$ vanishes. In the present context such a manifold will be called *real Lagrangian*. The local theory of such submanifolds is the main objective of this work.

If $\Phi : T^*(\mathbb{C}^n) \rightarrow T^*(\mathbb{C}^n)$ is a (local) holomorphic symplectic (or canonical) map, i.e., $\Phi^* \omega = \omega$, then it is also real symplectic, so $M^* = \Phi(M)$ is also real Lagrangian. We assume that M^* is also transverse to the fibers of the projection π . (This may fail in important cases, but they will not arise here). M^* is then the graph of a $(1,0)$ -form whose real part is d-closed by the real Lagrangian condition, and so locally equal to dr^* . The real function r^* , determined up to a constant, is the symplectic transform of r .

One of the first important works in holomorphic symplectic geometry is that of Lempert [4], in which a global version of the above process is used to construct interesting solutions to the homogeneous complex Monge-Ampere equation. This is based on the following. Let s denote the section of $T^*(\mathbb{C}^n)$ defined by ∂r . Then $\partial r = s^* \theta$, and $\bar{\partial} \partial r = d \partial r = s^* \omega = s^* \Phi^*(\omega|_{M^*}) = (\pi \circ \Phi \circ s)^*(\bar{\partial} \partial r^*)$. Thus, the null vectors of $\bar{\partial} \partial r$ correspond to those of $\bar{\partial} \partial r^*$ under $\pi \circ \Phi \circ s$, as we claimed above.

If the Levi form is non-degenerate, then there are no further invariants. In fact, all real analytic non-degenerate functions are locally equivalent under holomorphic symplectic transformation. This is a consequence of the real analytic Darboux theorem. Our first main result is the following formal power series generalization.

THEOREM 0.1. *Suppose that r is a real valued, real analytic function defined near 0 in \mathbb{C}^n , and that its Levi form has rank m , $0 \leq m \leq n$. Then near 0, r may be transformed into the formal power series form*

$$(0.3) \quad r = \sum_{j=1}^m |z^j|^2 + \sum_{\mu, \nu=m+1}^n H_{\mu\bar{\nu}} z^\mu \bar{z}^\nu,$$

by a formal symplectic map Φ as above. Here the functions $\bar{H}_{\mu\bar{\nu}} = H_{\nu, \bar{\mu}}$ are formal power series in all the variables (z, \bar{z}) without constant terms.

The original function r may of course be only a formal power series, in particular the Taylor series of a smooth function at 0. The partial normalization (0.3) prepares the way for the investigation of higher order invariants, which requires a study of the isotropy group of the form (0.3) and how it acts on the coefficients $H_{\mu\bar{\nu}}$. In case $m = n$, there are no terms $H_{\mu\bar{\nu}}$, and we have a convergent transformation Φ . If $m = n - 1$, there is precisely one third order invariant, which can take the values 0 or 1. It is 1 in the generic case, which

means that $(\partial\bar{\partial}r)^n$ vanishes along a real hypersurface which is transverse to the one-dimensional Levi null space.

Our second main result, also formal, concerns such generic degeneracies when $n = 1$.

THEOREM (0.2). *Let r be a real-valued, real analytic function of one complex variable z defined near $z = 0$. Suppose that $r_{z\bar{z}}(0) = 0$, $r_{z\bar{z}z}(0) \neq 0$. Then near 0, r may be formally transformed into the cubic function*

$$(0.4) \quad r = z^2\bar{z} + z\bar{z}^2,$$

by a formal power series symplectic map Φ .

Thus, Theorem 0.2 completely settles the problem of symplectic invariants in the case considered. Both theorems indicate the lack of invariants; that is, any smooth function can be so normalized to arbitrarily high order.

The proof of Theorem 0.1 proceeds rather directly via a formal power series construction of a generating function for the transformation Φ . This is done in Section 2 after the preliminary considerations of Section 1. The proof of Theorem 0.2, which may be viewed as a symplectic analogue for the normal form in Moser-Webster [5], is much more indirect. It begins in Section 3 where, as in [5], we pass to the complexification M^c of the surface M . On M^c there is induced a pair of holomorphic involutions τ_1, τ_2 in addition to the holomorphic two-form ω . The study of M is reduced to the study of this triple $\{\tau_1, \tau_2, \omega\}$, together with the anti-holomorphic involution of complex conjugation. In Section 4, we show that the pair τ_1, τ_2 , which has a parabolic character, can be formally linearized by a change of coordinates on M^c . In Section 5 we apply a further coordinate change on M^c to normalize the invariant two-form ω .

1. - Complex tangents and Levi form degeneracy

Let M be defined as in (0.2) with the real valued function r . Its real tangent planes and their complex “envelopes” are given by

$$T(M) : dR_\alpha = 0, \quad E_x(M) = T_x(M) + iT_x(M).$$

The holomorphic tangent “bundle” is given by

$$H(M) : \partial R_\alpha = 0, \quad \partial\bar{R}_\alpha \equiv - \sum \partial_\beta \partial_{\bar{\alpha}} r dz^\beta = 0.$$

Thus, a vector $v = (dz, dp)$ is in $H(M)$ if and only if its projection $\pi(v) = (dz)$ is in the Levi null space of r . In particular, M is totally real if and only if the Levi form of r is non-degenerate.

$H_*(M)$ may also be characterized in symplectic terms as the ω -isotropic subspace of $T_x(M)$,

$$\perp_\omega(T_x) = \{v \in T_x : \omega(v, u) = 0, \forall u \in T_x\}.$$

In fact, since $\operatorname{Re} \omega = 0$ on T_x , and H_x is i -invariant, it follows that $H_x \subset \perp_\omega(T_x)$. But $\perp_\omega(T_x) \subset \perp_{\operatorname{Re} \omega}(T_x) = T_x$, so $\perp_\omega(T_x) \subset H_x$. Hence, $H_x = \perp_\omega(T_x) = \perp_\omega(E_x)$. It follows that T_x is totally real if and only if $\omega|_{T_x} \equiv \operatorname{Im} \omega|_{T_x}$ is a real (non-degenerate) symplectic form on T_x . This leads to the following.

PROPOSITION 1.1. *All real analytic, totally real, real Lagrangian submanifolds of $T^*(\mathbb{C}^n)$ are locally equivalent under holomorphic symplectic transformation.*

For the proof, suppose we are given two such, M and M^* . Then the theorem of Darboux gives a locally real analytic map $\varphi : M \rightarrow M^*$ with $\varphi^*(\omega|_{M^*}) = \omega|_M$. The complexification $\Phi : T^*(\mathbb{C}^n) \rightarrow T^*(\mathbb{C}^n)$ of φ is locally biholomorphic. Both the real and the imaginary parts of the holomorphic $(2, 0)$ -form $\Phi^*\omega - \omega$ vanish when restricted to M . It readily follows that $\Phi^*\omega = \omega$ on \mathbb{C}^n .

We note that locally each manifold of Proposition 1.1 is the fixed point set of a locally defined anti-holomorphic involution ρ for which $\rho^*\omega = -\bar{\omega}$. The proposition is analogous to the result in real symplectic geometry [1] which states that, locally, intrinsic equivalence implies extrinsic equivalence for all submanifolds of a real symplectic space. When M is not totally real, as in Section 3 below, additional conditions will generally be required.

We add a remark on Poisson brackets. For any smooth function f , the real and complex Hamiltonians H_f, X_f are defined respectively by ($\iota =$ interior product)

$$df = \iota_H \operatorname{Re} \omega, \quad \partial f = \iota_{X_f} \omega.$$

In case f is real-valued, one has $2X_f = H_f^{(1,0)}$. The real and complex Poisson brackets are given by

$$(f, g)_r = H_f g, \quad (f, g)_c = X_f g.$$

If both f and g are real, $(f, g)_r = 4 \operatorname{Re}(f, g)_c$. Suppose $f^1 = 0, \dots, f^{2n} = 0$ are independent equations defining $M^{2n} \subset T^*(\mathbb{C}^n)$. Then M is real Lagrangian if and only if $(f^i, f^j)_r = 0$ on M . However, the brackets $(f^i, f^j)_c$ need not vanish. Let the f^i be the real and imaginary parts of R_α in (0.2). Then

$$(R_\alpha, R_\beta)_c = 0, \quad (R_\alpha, \bar{R}_\beta)_c = -\partial_\alpha \bar{\partial}_\beta r.$$

Thus, the complex Poisson brackets will vanish on M if and only if r is pluri-harmonic. M is then a holomorphic Lagrangian submanifold.

Every holomorphic point mapping of \mathbb{C}^n induces a holomorphic symplectic mapping of $T^*(\mathbb{C}^n)$, which also preserves the 1-form θ . Any holomorphic

function h on \mathbb{C}^n generates a symplectic mapping by

$$z^{*\alpha} = z^\alpha, \quad p_\alpha^* = p_\alpha + \partial_\alpha h,$$

which takes the surface $p_\alpha = \partial_\alpha r$ into $p_\alpha = \partial_\alpha(r - h - \bar{h})$. It allows us to remove the purely holomorphic and anti-holomorphic terms from the Taylor expansion of r to any order. Therefore, we assume that r has the form

$$(1.1) \quad r = b + H, \quad b = \sum_{\alpha, \beta=1}^n b_{\alpha\bar{\beta}} z^\alpha \bar{z}^\beta, \quad H = O(3) \equiv O(|z|^3),$$

where b is an Hermitian form.

The sign of any non-zero eigenvalue of b may be changed as follows. Suppose $b = \sum \lambda_\alpha |z_\alpha|^2$ is diagonalized and $\lambda_1 \neq 0$. The involution

$$z_1 = p_1^*, \quad p_1 = -z_1^*; \quad z_j = z_j^*, \quad p_j = p_j^*, \quad 2 \leq j \leq n,$$

gives in (0.2)

$$R_1 = p_1 - \lambda_1 \bar{z}_1 - H_1 = -z_1^* - \lambda_1 \bar{p}_1^* + \dots$$

Solving for p^* , we get the new eigenvalues $\lambda_1^* = -\lambda_1^{-1}$, $\lambda_j^* = \lambda_j$. Thus we may assume

$$(1.2) \quad b = \sum \lambda_\alpha |z_\alpha|^2; \quad \lambda_j = 1, \quad j \leq m; \quad \lambda_\mu = 0, \quad \mu > m.$$

With these normalizations and index ranges we have $(z, p) = 0 \in M$ and

$$(1.3) \quad \begin{cases} E_0(M) : dp_\mu = 0, \\ T_0(M) : dp_\mu = 0, \quad dp_j = d\bar{z}_j, \\ H_0(M) : dp_\mu = 0, \quad dp_j = 0, \quad dz_j = 0. \end{cases}$$

The linearized symplectic isotropy group of M , thus normalized at 0, is still rather large and acts on the higher order terms in r in a complicated way. Therefore, in the next section we consider maps Φ with $\Phi(0) = 0$, $d\Phi(0) = I$.

2. - The generating function

The mapping $(z^*, p^*) = \Phi(z, p)$ is symplectic if and only if

$$\begin{aligned} 0 &= \omega - \Phi^* \omega = dp_\alpha \wedge dz^\alpha - dp_\alpha^* \wedge dz^{*\alpha} \\ &= d\{p \cdot dz + z^* \cdot dp^*\}, \quad p \cdot dz = \sum p_\alpha dz^\alpha, \text{ etc.} \end{aligned}$$

This will hold if the graph of Φ is an integral submanifold of the Pfaffian equation

$$p \cdot dz + z^* \cdot dp^* = d\tilde{S},$$

for some generating function \tilde{S} on the product space. Choosing

$$\tilde{S} = z \cdot p^* + S(z, p^*), \quad S = O(3),$$

we get

$$(2.1) \quad \Phi : \begin{cases} z^* = z + S_{p^*}(z, p^*) \\ p^* = p - S_z(z, p^*). \end{cases}$$

This describes Φ implicitly; the second equation is solved for p^* and substituted into the first. It follows that $\Phi(0) = 0$, $d\Phi(0) = I$, and that therefore Φ preserves the form (0.2), (1.1), (1.2) of M . If we substitute (2.1) into the corresponding equation (0.2) for $M^* = \Phi(M)$, we get (dropping indices)

$$\begin{aligned} R^* \circ \Phi &= p^* - b\bar{z}^* - H_{z^*}^*(z^*) \\ &= p - S_z - b(\bar{z} + \bar{S}_{p^*}) - H_{z^*}^*(z^*). \end{aligned}$$

When restricted to M ; i.e., when

$$(2.2) \quad p = b\bar{z} + H_z(z),$$

$R^* \circ \Phi$ vanishes, and we get the basic functional relation

$$(2.3) \quad H_{z^*}^*(z^*) = H_z(z) - S_z(z, p^*) - b\bar{S}_{p^*}(z, p^*).$$

Here (z^*, p^*) are eliminated via (2.1), p by (2.2) and an analytic, or formal power series, equation in the variables (z, \bar{z}) results.

With a standard multi-index notation ($A = (\alpha_1, \dots, \alpha_k)$, $z^A = z^{\alpha_1} \dots z^{\alpha_k}$, $H_{A\bar{B}}$ symmetric in the indices A and in the indices B), we write

$$(2.4) \quad \begin{aligned} H &= \sum_{s=3}^{\infty} H^s, \\ H^s(z, \bar{z}) &= \sum_{|A|+|B|=s} H_{A\bar{B}} z^A \bar{z}^B, \quad \bar{H}_{A\bar{B}} = H_{B\bar{A}}. \end{aligned}$$

We assume that for a fixed s in \mathbb{Z}^+ , the terms H^t , $3 \leq t < s$, have already been normalized. Then we make a transformation (2.1) with

$$(2.5) \quad S \equiv S^s(z, p^*) = \sum_{|A|+|B|=s} S_A^B z^A p_B^*,$$

a homogeneous polynomial of degree s . This gives $\Phi = I + O(s - 1)$, and we must compare terms up to degree $s - 1$ in (2.3). Since

$$\partial_z H^{*t}(z^*, \bar{z}^*) = \partial_z H^t(z, \bar{z}) + O(s), \quad 3 \leq t < s,$$

and there are no constant terms, and H, H^* satisfy the reality condition in (2.4), we get

$$(2.6) \quad H^{*t}(z, \bar{z}) = H^t(z, \bar{z}), \quad 3 \leq t < s.$$

Also,

$$S_z(z, p^*) = S_z(z, p) + O(s) = S_z(z, b\bar{z}) + O(s),$$

and

$$b\overline{S_{p^*}}(z, p^*) = b\overline{S_p}(z, b\bar{z}) + O(s) = \partial_z[\overline{S}(z, b\bar{z})] + O(s).$$

Hence,

$$\partial_z H^{*s}(z, \bar{z}) = \partial_z[H^s(z, \bar{z}) - 2 \operatorname{Re} S(z, b\bar{z})],$$

and again by the reality of the series,

$$(2.7) \quad H^{*s}(z, \bar{z}) = H^s(z, \bar{z}) - 2 \operatorname{Re} S^s(z, b\bar{z}).$$

The substitution $p^* = b\bar{z}$ gives

$$(2.8) \quad \begin{aligned} S^s(z, b\bar{z}) &= \sum S_{AB} z^A \bar{z}^B, \quad S_{AB} = S_A^C b_{C\bar{B}}, \\ b_{C\bar{B}} &= b_{\gamma_1 \bar{\beta}_1} \cdots b_{\gamma_k \bar{\beta}_k}; \end{aligned}$$

so (2.7) is equivalent to

$$(2.9) \quad H_{AB}^* = H_{AB} - S_{AB} - \overline{S_{BA}}.$$

We assume that b is diagonalized as in (1.2). In the non-degenerate case ($m = n$) the choice $S_A^B \equiv S_{AB} = \frac{1}{2} H_{AB}$ makes $H_{AB}^* = 0$. Otherwise, we take S independent of p_μ^* , $m < \mu \leq n$, and set

$$(2.10) \quad \begin{aligned} S_{I\bar{J}} &= \frac{1}{2} H_{I\bar{J}}, \quad \text{if } I \cap (m, n] = J \cap (m, n] = \emptyset, \\ S_{A\bar{J}} &= H_{A\bar{J}}, \quad \text{if } A \cap (m, n] \neq \emptyset, J \cap (m, n] = \emptyset. \end{aligned}$$

This makes $H_{AB}^* = 0$, unless both multi-indices contain at least one element greater than m , in which case there is no change: $H_{AB}^* = H_{AB}$. This is the normalization required in (0.3).

We make a sequence $\Phi_s = I + O(s - 1)$, $s = 3, 4, 5, \dots$, of such transfor-

mations with polynomial generating functions and consider the composition

$$(2.11) \quad \Psi_s = \Phi_s \circ \Psi_{s-1} = \Psi_{s-1} + O(s-1), \quad s \geq 3.$$

It is clear that $\Psi_\infty = \lim \Psi_s$ determines a formal power series symplectic map which takes the given series $r(z, \bar{z})$ into the form (0.3), and Theorem 0.1 is proved.

Because of the way the unknown function S enters the functional equation (2.3), (via (2.1) and (2.2)), it is not clear how to set up a majorant problem to prove convergence. There is a good deal of similarity between the present problem and one treated in [7]. Moreover, the solution to the linearized problem, given by (2.10), involves no small divisors as in the case in [7]. However, if one attempts a KAM argument as in [7], serious difficulties arise in trying to control the size and shape of the surface M under iteration. At this point we cannot say whether one should expect convergence or divergence in Theorem 0.1.

We consider the third order terms in (0.3) in the case $m = n - 1$,

$$r = \sum_{i,j=1}^{n-1} b_{i\bar{j}} z^i \bar{z}^j + z^n \bar{z}^n \operatorname{Re} \left(\sum_{j=1}^n a_j z^j \right) + \dots$$

Since $\det(b_{i\bar{j}}) = +1$, an easy computation shows that $\partial_{z^n}(\det \partial_{\alpha\bar{\beta}} r)_{z=0} = 2a_n$. The Levi-form degeneracy at 0 is called *generic* if $a_n \neq 0$. This means that the z^n -axis, the Levi null space at 0, is transverse to the smooth real hypersurface of degenerate points. The intersection determines an invariant real line. By a linear change we may make $a_j = 0$, $j < n$, $a_n = 1$. This may disturb the normalization (0.3), but we then repeat the above process for $s = 3$ without altering these new conditions. Thus, we may arrange

$$(2.12) \quad r = \sum_{j=1}^{n-1} |z^j|^2 + (z^n + \bar{z}^n) z^n \bar{z}^n + O(4),$$

which is the starting point for the further normalizations in the following sections.

3. - Generic Levi-form degeneracies of a function of one complex variable

We now consider submanifolds of $T^*(\mathbb{C}) \cong \mathbb{C}^2$, with coordinates (z, p) , relative to the two form $\omega = dp \wedge dz$. All holomorphic curves are Lagrangian; however, the real Lagrangian surfaces are still rather special. The set of all $\operatorname{Re} \omega$ -isotropic two planes contains as a codimension-one submanifold the set of complex lines, and is itself a codimension-one sub-manifold of the Grassmannian of all real two-planes in \mathbb{C}^2 . Generic surfaces as studied in [2] and [5] have

isolated complex tangents, while those considered here have complex tangents along a real curve.

A real function $r(z, \bar{z})$ has a generic Levi-form degeneracy at 0 if $r_{z\bar{z}}(0) = 0$, $r_{z\bar{z}\bar{z}}(0) \neq 0$. By (2.12) we may assume

$$(3.1) \quad r = z^2\bar{z} + z\bar{z}^2 + H(z, \bar{z}), \quad H = \bar{H} = O(4).$$

The corresponding surface is

$$(3.2) \quad M : \begin{cases} p = r_z = 2z\bar{z} + \bar{z}^2 + H_z(z, \bar{z}), \\ \bar{p} = r_{\bar{z}} = 2z\bar{z} + z^2 + H_{\bar{z}}(z, \bar{z}), \end{cases}$$

and

$$(3.3) \quad \omega|_M = -r_{z\bar{z}}dz \wedge d\bar{z},$$

which degenerates along the real curve $r_{z\bar{z}} = 0$. One could, in theory, proceed to the further normalization of (3.1) via a generating function S as in Section 2; however, this leads to some very complicated linear algebra.

Thus, we proceed somewhat as in [5], which requires the complexification of M . For this we replace (z, p, \bar{z}, \bar{w}) by $X \in \mathbb{C}^4$,

$$(3.4) \quad \begin{aligned} X &= (z, p, \bar{w}, \bar{q}), & \pi_1(X) &= (z, p), & \pi_2(X) &= (\bar{w}, \bar{q}), \\ \rho(X) &= (w, q, \bar{z}, \bar{p}), \end{aligned}$$

where (\bar{w}, \bar{q}) are given complex variables and (w, q) their complex conjugates. The original \mathbb{C}^2 sits in \mathbb{C}^4 as the diagonal $\Delta = \{w = z, q = p\}$, which is the fixed-point set of the reflection ρ . The reality condition on r , from now on assumed real analytic, gives

$$(3.5) \quad \begin{aligned} \overline{r(z, \bar{w})} &= r(w, \bar{z}), & r_{\bar{w}}(z, \bar{w}) &= \overline{r_w(w, \bar{z})}, \\ \overline{s(z, \bar{w})} &= s(w, \bar{z}), & s(z, \bar{w}) &\equiv r_{z\bar{w}}(z, \bar{w}), \end{aligned}$$

or $r \circ \rho = \bar{r}$, $s \circ \rho = \bar{s}$. The complexification of M is

$$(3.6) \quad M^c : \begin{cases} p = r_z(z, \bar{w}) = 2z\bar{w} + \bar{w}^2 + H_z(z, \bar{w}), \\ \bar{q} = r_{\bar{w}}(z, \bar{w}) = 2z\bar{w} + z^2 + H_{\bar{w}}(z, \bar{w}), \end{cases}$$

which is a 2-dimensional complex surface in \mathbb{C}^4 . By (3.5) it is invariant under ρ and contains $M \cong M^c \cap \Delta$. The 2-form (3.3) extends holomorphically to M^c ,

$$(3.7) \quad \omega = -s(z, \bar{w})dz \wedge d\bar{w} = dp \wedge dz = -d\bar{q} \wedge d\bar{w},$$

by the real Lagrangian condition; and

$$(3.8) \quad \rho^* \omega = -\bar{\omega}.$$

The two projections in (3.4) restricted to M^c are two-fold branched coverings. To see this we use (z, \bar{w}) as holomorphic coordinates on M^c and suppose $\pi_1(z', \bar{w}') = \pi_1(z, \bar{w})$. Then $z' = z$ and if $\bar{w}' \neq \bar{w}$, $p' = p$ gives

$$(3.9) \quad \tau_1 : \begin{cases} z' = z \\ \bar{w}' = -2z - \bar{w} - \frac{1}{\bar{w}' - \bar{w}} \{H_z(z, \bar{w}') - H_z(z, \bar{w})\}, \end{cases}$$

which defines (implicitly) the non-trivial covering transformation of π_1 . Similarly, the covering transformation for π_2 is

$$(3.10) \quad \tau_2 : \begin{cases} z' = -z - 2\bar{w} - \frac{1}{z' - z} \{H_{\bar{w}}(z', \bar{w}) - H(z, \bar{w})\} \\ \bar{w}' = \bar{w}. \end{cases}$$

We have $\pi_i \circ \tau_i = \pi_i$ and by (3.7)

$$(3.11) \quad \tau_i^* \omega = \omega, \quad i = 1, 2.$$

The reality condition gives $\pi_1 \circ \rho = c \circ \pi_2$, where $c(x, y) = (\bar{x}, \bar{y})$, and

$$(3.12) \quad \rho \circ \tau_1 = \tau_2 \circ \rho.$$

Letting (z', \bar{w}') tend to (z, \bar{w}) in (3.9), (3.10) we see that τ_1, τ_2 have the common, nonsingular curve of fixed points

$$(3.13) \quad C : 0 = s(z, \bar{w}) = 2z + 2\bar{w} + H_{z\bar{w}}(z, \bar{w}),$$

which is of course the common branch locus of π_1 and π_2 , and curve of degeneracy of ω .

Suppose $F : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ is symplectic and transforms M into $M' = F(M)$, both real analytic and of the form (3.2). Let $\{\tau'_i, \rho', \omega'\}$ denote the corresponding data on M'^c . Let $f : M^c \rightarrow M'^c$ be the restriction of the complexified map $(z, p, \bar{w}, \bar{q}) \rightarrow (F(z, p), \bar{F}(\bar{w}, \bar{q}))$, or equivalently, the extension of $F : M \rightarrow M'$. Then,

$$(3.14) \quad \tau'_i \circ f = f \circ \tau_i, \quad \rho' \circ f = f \circ \rho, \quad f^* \omega' = \omega.$$

We want to show, conversely, that the symplectic theory of M in \mathbb{C}^2 reduces to a study of the quadruple $\{\tau_1, \tau_2, \rho, \omega\}$ on $M^c \cong \mathbb{C}^2$.

LEMMA 3.1. *Let M and M' be as in (3.1) and (3.2). Suppose $f : M^c \rightarrow M'^c$, $f(0) = 0$, is biholomorphic and satisfies (3.14). Then there exists a (local) holomorphic symplectic map $F : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ with $F(M) = M'$ inducing f .*

To prove this we define F as follows. Fix (z, p) near $0 \in \mathbb{C}^2$ and find $X \in M^c$ with $\pi_1(X) = (z, p)$. Then define $F(z, p) = \pi'_1 \circ f(X)$. F is well-defined by (3.14) and clearly bounded and holomorphic off the branch locus $\pi_1(C)$, a thin set (eliminate \bar{w} between (3.13) and the first equation in (3.6)). By the Riemann extension theorem f is holomorphic in a neighbourhood of 0, and by definition $\pi'_1 \circ f = F \circ \pi_1$. To see that F is symplectic, note that $\pi_1^* F^* = (F \circ \pi_1)^* = (\pi'_1 \circ f)^* = f^* \circ \pi_1'^*$, and

$$f^* \circ \pi_1'^*(dp' \wedge dz') = f^* \omega' = \omega = \pi_1^*(dp \wedge dz).$$

Hence, off the branch locus, and by continuity everywhere, $F^*(dp' \wedge dz') = dp \wedge dz$. In particular, F is (locally) biholomorphic. By the second equation in (3.14) $f(M^c \cap \Delta) = M'^c \cap \Delta'$, and

$$F(M) = F \circ \pi_1(M^c \cap \Delta) = \pi'_1 \circ f(M^c \cap \Delta) = M'.$$

Clearly F induces f , and the proof is complete.

It is convenient to make a change of coordinates

$$(3.15) \quad \begin{aligned} x_1 &= z + \bar{w}, \\ x_2 &= z - \bar{w}. \end{aligned}$$

We readily find

$$(3.16) \quad \rho(x_1, x_2) = (\bar{x}_1, -\bar{x}_2),$$

and

$$(3.17) \quad \begin{cases} \tau_i(x) = T_i x + H_i(x), & H_i = O(2), \quad i = 1, 2, \\ T_1 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \quad T_2 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \quad H_i = \begin{bmatrix} H_{i1} \\ H_{i2} \end{bmatrix}, \\ T_i^2 = I, \quad T_i H_i + H_i \circ \tau_i = 0; \end{cases}$$

the latter being equivalent to $\tau_i^2 = I$. From (3.7) and (3.15), $\omega = (x_1 + s_1(x)) dx_1 \wedge dx_2$, $s_1 \circ \rho = \bar{s}_1 = O(2)$. The change $f(x) = (x_1 + s_1(x), x_2)$ transforms C into the x_2 -axis, and $\rho \circ f = f \circ \rho$. So f preserves the forms (3.16) and (3.17). From $\tau_i(0, x_2) = (0, x_2)$, we get $H_i(0, x_2) = 0$, or

$$(3.18) \quad H_i(x) = x_1 \tilde{H}_i(x), \quad \tilde{H}_i = O(1), \quad i = 1, 2.$$

Also,

$$(3.19) \quad \omega = E(x)dx_1 \wedge dx_2, \quad E(x) = x_1 \tilde{E}(x), \quad \tilde{E}(0) = 1.$$

4. - Parabolic pairs of involutions

The pair of involutions τ_i given by (3.17), (3.18) with common fixed points along the x_2 -axis, has a certain parabolic character. Their theory is somewhat different from those in [5], which are either hyperbolic or elliptic in nature. The linearization of a single holomorphic involution by a coordinate change is a simple matter. The simultaneous treatment of a pair τ_1, τ_2 is more involved and requires a consideration of the commutator $\tau_2 \tau_1 \tau_2^{-1} \tau_1^{-1} = (\tau_2 \tau_1)^2$. As in [5] this leads us to the map $\sigma = \tau_2 \tau_1$,

$$(4.1) \quad \begin{aligned} \sigma(x) &= Sx + G(x), \quad G = T_2 H_1 + H_2 \circ \tau_1 \equiv x_1 \tilde{G}, \\ S &= T_2 T_1 = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}, \quad \tilde{G} = \begin{bmatrix} \tilde{G}_1 \\ \tilde{G}_2 \end{bmatrix} = O(1). \end{aligned}$$

We consider holomorphic coordinate changes f of the form

$$(4.2) \quad f(x) = x + F(x), \quad F = \begin{bmatrix} x_1 \tilde{F}_1 \\ \tilde{F}_2 \end{bmatrix} = O(2),$$

which preserve the fixed point set. We have

$$(4.3) \quad \tau_i \circ f = f \circ \tau_i^*, \quad \sigma \circ f = f \circ \sigma^*,$$

where τ_i^*, σ^* , the new coordinate forms, have the same linear parts as in (3.17), (4.1), and non-linear terms H_i^*, S^* . More explicitly (4.3) is written

$$(4.4) \quad F(Sx + G^*(x)) - SF(x) = G(x + F(x)) - G^*(x),$$

$$(4.5) \quad F(T_i x + H_i^*(x)) - T_i F(x) = H_i(x + F(x)) - H_i^*(x).$$

THEOREM 4.1. *There exists a formal power series transformation f (4.2) which takes the pair of involutions τ_i in (3.17), (3.18) into the linear pair $\tau_i^* = T_i$. If in addition $\rho \circ \tau_1 = \tau_2 \circ \rho$, where ρ is given by (3.16), then $\rho \circ f = f \circ \rho$.*

For the proof we decompose the power series H_i, G into homogeneous

terms,

$$(4.6) \quad \begin{aligned} H_i &= \sum_{s=1}^{\infty} H_i^{s+1}, & H_i^{s+1} &= x_1 \tilde{H}_i^s, \\ G &= \sum_{s=1}^{\infty} G^{s+1}, & G^{s+1} &= x_1 \tilde{G}^s, \end{aligned}$$

where the superscript s denotes the degree. We assume inductively that $H_i^{t+1} = G^{t+1} = 0$, $t < s$, and try to find $F = F^{s+1}$,

$$(4.7) \quad \begin{aligned} \tilde{F}_1^s &= \sum_{j=0}^s a_{1j} x_1^{s-j} x_2^j, \\ F_2^{s+1} &= \sum_{j=0}^{s+1} a_{2j} x_1^{s+1-j} x_2^j, \end{aligned}$$

for which $H_i^{s+1} = G^{s+1} = 0$. Comparing terms of degree $s+1$ in (4.4) gives

$$F^{s+1} \circ S - S F^{s+1} = G^{s+1} - G^{*s+1},$$

or

$$(4.8) \quad \begin{aligned} \tilde{F}_1^s \circ S - \tilde{F}_1^s &= \tilde{G}_1^s - \tilde{G}_1^{*s}, \\ F_2^{s+1} \circ S - F_2^{s+1} &= x_1(4\tilde{F}_1^s + \tilde{G}_2^s - \tilde{G}_2^{*s}). \end{aligned}$$

Thus, the basic equation is

$$(4.9) \quad \begin{aligned} K(a) &\equiv a \circ S - a = b, \\ a &= \sum_{k=0}^s a_k x_1^{s-k} x_2^k, & b &= \sum_{k=0}^s b_k x_1^{s-k} x_2^k, \end{aligned}$$

which is readily reduced to

$$(4.10) \quad \begin{aligned} 4^j b_j &= \sum_{k=j+1}^s 4^k a_k B_j^k, & 0 \leq j &\leq s-1, \\ b_s &= 0, \end{aligned}$$

where $B_j^k = \binom{k}{j}$ is the binomial coefficient. It is clear that the null space of K is all $\{a_0 x_1^s\}$, and the set $\{b_s x_2^s\}$ is a complement of the range of K . It follows that we can find an \tilde{F}_1^s so that $\tilde{G}_1^{*s} = b_s^* x_2^s$. We can then choose F_2^{s+1} to make

$\tilde{G}_2^{*s} = 0$. We assume that these normalizations have already been performed on G^{s+1} . We must then restrict (4.7) to the form

$$(4.11) \quad \begin{aligned} \tilde{F}_1^s &= a_{10}x_1^s, \\ F_2^{s+1} &= a_{20}x_1^{s+1} + a_{21}x_1^s x_2, \quad a_{21} = a_{10}. \end{aligned}$$

We claim that we now actually have $G^{s+1} = 0$. From (3.17), (4.1) we get

$$\begin{aligned} T_2 H_1^{s+1} + H_2^{s+1} \circ T_1 &= G^{s+1}, \\ T_i H_i^{s+1} + H_i^{s+1} \circ T_i &= 0, \end{aligned}$$

or

$$(4.12) \quad \begin{aligned} T_2 \tilde{H}_1^s - \tilde{H}_2^s \circ T_1 &= \tilde{G}^s, \\ T_i \tilde{H}_i^s - \tilde{H}_i^s \circ T_i &= 0. \end{aligned}$$

The first component of the second equation gives $-2\tilde{H}_{i1}^s(0, x_2) = 0$. So the first component of the right-hand side of the first equation has no pure x_2 term, proving our claim:

$$(4.13) \quad T_2 \tilde{H}_1^s = \tilde{H}_2^s \circ T_1.$$

From (4.12), (4.13), and (3.17) we get $\tilde{H}_2^s \circ S = \tilde{H}_2^s \circ T_2 \circ T_1 = \tilde{H}_1^s$. Also, we have $\tilde{H}_2^s = T_2 \tilde{H}_2^s \circ T_1 = S \tilde{H}_1^s$. From this follows

$$(4.14) \quad \tilde{H}_i^s \circ S = S^{-1} \tilde{H}_i^s, \quad i = 1, 2.$$

Comparing components in (4.14) and using the properties of K (4.9), we get ($i = 1, 2$)

$$\begin{aligned} \tilde{H}_{i1}^s &= h_{i10}x_1^s, \\ \tilde{H}_{i2}^s &= h_{i20}x_1^s + h_{i21}x_1^{s-1}x_2, \quad h_{i21} = -h_{i10}. \end{aligned}$$

Now (4.12) gives

$$\begin{aligned} h_{i10} &= (-1)^{s-1} h_{i10}, \quad h_{i21} = (-1)^{s-1} h_{i21}, \\ \pm 2h_{i10} + h_{i20} &= (-1)^s (h_{i20} \mp 2h_{i21}); \end{aligned}$$

where the upper sign is taken for $i = 1$, the lower for $i = 2$. Now (4.13) gives

$$h_{110} = (-1)^{s-1} h_{210}, \quad h_{121} = (-1)^{s-1} h_{221}, \quad 2h_{110} + h_{120} = (-1)^s (h_{220} - 2h_{221}).$$

Thus, for s even we have $h_{110} = h_{120} = 0$, while for s odd we have $h_{120} = -2h_{110}$.

We now make a further transformation f with (4.11) holding. This preserves all the above normalizations, and via (4.5) gives

$$F^{s+1} \circ T_i - T_i F^{s+1} = H_i^{s+1} - H_i^{*s+1}.$$

Comparing coefficients for $i = 1$ gives

$$\begin{aligned} [(-1)^{s+1} - 1]a_{20} + 2[(-1)^s - 1]a_{10} &= h_{120}h_{120}^*, \\ [(-1)^s - 1]a_{10} &= h_{121} - h_{121}^*, \\ [(-1)^s - 1]a_{10} &= -h_{110} + h_{120}^*. \end{aligned}$$

For s even we set $a_{10} = 0$ and choose a unique a_{20} to make $h_{120}^* = 0$. For s odd, we set $a_{20} = 0$ and choose a_{10} uniquely to make $h_{120}^* = 0$. By the preceding paragraph we now have $\tilde{H}_1^s = \tilde{H}_2^s = 0$ in both cases. This completes the inductive step for the first statement of the theorem.

The normalizations just made on F^{s+1} (4.7) are equivalent to

$$(4.15) \quad \tilde{F}_1^s(x_1, 0) = 0, \quad s \text{ even}; \quad F_2^{s+1}(x_1, 0) = 0, \quad s \text{ odd}.$$

With them F^{s+1} is uniquely determined. We define $\hat{f} = \rho \circ f \circ \rho$ (see (3.16)). Then

$$\hat{f}(x) = x + \rho \circ F \circ \rho(x) = (x_1 + x_1 \overline{\tilde{F}_1^s(\bar{x}_1, -\bar{x}_2)}, -\overline{F_2^{s+1}(\bar{x}_1, -\bar{x}_2)}).$$

Setting $x_2 = 0$, we see that \hat{f} is normalized if f is. Now we suppose that (3.12) holds, and that the τ_i have been linearized to order $s+2$ by a normalized $f = I + F^{s+1}$. Then

$$\tau_1 \circ \hat{f} = \rho \circ \tau_2 \circ f \circ \rho \equiv \rho \circ f \circ T_2 \circ \rho = \hat{f} \circ T_1,$$

where \equiv denotes equality mod $O(s+2)$. Similarly,

$$\tau_2 \circ \hat{f} \equiv \hat{f} \circ T_2.$$

By the uniqueness it follows that $f = \hat{f}$, or $\rho \circ f = f \circ \rho$.

As in the proof of Theorem (0.1) we construct the formal map

$$f_\infty = \lim_{s \rightarrow \infty} f^s \circ \cdots \circ f^1, \quad f^s = I + O(s+1),$$

each f^s normalized. Under the reality condition we have $f_\infty \circ \rho = \rho \circ f_\infty$ as formal power series. This completes the proof of Theorem (4.1).

5. - Invariant two-forms

We now take into consideration the τ_i -invariant two-form ω which was ignored in Section 4. Under the transformation (4.2), (4.3) we have $\omega = f^*\omega^*$, where ω^* is τ_i^* -invariant. If $\rho \circ f = f \circ \rho$ and (3.8) holds for ω , then it also holds for ω^* . In view of Theorem 4.1 we assume that the $\tau_i = T_i$ are linear. We must determine the most general invariant ω . From (3.19) and (3.17)

$$(5.1) \quad \tilde{E} \circ T_i = \tilde{E}, \quad \tilde{E} \circ S = \tilde{E}.$$

As in the last section we readily see that $\tilde{E}(x) = \tilde{E}(x_1)$, $\tilde{E}(-x_1) = \tilde{E}(x_1)$; so we set

$$(5.2) \quad \tilde{E} = 1 + v(t), \quad t = x_1^2, \quad v(0) = 0.$$

If the condition (3.8) holds, it follows that v is a power series with real coefficients.

To simplify the coefficient \tilde{E} , we apply an automorphism g of the pair T_i which fixes the origin,

$$(5.3) \quad T_i g = g \circ T_i, \quad S g = g \circ S.$$

It follows easily from Section 4 that g has the form

$$(5.4) \quad \begin{aligned} g(x) &= (x_1 B(x_1), A(x_1) + x_2 B(x_1)), \\ A(-x_1) &= A(x_1), \quad B(-x_1) = B(x_1), \quad B(0) \neq 0, \quad A(0) = 0. \end{aligned}$$

For such g we have

$$g^*\omega = x_1 B^2 \tilde{E}(x_1 B)(B + x_1 B'(x_1)) dx_1 \wedge dx_2.$$

Thus we take $A = 0$, and

$$B = 1 + b(t), \quad t = x_1^2,$$

and try to make the new \tilde{E} equal to 1. This gives the initial value problem for b

$$\begin{aligned} 1 &= (1 + b(t))^2 (1 + v(tB^2))(1 + b(t) + 2tb'(t)), \\ 0 &= b(0). \end{aligned}$$

Simplifying, we get

$$b + 2tb' = K(1 - K)^{-1}, \quad K \equiv -2b - b^2 - v(1 + b)^2$$

or, for a suitable φ, a_0 ,

$$(5.5) \quad \begin{aligned} 3b + 2tb' &= a_0t + \varphi(t, b), \quad b(0) = 0, \\ \varphi(t, b) &= \sum_{i+j \geq 2} a_{ij}t^i b^j. \end{aligned}$$

If the reality condition holds, then a_0 and the a_{ij} are real. In case ω is holomorphic (after linearizing the $\tau_i!$), then φ is holomorphic.

Substitution of the series

$$(5.6) \quad b = \sum_{j=1}^{\infty} b_j t^j$$

into (5.5) and comparing coefficients gives

$$\begin{aligned} 5b_1 &= a_0, \\ (3 + 2j)b_j &= P_j(a_0, a_{ik}; b_k), \quad j \geq 2, \end{aligned}$$

where the P_j are certain polynomials with non-negative coefficients, which involve b_k only for $k < j$. Thus (5.5) has a unique formal solution (5.6), which has real coefficients under the reality assumption. In the convergent case $a_0t + \varphi$ is majorized by the series for

$$A \left(t + \frac{t^2 + b^2}{1 - t - b} \right),$$

if the constant $A > 0$ is sufficiently large. $b(t)$ is then majorized by the solution $\hat{b}(t)$ to

$$5\hat{b} = A \left(t + \frac{t^2 + \hat{b}^2}{1 - t - \hat{b}} \right), \quad \hat{b}(0) = 0,$$

which exists and is convergent by the holomorphic implicit function theorem. This proves the following.

PROPOSITION 5.1. *Let ω in (3.19) be invariant under the linear involutions T_i in (3.17). Then there exists an automorphism g (5.4) of the pair T_i taking ω into the form*

$$(5.7) \quad \omega = x_1 dx_1 \wedge dx_2.$$

Furthermore, $g \circ \rho = \rho \circ g$ if (3.8) holds, and g is holomorphic if ω is.

We may now conclude the proof of Theorem 0.2. Given any power series (3.1) we may truncate it to get a polynomial surface $M \subset \mathbb{C}^2$ and the data $\{\tau_i, \rho, \omega\}$ on M^c . By combining the arguments of Theorem 4.1 and Proposi-

tion 5.1, we may find a map f of M^c which takes the data to that induced by the cubic (0.4), to a given order. As in Section 3 this induces a symplectic map F of M onto a surface, osculating that corresponding to (0.4), to high order. Taking a composition of such maps as in the proof of Theorem 0.1 gives the map required in Theorem 0.2.

A convergence proof of Theorem 4.1 would, in view of Proposition 5.1, immediately give a convergence result for Theorem 0.2. In view of results of Siegel [6] on divergence in the normal form for a symplectic vector field, positive results are not assured. The main difficulty at this point is in obtaining estimates for the solution of the linearized problem (4.9).

REFERENCES

- [1] V.I. ARNOL'D - A.B. GIVENTAL, *Symplectic geometry*, in "Dynamical Systems IV, EMS." vol. 4, Springer-Verlag, Berlin, 1990.
- [2] E. BISHOP, *Differentiable manifolds in complex Euclidean space*, Duke Math. J. **32** (1965), 1-22.
- [3] S.S. CHERN - J.K. MOSER, *Real hypersurfaces in complex manifolds*, Acta Math. **133** (1974), 219-271.
- [4] L. LEMPERT, *Symmetries and other transformations of the complex Monge-Ampere equation*, Duke Math. J. **52** (1985), 869-885.
- [5] J.K. MOSER - S.M. WEBSTER, *Normal forms for real surfaces in \mathbb{C}^2 near complex tangents and hyperbolic surface transformations*, Acta Math. **150** (1983), 255-296.
- [6] C.L. SIEGEL - J.K. MOSER, "*Lectures on Celestial Mechanics*", Springer-Verlag, New York, 1971.
- [7] S.M. WEBSTER, *A normal form for a singular first order partial differential equation*, Amer. J. Math. **109** (1987), 807-833.

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