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Hyperbolic Systems of Partial Differential Inclusions

JEAN-PIERRE AUBIN - HÉLÈNE FRANKOWSKA

0. - Introduction

Let X, Y, Z denote finite dimensional vector-spaces, $f : X \times Y \mapsto X$ be a single-valued map, $G : X \times Y \rightsquigarrow Y$ be a set-valued map and $A \in \mathcal{L}(Y, Y)$ a linear operator. We set throughout this paper $\lambda = \min_{\|x\|=1} \langle Ax, x \rangle$.

We recall that the contingent cone $T_K(x)$ to a subset $K \subset X$ at $x \in K$ is defined by

$$T_K(x) := \left\{ v \in X \,|\, \liminf_{h \to 0+} \frac{d(x + hv, K)}{h} = 0 \right\}.$$

and that the *contingent derivative* $DR(x, y)$ of a set-valued map $R : X \rightsquigarrow Y$ at $(x, y) \in \text{Graph}(R)$ is defined by

$$\text{Graph}(DR(x, y)) := T_{\text{Graph}(R)}(x, y).$$

When $R = r$ is single-valued, we set $Dr(x) := Dr(x, r(x))$. Naturally, $Dr(x)(u) = r'(x)u$ whenever r is differentiable at x.

Usually, a Lipschitz map r is not differentiable, but *contingently differentiable* in the sense that its contingent derivative has nonempty values. In this case, it associates to every direction $u \in X$ the subset

$$Dr(x)(u) := \left\{ v \in Y \,|\, \liminf_{h \to 0+} \left\| v - \frac{r(x + hu) - r(x)}{h} \right\| = 0 \right\}.$$

See [8, Chapter 5] for more details on differential calculus of set-valued maps.

In this paper, we shall look for single-valued and set-valued *contingent solutions* to hyperbolic systems of partial differential inclusions, i.e., single-valued maps $r : X \mapsto Y$ *with closed graph* satisfying

$$\forall x \in X, \;\; Ar(x) \in Dr(x)(f(x, r(x))) - G(x, r(x))$$

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and set-valued maps $R : X \rightsquigarrow Y$ with *closed graph* satisfying

$$\forall x \in X,\ \forall y \in R(x),\ Ay \in DR(x,y)(f(x,y)) - G(x,y).$$

We observe that when r is differentiable, the contingent differential inclusion boils down to a *quasi-linear hyperbolic system of first-order partial differential equations*[1]

$$\forall j = 1,\ldots,m,\ \sum_{k=1}^{m} a_j^k r_k(x) = \sum_{i=1}^{n} \frac{\partial r_j}{\partial x_i} f_i(x, r(x)) - g_j(x, r(x)).$$

Motivations: Tracking Property — Consider the *system of differential inclusions*

$$(1)\qquad \begin{cases} x'(t) = f(x(t), y(t)) \\ y'(t) \in Ay(t) + G(x(t), y(t)) \end{cases}$$

The solutions to the inclusion

$$\forall x \in X,\ Ar(x) \in Dr(x)(f(x, r(x))) - G(x, r(x))$$

are the maps $r : X \mapsto Y$, regarded as *observation maps*, satisfying what is called the *tracking property*: for every $x_0 \in X$, there exists a solution $(x(\cdot), y(\cdot))$ to this system of differential inclusions (1) starting at $(x_0, y_0 = r(x_0))$ and satisfying

$$\forall t \geq 0,\ y(t) = r(x(t)).$$

One can also look for set-valued contingent solutions $R : X \rightsquigarrow Y$ to the inclusion

$$(2)\qquad \forall (x,y) \in \mathrm{Graph}(R),\ Ay \in DR(x,y)(F(x,y)) - G(x,y)$$

characterizing the *tracking property*: for every $x_0 \in \mathrm{Dom}(R)$ and every $y_0 \in R(x_0)$, there exists a solution $(x(\cdot), y(\cdot))$ to the system of differential inclusions

$$\begin{cases} x'(t) \in F(x(t), y(t)) \\ y'(t) \in Ay(t) + G(x(t), y(t)) \end{cases}$$

starting at (x_0, y_0) and satisfying

$$\forall t \geq 0,\ y(t) \in R(x(t)).$$

[1] For several special types of systems of differential equations, the graph of such a map r (satisfying some additional properties) is called a *center manifold*.

Motivations: Inclusions governing feedback controls — The partial differential inclusions governing the feedback controls $r : K \mapsto Y$ regulating solutions of a control system (U, f):

$$(3) \quad \begin{cases} \text{i)} & x'(t) = f(x(t), u(t)) \text{ for almost all } t \geq 0 \\ \text{ii)} & u(t) \in U(x(t)) \end{cases}$$

belong to the class studied in this paper, as it was mentioned in [9,11,12]. Here, $U : X \rightsquigarrow Y$ is a closed set-valued map, $f : \mathrm{Graph}(U) \mapsto X$ a continuous (single-valued) map with linear growth and $K = \mathrm{Dom}(U)$. Let $\varphi : \mathrm{Graph}(U) \mapsto \mathbb{R}_+$ be a nonnegative continuous function with linear growth (in the sense that $\varphi(x, u) \leq c(\|x\| + \|u\| + 1)$).

We look for feedback controls r satisfying the following property: *for any $x_0 \in K$, there exists a solution to the differential equation*

$$x'(t) = f(x(t), r(x(t))) \;\&\; x(0) = x_0$$

such that $u(t) := r(x(t)) \in U(x(t))$ is absolutely continuous and fulfils the growth condition

$$\|u'(t) - Au(t)\| \leq \varphi(x(t), u(t))$$

for almost all t. Such feedback controls r are *solutions to the following contingent differential inclusion*

$$\forall x \in K, \;\; Ar(x) \in Dr(x)(f(x, r(x))) - \varphi(x, r(x))B$$

satisfying the constraints

$$\forall x \in K, \;\; r(x) \in U(x).$$

Outline — We extend in the first section Hadamard's formula of solutions to linear hyperbolic differential equations to the set-valued case. Namely, we shall prove the existence of a set-valued contingent solutions $R_\star$ to the *decomposable system*

$$\forall (x, y) \in \mathrm{Graph}(R_\star), \;\; Ay \in DR_\star(x, y)(\Phi(x)) - \Psi(x)$$

where $\Phi : K \rightsquigarrow X$ and $\Psi : K \rightsquigarrow Y$ are two Marchaud maps[2], $K \subset X$ is closed and $A \in \mathcal{L}(Y, Y)$.

If we denote by $S_\Phi(x, \cdot)$ the set of solutions $x(\cdot)$ to the differential inclusion $x'(t) \in \Phi(x(t))$ starting at x, then the set-valued map $R_\star : X \rightsquigarrow Y$ defined by

$$\forall x \in X, \;\; R_\star(x) := -\int_0^\infty e^{-At}\Psi(S_\Phi(x, t))dt$$

2 A Marchaud map $\Phi : K \rightsquigarrow Y$ is an upper semicontinuous set-valued map with nonempty compact convex images and with linear growth.

is the *largest contingent solution with linear growth* to this partial differential inclusion when $\lambda := \min_{\|x\|=1}\langle Ax, x\rangle > 0$ is large enough. We also show that it is Lipschitz whenever Φ and Ψ are Lipschitz and compare the solutions associated with maps Φ_i and Ψ_i $(i = 1, 2)$.

We then turn our attention in the second section to partial differential inclusions of the form

$$\forall x \in X,\ \ Ar(x) \in Dh(x)(f(x, h(x))) - G(x, h(x))$$

when $\lambda > 0$ is large enough, $f : X \times Y \mapsto X$ is Lipschitz, $G : X \leadsto Y$ is Lipschitz with nonempty convex compact values and satisfies[3]

$$\forall x, y,\ \ \|G(x, y)\| \leq c(1 + \|y\|).$$

When G is single-valued, we obtain a global *Center Manifold Theorem*, stating the existence and uniqueness of an invariant manifold for systems of differential equations with Lipschitz right-hand sides (existence and uniqueness of a contingent solution r has been proved by viscosity methods in [6,7] when $A = \lambda\mathbf{1}$).

We end this paper with comparison theorems between single-valued and set-valued solutions to such partial differential inclusions, using both the extension of Hadamard's formula and some kind of maximum principle.

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Notations — If $r : X \mapsto Y$, we set

$$\|r\|_\infty := \sup_{x\in X}\|r(x)\| \in [0, \infty]\ \ \&\ \ \|r\|_\Lambda := \sup_{x\neq y}\frac{\|r(x) - r(y)\|}{\|x - y\|} \in [0, \infty]$$

and we denote by $\mathcal{C}_\Lambda(X, Y)$ the set of all Lipschitz maps from X to Y.

When $G : X \leadsto Y$ is Lipschitz with nonempty closed images, we denote by $\|G\|_\Lambda$ its Lipschitz constant, the smallest of the constants l satisfying

$$\forall z_1, z_2 \in X,\ \ G(z_1) \subset G(z_2) + l\|z_1 - z_2\|B$$

where B is the closed unit ball in Y.

When $L \subset X$ and $M \subset X$ are two closed subsets of a metric space, we denote by

$$\Delta(L, M) := \sup_{y\in L}\inf_{z\in M} d(y, z) = \sup_{y\in L} d(y, M)$$

3 We set $\|K\| := \sup_{x\in K}\|x\|$ when $K \subset X$. It is equal to $-\infty$ whenever K is empty.

their *semi-Hausdorff distance*[4], and recall that $\Delta(L, M) = 0$ if and only if $L \subset M$. If Φ and Ψ are two set-valued maps from X to Y, we set

$$\Delta(\Phi, \Psi)_\infty = \sup_{x \in X} \Delta(\Phi(x), \Psi(x)) := \sup_{x \in X} \sup_{y \in \Phi(x)} d(y, \Psi(x)).$$

We recall that solutions are always understood as set-valued or single-valued maps *with closed graph.*

1. - Contingent Solutions to Decomposable Systems

We need first to establish some properties of contingent set-valued solutions to decomposable systems.

Let $K \subset X$ be a closed subset, $\Phi : K \rightsquigarrow X$ and $\Psi : K \rightsquigarrow Y$ be two Marchaud maps and $A \in \mathcal{L}(Y, Y)$. We say that K is a *viability domain* of Φ if

$$\forall x \in K, \;\; \Phi(x) \cap T_K(x) \neq \emptyset.$$

We set

$$\lambda := \inf_{\|x\|=1} \langle Ax, x \rangle$$

and we observe that

$$\forall y \in Y, \;\; \left\| e^{-At} y \right\| \leq e^{-\lambda t} \|y\|.$$

We look for a solution $R_\star : K \rightsquigarrow Y$ to the *decomposable system*

$$\text{(4)} \qquad \forall (x, y) \in \operatorname{Graph}(R_\star), \;\; Ay \in DR_\star(x, y)(\Phi(x)) - \Psi(x).$$

Denote by $S_\Phi(x, \cdot)$ the set of solutions $x(\cdot)$ to the differential inclusion $x'(t) \in \Phi(x(t))$ starting at x viable in K (in the sense that $x(t) \in K$ for all $t \geq 0$), which exists thanks to the Viability Theorem (see [2,3]).

[4] The *Hausdorff distance* between L and M is $\max\{\Delta(L, M), \Delta(M, L)\}$, which may be equal to ∞.

We introduce the set-valued map $R_\star : K \rightsquigarrow Y$ defined[5] by

$$\forall x \in K, \;\; R_\star(x) := -\int_0^\infty e^{-At}\Psi(S_\Phi(x,t))dt. \tag{5}$$

THEOREM 1.1. *Assume that* $\Phi : K \rightsquigarrow X$ *and* $\Psi : K \rightsquigarrow Y$ *are Marchaud maps and that* K *is a closed viability domain of* Φ. *If* λ *is large enough, then* $R_\star : K \rightsquigarrow Y$ *defined by* (5) *is the* largest contingent solution to inclusion (4) with linear growth *and is bounded whenever* Ψ *is bounded.*

More precisely, if there exist positive constants α, β *and* γ *such that*

$$\forall x \in K, \;\; \|\Phi(x)\| \leq \alpha(\|x\| + 1) \;\; \& \;\; \|\Psi(x)\| \leq \beta + \gamma\|x\|$$

and if $\lambda > \alpha$, *then*

$$\forall x \in K, \;\; \|R_\star(x)\| \leq \frac{\beta}{\lambda} + \frac{\gamma}{\lambda - \alpha}(\|x\| + 1). \tag{6}$$

Furthermore, if $K := X$ *and* Φ, Ψ *are Lipschitz, then* $R_\star : X \rightsquigarrow Y$ *is also Lipschitz (with nonempty values) whenever* λ *is large enough:*

$$\text{If } \lambda > \|\Phi\|_\Lambda, \;\; R_\star(x_1) \subset R_\star(x_2) + \frac{\|\Psi\|_\Lambda}{\lambda - \|\Phi\|_\Lambda}\|x_1 - x_2\|B$$

for every $x_1, x_2 \in X$.

Formula (5) shows also that the graph of $R_\star$ is convex (respectively a convex cone) whenever the graphs of the set- valued maps Φ and Ψ are convex (respectively are convex cones).

PROOF.

1. — We prove first that the graph of $R_\star$ satisfies contingent inclusion (4).

Indeed, choose an element y in $R_\star(x)$. By definition of the integral of a set-valued map, this means that there exist a solution $x(\cdot) \in S_\Phi(x, \cdot)$ to the

5 By definition of the integral of a set-valued map (see [8, Chapter 8] for instance), this means that for every $y \in R_\star(x)$, there exist a solution $x(\cdot) \in S_\Phi(x, \cdot)$ to the differential inclusion $x'(t) \in \Phi(x(t))$ starting at x and $z(t) \in \Psi(x(t))$ such that

$$y := -\int_0^\infty e^{-At}z(t)dt.$$

differential inclusion $x'(t) \in \Phi(x(t))$ starting at x which is viable in K and $z(t) \in \Psi(x(t))$ such that

$$y := -\int_0^\infty e^{-At} z(t)dt \in R_\star(x).$$

We check that for every $\tau > 0$

$$-\int_0^\infty e^{-At} z(t+\tau)dt \in R_\star(x(\tau)) = R_\star\left(x + \tau\left(\frac{1}{\tau}\int_0^\tau x'(t)dt\right)\right).$$

By observing that

$$\begin{cases} \dfrac{1}{\tau}\displaystyle\int_0^\infty e^{-At}(z(t) - z(t+\tau))dt \\ = -\dfrac{e^{A\tau}-1}{\tau}\displaystyle\int_0^\infty e^{-At}z(t)dt + \dfrac{e^{A\tau}}{\tau}\displaystyle\int_0^\tau e^{-At}z(t)dt \end{cases}$$

we deduce that

$$\begin{cases} y + \tau\left(-\dfrac{e^{A\tau}-1}{\tau}\displaystyle\int_0^\infty e^{-At}z(t)dt + \dfrac{e^{A\tau}}{\tau}\displaystyle\int_0^\tau e^{-At}z(t)dt\right) \\ \qquad \in R_\star\left(x + \tau\left(\dfrac{1}{\tau}\displaystyle\int_0^\tau x'(t)dt\right)\right). \end{cases}$$

Since Φ is upper semicontinuous, we know that for any $\varepsilon > 0$ and t small enough, $\Phi(x(t)) \subset \Phi(x) + \varepsilon B$, so that $x'(t) \in \Phi(x) + \varepsilon B$ for almost all small t. Therefore, $\Phi(x)$ being closed and convex, we infer that for $\tau > 0$ small enough, $\frac{1}{\tau}\int_0^\tau x'(t)dt \in \Phi(x) + \varepsilon B$ thanks to the Mean-Value Theorem. This latter set being compact, there exists a sequence of $\tau_n > 0$ converging to 0 such that $\frac{1}{\tau_n}\int_0^{\tau_n} x'(t)dt$ converges to some $u \in \Phi(x)$.

In the same way, Ψ being upper semicontinuous, $\Psi(x(t)) \subset \Psi(x) + \varepsilon B$ for any $\varepsilon > 0$ and t small enough, so that $z(t) \in \Psi(x) + \varepsilon B$ for almost all small t. The Mean-Value Theorem implies that

$$\forall n > 0, \ z_n := \frac{1}{\tau_n}\int_0^{\tau_n} z(t)dt \in \Psi(x) + \varepsilon B$$

since this set is compact and convex. Furthermore, there exists a subsequence of z_n converging to some $z_0 \in \Psi(x)$. Hence, since

$$\frac{1}{\tau_n}\int_0^{\tau_n}\left(e^{-At}-1\right)z(t)dt \to 0$$

we infer that

$$Ay + z_0 \in DR_\star(x,y)(u)$$

so that $Ay \in DR_\star(x,y)(\Phi(x)) - \Psi(x)$.

2. — Let us prove now that the graph of $R_\star$ is closed when λ is large enough. Consider for that purpose a sequence of elements (x_n, y_n) of the graph of $R_\star$ converging to (x,y). There exist solutions $x_n(\cdot) \in S_\Phi(x_n,\cdot)$ to the differential inclusion $x' \in \Phi(x)$ starting at x_n, viable in K and measurable selections $z_n(t) \in \Psi(x_n(t))$ such that

$$y_n := -\int_0^{\infty} e^{-At}z_n(t)dt \in R_\star(x_n).$$

The growth of Φ being linear, there exists $\alpha > 0$ such that the solutions $x_n(\cdot)$ obey the estimate

$$\|x_n(t)\| \le (\|x_n\|+1)e^{\alpha t} - 1 \qquad \& \qquad \|x_n'(t)\| \le \alpha(\|x_n\|+1)e^{\alpha t}.$$

By [8, Theorem 10.1.9], we know that there exists a subsequence (again denoted by) $x_n(\cdot)$ converging uniformly on compact intervals to a solution $x(\cdot) \in S_\Phi(x,\cdot)$.

The growth of Ψ being also linear, we deduce that, setting $u_n(t) := e^{-At}z_n(t)$,

$$\|z_n(t)\| \le \beta + \gamma(\|x_n\|+1)e^{\alpha t}$$
$$\|u_n(t)\| \le \beta e^{-\lambda t} + \gamma(\|x_n\|+1)e^{-(\lambda-\alpha)t}.$$

When $\lambda > \alpha$, Dunford-Pettis' Theorem implies that a subsequence (again denoted by) $u_n(\cdot)$ converges weakly to some $u(\cdot) \in L^1(0,\infty;Y)$. This implies that $z_n(\cdot)$ converges weakly to some $z(\cdot)$ in the space $L^1(0,\infty;Y;e^{-\lambda t}dt)$. The Convergence Theorem [8, Therem 7.2.2] states that $z(t) \in \Psi(x(t))$ for almost every t. Since the integrals y_n converge to $-\int_0^{\infty} e^{-At}z(t)dt$, we have proved that

$$y = -\int_0^{\infty} e^{-At}z(t)dt \in R_\star(x).$$

3. — Estimate (6) is obvious since any solution $x(\cdot) \in S_{\Phi}(x, \cdot)$ satisfies

$$\forall t \geq 0, \ \ \|x(t)\| \leq (\|x\| + 1)e^{\alpha t}$$

so that, if $\lambda > \alpha$,

$$\|R_{\star}(x)\| \leq \int_{0}^{\infty} e^{-\lambda t} \left(\beta + \gamma(\|x\| + 1)e^{\alpha t}\right) dt = \frac{\beta}{\lambda} + \frac{\gamma}{\lambda - \alpha}(\|x\| + 1).$$

Assume now that $M : K \rightsquigarrow Y$ is any set-valued contingent solution to inclusion (4) with linear growth: there exists $\delta > 0$ such that for all $x \in K$, $\|M(x)\| \leq \delta(\|x\| + 1)$. Since Graph($M$) enjoys the viability property for the set-valued map $(x, y) \rightsquigarrow (\Phi(x), Ay + \Psi(x))$, we know that for any $(x, y) \in \text{Graph}(M)$, there exists a solution $(x(\cdot), y(\cdot))$ to the system of differential inclusions

$$(7) \qquad \begin{cases} \text{i)} & x'(t) \in \Phi(x(t)) \\ \text{ii)} & y'(t) - Ay(t) \in \Psi(x(t)) \end{cases}$$

starting at (x, y) such that $y(t) \in M(x(t))$ for all $t \geq 0$. We also know that $\|x(t)\| \leq (\|x\| + 1)e^{\alpha t}$ so that $\|y(t)\| \leq \delta(1 + (\|x\| + 1)e^{\alpha t})$. The second differential inclusion of the above system implies that

$$t \mapsto z(t) := y'(t) - Ay(t)$$

is a measurable selection of $\Psi(x(\cdot))$ satisfying the growth condition

$$\|z(t)\| \leq \beta + \gamma(\|x\| + 1)e^{\alpha t}.$$

Therefore, if $\lambda > \alpha$, the function $e^{-At}z(t)$ is integrable. On the other hand, integrating by parts $e^{-At}z(t) := e^{-At}y'(t) - e^{-At}Ay(t)$, we obtain

$$e^{-AT}y(T) - y = \int_{0}^{T} e^{-At}z(t)dt$$

which implies that

$$y = -\int_{0}^{\infty} e^{-At}z(t)dt \in R_{\star}(x)$$

by letting $T \mapsto \infty$. Hence we have proved that[6] $M(x) \subset R_{\star}(x)$.

4. — Assume now that $K = X$ and that Φ and Ψ are Lipschitz, take any

[6] This proof actually implies that any set-valued contingent solution M with polynomial

pair of elements x_1 and x_2 and $y_1 = -\int_0^\infty e^{-At} z_1(t)dt \in R_\star(x_1)$, where

$$\text{for some } x_1(\cdot) \in S_\Phi(x_1, \cdot) \ \& \ z_1(t) \in \Psi(x_1(t)) \text{ a.e. in } [0, +\infty[.$$

By the Filippov Theorem[7] there exists a solution $x_2(\cdot) \in S_\Phi(x_2, \cdot)$ such that

$$\forall t \geq 0, \ \|x_1(t) - x_2(t)\| \leq e^{\|\Phi\|_\Lambda t} \|x_1 - x_2\|.$$

We denote by $z_2(t)$ the projection of $z_1(t)$ onto the closed convex set $\Psi(x_2(t))$, which is measurable thanks to [8, Corollary 8.2.13] and which satisfies

$$\forall t \geq 0, \ \|z_1(t) - z_2(t)\| \leq \|\Psi\|_\Lambda \|x_1(t) - x_2(t)\| \leq \|\Psi\|_\Lambda e^{\|\Phi\|_\Lambda t} \|x_1 - x_2\|.$$

growth in the sense that for some $\rho \geq 0$,

$$\forall x \in X, \ \|M(x)\| \leq \delta(\|x\|^\rho + 1)$$

is contained in $R_\star$ whenever $\lambda > \alpha\rho$, i.e., that there is no contingent solution with polynomial growth other than with linear growth (and bounded when $\gamma = 0$).

7 Adapted to the case of solutions defined on $[0, \infty[$. Filippov's Theorem (see [5, Theorem 2.4.1] for instance), yields an estimate on any finite interval $[0, T]$: If Φ is c-Lipschitz with nonempty closed values, and if an absolutely continuous function $y(\cdot)$ and an initial state x_0 are given, then there exists a solution $x(\cdot)$ to the differential inclusion (7)i) defined on $[0, T]$ starting at x_0 and satisfying the estimate

$$\|x(t) - y(t)\| \leq e^{ct}\left(\|x_0 - y(0)\| + \int_0^t d(y'(s), \Phi(y(s)))e^{-cs}ds\right). \tag{8}$$

We can extend it to the interval $[0, +\infty[$. Indeed, there exists a solution $x(\cdot)$ to the differential inclusion defined on $[0, T]$ starting at x_0 satisfying estimate (8) and in particular

$$\|x(T) - y(T)\| \leq e^{cT}\left(\|x_0 - y(0)\| + \int_0^T d(y'(s), \Phi(y(s)))e^{-cs}ds\right).$$

There also exists a solution $z(\cdot)$ to the differential inclusion (7)i) starting at $x(T)$ estimating the function $t \mapsto y(t + T)$ and satisfying

$$\|z(t) - y(t+T)\| \leq e^{ct}\left(\|z(0) - y(T)\| + \int_0^t d(y'(s+T), \Phi(y(s+T)))e^{-cs}ds\right).$$

Hence we can extend $x(\cdot)$ on the interval $[0, 2T]$ by concatenating it with the function $t \mapsto x(t) := z(t - T)$ on the interval $[T, 2T]$, we check that the above estimates yield (8) for $t \in [0, 2T]$ and we reiterate this process. See the forthcoming monograph [23].

Therefore, if $\lambda > \|\Phi\|_\Lambda$, $y_2 = -\int_0^\infty e^{-At} z_2(t)dt$ belongs to $R_\star(x_2)$ and satisfies

$$\|y_1 - y_2\| \leq \int_0^\infty \|\Psi\|_\Lambda e^{-t(\lambda - \|\Phi\|_\Lambda)} \|x_1 - x_2\| dt \leq \frac{\|\Psi\|_\Lambda}{\lambda - \|\Phi\|_\Lambda} \|x_1 - x_2\| \qquad \square$$

THEOREM 1.2. *Consider now two pairs* (Φ_1, Ψ_1) *and* (Φ_2, Ψ_2) *of Marchaud maps defined on* X *and their associated solutions*

$$\forall x \in X, \; R_{\star i}(x) := -\int_0^\infty e^{-At} \Psi_i(S_{\Phi_i}(x,t))dt \qquad (i = 1, 2)$$

to inclusion (4). *If the set-valued maps* Φ_2 *and* Ψ_2 *are Lipschitz, and if* $\lambda > \|\Phi_2\|_\Lambda$, *then*

$$\Delta(R_{\star_1}, R_{\star_2})_\infty \leq \frac{1}{\lambda} \Delta(\Psi_1, \Psi_2)_\infty + \frac{\|\Psi_2\|_\Lambda}{\lambda(\lambda - \|\Phi_2\|_\Lambda)} \Delta(\Phi_1, \Phi_2)_\infty.$$

PROOF. Choose $y_1 = -\int_0^\infty e^{-At} z_1(t)dt \in R_{\star 1}(x)$ where

$$x_1(\cdot) \in S_{\Phi_1}(x, \cdot) \;\&\; z_1(t) \in \Psi_1(x_1(t)).$$

In order to compare $x_1(\cdot)$ with the solution-set $S_{\Phi_2}(x, \cdot)$ via the Filippov Theorem, we use the estimate

$$d(x_1'(t), \Phi_2(x_1(t))) \leq \sup_{z \in \Phi_1(x_1(t))} d(z, \Phi_2(x_1(t))) \leq \Delta(\Phi_1, \Phi_2)_\infty.$$

Therefore, there exists a solution $x_2(\cdot) \in S_{\Phi_2}(x, \cdot)$ such that

$$\forall t \geq 0, \; \|x_1(t) - x_2(t)\| \leq \Delta(\Phi_1, \Phi_2)_\infty \frac{e^{t\|\Phi_2\|_\Lambda} - 1}{\|\Phi_2\|_\Lambda}$$

by Filippov's Theorem. As before, we denote by $z_2(t)$ the projection of $z_1(t)$ onto the closed convex set $\Psi_2(x_2(t))$, which is measurable and satisfies

$$\begin{cases} \forall t \geq 0, \; \|z_1(t) - z_2(t)\| \leq \Delta(\Psi_1, \Psi_2)_\infty + \|\Psi_2\|_\Lambda \|x_1(t) - x_2(t)\| \\ \leq \Delta(\Psi_1, \Psi_2)_\infty + \|\Psi_2\|_\Lambda \Delta(\Phi_1, \Phi_2)_\infty \left(e^{t\|\Phi_2\|_\Lambda} - 1\right) / \|\Phi_2\|_\Lambda. \end{cases}$$

Therefore, if $\lambda > \|\Phi_2\|_\Lambda$, $y_2 = -\int_0^\infty e^{-At} z_2(t)dt$ belongs to $R_{\star 2}(x)$ and satisfies

$$\begin{cases} \|y_1 - y_2\| \\ \leq \int_0^\infty e^{-\lambda t} \Delta(\Psi_1, \Psi_2)_\infty dt + \|\Psi_2\|_\Lambda \Delta(\Phi_1, \Phi_2)_\infty \int_0^\infty \frac{e^{t\|\Phi_2\|_\Lambda} - 1}{\|\Phi_2\|_\Lambda} e^{-\lambda t} dt \\ \leq \frac{\Delta(\Psi_1, \Psi_2)_\infty}{\lambda} + \frac{\|\Psi_2\|_\Lambda}{\lambda(\lambda - \|\Phi_2\|_\Lambda)} \Delta(\Phi_1, \Phi_2)_\infty. \quad \square \end{cases}$$

When $\Phi := \varphi$, $\Psi := \psi$ are single-valued, we obtain:

PROPOSITION 1.3. *Assume that $\varphi : X \mapsto X$ and $\psi : X \mapsto Y$ are Lipschitz and that ψ is bounded. Then when $\lambda > 0$, the map $r := \Gamma(\varphi, \psi)$ defined by*

$$r(x) = -\int_0^\infty e^{-At} \psi(S_\varphi(x, t)) dt$$

is the unique bounded single-valued solution to the contingent inclusion

$$Ar(x) \in Dr(x)(\varphi(x)) - \psi(x) \tag{9}$$

and satisfies

$$\|r\|_\infty \leq \frac{\|\psi\|_\infty}{\lambda} \ \& \ \text{ if } \lambda > \|\varphi\|_\Lambda, \ \|r\|_\Lambda \leq \frac{\|\psi\|_\Lambda}{\lambda - \|\varphi\|_\Lambda}. \tag{10}$$

Furthermore, for all Lipschitz single-valued maps $\varphi_i : X \mapsto X$, $\psi_i : X \mapsto Y$, $i = 1, 2$ such that ψ_1, ψ_2 are bounded and all $\lambda > \|\varphi_2\|_\Lambda$

$$\|\Gamma(\varphi_1, \psi_1) - \Gamma(\varphi_2, \psi_2)\|_\infty \leq \frac{\|\psi_1 - \psi_2\|_\infty}{\lambda} + \frac{\|\psi_2\|_\Lambda}{\lambda(\lambda - \|\varphi_2\|_\Lambda)} \|\varphi_1 - \varphi_2\|_\infty. \tag{11}$$

The proof can be derived from Theorems 1.1 and 1.2 or directly from the properties of linear systems of hyperbolic equations established in [7].

2. - Existence of a Lipschitz contingent solution

We shall now prove the existence of a contingent single-valued solution to inclusion

$$\forall x \in X, \ Ar(x) \in Dr(x)(f(x, r(x))) - G(x, r(x)). \tag{12}$$

THEOREM 2.1. *Assume that the map* $f : X \times Y \mapsto X$ *is Lipschitz, that* $G : X \leadsto Y$ *is Lipschitz with nonempty convex compact values and that*

$$\forall x, y, \ \ \|G(x, y)\| \leq c(1 + \|y\|)$$

for some $c > 0$.

Then if $\lambda > \max(c, 4\nu\|f\|_\Lambda\|G\|_\Lambda)$ *(where* ν *is the dimension of* X*), there exists a bounded Lipschitz contingent solution to the partial differential inclusion* (12).

PROOF. Since for every Lipschitz single-valued map $s(\cdot)$, the set-valued map $x \leadsto G(x, s(x))$ is Lipschitz (with constant $\|G\|_\Lambda(1 + \|s\|_\Lambda)$ and has convex compact values, [8, Theorem 9.4.1] implies that the subset G_s of Lipschitz selections ψ of the set-valued map $x \leadsto G(x, s(x))$ with Lipschitz constant not larger than $\nu\|G\|_\Lambda(1 + \|s\|_\Lambda)$ is not empty (where ν denotes the dimension of X). We denote by φ_s the Lipschitz map defined by $\varphi_s(x) := f(x, s(x))$, with Lipschitz constant equal to $\|f\|_\Lambda(1 + \|s\|_\Lambda)$.

The solutions r to inclusion (12) are the fixed points to the set-valued map $\mathcal{H} : \mathcal{C}_\Lambda(X, Y) \leadsto \mathcal{C}(X, Y)$ defined by

$$\mathcal{H}(s) := \{\Gamma(\varphi_s, \psi)\}_{\psi \in G_s}. \tag{13}$$

Indeed, if $r \in \mathcal{H}(r)$, there exists a selection $\psi \in G_r$ such that

$$Ar(x) \in Dr(x)(f(x, r(x))) - \psi(x) \subset Dr(x)(f(x, r(x)))G(x, r(x)).$$

Since $\|G(x, y)\| \leq c(1 + \|y\|)$, we deduce that any selection $\psi \in G_s$ satisfies

$$\|\psi\|_\infty \leq c(1 + \|s\|_\infty).$$

Therefore, Proposition 1.3 implies that if λ is large enough,

$$\forall r \in \mathcal{H}(s), \ \ \|r\|_\infty \leq \frac{c}{\lambda}(1 + \|s\|_\infty) \ \& \ \|r\|_\Lambda \leq \frac{\nu\|G\|_\Lambda(1 + \|s\|_\Lambda)}{\lambda - \|f\|_\Lambda(1 + \|s\|_\Lambda)}.$$

We first observe that when $\lambda > c$,

$$\forall s \in \mathcal{C}_\Lambda(X, Y) \text{ such that } \|s\|_\infty \leq \frac{c}{\lambda - c}, \ \forall r \in \mathcal{H}(s), \ \ \|r\|_\infty \leq \frac{c}{\lambda - c}.$$

When $\lambda > 4\nu\|f\|_\Lambda\|G\|_\Lambda$, we denote by

$$\rho(\lambda) := \frac{\lambda - \|f\|_\Lambda - \nu\|G\|_\Lambda\sqrt{\lambda^2 - 2\lambda(\|f\|_\Lambda + \nu\|G\|_\Lambda) + (\|f\|_\Lambda - \nu\|G\|_\Lambda)^2}}{2\|f\|_\Lambda}$$

the smallest root of the equation

$$\lambda\rho = \|f\|_\Lambda\rho^2 + (\|f\|_\Lambda + \nu\|G\|_\Lambda)\rho + \nu\|G\|_\Lambda$$

which is positive. We observe that

$$\lim_{\lambda\to+\infty} \lambda\rho(\lambda) = \nu\|G\|_\Lambda$$

and infer that

$$\forall s \in \mathcal{C}_\Lambda(X,Y) \text{ such that } \|s\|_\Lambda \le \rho(\lambda),\ \ \forall r \in \mathcal{H}(s),\ \ \|r\|_\Lambda \le \rho(\lambda)$$

because r being of the form $\Gamma(\varphi_s, \psi_s)$, satisfies by Proposition 1.3:

$$\|r\|_\Lambda \le \frac{\|\psi_s\|_\Lambda}{\lambda - \|\varphi_s\|_\Lambda} \le \frac{\nu\|G\|_\Lambda(1+\|s\|_\Lambda)}{\lambda - \|f\|_\Lambda(1+\|s\|_\Lambda)} \le \frac{\nu\|G\|_\Lambda(1+\rho(\lambda))}{\lambda - \|f\|_\Lambda(1+\rho(\lambda))} = \rho(\lambda).$$

Let us denote by $B^1_\infty(\lambda)$ the subset of $\mathcal{C}_\Lambda(X,Y)$ defined by

$$B^1_\infty(\lambda) := \left\{ r \in \mathcal{C}_\Lambda(X,Y) \mid \|r\|_\infty \le \frac{c}{\lambda - c}\ \&\ \|r\|_\Lambda \le \rho(\lambda) \right\}$$

which is compact (for the compact convergence topology) thanks to Ascoli's Theorem.

We have therefore proved that if $\lambda > \max(c, 4\nu\|f\|_\Lambda\|G\|_\Lambda)$, the set-valued map $\mathcal{H}$ sends the compact subset $B^1_\infty(\lambda)$ to itself.

It is obvious that the values of $\mathcal{H}$ are convex. Kakutani's Fixed-Point Theorem implies the existence of a fixed point $r \in \mathcal{H}(r)$ if we prove that the graph of $\mathcal{H}$ is closed.

Actually, the graph of the restriction of $\mathcal{H}$ to $B^1_\infty(\lambda)$ is compact. Indeed, let us consider any sequence $(s_n, r_n) \in \mathrm{Graph}(\mathcal{H})$ such that $s_n \in B^1_\infty(\lambda)$. Since $B^1_\infty(\lambda)$ is compact, a subsequence (again denoted by) (s_n, r_n) converges to some function

$$(s,r) \in B^1_\infty(\lambda) \times B^1_\infty(\lambda).$$

But there exist bounded Lipschitz selections $\psi_n \in G_{s_n}$ with Lipschitz constant $\nu\|G\|_\Lambda(1+\rho(\lambda))$ such that

$$\forall n \ge 0,\ \ r_n = \Gamma(\varphi_{s_n}, \psi_n).$$

Therefore a subsequence (again denoted by) ψ_n converges to some function $\psi \in G_s$. Since φ_{s_n} converges obviously to φ_s, we infer that r_n converges to $\Gamma(\varphi_s, \psi)$, i.e., that $r \in \mathcal{H}(s)$, since Γ is continuous by formula (11) of Proposition 1.3.

3. - Comparison Results

The point of this section is to compare two solutions to inclusion (12), or even, a single-valued solution and a contingent set-valued solution $M : X \rightsquigarrow Y$.

We first deduce from Theorem 1.2 the following "localization property":

THEOREM 3.1. *We posit the assumptions of Theorem* 2.1 *with* $A \in \mathcal{L}(Y, Y)$ *such that* $\lambda > \max(c, 4\nu\|f\|_\Lambda\|G\|_\Lambda)$ *(where* ν *is the dimension of* X*). Let* $\Phi : X \rightsquigarrow X$ *and* $\Psi : X \rightsquigarrow Y$ *be two Lipschitz and Marchaud maps with which we associate the set-valued map* $R_\star$ *defined by*

$$\forall x \in X, \;\; R_\star(x) := -\int_0^\infty e^{-At}\Psi(S_\Phi(x,t))dt.$$

Then any single-valued contingent solution $r(\cdot)$ *to inclusion* (12) *having linear growth satisfies the following estimate*

$$\begin{cases} \forall x \in X, \;\; d(r(x), R_\star(x)) \leq \\ \dfrac{1}{\lambda} \sup\limits_{x \in X} \Delta(G(x, r(x)), \Psi(x)) + \dfrac{\|\Psi\|_\Lambda}{\lambda(\lambda - \|\Phi\|_\Lambda)} \sup\limits_{x \in X} d(f(x, r(x)), \Phi(x)). \end{cases}$$

In particular, if we assume that

$$\forall y \in Y, \;\; f(x,y) \in \Phi(x) \;\; \& \;\; G(x,y) \subset \Psi(x)$$

then all single-valued contingent solutions $r(\cdot)$ *to inclusion* (12) *with linear growth are selections of* $R_\star$.

PROOF. Let r be any single-valued contingent solution to inclusion (12) with linear growth. One can show that r can be written in the form

$$r(x) = -\int_0^\infty e^{-At}z(t)dt \text{ where } z(t) \in G(x(t), r(x(t)))$$

by using the same arguments as in the third part of the proof of Theorem 1.1.

We also adapt the proof of Theorem 1.2 with $\Phi_1 := f(x, r(x))$, $z_1(t) := z(t)$, $\Phi_2 := \Phi$ and $\Psi_2 := \Psi$, to show that the estimates stated in the theorem hold true. $\square$

The next comparison results are consequences of the following kind of *maximum principle.*

We recall that when M is Lipschitz around x and $y \in M(x)$, its *adjacent derivative* $D^\flat M(x,y) \subset DM(x,y)$ is defined by

$$v \in D^\flat M(x,y)(u) \text{ if and only if } \lim_{h \to 0+} d\left(v, \frac{M(x+hu) - y}{h}\right) = 0.$$

A set-valued map M is said to be *derivable* at $(x,y) \in \text{Graph}(M)$ if the contingent and adjacent derivatives coincide at (x,y) and derivable if it is derivable at every point of its graph. See [8, Chapter 5] for more details.

LEMMA 3.2. (MAXIMUM PRINCIPLE) *We posit the assumptions of Theorem* 2.1 *with* $A \in \mathcal{L}(Y,Y)$ *such that* $\lambda > \max(c, 4\nu\|f\|_\Lambda\|G\|_\Lambda)$. *Let* M *be a Lipschitz set-valued map such that* $D^\flat M(x,y)(f(x,y))$ *is nonempty for every* $(x,y) \in \mathrm{Graph}(M)$. *Let* r *be any Lipschitz single-valued solution to* (12) *and set*

$$\Gamma(x) := G(x, r(x)) \cap (Dr(x)(f(x, r(x))) - Ar(x)).$$

If the supremum

$$\delta := \sup_{(x,y)\in\mathrm{Graph}(M)} \|r(x) - y\|$$

is finite, then

$$\delta \leq \frac{1}{\lambda} \sup_{(x,y)\in\mathrm{Graph}(M)} d\left(\Gamma(x), \overline{co}(D^\flat M(x,y)(f(x,r(x)))) - Ay\right).$$

The same conclusion holds true if we assume that the solution r *is derivable and when we replace the adjacent derivative of* M *by its contingent derivative.*

PROOF. It is sufficient to consider the case when the supremum

$$\delta := \sup_{(x,y)\in\mathrm{Graph}(M)} \|r(x) - y\| = \|r(\bar{x}) - \bar{y}\|$$

is achieved[8] at some $(\bar{x}, \bar{y})$ of the graph of M and when $\delta > 0$.

Let us take $\psi := v - Ar(\bar{x})$ in the set

$$G(\bar{x}, r(\bar{x})) \cap (Dr(\bar{x})(f(\bar{x}, r(\bar{x}))) - Ar(\bar{x})).$$

Set $u := f(\bar{x}, r(\bar{x}))$. Since r is Lipschitz, there exists a sequence $h_n > 0$ converging to 0 such that

$$\frac{r(\bar{x} + h_n u) - r(\bar{x})}{h_n} \text{ converges to } v.$$

Since M is Lipschitz, we deduce that for any $w \in D^\flat M(\bar{x}, \bar{y})(u)$, there exists a sequence w_n converging to w such that $\bar{y} + h_n w_n \in M(\bar{x} + h_n u)$. Thus

$$\|r(\bar{x}) - \bar{y}\| \geq \left\| r(\bar{x}) - \bar{y} + h_n \left(\frac{r(\bar{x} + h_n u) - r(\bar{x})}{h_n} - w_n \right) \right\|.$$

Therefore,

$$\forall w \in D^\flat M(\bar{x}, \bar{y})(u), \langle r(\bar{x}) - \bar{y}, v - w \rangle \leq 0$$

8 If the nonnegative bounded function $\chi(x,y) := \|r(x) - y\|$ does not achieve its maximum, we use a standard argument which can be found in [17,26] for instance. One can find approximate maxima (x_n, y_n) such that $\chi(x_n, y_n)$ converges to $\sup_{(x,y)\in\mathrm{Graph}(M)}\chi(x,y)$ and $\chi'(x_n, y_n)$ converges to 0.

and we infer that

$$\forall w \in \overline{co}(D^{\flat}M(\bar{x},\bar{y})(f(\bar{x},r(\bar{x})))),\ \langle r(\bar{x})-\bar{y}, A(r(\bar{x})-\bar{y})+A\bar{y}+\psi-w\rangle \leq 0$$

from which we obtain the estimate

$$\begin{cases} \lambda\|r(\bar{x})-\bar{y}\| \\ \leq \inf_{\psi\in\Gamma(\bar{x}),\, w\in\overline{co}(D^{\flat}M(\bar{x},\bar{y})(f(\bar{x},r(\bar{x}))))} \|A\bar{y}+\psi-w\| \\ = d\Big(\Gamma(\bar{x}), \overline{co}(D^{\flat}M(\bar{x},\bar{y})(f(\bar{x},r(\bar{x}))))-A\bar{y}\Big). \end{cases} \qquad \square$$

We use this Lemma to compare two solutions to inclusion (12):

THEOREM 3.3. *We posit the assumptions of Theorem* 2.1. *Let* r_1 *and* r_2 *be two Lipschitz contingent solutions to* (12). *If* r_2 *is differentiable and if* $\lambda > \|r_2\|_\Lambda\|f\|_\Lambda$, *then*

$$\|r_1 - r_2\|_\infty \leq \sup_{x\in X} \frac{\|G(x,r_1(x)) - G(x,r_2(x))\|}{\lambda - \|r_2\|_\Lambda\|f\|_\Lambda}.$$

When f *does not depend on* y, *we can take* $\|f\|_\Lambda = 0$ *in the above estimate.*
In particular, when G *does not depend on* y, *we deduce that*

$$\|r_1 - r_2\|_\infty \leq \sup_{x\in X} \frac{\mathrm{Diam}(G(x))}{\lambda - \|r_2\|_\Lambda\|f\|_\Lambda}.$$

More generally, let us consider a set-valued contingent solution $M : X \rightsquigarrow Y$ to the inclusion

$$(14) \qquad \forall (x,y) \in \mathrm{Graph}(M),\ \ Ay \in DM(x,y)(f(x,y)) - G(x,y).$$

THEOREM 3.4. *We posit the assumptions of Theorem* 2.1. *Let* r *be a Lipschitz contingent solution to* (12) *and* M *be a Lipschitz set-valued contingent solution to inclusion* (14) *in the stronger sense that for every* $(x,y) \in \mathrm{Graph}(M)$, *there exists a Lipschitz closed convex process* $E(x,y) \subset \overline{co}(D^{\flat}M(x,y))$ *satisfying*

$$\forall (x,y) \in \mathrm{Graph}(M),\ \ Ay \in E(x,y)(f(x,y)) - G(x,y)$$

and

$$\|E\|_\Lambda := \sup_{(x,y)\in\mathrm{Graph}(M)} \|E(x,y)\|_\Lambda < +\infty.$$

Assume also that the supremum

$$\delta := \sup_{(x,y)\in\mathrm{Graph}(M)} \|r(x) - y\|$$

is finite and that $\lambda > \|E\|_\Lambda \|f\|_\Lambda$. *Then*

$$\sup_{(x,y)\in \mathrm{Graph}(M)} \|r(x)-y\| \le \sup_{(x,y)\in \mathrm{Graph}(M)} \frac{\|G(x,r(x))-G(x,y)\|}{\lambda - \|E\|_\Lambda \|f\|_\Lambda}$$

or, equivalently,

$$\forall (x,y)\in \mathrm{Graph}(M),\;\; M(x)\subset r(x)+\sup_{(x,y)\in \mathrm{Graph}(M)} \frac{\|G(x,r(x))-G(x,y)\|}{\lambda - \|E\|_\Lambda \|f\|_\Lambda} B.$$

When f *does not depend on* y, *we can take* $\|f\|_\Lambda = 0$ *in the above estimates.*
In particular, when G *does not depend on* y, *we deduce that*

$$\forall (x,y)\in \mathrm{Graph}(M),\;\; M(x)\subset r(x)+\sup_{x\in \mathrm{Dom}(M)} \frac{\mathrm{Diam}(G(x))}{\lambda - \|E\|_\Lambda \|f\|_\Lambda} B.$$

PROOF. By Lemma 3.2, it is enough to show that for every $(x,y) \in \mathrm{Graph}(M)$ and

$$\psi \in G(x,r(x)) \cap \Big(Dr(x)(f(x,r(x))) - Ar(x)\Big)$$

there exists

$$w \in \overline{co}\Big(D^\flat M(x,y)(f(x,r(x)))\Big)$$

such that

$$\|\psi - (w - Ay)\| \le \|G(x,r(x)) - G(x,y)\| + \|E\|_\Lambda \|f\|_\Lambda \delta.$$

Take any such ψ. By assumption, we know that the norms of the closed convex processes $E(x,y)$ are bounded by $\|E\|_\Lambda$ and that

$$\begin{cases} Ay \in E(x,y)(f(x,y)) - G(x,y) \\ \subset E(x,y)(f(x,r(x))) + E(x,y)(f(x,y) - f(x,r(x))) - G(x,y). \end{cases}$$

Then there exist

$$w \in E(x,y)(f(x,r(x))) \subset \overline{co}\Big(D^\flat M(x,y)(f(x,r(x)))\Big)$$

and $\psi' \in G(x,y)$ satisfying

$$\|Ay - w + \psi'\| \le \|E\|_\Lambda \|f\|_\Lambda \|r(x) - y\| \le \|E\|_\Lambda \|f\|_\Lambda \delta.$$

Hence

$$\begin{cases} \|\psi - (w - Ay)\| \leq \|Ay - w + \psi'\| + \|\psi - \psi'\| \\ \leq \|E\|_\Lambda \|f\|_\Lambda \delta + \|G(x, r(x)) - G(x, y)\| \\ \leq \|E\|_\Lambda \|f\|_\Lambda \delta + \sup\limits_{(x,y)\in\text{Graph}(M)} \|G(x, r(x)) - G(x, y)\| \end{cases}$$

from which the conclusion of Theorem 3.4 follows. □

Uniqueness follows when λ is large enough and when we assume the existence of a set-valued map M the graph of which is an *invariance domain* of the set-valued map $(x, y) \rightsquigarrow f(x, y) \times (Ay + G(x, y))$, in the sense that[9]

$$\forall (x, y) \in \text{Graph}(M), \;\; G(x, y) + Ay \subset DM(x, y)(f(x, y)).$$

We need to use the *circatangent derivative* $CM(x, y)$ of M at (x, y) defined by

$$v \in CM(x, y)(u) \text{ if and only if } \lim_{\substack{(x', y') \to_G (x, y) \\ h \to 0+}} d\left(v, \frac{M(x' + hu) - y'}{h}\right) = 0$$

where $\to_G$ denotes the convergence in the graph of G. See [8, Chapter 4] for more details.

THEOREM 3.5. *We posit the assumptions of Theorem* 2.1. *Assume that the graph of the Lipschitz set-valued map M is an invariance domain of $(x, y) \rightsquigarrow f(x, y) \times (Ay + G(x, y))$ and that there exists Lipschitz closed convex process E satisfying*

$$\forall (x, y) \in \text{Graph}(M), \;\; CM(x, y) \subset E(x, y) \subset \overline{co}(D^\flat M(x, y))$$

and that

$$\|E\|_\Lambda := \sup_{(x,y)\in\text{Graph}(M)} \|E(x, y)\|_\Lambda < +\infty.$$

[9] One can prove that when F is Lipschitz with closed values and the graph of M is closed, then Graph(M) is an *invariance domain* if and only if it is invariant in the sense that for any $(x_0, y_0) \in$ Graph(M), every solution to the system of differential inclusions

$$\begin{cases} x'(t) = f(x(t), y(t)) \\ y'(t) \in Ay(t) + G(x(t), y(t)) \end{cases}$$

starting at (x_0, y_0) satisfies

$$\forall t \geq 0, \;\; y(t) \in M(x(t)).$$

If λ is large enough, then $M(x) = \{r(x)\}$ for any (single-valued) contingent solution r to inclusion (12) *such that the supremum*

$$\delta := \sup_{(x,y)\in \mathrm{Graph}(M)} \|r(x) - y\|$$

is finite.

PROOF. Since f and G are lower semicontinuous, we know from [8, Theorem 4.1.9] that inclusion

$$\forall (x,y) \in \mathrm{Graph}(M), \;\; G(x,y) + Ay \subset DM(x,y)(f(x,y))$$

holds true with the circatangent derivative $CM(x,y)$ (which is a closed convex process), so that

$$\forall (x,y) \in \mathrm{Graph}(M), \;\; G(x,y) + Ay \subset CM(x,y)(f(x,y)) \subset E(x,y)(f(x,y)).$$

Observe that it is sufficient to prove that

$$\lambda\delta \leq \|G\|_\Lambda \delta + \|E\|_\Lambda \|f\|_\Lambda \delta$$

which implies that $\delta = 0$ whenever $\lambda > \|G\|_\Lambda + \|E\|_\Lambda \|f\|_\Lambda$.

By Lemma 3.2, it is enough to show that for every $(x,y) \in \mathrm{Graph}(M)$ and

$$\psi \in G(x, r(x)) \cap \Big(Dr(x)(f(x, r(x))) - Ar(x) \Big)$$

there exists

$$w \in \overline{co}\Big(D^\flat M(x,y)(f(x, r(x))) \Big)$$

such that

$$\|\psi - (w - Ay)\| \leq \|G\|_\Lambda \delta + \|E\|_\Lambda \|f\|_\Lambda \delta.$$

Take any such ψ. Since G is Lipschitz, we infer that

$$\psi \in G(x, r(x)) \subset G(x,y) + \|G\|_\Lambda \|r(x) - y\| B \subset G(x,y) + \|G\|_\Lambda \delta B.$$

Therefore,

$$Ay + \psi \in Ay + G(x,y) + \|G\|_\Lambda \delta B \subset CM(x,y)(f(x,y)) + \|G\|_\Lambda \delta B$$

and, $E(x,y)$ being a closed convex process with a norm less than or equal to $\|E\|_\Lambda$,

$$\begin{cases} E(x,y)(f(x,y)) \subset E(x,y)(f(x,r(x))) + E(x,y)(f(x,y) - f(x,r(x))) \\ \subset E(x,y)(f(x,r(x))) + \|E\|_\Lambda \|f\|_\Lambda \delta. \end{cases}$$

Hence there exists

$$w \in E(x,y)(f(x,r(x))) \subset \overline{co}\Big(D^{\flat}M(x,y)(f(x,r(x)))\Big)$$

such that

$$\|Ay + \psi - w\| \leq \|G\|_{\Lambda}\delta + \|E\|_{\Lambda}\|f\|_{\Lambda}\delta. \qquad \square$$

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