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On the Existence of Two-dimensional Invariant Tori for Scalar Parabolic Equations with Time Periodic Coefficients

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In this paper, we consider the equation

$$(1)\qquad \begin{aligned} &\frac{\partial u}{\partial t} = \Delta u + g(t,x,u) && \text{in } \Omega \times [0,\infty) \\ &u = 0 && \text{on } \partial\Omega \times [0,\infty) \end{aligned}$$

where $g : [0,\infty)\times\overline{\Omega}\times\mathbb{R} \to \mathbb{R}$ is T-periodic in t. (We also consider other boundary conditions). If g is independent of t, then there is a Liapounov functional for the semiflow and thus, under very general hypotheses, any solution bounded in a suitable norm for $t \geq 0$ converges to the set of stationary (that is, time-independent) solutions as $t \to \infty$. (See for example Henry [7]).

Now, if g depends T-periodically on time, the natural analogue of the stationary solutions are the solutions which are T-periodic in t. Thus the natural conjecture is that, if g is T-periodic in t, then every solution of (1), which is bounded for $t \geq 0$, must approach the T-periodic solutions as $t \to \infty$. The main result of this paper is that this conjecture is false if $\dim\Omega > 1$ or if Ω is 1-dimensional and we use periodic (in x) boundary conditions. More precisely, under the conditions above, we obtain a hyperbolic invariant 2-torus which contains no periodic solutions. We do not know whether higher-dimensional tori or chaotic behaviour can occur. (We suspect so). Brunovsky and Polačik and Sanstede [2] have recently used lap number ideas to prove that the expected result is true if Ω is one-dimensional and we use Dirichlet or Neumann boundary conditions. If g is independent of x, this was proved earlier in Chen and Matano [3] who also showed that, in this case, it is still true for periodic boundary conditions.

Note that it is not surprising that relatively complicated dynamics can occur when g has ∇u dependence (cp. Polačik [14]) even if g is independent of t. The interest in our work is that we do not need any terms in ∇u.

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As is frequently the case in nonlinear analysis, the key step is a linear one. We prove that the time T-map for

$$\begin{aligned} \frac{\partial u}{\partial t} &= \Delta u + a(x,t)u && \text{in } \Omega \times [0,\infty) \\ u &= 0 && \text{on } \partial\Omega \times [0,T] \end{aligned}$$

can have complex eigenvalues. This appears to be new. The main technical difficulty in this is to obtain a good "generic" case (which does not seem to include any simple explicitly solvable examples) and then make a careful use of perturbation theory. Thus many of our ideas could be used to obtain generic results for the linear problem.

As we mention in §3, similar behaviour can occur for some ordinary differential equations.

In §1, we study the linear problem in more than one space dimension while in §2 we obtain our main results. Finally, in §3, we study the case of one space dimension and periodic boundary conditions.

1. - The linear problem

The main result of this section is to prove that there is a smooth T-periodic function $a(x,t)$ on a convex set Ω in $\mathbb{R}^n$ such that the time T-map W for the initial-value problem

$$(2) \qquad \begin{aligned} \frac{\partial u}{\partial t} &= \Delta u + a(x,t)u && \text{in } \Omega \times \mathbb{R}^+ \\ u &= 0 && \text{on } \partial\Omega \times \mathbb{R}^+ \end{aligned}$$

has complex eigenvalues. Actually, we will prove a slightly more precise result. We can always assume that $T = 2\pi$ by a simple rescaling.

As a first step, we show that we can consider the eigenvalues of the period-boundary value problem

$$(3) \qquad \begin{aligned} \frac{\partial u}{\partial t} &= \Delta u + a(x,t)u + \lambda u && \text{in } \Omega \times [0,2\pi] \\ u(x,t) &= 0 && \text{on } \partial\Omega \times [0,2\pi] \\ u(x,0) &= u(x,2\pi). \end{aligned}$$

By a well-known result (cp. Hess [8], Lemma 2.2), α is a non-zero eigenvalue of W if and only if $\alpha = e^{-2\pi\lambda}$, where λ is an eigenvalue of the problem (3). Thus, it suffices to find complex eigenvalues λ of (3) with $|\operatorname{Im}\lambda| < \frac{1}{2}$. This new problem is more convenient because it is local.

We will construct complex eigenvalues of (3) by perturbing them from a double real eigenvalue by varying a. Thus we consider the problem

$$(4)\qquad \begin{aligned} &\frac{\partial u}{\partial t} = \Delta u + a(x,t)u + \varepsilon b(x,t)u + \lambda u && \text{in } \Omega \times [0,2\pi] \\ &u = 0 && \text{on } \partial\Omega \times [0,2\pi] \\ &u \text{ is } 2\pi \text{ periodic in } t. \end{aligned}$$

Before we consider this problem, we need to know that the adjoint problem to (4) is the problem

$$\begin{aligned} &-\frac{\partial u}{\partial t} = \Delta u + a(x,t)u + \varepsilon b(x,t)u + \lambda u && \text{in } \Omega \times [0,2\pi] \\ &u = 0 && \text{on } \partial\Omega \times [0,2\pi] \\ &u \text{ is } 2\pi \text{ periodic in } t. \end{aligned}$$

Here, we need to be a little more precise. Our two operators are defined to have dense domain

$$\left\{u \in L^2(\Omega \times [0,2\pi]) : \frac{\partial u}{\partial t},\ \Delta u \in L^2(\Omega \times [0,2\pi]),\ u(x,t) = 0 \right.$$
$$\left. \text{if } x \in \partial\Omega,\ u \text{ is } 2\pi \text{ periodic in } t\right\}$$

where the derivatives are distributional derivatives and the boundary conditions make sense because $\Delta u \in L^2$ and $\frac{\partial u}{\partial t} \in L^2$. Since the other terms are bounded and self-adjoint, it suffices to prove these results for $\frac{\partial u}{\partial t} - \Delta u$. This operator is closed and invertible since we can easily write down an eigenfunction expansion (Fourier in t, eigenfunctions for the Laplacian in x) to obtain a continuous inverse. Eigenfunction expansion also imply that the domain is dense. We can use finite parts of this eigenfunction expansions (and closures) to justify the formula for the adjoint of (4). (Note that the domain of the adjoint can be no larger because the adjoint must be invertible when a, b, λ all vanish). Since $\frac{\partial u}{\partial t} - \Delta u$ has compact resolvent, so must (3).

PROPOSITION. *Assume that, if* $\varepsilon = 0$, λ_0 *is a real double eigenvalue of* (4) *with two linearly independent eigenfunctions* e_1, e_2. *Let* f_1, f_2 *denote the eigenfunctions of the adjoint problem to* (3) *for* $\lambda = \lambda_0$ *chosen so that* $\langle f_i, e_j\rangle = \delta_{ij}$ *where* $\langle\ ,\ \rangle$ *denote the usual scalar product on* L^2. *If the four functions* $f_i(x,t)e_j(x,t)$ *are linearly independent functions on* $\Omega \times [0,2\pi]$, *then, for a suitable* b, (4) *has complex eigenvalues with small imaginary part for small non-zero* ε.

PROOF. Standard perturbation theory (cp. Kato [11], p. 81) shows that the eigenvalues of (4) near λ_0, when ε is small, are largely determined by the 2×2

matrix $A = (\tilde{a}_{ij})$ where

$$\tilde{a}_{ij} = \int_{\Omega\times[0,2\pi]} f_j(x,t)b(x,t)e_i(x,t)\,dx\,dt.$$

In particular, if A has distinct eigenvalues $\hat{\lambda}_1$, $\hat{\lambda}_2$, then the eigenvalues of (4) near λ_0 are of the form $\lambda_0 - \varepsilon\hat{\lambda}_i + o(\varepsilon)$ (for $i = 1, 2$) for small ε. Hence, we see that, if we can choose b so that A has distinct complex eigenvalues, then we will have proved the proposition. Hence it suffices to prove that the linear map $b \to A$ is onto when considered as a mapping of $C_p^\infty(\Omega \times [0, 2\pi])$ to $M_{2,2}$. Here $M_{2,2}$ denotes the set of real 2×2 matrices and $C_p^\infty(\Omega \times [0, 2\pi])$ denotes the space of smooth functions on $\Omega \times [0, 2\pi]$ which are 2π-periodic in t. If the map is not onto, there exist numbers c_{ij} such that $\sum_{i,j=1}^{2} c_{ij}\tilde{a}_{ij} = 0$ for all b's, that is, such that

$$\sum_{i,j=1}^{2} \int_{\Omega\times[0,2\pi]} c_{ij} f_j(x,t)b(x,t)e_i(x,t)\,dx\,dt = 0$$

for all $b \in C_p^\infty(\Omega \times [0, 2\pi])$. By density, this equality must hold for all $b \in L^2(\Omega \times [0, 2\pi])$. Hence we see that

$$\int_{\Omega\times[0,2\pi]} b(x,t)\left(\sum_{i,j=1}^{2} c_{ij} f_j(x,t)e_i(x,t)\,dx\,dt = 0\right)$$

for all $b \in L^2(\Omega \times [0, 2\pi])$ and hence $\sum_{i,j=1}^{2} c_{ij} f_j(x,t)e_i(x,t) = 0$ a.e. in $\Omega \times [0, 2\pi]$. Since eigenfunctions are continuous by standard regularity theory, this last equality must hold everywhere on $\Omega \times [0, 2\pi]$. Since this is impossible by our linear independence assumption, the result is proven.

REMARK 1. The proof and standard perturbation theory show that there is a complex eigenvalue $\lambda(\varepsilon)$ for all small non-zero ε such that $\lambda(\varepsilon)$ depends continuously on ε and $\lambda(\varepsilon) \to \lambda_0$ as $\varepsilon \to 0$.

REMARK 2. Since A can be any 2×2 matrix, we can choose A so that an eigenvector of (4) corresponding to an eigenvalue close to λ_0 is close to any given combination of eigenvectors in the eigenspace corresponding to λ_0. Using a variant of Lemma 2.2 in [8] again, this means we can choose A so that the time 2π map of (4) has an eigenvector corresponding to an eigenvalue close to $e^{-2\pi\lambda_0}$ which is close to any given linear combination of the eigenvectors of the time 2π map corresponding to the eigenvalue $e^{-2\pi\lambda_0}$ (for (3)). In particular, this means that we can choose A so that the time 2π map for (4) has complex eigenvalues near $e^{-2\pi\lambda_0}$ with the corresponding eigenvector having real and imaginary parts close to an orthonormal basis for $\mathbb{R}^2$. This means that, if we choose a linear change of coordinates so that, for some small ε, the time 2π map is a complex rotation on the span of the eigenspace corresponding to eigenvalues near λ_0 (as in [10], p. 27), then the change of coordinates is close to the identity. This is useful in §2.

Hence, it suffices to construct an example of a double eigenvalue λ_0 such that our linear independence condition holds. Unfortunately, in all the standard examples where one can calculate the eigenfunctions explicitly, it seems that the linear independence condition fails. Hence we have to proceed indirectly. Once again, we will obtain our independence by a perturbation argument. We use symmetries to retain the double eigenvalue.

However, before doing this, it is convenient to examine the linear dependence relation

$$\sum_{i,j=1}^{2} c_{ij}\, e_i(x,t) f_j(x,t) \equiv 0.$$

Suppose $\tilde{A} \subseteq \Omega \times [0, 2\pi]$ and e_1 vanishes on $\tilde{A}$. Then it follows that $e_2(x,t)(c_{21}f_1(x,t) + c_{22}f_2(x,t)) = 0$. Thus, we see that, if we can ensure that $e_2(x,t) \not\equiv 0$ on $\tilde{A}$, then an eigenfunction of the adjoint problem must vanish on $\tilde{A}$. A similar result holds if we interchange the roles of the e's and f's.

We now construct our example by a 2-stage perturbation process.

Step 1. We first assume that $n = 2$, Ω is the unit ball B and $a(x,t) = a(r)$ where $r = \|x\|$. We choose λ_0 so that we have eigenfunctions of the form $e_1(x,t) = e(r)\cos\theta$, $e_2(x,t) = e(r)\sin\theta$. We can ensure that λ_0 is only a double eigenvalue. (For example, it is easy to see that this holds if a is constant). Note that e is a solution of

$$\frac{\partial u}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + a(r)u + \lambda_0 u - \frac{1}{r^2}u$$

$$u = 0 \quad \text{when } r = 1$$

$$u \text{ is } 2\pi \text{ periodic in } t.$$

The adjoint eigenfunctions will be of the form $f(r,t)\cos\theta$, $f(r,t)\sin\theta$, where f satisfies the same equation as e except that $\dfrac{\partial u}{\partial t}$ is replaced by $-\dfrac{\partial u}{\partial t}$. (Since a is independent of t, so are e and f). We distinguish between e and f because it is useful in later formulae.

In this case, it is easy to see that there is only one relation between the 4 function $f_j e_i$ and it is the relation $c_{11} = c_{22} = 0$, $c_{12} = -c_{21}$. (This comes about basically because of the θ dependence). We use here that $e_i f_j$ does not vanish identically. This follows because e and f have only simple zeros for $r > 0$ (being solutions of ordinary differential equations).

We now make a small perturbation $\varepsilon a_1(r,t)u$ to ensure that there is a zero of e which is not a zero of f. In the invariant subspace of functions even in θ, λ_0 has algebraic multiplicity 1. Hence, by standard perburbation theory (cp. Kato [11], p. 77–79), the perturbed eigenvalue in this subspace will be $\lambda_0 - \varepsilon\tilde{\lambda}_1 + o(\varepsilon)$, where

$$\tilde{\lambda}_1 = \left(\int_{B\times[0,2\pi]} f\, e \cos^2\theta \right)^{-1} \int_{B\times[0,2\pi]} a_1\, f\, e \cos^2\theta$$

and the corresponding eigenfunction will be of the form

$$e(r,t)\cos\theta + \varepsilon\tilde{e}_1(x,y,t) + o(\varepsilon),$$

where $\tilde{e}_1 = -L^{-1}(\tilde{\lambda}_1\, e \cos\theta - a_1\, e\cos\theta)$, L denotes the operator

$$\frac{\partial u}{\partial t} - \Delta u - au - \lambda_0 u$$

with the boundary conditions, and where the inverse is on the set of functions even in θ and orthogonal to $f \cos\theta$. Because of the symmetries and the evenness in θ, $\tilde{e}(x,y,t)$ is of the form $\tilde{e}(r,t)\cos\theta$ and in the equation for $\tilde{e}_1$ (and thus for $\tilde{e}$) we really only need to invert an ordinary differential operator rather than a partial differential operator.

Let $T = \left\{ a_1(r,t) : \int_0^{2\pi}\int_0^1 a_1(r,t)e^2(r)r\,\mathrm{d}r\,\mathrm{d}t = 0 \right\}$ and let P_1 be the natural orthogonal projection onto T. (This is also the spectral projection). If $a_1 \in T, \tilde{e}_1 = L^{-1}(a_1 e\cos\theta)$. Under the same perturbation, f_1 perturbs to an eigenfunction even in θ, $f_1 + \varepsilon\tilde{f}_1 + o(\varepsilon)$. Moreover, if $a_1 \in T$ (since $e_1 = f_1$) we see by similar arguments to above that

$$\tilde{f}_1 = (L^*)^{-1}\ (a_1\, e\cos\theta).$$

Since $a(r,t) = a(r)$ and thus $e(r,t) = e(r)$, the equation for e is an ordinary differential equation and thus the positive zeros of e are simple. (As usual, the equation is singular at $r = 0$). If r_0 is a positive zero of e, then for r near r_0 and $t \in [0,T]$,

$$e(r) + \varepsilon\tilde{e}(r,t) + o(\varepsilon) = e'(r_0)(r - r_0) + \varepsilon\tilde{e}(r_0,t) + o(\varepsilon,(r - r_0)).$$

Hence we see that the eigenfunction vanishes on a curve

$$r = r_0 - \varepsilon(e'(r_0))^{-1}\tilde{e}(r_0,t) + o(\varepsilon).$$

(This proof is easily made rigorous). A similar formula holds for the curve of zeros of the adjoint eigenfunction near $r = r_0$. Hence we see that if we can ensure that $\tilde{e}(r_0,t_0)$ and $\tilde{f}(r_0,t_0)$ have different values at (r_0,t_0), for some t_0, then nearby the curves of zeros of the two eigenfunctions will be distinct. This basically comes down to proving that the two operators L^{-1} and $(L^*)^{-1}$ are sufficiently different.

To prove this, we use eigenfunction expansions. Let $\{\phi_k(r)\cos\theta\}_{r=1}^{\infty}$ denote the eigenfunctions of the Laplacian of the form $\hat{f}(r)\cos\theta$ and let λ_k, $k = 1, \dots, \infty$, denote the corresponding eigenvalues. Assume $\lambda_{k_0} = \lambda_0$. One easily sees that

$$L^{-1}(a_1 e) = \sum (ij + \lambda_k - \lambda_0)^{-1} \alpha_{jk}\, e^{ijt} \phi_k(r),$$

where our summation is $j = -\infty$ to ∞, $k = 1$ to ∞ except we omit the term $j = 0$, $k = k_0$, where $a_1 e = \sum\limits_{j=-\infty}^{\infty} \sum\limits_{k=1}^{\infty} \alpha_{jk} e^{ijt} \phi_k(r)$ and where we have omitted the $\cos\theta$ factor on all terms to simplify the formula. The formula for $\tilde{f}$ is analogous except that we replace $(ij + \lambda_k - \lambda_0)^{-1}$ by $(-ij + \lambda_k - \lambda_0)^{-1}$.

By a simple calculation, we see that

$$\tilde{e}(r,t) - \tilde{f}(r,t) = \sum \frac{2ij\alpha_{jk}}{j^2 + (\lambda_k - \lambda_0)^2}\, e^{ijt} \phi_k(r). \tag{5}$$

We use this to prove that $\tilde{e}(r_0, t_0) \neq \tilde{f}(r_0, t)$, for some t_0, for suitable $a_1(r,t) \in T$. Now $\tilde{e} - \tilde{f} = S(a_1 e)$ where $S = L^{-1} - (L^*)^{-1}$. If p is sufficiently large, the regularity theory in [12] (cp. also [5, §5]) ensures that S is a continuous map of $L^p(B \times [0, 2\pi])$ into $C(B \times [0, 2\pi])$. Hence, if we prove that

$$\{a_1 e : a_1 \in T,\ a_1 \text{ is smooth and } 2\pi \text{ periodic in } t\}$$

is dense in

$$T_2 = \{w \in L^p(B \times [0, 2\pi]) : w \text{ is radially symmetric, } w \text{ is orthogonal to } e\},$$

it suffices to find a single $g \in T_2$ such that $S(g)(r_0, t_0)$ is non-zero for some t_0. If our expansion for g (as above) has only a finite number of terms, we can put $r = r_0$ in (5) and prove our claim. (Note that the α_{jk} in the expansion of g are arbitrary except that $\alpha_{0k_0} = 0$ and $\overline{\alpha_{jk}} = \alpha_{-j,k}$ and that completeness ensures that $\phi_k(r_0) \neq 0$ for some k).

Hence we will have proved our claim if we prove that

$$\left\{ a_1 e : a_1 \text{ is smooth and } 2\pi \text{ periodic}, \int_0^1 \int_0^{2\pi} a(r,t) e^2(r,t) r \,\mathrm{d}r\,\mathrm{d}t = 0 \right\}$$

is dense in

$$\left\{ v : \int_0^1 \int_0^{2\pi} |v(r,t)|^p r \,\mathrm{d}r\,\mathrm{d}t < \infty, \int_0^1 \int_0^{2\pi} v(r,t) e(r,t) r \,\mathrm{d}r\,\mathrm{d}t = 0 \right\}$$

for suitable $p \geq 2$. Here and below our L^p norms on $[0,1]$ have weight r. Since P_1 is easily seen to be continuous in the L^p norm, it suffices to prove that the subspace

$$W = \{a_1 e : a_1 \text{ is smooth and } 2\pi\text{-periodic}\}$$

is dense in $L^p([0,1] \times [0,2\pi])$ for suitable $p \geq 2$. Suppose not. Then there exists $h \in L^q([0,1] \times [0,2\pi]) \backslash \{0\}$ such that

$$\int_0^{2\pi} \int_0^1 h a_1 \, e r \, \mathrm{d}r \, \mathrm{d}t = 0$$

for all smooth 2π periodic a_1. Since $h\, e \in L^q$ and since the smooth 2π-periodic functions are dense in L^p, it follows that $\int_0^{2\pi} \int_0^1 h\, e\, v r \, \mathrm{d}r \, \mathrm{d}t = 0$ for all $v \in L^p$ and thus $h(r,t)e(r,t) = 0$ a.e.. Now the zero set of e has zero measure. (Recall that $e = e(r)$ and e has isolated zeros in r for $r > 0$). Hence $h = 0$ a.e.. Hence we have a contradiction and our claim follows.

Step 2. In this final perturbation step, we will use another perturbation which partially breaks the radial symmetry. (Whenever we have radial symmetry and a double eigenvalue, we always have a linear relation between the $f_i e_j$ and thus the assumptions of the proposition are not satisfied). The idea is to retain enough symmetry to ensure that the double eigenvalue does not split under the perturbation. We add a perturbation $\varepsilon a_2(x,t)u$ where a_2 is invariant under rotations of $\frac{1}{2}\pi$.

We first show that, under a perturbation of this type, the double eigenvalue $\tilde{\lambda}_0$ (that is λ_0 after the perturbation in Step 1) perturbs to a double real eigenvalue (also of geometric multiplicity 2). If not, it would have to perturb to 2 simple eigenvalues (possibly complex) or to a single eigenvalues of geometric multiplicity 1 and algebraic multiplicity 2. In each case, the space of eigenvectors corresponding to an eigenvalue will be one-dimensional, close to the original eigenspace and invariant under rotations through $\frac{\pi}{2}$. By taking the limits as $\varepsilon \to 0$, we deduce that the eigenspace corresponding to $\tilde{\lambda}_0$ for the original operator (that is, the one after Step 1) must have one-dimensional subspaces invariant under rotations through $\frac{\pi}{2}$. Because our original eigenspace is of the form $e(r,t)(a \cos\theta + b \sin\theta)$, it is easy to see that this is impossible. Hence $\tilde{\lambda}_0$ perturbs to a double real eigenvalue $\tilde{\lambda}_\varepsilon$. Let e_1^n, e_2^n denote the eigenfunctions after Step 1 and $\tilde{e}_1^n, \tilde{e}_2^n$ the perturbed eigenfunctions after our new perturbation.

If $e_1^n = e^n \cos\theta$ and $e_2^n = e^n \sin\theta$ (and thus $e_2^n = -\hat{T} e_1^n$, where $\hat{T}$ is the operation of rotation through $\frac{\pi}{2}$), then by perturbation theory again, $\tilde{e}_1^n = e^n \cos\theta + \varepsilon \hat{e}_1 + o(\varepsilon)$ and $\tilde{e}_2^n = e^n \sin\theta + \varepsilon \hat{e}_2 + o(\varepsilon)$, where $\hat{L}\hat{e}_i = P_2(a_2 e_i^n)$ and $\hat{e}_i$ is orthogonal to f_1^n, f_2^n. Here $\hat{L}$ is the same as L except that L is replaced by the corresponding operator after the perturbation in Step 1 and P_2 is the spectral projection onto the eigenspace of $\tilde{\lambda}_0$.

Since a_2 is $\hat{T}$ invariant and $e_2^n = -\hat{T}e_1^n$, $\hat{e}_2 = -\hat{T}\hat{e}_1$. Suppose that $(\tilde{r}_0, \tilde{t}_0)$ is a point where $e^n = 0$. By our earlier arguments and continuity, $\frac{\partial e^n}{\partial r}(\tilde{r}_0, \tilde{t}_0) \neq 0$. Hence, locally near $(\tilde{r}_0, \tilde{t}_0)$, the zeros of e^n form a smooth curve $r = h(t)$. We prove that we can choose a_2 so that $\hat{e}_1(\tilde{r}_0, \tilde{t}_0, \frac{1}{8}) \neq \hat{e}_2(\tilde{r}_0, \tilde{t}_0, \frac{1}{8})$ (for coordinates r, t, θ). A simple estimate for zeros similar to that in Step 1 shows that we can find points near $(\tilde{r}_0, \tilde{t}_0, \frac{1}{8})$ (in fact points with $\theta = \frac{1}{8}$) where $\tilde{e}_1^n = 0$ while $\tilde{e}_2^n \neq 0$. The result will follow easily from this. To construct our example, note that $\hat{L}(\hat{e}_1 - \hat{e}_2) = P_2(a_2(e_1^n - e_2^n))$. Now, in L^p,

$$
\begin{aligned}
&\overline{\{a_2(e_1^n - e_2^n) : a_2 \text{ is symmetric}\}} \\
&\quad = \overline{\{a_2\, e^n(r,t)(\cos\theta - \sin\theta) : a_2 \text{ is symmetric}\}} \\
&\quad \subseteq \overline{\{a_2\, e^n(r,t) : a_2 \text{ is symmetric}\}}(\cos\theta - \sin\theta) \\
&\quad = \{g(r,t,\theta)(\cos\theta - \sin\theta) : g \text{ is symmetric } g \in L^p\} \\
&\quad \equiv T_1.
\end{aligned}
$$

Here, by symmetric, we mean invariant under a rotation of $\frac{1}{2}\pi$. To prove the last step, we have used a simple Hahn-Banach theorem argument in $\{u \in L^p(B) : u$ is symmetric$\}$. (It is very similar to part of the argument in Step 1). Hence, if $(\hat{e}_1 - \hat{e}_2)(r_0, t_0, \frac{1}{8}) = 0$ always, it would follow by density that the solution of $\hat{L}w = P_2\hat{f}$ (with w orthogonal to $f^n(r,t)\cos\theta$, $f^n(r,t)\sin\theta$) vanishes at $(r_0, t_0, \frac{1}{8})$ for every $\hat{f}$ of the form $g(r,t,\theta)(\cos\theta - \sin\theta)$ with g symmetric. Recall that $\hat{L}$ is of the form $\frac{\partial u}{\partial t} - \Delta u - \tilde{a}(r,t)u$. Choose $\tilde{w}(r,t)$ smooth so that $\tilde{w}$ is 2π periodic in t, $\tilde{w}(r_0, t_0) \neq 0$, $\tilde{w}$ vanishes near $r = 0$ and $r = 1$ and $\tilde{w}$ is r orthogonal to f^n on $[0,1] \times [0, 2\pi]$. It is easy to see that $\tilde{w}(r,t)(\cos\theta - \sin\theta)$ is smooth, is in T_1 and is orthogonal to f_1^n and f_2^n and $\tilde{w}(r_0, t_0, \frac{1}{8}) \neq 0$. Since $\hat{L}$ is of the form $\frac{\partial u}{\partial t} - \Delta u - a(r,t)u$, an easy computation shows that $\hat{L}w \in T_1$, where $w(r,t,\theta) = \tilde{w}(r,t)(\cos\theta - \sin\theta)$. Since $\hat{L}w$ is in the range of $\hat{L}$, $P_2\hat{L}w = \hat{L}w$. Thus $P_2\hat{L}w$ has the required form. This contradicts our claim above and hence we can ensure there exists a point near $(r_0, t_0, \frac{1}{8})$ where $\tilde{e}_1^n = 0$ and $\tilde{e}_2^n \neq 0$.

We now complete the construction of the linear example when $n = 2$. By Step 1 in our construction, we have a real eigenvalue $\tilde{\lambda}_0$ and an (r_0, t_0) where $e^n(r_0, t_0) \neq 0$ and $f^n(r_0, t_0) = 0$ (where $e_1^n = e^n(r,t)\cos\theta$ and $e_2^n = e^n(r,t)\sin\theta$ are the eigenfunctions corresponding to $\tilde{\lambda}_0$ and f_1^n and f_2^n are the corresponding adjoint eigenfunctions). By Step 2 (applied to the adjoint), we obtain an example of a double eigenvalue $\tilde{\lambda}_0$ and a point $(\tilde{r}_0, \tilde{t}_0, \frac{1}{8})$ where $\tilde{e}_1^n$ and $\tilde{e}_2^n$ are not small (by continuity), $\tilde{f}_1^n(\tilde{r}_0, \tilde{t}_0, \frac{1}{8}) = 0$ while $\tilde{f}_2^n(\tilde{r}_0, \tilde{t}_0, \frac{1}{8}) \neq 0$. We show that this ensures our rank condition holds. Suppose not. Because our perturbations are small, continuity arguments and our earlier comments on the possible relations prior to our perturbations show that the only possible relation is

$$\sum_{i,j=1}^{2} c_{ij} \tilde{e}_i^n \tilde{f}_j^n = 0 \text{ on } \Omega \times [0, 2\pi]$$

where c_{12}, c_{21} are near 1 and c_{11}, c_{22} are small. At $(\tilde{r}_0, \tilde{t}_0, \frac{1}{8})$, $\tilde{f}_1^n$ vanishes while $\tilde{f}_2^n$ does not and hence our relation becomes $c_{12}\tilde{e}_1^n + c_{22}\tilde{e}_2^n = 0$. Since c_{12} is near 1 while c_{22} is small and the eigenfunctions are bounded, this implies that $\tilde{e}_1^n$ is small at $(\tilde{r}_0, \tilde{t}_0, \frac{1}{8})$. By our comments earlier in this paragraph this is impossible and hence we have our example if $n = 2$.

We now consider the case where $n > 2$. In this case we construct our example on a cylinder $B \times K$ where B is the two dimensional ball and K is a convex set in $\mathbb{R}^{n-2}$ with smooth boundary. If $a(\underline{x}, t)$ is the coefficient in the two-dimensional example which satisfies the assumptions of the proposition, we choose a coefficient of the form $a(\underline{x}, t) + r(z)$ (where $\underline{x} = (x, y)$ and $z \in K$). In this case we can find eigenfunctions of our periodic problem by separating variables. We find eigenvalues of the form $\lambda_0 + \lambda_1$ where λ_0 is an eigenvalue of our two dimensional problem above and λ_1 is an eigenvalue of problem

$$(6) \qquad -\Delta u = \lambda u \text{ on } K, \; u = 0 \text{ on } \partial K$$

and these are the only eigenvalues. By replacing K by αK where α is small and positive, we can ensure that the gap between the first and second eigenvalues is large. This ensures that, if we choose λ_0 to be the double eigenvalue of the two dimensional problem of the type in the proposition and λ_1 the first eigenvalue of (6), then $\lambda_0 + \lambda_1$ is a double eigenvalue with eigenfunctions of the form $e(\underline{x}, t)\phi_1(z)$ where e is an eigenfunction of the two-dimensional problem and ϕ_1 is the first eigenfunction of (6). Since a similar formula holds for the adjoint eigenfunctions (with the same ϕ_1), we easily see that the assumptions of the proposition are satisfied for the n-dimensional problem on $B \times K$. Hence the proposition implies that we have an n-dimensional example.

Once we have one example, a minor variation of the theory in §5 of [5] ensures that the complex eigenvalues persist for quite general domain variations. Thus we have examples for many different domains. In particular, we have examples for each $n \geq 2$ in strongly convex domains with smooth boundaries. Note also that our examples are examples where a has quite small dependence upon t.

Note that our methods are not very dependent on boundary conditions. In particular, we obtain similar examples for Neumann or Robin boundary conditions. Lastly note that our method constructs a family of linear operators depending analytically on ε so that the time T-map has a double real eigenvalue $\tilde{\lambda}_0$ for $\varepsilon = 0$ and two complex eigenvalues $\lambda_1(\varepsilon)$, $\lambda_2(\varepsilon)$ for small ε. Note that $\lambda_1(\varepsilon)$ and $\lambda_2(\varepsilon) = \overline{\lambda_1(\varepsilon)}$ have imaginary part tending to zero as $\varepsilon \to 0$. By adding a constant depending on ε on to a, we can assume that Re $\lambda_1(\varepsilon) = 0$ for all ε (at the expense of losing smoothness in ε) or that Re $\lambda_1(\varepsilon_0) = 0$ at any given ε_0.

One final comment. Our time T-map is a small perturbation of the time T-map for an autonomous equation with a double eigenvalue τ_0. Because the unperturbed equation is autonomous, the time T-map for it has only real spectrum. Hence we see by continuity of the eigenvalues under perturbation that no other eigenvalue in our example has the same real part as the two complex eigenvalues near τ_0.

2. - The existence of the hyperbolic torus

In this section, we use the construction of the last section to produce our hyperbolic torus. More precisely we construct a hyperbolic invariant circle containing no periodic points for the time T-map. To do this, we first use a centre manifold reduction and then a variant of a theorem of Ruelle-Takens [15]. We have to be a little careful because we are using centre manifolds of perturbed operators. As before, we can change variables so that $T = 2\pi$.

We start off with

$$\text{(7)} \qquad \begin{aligned} &\frac{\partial u}{\partial t} = \Delta u + au - u^3 \\ &u = 0 \qquad \text{on } \Omega \times \mathbb{R}^+ \end{aligned}$$

for suitably chosen constant a. We showed in the last section there is a family $a_\varepsilon(x,t)$ with 2π-periodic coefficients depending smoothly on ε and C^∞ close to a so that the time 2π-map of the linearized part of (7) at $u = 0$ (with a replaced by a_ε) has a pair of simple complex eigenvalues for $\varepsilon > 0$ small which collapse to a real eigenvalue τ_0 of the time 2π-map for (4) at $\varepsilon = 0$. (The perturbation is C^∞ close because at each of the 3 stages of the perturbation we are adding a C^∞ small function). We have added a constant to a to ensure that $\tau_0 = 1$. Let W_ε denote the time 2π map for

$$\text{(8)} \qquad \begin{aligned} &\frac{\partial u}{\partial t} = \Delta u + a_\varepsilon(x,t)u - u^3 \quad \text{in } \Omega \times [0, 2\pi] \\ &u = 0 \qquad\qquad\qquad\qquad \text{on } \partial\Omega \times [0, 2\pi]. \end{aligned}$$

We choose $p > \frac{1}{2}n$ and $\alpha \in (0,1)$ so that the fractional power space

$D(A_1^\alpha) \subseteq C(\Omega)$ where A_1 denotes $-\Delta$ with Dirichlet boundary conditions on $L^p(\Omega)$ and A_1^α is the fractional power in the sense of [7], §1.3. Note that standard reqularity theory ensures that the spectrum of W_ε is independent of p for $p > 1$. By the theory in §3.4 of [7], there is a neighbourhood Y of zero in $D(A_1^\alpha)$ such that W_ε is a smooth map of Y into $D(A_1^\alpha)$ and depends smoothly on ε. Moreover W_ε is compact on Y. In addition since a_ε is C^∞ close to a, we see that W_ε is C^∞ close to $\hat{W}$, the time 2π-map for (7). (This does not follow directly from the theorems but follows exactly in the same way as 3.4.4 and 3.4.5 are proved in [7]). Now, by our construction, $\hat{W}'(0)$ has a double eigenvalue $\tilde{\tau}_0$ and no other eigenvalue has the same real part. Hence, if $k > 0$, there is a two dimensional C^k centre manifold M_ε which is tangent to the space spanned by the eigenvectors corresponding to the eigenvalues near $\tilde{\tau}_0$. Moreover M_ε depends C^k smoothly upon ε and, at 0, is C^k close to the centre manifold $\hat{M}$ for $\hat{W}$. (By the latter, we mean that the k derivatives at zero of M_ε are close to those of $\hat{M}$ in the natural parametrization).

We need to explain a little the last claims. Firstly, we construct a centre unstable manifold $\tilde{N}_\varepsilon$ (which is finite-dimensional). To do this, we use Theorem 5 in [17] (with the truncation trick of Theorem 7 in [17]). We apply this each time with the same linear part (that of $\hat{W}$) and the same truncation. We use the usual trick of adding an extra equation $\dot{\varepsilon} = 0$ to ensure that the manifold depends C^k upon ε. To see that the derivatives of M_ε at 0 are close to those of the centre-stable manifold $\hat{N}$ for $\hat{W}$, we use the usual formula for the derivatives at 0. Finally, to obtain a centre manifold rather than a centre unstable manifold we use the Hirsch-Pugh-Shub [9] trick (cp. p. 49) of considering the inverse of W_ε on N_ε and using the idea above a second time. (This works because N_ε is finite-dimensional and W_ε is a local diffeomorphism on N_ε).

Hence we see that the discrete flow $W_\varepsilon | N_\varepsilon$ depends smoothly on ε and is C^k close at zero to $\hat{W}|\hat{N}$. Note that, if $\lambda(\varepsilon) = \lambda_1(\varepsilon) + i\lambda_2(\varepsilon)$ is an eigenvalue of (4), then, as in §1, the corresponding eigenvalue of the time 2π-map is $e^{-2\pi\lambda(\varepsilon)}$. This has absolute value $e^{-2\pi\lambda_1(\varepsilon)}$ and argument $-2\pi\lambda_2(\varepsilon)$. By adding a term $\delta(\varepsilon)u$ to our equation (4) with $\delta(\varepsilon)$ small and smooth in ε, we can arrange that the pair of complex eigenvalues move through the unit circle at any small ε we wish and with non-zero speed. We can do this without disturbing what we have already achieved. In addition, since the imaginary part of $\lambda(\varepsilon)$ varies with ε, we can ensure that, when the eigenvalues move through the unit circle, they are not roots of unity (and satisfy the diophantine conditions for twist theorems).

We now show that $\hat{W}|\hat{N}$ has good cubic terms. It is convenient to do this indirectly. But, before we do this, it is convenient to look at the symmetries in our situation. Our equation (7) is invariant under an 0(2) symmetry generated by rotations and reflections on the ball B. Hence the time map-$\hat{W}$ is also 0(2)-invariant and thus the centre manifold $\hat{M}$ is also 0(2)-invariant (cp. [6]). Hence $\hat{W}|\hat{M}$ is 0(2)-invariant. In this case, our equation is autonomous and hence we can construct $\hat{W}$ on the centre manifold $\hat{M}$ as the time 2π-map for the flow on the centre manifold for the corresponding differential equation. Thus we can use the arguments in Chossat and Golubitsky [4] to deduce that the cubic term of the differential equations on the centre manifold can be found by calculating the cubic term of the bifurcation equation for the problem $u' = w(\Delta u + au - u^3)$ in the space of 2π-periodic functions and w near 1. (We are using an infinite-dimensional version of the result in [4] and in a case where the linearization has eigenvalues with positive real part but the proof is the same. Note that their condition that the quadratic term vanishes is easily checked since our maps have to be odd). Hence we need to calculate the cubic term in this bifurcation equation. Because the eigenspace of the linear part is independent of t, we see that the bifurcation equation is the same as that in the space of functions constant in time. By a standard simple computation using the oddness, one sees that the bifurcation equation is of the form $c\|x\|^2 x +$ h.o.t. $= 0$, where $c = \int_B h_1^4 \, dx$ (h_1 is normalized so that $\int_B h_1^2 \, dx = 1$). To see the form of the bifurcation equation, note that in the space of time independent functions our equation is a gradient system. Hence the bifurcation equation is a gradient and is 0(2)-invariant. It follows easily that the cubic term has the above form and it is easy to calculate c. Note that $c > 0$ and hence the bifurcation equation (and thus the differential equation on the centre manifold in normal form) has a non-trivial cubic term. It follows that the time 2π-map $\hat{W}$ has a non-trivial cubic term of the form $c\|x\|^2 x$. To see the last claim, we note that the 1-dimensional subspace of T which is even in θ is $\hat{W}$-invariant (and in fact is invariant under the continuous time flow). This follows because the system is 0(2)-invariant and the set of functions even in θ is the fixed point set of the obvious reflection in 0(2). Thus, we are down to the one dimensional case where the claim can be checked by a simple explicit integration. Hence our claim follows. Hence our claim that $\hat{W}$ is a non-trivial cubic term follows. In fact, by the symmetry $\hat{W}(x) = x - c\|x\|^2 x+$ h.o.t. This proves our claim.

Now the cubic terms in W_ε will be C^3 close to those of $\hat{W}$. Moreover, W_ε is odd in x and hence has no quadratic terms. Hence the formulae in equations (12) and (15) on p. 30 of [10] and the proof of Lemma 1 there show that the cubic term in W_ε in canonical form is never zero. (By Remark 2 after Proposition 1, we can arrange that at ε_0 the linear coordinate change in [10], p. 27, to put the linear part in canonical form is close to the identity and hence this change of variable will not eliminate the cubic term $\mu|z|^2 z$ with μ real which gives the non-trivial cubic in the canonical form). We now apply Theorem 1 in [10] to obtain an invariant circle S_ε for W_ε, for ε near ε_0, where ε_0 is chosen so that the eigenvalues cross the unit circle at ε_0. Lastly S_ε is hyperbolic on N_ε by the construction on [10]. (Note that, up to some rescalings, it is a small perturbation of an invariant circle which is easy to check to be hyperbolic). Thus it is hyperbolic in the whole space by properties of centre manifolds.

Lastly, we ensure that, for suitable ε near ε_0, there are no periodic points of W_ε on the invariant circle. To see this, we do our earlier construction so that at ε_0 the eigenvalues of W_ε on the unit circle are not roots of unity. By Theorem 3 on p. 49 of [10], the rotation number ℓ_ε of W_ε on the invariant circle is continuous in ε and equals θ_0 at ε_0, where θ_0 is the argument of the roots of $W'_{\varepsilon_0}(0)$ on the unit circle. Since θ_0 is irrational by construction, there must exist positive ε near ε_0 where ℓ_ε is irrational and hence there are no periodic points on the invariant circle (by the theorem on p. 48 of [10]).

We can use the same trick as at the end of §1 to obtain examples where $n > 2$. Once again, the boundary conditions are not important.

There is one last comment. In this section, we have to be a little careful to obtain centre manifolds close to $\hat{N}$. We used this as a device to keep track of the leading terms on the centre manifold. There is an alternative which is shorter but gives a weaker result. Assume that $f_\varepsilon : N \to \mathbb{R}^2$ is C^k (in ε and x) where k is large and N is a neighbourhood of zero. We assume that $f_\varepsilon(0) = 0$ for all ε and at ε_0 an eigenvalue $e^{i\theta_0}$ cross through the unit circle and θ_0 satisfies the Diophantine conditions in [16], p. 18. Suppose $\tilde{B}$ is a small ball in $\mathbb{R}^2$. There are 3 possibilities: (i) there is a closed curve C containing 0 in its interior and $C \subseteq \tilde{B}$ such that $f_{\varepsilon_0}(C)$ is in the interior of C; (ii) as for (i) except that C is in the interior of $f_{\varepsilon_0}(C)$; or (iii) for every closed curve $C \subseteq \tilde{B}$ with $0 \in \operatorname{int} C$, $f_{\varepsilon_0}(C)$ intersects C. In case (iii), the Moser twist theorem (cp. [16], Theorem 11.1 and §13) implies that f_{ε_0} has an invariant circle $\tilde{C}$ with $0 \in \operatorname{int} \tilde{C}$ and $\tilde{C} \subseteq \tilde{B}$. (In fact, there are infinitely many invariant circles). Assume 0 is unstable for $\varepsilon > \varepsilon_0$ (and thus zero is stable for $\varepsilon < \varepsilon_0$). In case (i), $f_\varepsilon(C)$ is in the interior of C if ε is close to ε_0. If ε is also larger than ε_0, then if $\tilde{M}$ is a suitable small contractible neighbourhood of zero $f_\varepsilon(\partial\tilde{M})$ does not intersect the closure of $\tilde{M}$. (Remember that both eigenvalues are greater than 1 in absolute value). Thus $M = \overline{\operatorname{int} C} \setminus \tilde{M}$ is f_ε invariant. Hence $\hat{M}_\varepsilon = \bigcap_{n=1}^{\infty} f_\varepsilon^n(M)$ is compact and invariant. Thus, for every ε larger than ε_0 and close to ε_0, there is an invariant small "generalized annulus" $\hat{M}_\varepsilon$. We can think of $\hat{M}_\varepsilon$ as a "generalized annulus" because it has the Cech cohomology of a circle (in fact the shape of a circle in the sense of Mardesic and Segal [13]). In case (i), we obtain a similar result for $\varepsilon < \varepsilon_0$ by considering f_ε^{-1}. One disadvantage of this procedure (other than not knowing $\hat{M}_\varepsilon$ is a true circle) is that it is unclear if $\hat{M}_\varepsilon$ contains periodic points of f_ε. (If ε is close to ε_0, our choice of θ_0 ensures that they can only have very large minimal periods).

3. - The one-dimensional case with periodic boundary conditions

In this section, we show that similar behaviour occurs in the one-dimensional case with period boundary conditions. The arguments are similar and we stress the differences. The arguments are in fact a little easier.

We first consider the linear problem

$$\text{(9)} \qquad \begin{gathered} \frac{\partial u}{\partial t} = u_{xx} + a(x,t)u \\ u \text{ is 1 periodic in } x \end{gathered}$$

(or equivalently, if a is 1-periodic in x, $u(0,t) = u(1,t)$, $u_x(0,t) = u_x(1,t)$).

Assume that $u(x,t)$ is a 2π-periodic (in t) solution of (9). For each $t > 0$, there is defined a lap number of $u(x,t)$ on $[0,1]$, denoted by $\ell(u(\cdot,t))$ (cp. [1] or [3]). Thus $\ell(u(\cdot,t))$ is non-increasing in t and by [1], ℓ is strictly decreasing at $\bar{t}$ if there is an $\bar{x}$ such that $u(\bar{x},\bar{t}) = 0$ and $u_x(\bar{x},\bar{t}) = 0$. However, u is 2π-periodic in t and thus $\ell(u(\cdot,t))$ is periodic in t. Hence, if u is 2π-periodic in t, $u_x(x,t) \neq 0$ when $u(x,t) = 0$. Hence, if a is smooth, the implicit function theorem implies that the zero set of u in $[0,1] \times [0,2\pi]$ is a finite set of smooth non-intersecting curves $x = g_i(t)$ for $0 \le t < 2\pi$, $1 \le i \le k$. A second consequence of the above result is that, if u_1 and u_2 are linearly independent 2π-periodic solutions of (9), then they have no common zero. Thus follows because, if $u_1(x_0,t_0)$ and $u_2(x_0,t_0)$ were both zero, a suitable linear combination $w = c_1u_1 + c_2u_2$ would have $w(x_0,t_0) = w_x(x_0,t_0) = 0$. This is impossible by our comments above.

We now note that the analogue of the proposition of §1 holds. Thus, to construct a counterexample, we need only produce a double eigenvalue such as the eigenfunctions e_1, e_2 and the adjoint eigenfunctions f_1, f_2 have the property that the four functions e_if_j are linearly independent functions on $[0,1] \times [0,2\pi]$. (The arguments in §2 are unchanged). Suppose not, that is, suppose that

$$\sum_{i,j=1}^{2} c_{ij}\, e_i(x,t) f_j(x,t) = 0 \tag{10}$$

on $[0,1] \times [0,2\pi]$ where c_{ij} are not all zero. It follows easily that $c_{12}\,e_1 + c_{22}e_2$ vanishes on any curve $x = \hat{g}_i(t)$ where f_1 vanishes. Note that $c_{12} = c_{22} = 0$ is impossible because it would follow easily from (10) that $c_{11}\,e_1 + c_{21}\,e_2$ vanishes on $[0,1] \times [0,2\pi]$ (since the zeros of f_1 are nowhere dense). This is impossible because e_1 and e_2 are linearly independent. By linear independence, we also see that $c_{12}\,e_1 + c_{22}\,e_2$ does not vanish identically. Suppose we choose a to be invariant in x under the reflection R about $x = \frac{1}{2}$. Then we can choose f_1 to be even in x and f_2 odd in x (about $x = \frac{1}{2}$) with the corresponding symmetries for the e_i. Now since f_1 is even in x, if $f_1(x,t) = 0$, $c_{12}\,e_1 + c_{22}\,e_2$ vanishes at both (x,t) and (Rx,t). Since $e_1(Rx,t) = e_1(x,t)$ while $e_2(Rx,t) = -e_2(x,t)$ and e_1 and e_2 have no common-zeros, we see that this can only happen if $c_{12} = 0$ or $c_{22} = 0$. As we see below, we can ensure that $c_{12} \neq 0$ and hence $c_{22} = 0$. Thus all the zeros of f_1 are zeros of e_1. Since we can interchange the role of the e's and f's, e_1 and f_1 have the same zeros.

We will now use a perturbation idea similar to §1 to ensure this need not happen. We start with $a(x,t) = 1$ where we have a double eigenvalue λ_0 and then add a perturbation $\varepsilon a_1(x,t)$ where a_1 is invariant under R and under translations of $\frac{1}{4}$. (Thus a_1 is invariant under the dihedral group for the usual action of D_4 on S^1. Here we are identifying periodic functions on $[0,1]$ with functions on S^1). If $\varepsilon = 0$, a simple computation shows that $f_1 = e_1$, $f_2 = e_2$ and the only relation of the type (10) is $c_{11} = c_{22} = 0$, $c_{12} = -c_{21}$. As in §1, our symmetries ensure that the double eigenvalue persists under the perturbation. For the perturbed problem, the eigenvectors e_1^ε, e_2^ε and the adjoint eigenvectors f_1^ε, f_2^ε (where e_1, f_1 are even under R while e_2, f_2 are odd) either have no relation of the form (10) in which case we are finished or the only relation has c_{11}, c_{22} small, $c_{12} \doteq 1$, $c_{21} \doteq -1$. (As in §1, this follows by continuity). By our comments in the previous paragraph, such a relation ensures that f_1^ε and e_1^ε have the same zeros. We show that this does not occur if we choose a_1 suitably. Suppose that r_0 is a zero of e_1. By a similar argument to one in Step 1 in the argument in §1, it suffices to find a_1 orthogonal to e_1^2 so that the Fourier series for $t \in [0, 2\pi]$

$$\sum \frac{2ij\alpha_{jk}}{j^2 + (\lambda_k - \lambda_0)^2} e^{ijt} \phi_k(r_0) \tag{11}$$

is not trivial for some $t \in [0, 2\pi]$ where $\phi_k(r)$ are the eigenfunctions of the unperturbed problem, λ_k are the corresponding eigenvalues, α_{jk} are the coefficients in the expansion of $a_1 f_1$ on $[0, 1] \times [0, 2\pi]$ and the summation is over $j = -\infty$ to ∞, $k = 1, \ldots, \infty$ with $j = 0$, $\lambda_k = \lambda_0$ deleted. Here, as in §1, we need to choose a_1 so that the expansion of $a_1 f_1$ on $[0, 1] \times [0, 2\pi]$ contains only a finite number of non-zero terms. Much as in part of the argument in Step 2 in §1, the closure of $\{a_1 f_1\}$ in $L^p([0, 1] \times [0, 2\pi])$ includes gf_1, where g is any symmetric function in $L^p([0, 1] \times [0, 2\pi])$ for the D_4 action satisfying the orthogonality condition. Thus it suffices to prove that (11) is not trivial when α_{jk} are now the coefficients of the expansion of gf_1. Since we can think of g's as effectively any function on the part $[0, \frac{1}{8}]$ of the interval (in x) and satisfying the orthogonality condition, this is easy but tedious. (For example, we can choose $f_1(x) = \cos 6\pi x$, $r_0 = \frac{1}{12}$ and $g(x) = \cos 8\pi x \cos kt$ where k is a large positive integer). This completes the construction.

The arguments in §2 can be similarly extended.

REMARKS

1. It is easy to use this one dimensional example to produce examples on annular domains. (Essentially, we do the linear part by separation of variables). To use this approach to obtain examples with smooth coefficients on convex domains, we need to use an approximation argument and the results seem to be less precise than those of §1 and §2.

2. Another way one might try to proceed to find complex eigenvalues of the linear part is to try $a(x, t)$ which are step functions in t. This also appears tedious because one needs at least 2 jumps in t in $(0, 2\pi)$ to obtain complex eigenvalues.

3. A similar phenomena occurs for some ordinary differential equations. For $n \geq 2$, there exists a smooth function $W : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ which is 2π-periodic in t (where $W = W(x, t)$) such that there is a solution $z(t)$ of $x'(t) = \nabla_x W(x, t)$ which is bounded for $t \geq 0$ but which does not contain a 2π-periodic solution in its ω-limit set.

REFERENCES

[1] S. ANGENENT, *The zero set of a solution of a parabolic equation*, J. Reine Angew. Math. **396** (1988), 79–96.

[2] P. BRUNOVSKY - P. POLAČIK B. SANSTEDE, *Convergence in general parabolic equations in one space dimension*, preprint.

[3] X. CHEN - H. MATANO, *Convergence, asymptotic periodicity and finite time blow-up in one space dimension semilinear heat equations*, J. Differential Equations **78** (1989), 159–172.

[4] P. CHOSSAT - M. GOLUBITSKY, *Hopf bifurcation in the presence of symmetry, centre manifolds and Liapounov-Schmidt reduction*, p. 344–352 in "Oscillation, bifurcation and chaos", Amer Math Soc., Providence, 1987.

[5] E.N. DANCER, *The effect of domain shape on the number of positive solutions of certain nonlinear equations II*, J. Differential Equations, **87** (1990), 316–339.

[6] M. GOLUBITSKY - I. STEWART - D. SCHAEFFER, *Singularities and groups in bifurcation theory II*, Springer, Berlin, 1988.

[7] D. HENRY, *Geometric theory of semilinear parabolic equations*, Lecture Notes in Mathematics 840, Springer, Berlin, 1981.

[8] P. HESS, *On positive solutions of semilinear periodic parabolic problems*, in Infinite-dimensional systems, Lecture Notes in Mathematics 1076, Springer, Berlin, 1984.

[9] M. HIRSCH - C. PUGH - M. SHUB, *Invariant manifolds*, Lecture Notes in Mathematics, Vol. 583, Springer, Berlin, 1977.

[10] G. IOOSS, *Bifurcation of maps and application*, North Holland, Amsterdam, 1979.

[11] T. Kato, *Perturbation theory for linear operators*, Springer, Berlin, 1966.

[12] O. LADYZHENSKAYA - V. SOLONNIKOV - N. URALCEVA, *Linear and quasilinear equations of parabolic type*, Amer. Math. Soc., Providence, 1981.

[13] I. MARDESIC - J. SEGAL, *Shape theory*, North Holland, Amsterdam, 1982.

[14] P. POLAČIK, *Complicated dynamics in scalar semilinear parabolic equations in higher space dimension*, to appear.

[15] D. RUELLE - F. TAKENS, *On the nature of turbulence*, Comm. Math. Phys. **20** (1972), 167–192.

[16] S. STENBERG, *Celestial mechanics - Part II*, Benjamin, New York, 1969.

[17] A. VANDERBAUWHEDE, *Invariant manifolds in infinite dimensions*, 409–420 in "Dynamics of infinite dimensional systems", Springer, Berlin, 1987.

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