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Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 18, nº 2 (1991), p. 251-293

<http://www.numdam.org/item?id=ASNSP_1991_4_18_2_251_0>

Homogenization and Corrector for the Wave Equation in Domains with Small Holes

D. CIORANESCU - P. DONATO - F. MURAT - E. ZUAZUA

1. - Introduction

In this paper we study the homogenization of the wave equation with Dirichlet boundary conditions in perforated domains with small holes. Let Ω be a fixed bounded domain of $\mathbb{R}^n$ $(n \geq 2)$. Denote by Ω_ε the domain obtained by removing from Ω a set $S_\varepsilon = \bigcup_{i=1}^{N(\varepsilon)} S_\varepsilon^i$ of $N(\varepsilon)$ closed subsets of Ω (here, $\varepsilon > 0$ denotes a parameter which takes its values in a sequence which tends to zero while $N(\varepsilon)$ tends to infinity). Finally let $T > 0$ be fixed. We consider the wave equation

$$(1.1) \qquad \begin{cases} u_\varepsilon'' - \Delta u_\varepsilon = f_\varepsilon & \text{in } \Omega_\varepsilon \times (0,T) \\ u_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon \times (0,T) \\ u_\varepsilon(0) = u_\varepsilon^0, \ \ u_\varepsilon'(0) = u_\varepsilon^1 & \text{in } \Omega_\varepsilon. \end{cases}$$

Our aim is to describe the convergence of the solutions u_ε, to identify the equation satisfied by the limit u and to give corrector results.

In the whole of the present paper the sets Ω_ε will be assumed to satisfy the requirements of the abstract framework introduced by D. Cioranescu and F. Murat [6] (see assumption (2.1) below) for the study of the homogenization of elliptic problems in perforated domains with Dirichlet boundary data. The model case (see Figure 1 on the next page) is provided by a domain periodically perforated (with a period 2ε in the direction of each coordinate axis) by holes of size r_ε, where r_ε is asymptotically equal to or smaller than a "critical size"

Pervenuto alla Redazione il 29 Ottobre 1990.

a_ε. This critical size a_ε is given by

$$\begin{cases} a_\varepsilon = C_0 \varepsilon^{n/(n-2)} & \text{for } n \geq 3 \\ a_\varepsilon = \delta_\varepsilon \exp(-C_0/\varepsilon^2) & \text{for } n = 2 \end{cases}$$

where $C_0 > 0$ is fixed and $\varepsilon^2 \log \delta_\varepsilon \to 0$ as $\varepsilon \to 0$ (see Section 2 below).

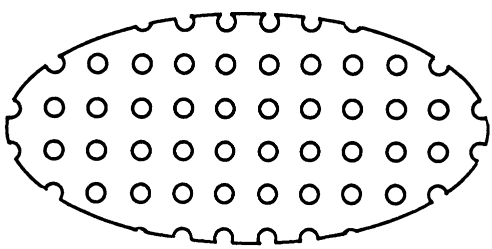

Figure 1

In this abstract framework, let v_ε be the solution of the problem

(1.2)
$$\begin{cases} -\Delta v_\varepsilon = g & \text{in } \Omega_\varepsilon \\ v_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon \end{cases}$$

where g is given in $H^{-1}(\Omega)$. Denote by $\tilde{v}_\varepsilon$ the extension of v_ε to the whole of Ω defined by

$$\tilde{v}_\varepsilon = \begin{cases} v_\varepsilon & \text{in } \Omega_\varepsilon \\ 0 & \text{in the holes } S_\varepsilon. \end{cases}$$

It has been shown in [6] that $\tilde{v}_\varepsilon$ weakly converges in $H_0^1(\Omega)$ to the solution v of the problem

(1.3)
$$\begin{cases} -\Delta v + \mu v = g & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases}$$

where μ is a nonnegative Radon measure belonging to $H^{-1}(\Omega)$. This measure appears in the abstract framework and relies on the behaviour of the capacity of the set S_ε as $\varepsilon \to 0$. In the model case described above, μ is a constant which is strictly positive when the size of the holes is the critical one. In such case the additional zero order term μv appears in the limit equation.

As a first hypothesis on the data in (1.1) we assume that

(1.4)
$$u_\varepsilon^0 \in H_0^1(\Omega_\varepsilon), \;\; u_\varepsilon^1 \in L^2(\Omega_\varepsilon), \;\; f_\varepsilon \in L^1(0,T; L^2(\Omega_\varepsilon))$$

$$(1.5)\qquad \begin{cases} \tilde{u}_\varepsilon^0 \rightharpoonup u^0 & \text{weakly in } H_0^1(\Omega) \\ \tilde{u}_\varepsilon^1 \rightharpoonup u^1 & \text{weakly in } L^2(\Omega) \\ \tilde{f}_\varepsilon \rightharpoonup f & \text{weakly in } L^1(0,T;L^2(\Omega)). \end{cases}$$

As it is naturally expected, the extension $\tilde{u}_\varepsilon$ of the solution u_ε of (1.1) weakly $*$ converges in $L^\infty(0,T;H_0^1(\Omega)) \cap W^{1,\infty}(0,T;L^2(\Omega))$ to the solution u of the problem

$$(1.6)\qquad \begin{cases} u'' - \Delta u + \mu u = f & \text{in } \Omega \times (0,T) \\ u = 0 & \text{on } \partial\Omega \times (0,T) \\ u(0) = u^0,\ \ u'(0) = u^1 & \text{in } \Omega. \end{cases}$$

This result is proved in Theorem 3.1 of Section 3.

In Section 4 we prove corrector results for the problem (1.1) by following ideas similar to those used by S. Brahim-Ostmane, G.A. Francfort and F. Murat [1], who adapted to the wave equation the ideas introduced by L. Tartar [18] in the elliptic case. Under a special assumption on the data (see (1.9) below) we prove in Theorem 4.1 that $\tilde{u}_\varepsilon$ can be decomposed in

$$(1.7)\qquad \tilde{u}_\varepsilon = u w_\varepsilon + R_\varepsilon.$$

In this decomposition u is the solution of (1.6), w_ε are functions which appear in the abstract framework (they are related to the capacitary potential of the holes) and the remainder R_ε satisfies the strong convergence property

$$(1.8a)\qquad R_\varepsilon \to 0 \quad \text{strongly in } C^0([0,T];W^{1,1}(\Omega));$$

we also prove that

$$(1.8b)\qquad \tilde{u}_\varepsilon' \to u' \quad \text{strongly in } C^0([0,T];L^2(\Omega)).$$

The term $u w_\varepsilon$ is then a good approximation ("the corrector") of the solution u_ε.

In order to obtain (1.7)-(1.8) we have to make special assumptions on the data; to be precise, we will assume that there exists $g_\varepsilon \in H^{-1}(\Omega)$ such that

$$(1.9)\qquad \begin{cases} -\Delta u_\varepsilon^0 = g_\varepsilon & \text{in } \mathcal{D}'(\Omega_\varepsilon) \text{ with } g_\varepsilon \to g \text{ strongly in } H^{-1}(\Omega) \\ \tilde{u}_\varepsilon^1 \to u^1 & \text{strongly in } L^2(\Omega) \\ \tilde{f}_\varepsilon \to f & \text{strongly in } L^1(0,T;L^2(\Omega)). \end{cases}$$

Note that the initial condition u_ε^0 has to satisfy the very special hypothesis (1.9a). The meaning of this hypothesis is that u_ε^0 has to be well adapted to the asymptotic behaviour of the holes. In general (1.9a) implies only the weak (and not strong) convergence in $H_0^1(\Omega)$ of $\tilde{u}_\varepsilon^0$ (see Remark 4.1 below).

Assumption (1.9) on the data turns to imply the convergence of the energies

$$\frac{1}{2}\int_{\Omega_\varepsilon} [|\nabla u_\varepsilon(x,t)|^2 + |u'_\varepsilon(x,t)|^2]\mathrm{d}x$$

to the energy of the limit problem

$$\frac{1}{2}\int_{\Omega} [|\nabla u(x,t)|^2 + |u'(x,t)|^2]\mathrm{d}x + \frac{1}{2}\int_{\Omega} |u(x,t)|^2\mathrm{d}\mu(x).$$

This convergence of energies is at the root of the proof of the corrector result (1.7)-(1.8).

We also consider in Section 6 the case of initial data which have a regularity weaker than (1.4)-(1.5), i.e., the case where

$$u_\varepsilon^0 \in L^2(\Omega_\varepsilon),\ \ u_\varepsilon^1 \in H^{-1}(\Omega_\varepsilon),\ \ f_\varepsilon \in L^1(0,T;L^2(\Omega_\varepsilon)). \tag{1.10}$$

We then prove that under suitable convergence assumptions on u_ε^0, u_ε^1 and f_ε, the extension $\tilde{u}_\varepsilon$ of the solution of (1.1) weakly $*$ converges in $L^\infty(0,T;L^2(\Omega))$ to the solution u of the problem (1.6) (see Theorem 6.2). In this setting we also obtain, under special assumptions on the data, the following corrector result (see Theorem 6.3):

$$\tilde{u}_\varepsilon \to u \qquad \text{strongly in } C^0([0,T];L^2(\Omega)). \tag{1.11}$$

Besides their own interest, the previous results have interesting applications to the exact boundary controllability problem for the wave equation in domains with small holes, see D. Cioranescu, P. Donato and E. Zuazua [4], [5], where Theorems 3.1, 4.1 and 5.1 are used as a tool combined to the "Hilbert Uniqueness Method" introduced by J.-L. Lions [14].

The present paper is only concerned with homogeneous Dirichlet boundary data. Let us mention that the case of homogeneous Neumann data leads to completely different results, the critical size being in this case $a_\varepsilon = \varepsilon$ (see D. Cioranescu and P. Donato [3] for the homogenization of this problem).

The present paper is organized as follows:

Section 2 is divided into three parts. We first recall (Subsection 2.1) the abstract framework of [6] on the geometry of the holes, as well as the results dealing with the homogenization of the elliptic problem in this setting. The counterparts of some of these results for the time dependent case are given in

Subsection 2.3, where a quasi-extension operator is also introduced. Subsection 2.2 presents some compactness results in the spaces $L^p(0,T;X)$.

In Section 3 we prove the weak $*$ convergence in $L^\infty(0,T;H_0^1(\Omega)) \cap W^{1,\infty}(0,T;L^2(\Omega))$ of the extension $\tilde{u}_\varepsilon$ of the solution u_ε of (1.1) to the solution u of (1.6). Lower semicontinuity of the corresponding energy is also proved.

In Section 4 the corrector result (1.7)-(1.8) is proved when the assumption (1.9) on the data is made.

In Section 5 we consider the case where the size of the holes is smaller than the critical one. In this situation $\mu = 0$ and the convergence of $\tilde{u}_\varepsilon$ to u is proved to take place in the strong topology of $L^\infty(0,T;H_0^1(\Omega)) \cap W^{1,\infty}(0,T;L^2(\Omega))$.

In Section 6 we prove the convergence of the solutions and the corrector result (1.11) in the case where the data only meet a weaker regularity assumption (see (1.10)).

Finally the Appendix is devoted to the proof of the density of $\mathcal{D}(\Omega)$ in $H_0^1(\Omega) \cap L^2(\Omega; d\mu)$ when μ is a nonnegative and finite Radon measure which belongs to $H^{-1}(\Omega)$.

2. - Geometric setting. A review of the elliptic case and preliminary results

This section is divided into three parts. In the first one (Subsection 2.1) we describe the geometry of the problem and the abstract framework introduced by D. Cioranescu and F. Murat [6] in which the present work will be carried out; we also recall the homogenization and corrector results obtained in this framework when dealing with elliptic problems. Subsection 2.2 deals with compactness results in the spaces $L^p(0,T;X)$. In Subsection 2.3 we introduce a "quasi-extension" operator and prove some pointwise (with respect to the time variable) lower semicontinuity results of the energy for the time dependent case. These latest results are in some sense the time dependent counterparts of the results presented in Subsection 2.1 for the elliptic case.

2.1 *The geometry of the problem. A review of homogenization and corrector results in the elliptic case.*

Let Ω be a bounded domain of $\mathbb{R}^n$ $(n \geq 2)$ (no regularity is assumed on the boundary $\partial\Omega$), and let Ω_ε be the domain obtained by removing from Ω a set $S_\varepsilon = \bigcup_{i=1}^{N(\varepsilon)} S_\varepsilon^i$ of $N(\varepsilon)$ closed subsets of $\mathbb{R}^n$ (the holes). Here ε is a parameter which takes its values in a sequence which tends to zero while $N(\varepsilon)$ is an integer which tends to infinity.

Instead of making direct geometric assumptions on the holes S_ε^i, we adopt here the abstract framework introduced by D. Cioranescu and F. Murat in [6] where the assumption on the geometry of the holes is made by assuming the existence of a suitable family of test functions. Precisely we will assume that

$$(2.1)\qquad \begin{cases} \text{there exists a sequence of test functions } w_\varepsilon \text{ such that} \\ \text{(i) } w_\varepsilon \in H^1(\Omega) \cap L^\infty(\Omega), \ \|w_\varepsilon\|_{L^\infty(\Omega)} \le M_0 \\ \text{(ii) } w_\varepsilon = 0 \text{ on } S_\varepsilon \\ \text{(iii) } w_\varepsilon \rightharpoonup 1 \text{ weakly in } H^1(\Omega) \text{ and a.e. in } \Omega \\ \text{(iv) } -\Delta w_\varepsilon = \mu_\varepsilon - \gamma_\varepsilon \text{ where } \mu_\varepsilon, \gamma_\varepsilon \in H^{-1}(\Omega), \\ \quad \mu_\varepsilon \to \mu \text{ strongly in } H^{-1}(\Omega) \\ \quad \langle \gamma_\varepsilon, v_\varepsilon \rangle_\Omega = 0 \text{ for any } v_\varepsilon \in H_0^1(\Omega) \text{ such that } v_\varepsilon = 0 \text{ on } S_\varepsilon. \end{cases}$$

In (2.1) and henceforth $\langle \cdot, \cdot \rangle_\Omega$ denotes the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$, while $\langle \cdot, \cdot \rangle_{\Omega_\varepsilon}$ will denote the duality pairing between $H^{-1}(\Omega_\varepsilon)$ and $H_0^1(\Omega_\varepsilon)$.

REMARK 2.1. Hypothesis (2.1) differs from the original framework proposed in [6] by the fact that in (2.1) μ is only assumed to belong to $H^{-1}(\Omega)$ while in [6] μ was assumed to belong to $W^{-1,\infty}(\Omega)$. Nevertheless the present variation allows one to prove the same type of results in the elliptic case (see H. Kacimi and F. Murat [12, Paragraphe 2]). Note that the framework adopted here is of the type (H5)′ in the notation of [6] (see [6, Remarque 1.6]) which for the present case is more convenient than the hypothesis (H5) of [6]. □

EXAMPLE 2.1. Let us present the typical example where assumption (2.1) is satisfied. Consider the case where Ω is periodically perforated, with a period 2ε in the direction of each coordinate axis, by small holes S_ε^i of form S and size a_ε obtained from a model hole S by a translation and an a_ε-homothety. To be precise S_ε^i is given by

$$(2.2)\qquad S_\varepsilon^i = a_\varepsilon S + 2\varepsilon \sum_{k=1}^{n} i_k e_k$$

where $(i_1, i_2, \dots, i_n)$ is a multi-index of $\mathbb{Z}^n$, $\{e_1, \dots, e_n\}$ is the canonical basis of $\mathbb{R}^n$, S is a closed set contained in the ball B_1 of radius 1 centered at the origin (in the case $n = 2$, S is assumed to contain a ball centered at the origin, see [12, Remarque 2.2]), and $a_\varepsilon < \varepsilon$ satisfies

$$(2.3)\qquad \begin{cases} \varepsilon^2 \log a_\varepsilon \to -C_0 & \text{if } n = 2 \\ a_\varepsilon \varepsilon^{-n/(n-2)} \to C_0 & \text{if } n \ge 3 \end{cases}$$

for a given $C_0 > 0$.

The model case is then

$$
(2.4)\qquad \begin{cases} a_\varepsilon = \delta_\varepsilon \exp\left(-\frac{C_0}{\varepsilon^2}\right) & \text{if } n = 2, \text{ with } \varepsilon^2 \log \delta_\varepsilon \to 0 \text{ as } \varepsilon \to 0 \\ a_\varepsilon = C_0 \varepsilon^{n/(n-2)} & \text{if } n \geq 3, \end{cases}
$$

the model hole S being choosen as the unit ball of $\mathbb{R}^n$.

When a_ε is given by (2.3) or (2.4), it is possible to construct "explicitely" w_ε (see [6, Exemple modèle 2.1]) on the cube P_ε^i of size 2ε of center $x_\varepsilon^i = 2\varepsilon \sum_{k=1}^{n} i_k e_k$: consider the function $w_\varepsilon \in H^1(P_\varepsilon^i)$ defined by

$$
(2.5)\qquad \begin{cases} \Delta w_\varepsilon = 0 & \text{in } B_\varepsilon^i \backslash S_\varepsilon^i \\ w_\varepsilon = 0 & \text{in } S_\varepsilon^i \\ w_\varepsilon = 1 & \text{in } P_\varepsilon^i \backslash B_\varepsilon^i \end{cases}
$$

where B_ε^i is the ball of center x_ε^i and radius ε. (When S is a ball the function w_ε can be easily computed in radial coordinates). The function w_ε defined by (2.5) in each P_ε^i satisfies (2.1) with

$$
(2.6)\qquad \begin{cases} \mu = \dfrac{1}{C_0} \dfrac{\pi}{2} & \text{if } n = 2 \\ \mu = \dfrac{C_0^{n-2}}{2^n} \operatorname{Cap}(S, \mathbb{R}^n) & \text{if } n \geq 3. \end{cases}
$$

For the proof see [6, Exemple modèle 2.1] and [12, Théorème 2.1]; in (2.6)

$$
\operatorname{Cap}(S, \mathbb{R}^n) = \inf_{\substack{\varphi \in \mathcal{D}(\mathbb{R}^n) \\ \varphi = 1 \text{ on } S}} \int_{\mathbb{R}^n} |\nabla \varphi|^2 \mathrm{d}x
$$

is the capacity in $\mathbb{R}^n$ of the closed set S.

We refer the reader to [6] and to H. Kacimi [11, Chapitre 1] for other examples of holes where assumption (2.1) is satisfied. □

REMARK 2.2. In the above Example 2.1, the size a_ε defined in (2.3) is critical in the following sense: when the size of the holes is r_ε with $r_\varepsilon << a_\varepsilon$, i.e., when

$$
(2.7)\qquad \begin{cases} \varepsilon^2 \log r_\varepsilon \to -\infty & \text{if } n = 2 \\ \varepsilon^{n/(n-2)}/r_\varepsilon \to +\infty & \text{if } n \geq 3, \end{cases}
$$

hypothesis (2.1) is satisfied, but in (iii) w_ε now converges *strongly* to 1 in $H^1(\Omega)$ and in (iv), μ_ε and γ_ε strongly converge to 0; thus $\mu = 0$ in this case. On the other hand, if $a_\varepsilon << r_\varepsilon$ (which corresponds to replace ∞ by 0 in (2.7)), it can be proved that there is no sequence satisfying assertions (i), (ii) and (iii) of (2.1). The size a_ε given by (2.3) is therefore the only one for which (2.1) holds with weak (and not strong) convergence of w^ε to 1 in (iii). □

In the abstract framework of hypothesis (2.1), the following results can be proved (see [6, Chapitre 1] and [12, Paragraphe 2]).

LEMMA 2.1. *If* (2.1) *holds true, the distribution* μ *which appears in* (iv) *is given by*

$$(2.8)\qquad \langle \mu, \varphi\rangle_\Omega = \lim_{\varepsilon\to 0}\int_\Omega \varphi|\nabla w^\varepsilon|^2 \mathrm{d}x \quad \textit{for all } \varphi\in\mathcal{D}(\Omega).$$

Thus μ *is a positive Radon measure as well as an element of* $H^{-1}(\Omega)$*; moreover* $\mu(\Omega)$ *is finite.*

A result of J. Deny [7] (see also H. Brézis and F.E. Browder [2]) then asserts that any function $v\in H_0^1(\Omega)$ is measurable with respect to the measure $\mathrm{d}\mu$ and belongs to $L^1(\Omega, \mathrm{d}\mu)$, namely

$$(2.9)\qquad \begin{cases} \text{if } \mu\in H^{-1}(\Omega),\ \ \mu\geq 0 \text{ and } v\in H_0^1(\Omega), \\ \text{then } v\in L^1(\Omega;\mathrm{d}\mu) \text{ and } \langle\mu, v\rangle_\Omega = \int_\Omega v\,\mathrm{d}\mu. \end{cases}$$

This allows one to define without ambiguity the space

$$(2.10)\qquad V = H_0^1(\Omega)\cap L^2(\Omega;\mathrm{d}\mu)$$

which is a Hilbert space for the scalar product

$$(2.11)\qquad a(u,v) = \int_\Omega \nabla u\nabla v\,\mathrm{d}x + \int_\Omega uv\,\mathrm{d}\mu.$$

Finally, for any $v\in L^2(\Omega_\varepsilon)$ define $\tilde{v}$ as the extension of v by zero outside Ω_ε, i.e.

$$(2.12)\qquad \tilde{v}(x) = \begin{cases} v(x) & \text{if } x\in\Omega_\varepsilon \\ 0 & \text{if } x\in S_\varepsilon. \end{cases}$$

Of course one has

$$(2.13)\qquad \|\tilde{v}\|_{L^2(\Omega)} = \|v\|_{L^2(\Omega_\varepsilon)} \quad \text{if } v\in L^2(\Omega_\varepsilon).$$

Note that $\tilde{v}$ belongs to $H_0^1(\Omega)$ if v belongs to $H_0^1(\Omega_\varepsilon)$ and that

$$(2.14)\qquad \|\tilde{v}\|_{H_0^1(\Omega)} = \|v\|_{H_0^1(\Omega_\varepsilon)} \quad \text{if } v\in H_0^1(\Omega_\varepsilon).$$

We recall the following result on the homogenization of elliptic problems.

THEOREM 2.2. *Assume that* (2.1) *holds true and consider the solutions* v_ε *of the Dirichlet problems*

$$\begin{cases} -\Delta v_\varepsilon = g_\varepsilon & \text{in } \mathcal{D}'(\Omega_\varepsilon) \\ v_\varepsilon \in H_0^1(\Omega_\varepsilon) \end{cases} \tag{2.15}$$

where $g_\varepsilon \in H^{-1}(\Omega)$ *is such that*

$$g_\varepsilon \to g \quad \text{strongly in } H^{-1}(\Omega). \tag{2.16}$$

The sequence $\tilde{v}_\varepsilon$ *(obtained from the solution* v_ε *of* (2.15) *by the extension* (2.12)*) satisfies*

$$\begin{cases} \tilde{v}_\varepsilon \rightharpoonup v & \text{weakly in } H_0^1(\Omega) \\ \int\limits_{\Omega_\varepsilon} |\nabla v_\varepsilon|^2 \mathrm{d}x \to \int\limits_{\Omega} |\nabla v|^2 \mathrm{d}x + \int\limits_{\Omega} |v|^2 \mathrm{d}\mu \end{cases} \tag{2.17}$$

where $v = v(x)$ *is the unique solution of*

$$\begin{cases} -\Delta v + \mu v = g & \text{in } \mathcal{D}'(\Omega) \\ v \in V \end{cases} \tag{2.18}$$

(see Remark 2.3 *below). Moreover*

$$\begin{cases} \tilde{v}_\varepsilon = w_\varepsilon v + r_\varepsilon \\ r_\varepsilon \to 0 \quad \text{strongly in } W_0^{1,1}(\Omega). \end{cases} \tag{2.19}$$

Finally if v *belongs to* $H_0^1(\Omega) \cap C^0(\overline{\Omega})$, *the convergence of* r_ε *in* (2.19) *takes place in the strong topology of* $H_0^1(\Omega)$.

REMARK 2.3. Note that the variational formulation associated to (2.18) is (see (2.10), (2.11) for the definitions of V and a)

$$\begin{cases} a(v, y) = \langle g, y \rangle_\Omega, & \text{for all } y \in V \\ v \in V. \end{cases} \tag{2.20}$$

□

REMARK 2.4. Assertion (2.19) is a corrector result for the solution v_ε of the Dirichlet problem (2.15), since it allows one to replace $\tilde{v}_\varepsilon$ by the "explicit" expression $w_\varepsilon v$, up to the remainder r_ε which strongly converges to zero. □

It is finally worth mentioning the following lower semicontinuity of the energy.

THEOREM 2.3. *Assume that* (2.1) *holds true and consider a sequence* z_ε *such that*

(2.21)
$$\begin{cases} z_\varepsilon \in H_0^1(\Omega) & \\ z_\varepsilon = 0 & \textit{on } S_\varepsilon \\ z_\varepsilon \rightharpoonup z & \textit{weakly in } H_0^1(\Omega). \end{cases}$$

Then

(2.22)
$$\begin{cases} z \in V \\ \liminf\limits_{\varepsilon \to 0} \int_\Omega |\nabla z_\varepsilon|^2 \mathrm{d}x \geq \int_\Omega |\nabla z|^2 \mathrm{d}x + \int_\Omega |z|^2 \mathrm{d}\mu. \end{cases}$$

Moreover when z_ε *also satisfies*

(2.23)
$$\int_\Omega |\nabla z_\varepsilon|^2 \mathrm{d}x \to \int_\Omega |\nabla z|^2 \mathrm{d}z + \int_\Omega |z|^2 \mathrm{d}\mu$$

one has

(2.24)
$$\begin{cases} z_\varepsilon = w_\varepsilon z + r_\varepsilon \\ r_\varepsilon \to 0 \quad \textit{strongly in } W_0^{1,1}(\Omega). \end{cases}$$

Finally if z *belongs to* $H_0^1(\Omega) \cap C^0(\overline{\Omega})$, *the convergence of* r_ε *in* (2.24) *takes place in the strong topology of* $H_0^1(\Omega)$.

REMARK 2.5. Note that, in view of (2.9), any element of $H_0^1(\Omega)$ belongs to $L^1(\Omega; \mathrm{d}\mu)$; the first assertion of (2.22) thus claims that z actually belongs to $L^2(\Omega; \mathrm{d}\mu)$. □

2.2 *Some compactness lemmas.*

Let X and Y be two reflexive Banach spaces such that $X \subset Y$ with continuous and dense embedding. Denote by X' (resp. Y') the dual space of X (resp. Y) and by $\langle \cdot, \cdot \rangle_{X,X'}$ (resp. $\langle \cdot, \cdot \rangle_{Y,Y'}$) the duality pairing between X and X' (resp. Y and Y'). We will use the following space introduced in [16, Chapitre 3, Paragraphe 8.4]:

$$C_s^0(0,T;Y) = \{f \in L^\infty(0,T;Y) : t \mapsto \langle f(t), v\rangle_{Y,Y'} \text{ is continuous from } [0,T] \text{ into } \mathbb{R} \text{ for any fixed } v \in Y'\}.$$

LEMMA 2.4. *Consider a sequence* g_ε *such that*

(2.25)
$$\begin{cases} g_\varepsilon \rightharpoonup g & \textit{weakly } * \textit{ in } L^\infty(0,T;X) \\ g_\varepsilon \to g & \textit{strongly in } C^0([0,T];Y). \end{cases}$$

Then g_ε strongly converges to g in $C^0_s([0,T];X)$, i.e., for every $v \in X'$ the function

$$h_\varepsilon : t \longmapsto \langle g_\varepsilon(t), v\rangle_{X,X'} \tag{2.26}$$

belongs to $C^0([0,T])$ and satisfies

$$h_\varepsilon \to h \quad \text{strongly in } C^0([0,T]) \tag{2.27}$$

where h is defined by

$$h : t \longmapsto \langle g(t), v\rangle_{X,X'}. \tag{2.28}$$

PROOF. According to a result of J.-L. Lions and E. Magenes [16, Lemme 8.1, p. 297] we have

$$L^\infty(0,T;X) \cap C^0_s([0,T];Y) = C^0_s([0,T];X).$$

Hence g_ε belongs to $C^0_s([0,T];X)$ by (2.25) and therefore h_ε defined by (2.26) lies in $C^0([0,T])$.

To prove (2.27), it is sufficient to show that h_ε is a Cauchy sequence in $C^0([0,T])$. For a given $\hat{v} \in Y'$ introduce the function

$$\hat{h}_\varepsilon : t \longmapsto \langle g_\varepsilon(t), \hat{v}\rangle_{Y,Y'}.$$

We have

$$\left\{ \begin{aligned} |h_\varepsilon(t) - h_{\varepsilon'}(t)| &\le |h_\varepsilon(t) - \hat{h}_\varepsilon(t)| + |\hat{h}_\varepsilon(t) - \hat{h}_{\varepsilon'}(t)| \\ &\qquad + |\hat{h}_{\varepsilon'}(t) - h_{\varepsilon'}(t)| \\ &= |\langle g_\varepsilon(t), v - \hat{v}\rangle_{X,X'}| + |\langle g_\varepsilon(t) - g_{\varepsilon'}(t), \hat{v}\rangle_{Y,Y'}| \\ &\qquad + |\langle g_{\varepsilon'}(t), v - \hat{v}\rangle_{X,X'}| \\ &\le (\|g_\varepsilon\|_{L^\infty(0,T;X)} + \|g_{\varepsilon'}\|_{L^\infty(0,T;X)})\|v - \hat{v}\|_{X'} \\ &\qquad + \|g_\varepsilon - g_{\varepsilon'}\|_{C^0([0,T];Y)}\|\hat{v}\|_{Y'}. \end{aligned} \right. \tag{2.29}$$

Combining (2.25), (2.29) and the density of Y' in X', we easily deduce that h_ε is a Cauchy sequence in $C^0([0,T])$. □

PROPOSITION 2.5. *Assume further that the embedding $X \subset Y$ is compact. Let g_ε be a sequence such that*

$$g_\varepsilon \rightharpoonup g \quad \text{weakly in } L^1(0,T;X) \tag{2.30}$$

$$g'_\varepsilon \rightharpoonup g' \quad \text{weakly in } L^1(0,T;Y). \tag{2.31}$$

Then

$$g_\varepsilon \to g \quad \text{strongly in } C^0([0,T];Y). \tag{2.32}$$

PROOF. In order to obtain the result it is sufficient to prove that (see, e.g. J. Simon [17, Theorem 3])

$$\|g_\varepsilon(\cdot+h)-g_\varepsilon(\cdot)\|_{L^\infty(0,T-h;Y)}\to 0 \quad \text{as } h\to 0 \quad \text{uniformly in } \varepsilon. \tag{2.33}$$

On the first hand

$$\|g_\varepsilon(\cdot+h)-g_\varepsilon(\cdot)\|_{L^\infty(0,T-h;Y)}\le \sup_{t\in[0,T-h]}\int_t^{t+h}\|g'_\varepsilon(s)\|_Y ds. \tag{2.34}$$

On the other hand Theorem 4 of J. Diestel and J.J. Uhl, [8, p. 104] states that the norm (in Y) of a sequence which converges weakly in $L^1(0,T;Y)$ is uniformly integrable on $[0,T]$. This implies that the right hand side of (2.34) converges to zero as h converges to 0, uniformly in ε, which yields (2.33) and completes the proof. □

REMARK 2.6. The convergence (2.32) is proved in [17, Corollary 4] under the further assumption (compare with (2.31)) that for some $p>1$

$$g'_\varepsilon \rightharpoonup g' \qquad \text{weakly in } L^p(0,T;Y). \tag{2.35}$$

Observe also that compactness (2.32) is false in general when (2.31) is replaced by the boundedness of g'_ε in $L^1(0,T;Y)$: consider, e.g. the case where $X=Y=\mathbb{R}$, $g_\varepsilon(t)=t/\varepsilon$ if $0\le t\le\varepsilon$, $g_\varepsilon(t)=1$ if $\varepsilon\le t\le 1$ and $g(t)=1$ if $0\le t\le 1$. □

As a consequence of Lemma 2.4 and Proposition 2.5 we have the following result.

COROLLARY 2.6. *Assume that the embedding $X\subset Y$ is compact. Let g_ε be a sequence satisfying*

$$\begin{cases} g_\varepsilon \rightharpoonup g & \textit{weakly } * \textit{ in } L^\infty(0,T;X) \\ g'_\varepsilon \rightharpoonup g' & \textit{weakly in } L^1(0,T;Y). \end{cases} \tag{2.36}$$

Then g_ε strongly converges to g in $C^0_s([0,T];X)$, i.e.

$$\langle g_\varepsilon(\cdot),v\rangle_{X,X'}\to\langle g(\cdot),v\rangle_{X,X'} \quad \textit{strongly in } C^0([0,T]) \tag{2.37}$$

for all $v\in X'$.

REMARK 2.7. Note that (2.37) implies in particular that

$$g_\varepsilon(t)\rightharpoonup g(t) \qquad \text{weakly in } X \text{ for any fixed } t\in[0,T]$$

and not only almost everywhere in $[0,T]$. □

2.3 *Counterparts of some elliptic results. A "quasi-extension" operator.*

PROPOSITION 2.7. *Assume that* (2.1) *holds true and consider a sequence of functions* v_ε *in* $L^\infty(0,T;H_0^1(\Omega_\varepsilon))\cap W^{1,\infty}(0,T;L^2(\Omega_\varepsilon))$ *satisfying*

$$(2.38)\qquad \begin{cases} \tilde{v}_\varepsilon \rightharpoonup v & \textit{weakly} * \textit{ in } L^\infty(0,T;H_0^1(\Omega)) \\ \tilde{v}'_\varepsilon \rightharpoonup v' & \textit{weakly} * \textit{ in } L^\infty(0,T;L^2(\Omega)). \end{cases}$$

Then

$$(2.39)\qquad \langle \theta, \tilde{v}_\varepsilon(\cdot)\rangle_\Omega \to \langle \theta, v(\cdot)\rangle_\Omega \textit{ strongly in } C^0([0,T]) \textit{ for any } \theta \in H^{-1}(\Omega)$$

and on the other hand

$$(2.40)\qquad v \in L^\infty(0,T;V)\cap W^{1,\infty}(0,T;L^2(\Omega)).$$

REMARK 2.8. From (2.38) we know that

$$v \in L^\infty(0,T;H_0^1(\Omega))\cap W^{1,\infty}(0,T;L^2(\Omega));$$

property (2.40) then asserts that the limit v further satisfies

$$v \in L^\infty(0,T;L^2(\Omega;\mathrm{d}\mu)). \qquad \square$$

PROOF OF PROPOSITION 2.7. Convergence (2.39) is a direct consequence of Corollary 2.6 applied to $X = H_0^1(\Omega)$ and $Y = L^2(\Omega)$.

In view of Remark 2.7, we have in particular

$$\tilde{v}_\varepsilon(t) \rightharpoonup v(t) \qquad \text{weakly in } H_0^1(\Omega) \text{ for any fixed } t \in [0,T].$$

Applying Theorem 2.3 to $\tilde{v}_\varepsilon(t)$ and $v(t)$ we obtain that for any fixed $t \in [0,T]$

$$\begin{cases} v(t) \in V \\ \|v(t)\|_V^2 = \left\{\int_\Omega |\nabla v(x,t)|^2 \mathrm{d}x + \int_\Omega |v(x,t)|^2 \mathrm{d}\mu(x)\right\} \le \liminf_{\varepsilon\to 0} \int_\Omega |\nabla \tilde{v}_\varepsilon(x,t)|^2 \mathrm{d}x. \end{cases}$$

Since for some $C_0 < +\infty$ we have

$$\sup_{t\in[0,T]} \mathrm{ess} \int_\Omega |\nabla \tilde{v}_\varepsilon(x,t)|^2 \mathrm{d}x = \|\tilde{v}_\varepsilon\|_{L^\infty(0,T;H_0^1(\Omega))} \le C_0,$$

we have proved that $v(t)$ belongs to V for any t and that

$$\sup_{t\in[0,T]} \mathrm{ess}\, \|v(t)\|_V^2 \le C_0.$$

This implies (2.40) once the measurability of the function $v : [0,T] \longmapsto L^2(\Omega; \mathrm{d}\mu)$ is proved, since we know from (2.38) that v belongs to $L^\infty(0,T; H_0^1(\Omega)) \cap W^{1,\infty}(0,T; L^2(\Omega))$.

Since $L^2(\Omega; \mathrm{d}\mu)$ is separable, it is sufficient to prove, using Pettis' measurability Theorem (see [8, Theorem 2, p. 42]), that v is weakly measurable, i.e. that for any $\varphi \in L^2(\Omega; \mathrm{d}\mu)$, the function $t \longmapsto \int_\Omega v(x,t)\varphi(x)\mathrm{d}\mu(x)$ is measurable. We already know that v belongs to $L^\infty(0,T; H_0^1(\Omega))$ and therefore, by (2.9), to $L^\infty(0,T; L^1(\Omega; \mathrm{d}\mu))$. Thus the function $t \longmapsto \int_\Omega v(x,t)\psi(x)\mathrm{d}\mu(x)$ is measurable for any $\psi \in C^0(\overline{\Omega})$. Approximate now $\varphi \in L^2(\Omega; \mathrm{d}\mu)$ by a sequence $\psi_n \in C^0(\overline{\Omega})$. Since $v(t)$ belongs to $L^2(\Omega; \mathrm{d}\mu)$ for any $t \in [0,T]$, we have

$$\int_\Omega v(x,t)\psi_n(x)\mathrm{d}\mu(x) \to \int_\Omega v(x,t)\varphi(x)\mathrm{d}\mu(x) \qquad \text{for any } t \in [0,T].$$

This proves the measurability of the function $t \longmapsto \int_\Omega v(x,t)\varphi(x)\mathrm{d}\mu(x)$ and completes the proof of Proposition 2.7. □

In the following Proposition we prove the existence of some quasi-extension operators that will be useful in the sequel.

PROPOSITION 2.8. *Assume that* (2.1) *holds true and define the operator* P_ε *by*

$$P_\varepsilon\psi = w_\varepsilon\tilde{\psi} \text{ in } \Omega \quad \text{for all } \psi \in L^2(\Omega_\varepsilon) \tag{2.41}$$

where $\tilde{\psi}$ *is the extension of* ψ *by zero in the holes* S_ε *defined by* (2.12). *Then*

$$\begin{cases} P_\varepsilon \in \mathcal{L}(L^2(\Omega_\varepsilon); L^2(\Omega)) \\ \|P_\varepsilon\|_{\mathcal{L}(L^2(\Omega_\varepsilon); L^2(\Omega))} \leq M_0. \end{cases} \tag{2.42}$$

Moreover the operator P_ε *extends to an operator defined on* $H^{-1}(\Omega_\varepsilon)$, *and for any* $\varepsilon > 0$ *and any* $q \in (1, n/(n-1))$

$$\begin{cases} P_\varepsilon \in \mathcal{L}(H^{-1}(\Omega_\varepsilon); W^{-1,q}(\Omega)) \\ \|P_\varepsilon\|_{\mathcal{L}(H^{-1}(\Omega_\varepsilon); W^{-1,q}(\Omega))} \leq C_q. \end{cases} \tag{2.43}$$

REMARK 2.9. The operator P_ε is not an extension operator since $P_\varepsilon\psi$ does not coincide with ψ in Ω_ε: indeed w_ε does not coincide with 1 on this set. However for any φ in $L^2(\Omega)$ we have

$$P_\varepsilon\varphi \to \varphi \qquad \text{strongly in } L^2(\Omega) \tag{2.44}$$

which shows that P_ε acts as a "quasi-extension" operator. □

PROOF OF PROPOSITION 2.8. In view of (2.1), the operator P_ε defined by (2.41) clearly enjoys properties (2.42) and (2.44) (use Lebesgue's dominated convergence Theorem to prove (2.44)).

Let now $p > n$ be fixed; define on $W_0^{1,p}(\Omega)$ the operator

$$R_\varepsilon\varphi = \varphi w_\varepsilon|_{\Omega_\varepsilon} \quad \text{for all } \varphi \in W_0^{1,p}(\Omega). \tag{2.45}$$

Since $\nabla(R_\varepsilon\varphi) = w_\varepsilon\nabla\varphi + \varphi\nabla w_\varepsilon$ in Ω_ε, we have

$$\begin{cases} \int_{\Omega_\varepsilon} |\nabla(R_\varepsilon\varphi)|^2 \mathrm{d}x & \leq 2\int_{\Omega_\varepsilon} |w_\varepsilon|^2|\nabla\varphi|^2\mathrm{d}x + 2\int_{\Omega_\varepsilon} |\varphi|^2|\nabla w_\varepsilon|^2\mathrm{d}x \\ & \leq 2M_0^2 \int_{\Omega_\varepsilon} |\nabla\varphi|^2\mathrm{d}x + 2\|\varphi\|^2_{L^\infty(\Omega)}\int_\Omega |\nabla w_\varepsilon|^2\mathrm{d}x \\ & \leq C_p\|\varphi\|^2_{W_0^{1,p}(\Omega)} \end{cases} \tag{2.46}$$

where the constant C_p does not depend on ε. We thus have

$$\begin{cases} R_\varepsilon \in \mathcal{L}(W_0^{1,p}(\Omega); H_0^1(\Omega_\varepsilon)) \\ \|R_\varepsilon\|_{\mathcal{L}(W_0^{1,p}(\Omega);H_0^1(\Omega_\varepsilon))} \leq C_p. \end{cases} \tag{2.47}$$

Consider the operator R_ε^* defined on $H^{-1}(\Omega_\varepsilon)$ by

$$\begin{gathered} \langle R_\varepsilon^*\psi, \varphi\rangle_{W^{-1,q}(\Omega),W_0^{1,p}(\Omega)} = \langle\psi, R_\varepsilon\varphi\rangle_{H^{-1}(\Omega_\varepsilon),H_0^1(\Omega_\varepsilon)} \\ \text{for all } \psi \in H^{-1}(\Omega_\varepsilon) \text{ and } \varphi \in W_0^{1,p}(\Omega), \end{gathered} \tag{2.48}$$

where q is given by $\frac{1}{p} + \frac{1}{q} = 1$. The identity

$$\langle\psi, R_\varepsilon\varphi\rangle_{H^{-1}(\Omega_\varepsilon),H_0^1(\Omega_\varepsilon)} = \int_{\Omega_\varepsilon} \psi\varphi w_\varepsilon \mathrm{d}x = \int_{\Omega_\varepsilon} P_\varepsilon\psi\varphi\,\mathrm{d}x$$

$$\text{for all } \psi \in L^2(\Omega_\varepsilon), \text{ and } \varphi \in W_0^{1,p}(\Omega),$$

proves that $R_\varepsilon^* = P_\varepsilon$ is an extension of P_ε defined by (2.41) and (2.47) immediately implies (2.43). □

REMARK 2.10. An interesting consequence of the construction of the operators P_ε is that it makes possible to perform the homogenization of the elliptic problem (2.15) when the sequence g_ε only satisfies

$$g_\varepsilon \in H^{-1}(\Omega_\varepsilon), \ \|g_\varepsilon\|_{H^{-1}(\Omega_\varepsilon)} \leq C, \tag{2.49}$$

where $C > 0$ is a constant independent of ε. Under this hypothesis the extension $\tilde{v}_\varepsilon$ of the solution v_ε of (2.15) is still bounded in $H_0^1(\Omega)$.

On the other hand, from Proposition 2.8 we deduce that $P_\varepsilon g_\varepsilon$ is uniformly bounded in $W^{-1,q}(\Omega)$. Thus, by passing to a subsequence (denoted by ε'), we have

$$P_{\varepsilon'} g_{\varepsilon'} \rightharpoonup g^* \quad \text{weakly in } W^{-1,q}(\Omega). \tag{2.50}$$

Using $w_\varepsilon \varphi$ (with $\varphi \in \mathcal{D}(\Omega)$ and w_ε defined by (2.1)) as test function in the variational formulation of (2.15) and following exactly the proof of [6, Théorème 1], one easily proves that

$$\tilde{v}_{\varepsilon'} \rightharpoonup v \quad \text{weakly in } H_0^1(\Omega), \tag{2.51}$$

where v solves

$$\begin{cases} \int\limits_\Omega \nabla v \nabla \varphi \, dx + \int\limits_\Omega v\varphi \, d\mu = \langle g^*, \varphi \rangle, & \text{for all } \varphi \in \mathcal{D}(\Omega) \\ v \in V. \end{cases} \tag{2.52}$$

Since $\mathcal{D}(\Omega)$ is dense in V (see Appendix below), (2.52) actually holds for any φ in V. Since the left hand side of (2.52) is a continuous linear form on $\varphi \in V$, we actually have

$$g^* \in V' \tag{2.53}$$

and not only in $W^{1,q}(\Omega)$ for $q \in (1, n/(n-1))$ as obtained in (2.50).

Finally note that in order to pass to the limit in (2.15), for some right hand side g_ε defined on Ω and bounded in $H^{-1}(\Omega)$ (this assumption is slightly stronger than (2.49) where g_ε is only defined on Ω_ε), one has to consider a subsequence ε' such that $P_{\varepsilon'} g_{\varepsilon'}$ weakly converges to some g^* in $H^{-1}(\Omega)$ (see (2.50)) and not a subsequence such that $g_{\varepsilon'}$ weakly converges to some g^{**} in $H^{-1}(\Omega)$. □

We conclude this Section with the following result, which proves that averaging in space provides some compactness in time.

PROPOSITION 2.9. *Assume that* (2.1) *holds true and let* P_ε *be the quasi-extension operator defined in Proposition* 2.8. *Consider a sequence* v_ε *in* $L^\infty(0,T; L^2(\Omega_\varepsilon)) \cap W^{1,1}(0,T; H^{-1}(\Omega_\varepsilon))$ *satisfying*

$$\begin{cases} \tilde{v}_\varepsilon \rightharpoonup v & \text{weakly } * \text{ in } L^\infty(0,T; L^2(\Omega)) \\ P_\varepsilon v'_\varepsilon \rightharpoonup v' & \text{weakly in } L^1(0,T; W^{-1,q}(\Omega)) \end{cases} \tag{2.54}$$

for some $q \in (1, n/(n-1))$. *Then for all* $\varphi \in L^2(\Omega)$

$$\int\limits_\Omega \tilde{v}_\varepsilon(x, \cdot)\varphi(x) dx \to \int\limits_\Omega v(x, \cdot)\varphi(x) dx \quad \textit{strongly in } C^0([0,T]). \tag{2.55}$$

PROOF. Combining (2.1), (2.42) and (2.54a) one can easily prove that

$$P_\varepsilon v_\varepsilon = w_\varepsilon \tilde{v}_\varepsilon \rightharpoonup v \quad \text{weakly } * \text{ in } L^\infty(0,T,L^2(\Omega)).$$

Corollary 2.6 applied to the sequence $g_\varepsilon = P_\varepsilon v_\varepsilon$ with $X = L^2(\Omega)$, $Y = W^{-1,q}(\Omega)$, therefore implies that

$$\int_\Omega w_\varepsilon(x)\tilde{v}_\varepsilon(x,\cdot)\varphi(x)\mathrm{d}x \to \int_\Omega v(x,\cdot)\varphi(x)\mathrm{d}x \quad \text{strongly in } C^0([0,T]) \tag{2.56}$$

for all $\varphi \in L^2(\Omega)$.

Convergence (2.55) is then deduced from (2.56) and from

$$\int_\Omega \tilde{v}_\varepsilon(\cdot)\varphi\,\mathrm{d}x = \int_\Omega \tilde{v}_\varepsilon(\cdot)\varphi w_\varepsilon\,\mathrm{d}x + \int_\Omega \tilde{v}_\varepsilon(\cdot)\varphi(1-w_\varepsilon)\mathrm{d}x$$

since

$$\left|\int_\Omega \tilde{v}_\varepsilon(\cdot)\varphi(1-w_\varepsilon)\mathrm{d}x\right| \leq \|\tilde{v}_\varepsilon\|_{L^\infty(0,T,L^2(\Omega))}\|\varphi(1-w_\varepsilon)\|_{L^2(\Omega)} \to 0$$

in view of (2.54), (2.1) and Lebesgue's dominated convergence Theorem. □

3. - The homogenization result for the wave equation

The goal of this Section is to prove the homogenization result for the wave equation (Theorem 3.1). Lower semicontinuity of the energy is also proved (Theorem 3.2).

Consider a bounded domain Ω of $\mathbb{R}^n$ $(n \geq 2)$ and the domain Ω_ε obtained by removing from Ω a set $S_\varepsilon = \bigcup_{i=1}^{N(\varepsilon)} S_\varepsilon^i$ of "small" holes for which hypothesis (2.1) holds true. Let $T > 0$ be a given real number. Consider the wave equation

$$\begin{cases} u_\varepsilon'' - \Delta u_\varepsilon = f_\varepsilon & \text{in } Q_\varepsilon = \Omega_\varepsilon \times (0,T) \\ u_\varepsilon = 0 & \text{on } \Sigma_\varepsilon = \partial\Omega_\varepsilon \times (0,T) \end{cases} \tag{3.1a}$$

$$\begin{cases} u_\varepsilon(0) = u_\varepsilon^0 & \text{in } \Omega_\varepsilon \\ u_\varepsilon'(0) = u_0^1 & \text{in } \Omega_\varepsilon \end{cases} \tag{3.1b}$$

where the data u_ε^0, u_ε^1, f_ε are assumed to satisfy

$$u_\varepsilon^0 \in H_0^1(\Omega_\varepsilon),\ u_\varepsilon^1 \in L^2(\Omega_\varepsilon),\ f_\varepsilon \in L^1(0,T;L^2(\Omega_\varepsilon)). \tag{3.2}$$

Classical results (see, e.g. [16] or [14]) provide the existence and uniqueness of a solution $u_\varepsilon = u_\varepsilon(x,t)$ of (3.1) which satisfies

(3.3) $$u_\varepsilon \in C^0([0,T]; H_0^1(\Omega_\varepsilon)) \cap C^1([0,T]; L^2(\Omega_\varepsilon)).$$

Moreover defining, for any $t \in [0,T]$, the energy $E_\varepsilon(\cdot)$ by

(3.4) $$E_\varepsilon(t) = \frac{1}{2}\|u'_\varepsilon(t)\|^2_{L^2(\Omega_\varepsilon)} + \frac{1}{2}\|\nabla u_\varepsilon(t)\|^2_{L^2(\Omega_\varepsilon)}$$

one has the following energy identity

(3.5) $$E_\varepsilon(t) = E_\varepsilon(0) + \int_0^t \int_{\Omega_\varepsilon} f_\varepsilon(x,s) u'_\varepsilon(x,s) \mathrm{d}x\, \mathrm{d}s.$$

Recall that $\tilde{\cdot}$ denotes the extension by zero outside of Ω_ε and that V is the space $H_0^1(\Omega) \cap L^2(\Omega; \mathrm{d}\mu)$ (see (2.12) and (2.10)). We have the following homogenization result for the wave equation (3.1).

THEOREM 3.1. *Assume that* (2.1) *holds true and consider a sequence of data which satisfy*

(3.6) $$\begin{cases} \tilde{f}_\varepsilon \rightharpoonup f & \text{weakly in } L^1(0,T; L^2(\Omega)) \\ \tilde{u}^0_\varepsilon \rightharpoonup u^0 & \text{weakly in } H_0^1(\Omega) \\ \tilde{u}^1_\varepsilon \rightharpoonup u^1 & \text{weakly in } L^2(\Omega). \end{cases}$$

The sequence u_ε of solutions of (3.1) *then satisfies*

(3.7) $$\begin{cases} \tilde{u}_\varepsilon \rightharpoonup u & \text{weakly } * \text{ in } L^\infty(0,T; H_0^1(\Omega)) \\ \tilde{u}'_\varepsilon \rightharpoonup u' & \text{weakly } * \text{ in } L^\infty(0,T; L^2(\Omega)) \end{cases}$$

where $u = u(x,t)$ is the unique solution of the homogenized wave equation

(3.8a) $$\begin{cases} u'' - \Delta u + \mu u = f & \text{in } Q = \Omega \times (0,T) \\ u = 0 & \text{on } \Sigma = \partial\Omega \times (0,T) \end{cases}$$

(3.8b) $$\begin{cases} u(0) = u^0 & \text{in } \Omega \\ u'(0) = u^1 & \text{in } \Omega \end{cases}$$

(3.8c) $$u \in C^0([0,T]; V) \cap C^1([0,T]; L^2(\Omega)).$$

REMARK 3.1. In view of definition (2.11) of the scalar product $a(\cdot, \cdot)$ of V, the variational formulation of the wave equation (3.8a) is

$$(3.9)\quad \begin{cases} \dfrac{d^2}{dt^2} \int\limits_\Omega u(x,t)v(x)dx + a(u(t), v) = \int\limits_\Omega f(x,t)v(x)dx \text{ in } \mathcal{D}'(0,T), \ \forall v \in V \\ u \in L^\infty(0,T;V) \cap W^{1,\infty}(0,T;L^2(\Omega)). \end{cases}$$

Note that according to Theorem 2.3, the function u^0 (which is the weak limit in $H_0^1(\Omega)$ of functions $\tilde{u}_\varepsilon^0$ vanishing on the holes S_ε) belongs to V, so there is no contradiction between the two assertions $u(0) = u^0$ and $u \in C^0([0,T];V)$.

On the other hand, observe that classical results (see e.g. [16]) provide existence and uniqueness of a solution of (3.8). In fact, uniqueness holds in the larger class $L^\infty(0,T;V) \cap W^{1,\infty}(0,T;L^2(\Omega))$.

Finally note that $\tilde{f}_\varepsilon$ is assumed to converge weakly in $L^1(0,T;L^2(\Omega))$, which is a stronger assumption than to be bounded in this space. □

PROOF OF THEOREM 3.1. We proceed in four steps.

First step: a priori estimates.

From (3.5) we have

$$E_\varepsilon(t) = E_\varepsilon(0) + \int_0^t \int_{\Omega_\varepsilon} f_\varepsilon(x,s)u_\varepsilon'(x,s)dx\, ds$$

$$\le E_\varepsilon(0) + \sqrt{2}\int_0^t \|f_\varepsilon(s)\|_{L^2(\Omega_\varepsilon)}(E_\varepsilon(s))^{1/2}ds$$

which by Gronwall's inequality implies

$$(3.10)\qquad (E_\varepsilon(t))^{1/2} \le (E_\varepsilon(0))^{1/2} + \frac{1}{\sqrt{2}}\int_0^t \|f_\varepsilon(s)\|_{L^2(\Omega_\varepsilon)}ds.$$

In view of (3.6), the right hand side of (3.10) is bounded independently of $t \in [0,T]$ and ε. Using the properties of the extension by zero outside Ω_ε (see (2.12)) this implies that $\tilde{u}_\varepsilon$ is bounded in $L^\infty(0,T;H_0^1(\Omega)) \cap W^{1,\infty}(0,T;L^2(\Omega))$. Extracting a subsequence (still denoted by ε, since in the fourth step the whole sequence will be proved to converge) one has

$$(3.11)\qquad \begin{cases} \tilde{u}_\varepsilon \rightharpoonup u & \text{weakly } * \text{ in } L^\infty(0,T;H_0^1(\Omega)) \\ \tilde{u}_\varepsilon' \rightharpoonup u' & \text{weakly } * \text{ in } L^\infty(0,T;L^2(\Omega)). \end{cases}$$

On the other hand, in view of Proposition 2.7 we have

$$(3.12)\qquad u \in L^\infty(0,T;V) \cap W^{1,\infty}(0,T;L^2(\Omega)).$$

Second step: passing to the limit in the wave equation (3.1a).

Using in (3.1a) the test function $\psi(t)\varphi(x)w_\varepsilon(x)$ where $\psi \in \mathcal{D}((0,T))$, $\varphi \in \mathcal{D}(\Omega)$ and w_ε is defined in hypothesis (2.1), we obtain after integration by parts

$$\int_{Q_\varepsilon} u_\varepsilon \psi'' \varphi w_\varepsilon \, dx\, dt + \int_{Q_\varepsilon} \nabla u_\varepsilon \nabla w_\varepsilon \psi\varphi \, dx\, dt + \int_{Q_\varepsilon} \nabla u_\varepsilon \nabla \varphi w_\varepsilon \psi \, dx\, dt = \int_{Q_\varepsilon} f_\varepsilon \psi\varphi w_\varepsilon \, dx\, dt.$$

Extension by zero outside Ω_ε, Fubini's Theorem and an integration by parts in the second term give, in view of the identity $-\Delta w_\varepsilon = \mu_\varepsilon - \gamma_\varepsilon$ (see (2.1)(iv))

$$(3.13)\quad \begin{cases} \int_\Omega \varphi w_\varepsilon \left(\int_0^T \psi'' \tilde{u}_\varepsilon \, dt \right) dx + \left\langle \mu_\varepsilon - \gamma_\varepsilon, \varphi \left(\int_0^T \psi \tilde{u}_\varepsilon \, dt \right) \right\rangle_\Omega \\ - \int_\Omega \nabla w_\varepsilon \nabla \varphi \left(\int_0^T \psi \tilde{u}_\varepsilon \, dx \right) dt + \int_\Omega w_\varepsilon \nabla \varphi \nabla \left(\int_0^T \psi \tilde{u}_\varepsilon dt \right) dx \\ \qquad = \int_Q \tilde{f}_\varepsilon \psi \varphi w_\varepsilon \, dx\, dt. \end{cases}$$

Consider the function $U_\varepsilon \in H_0^1(\Omega)$ defined by

$$U_\varepsilon(x) = \int_0^T \psi(t)\tilde{u}_\varepsilon(x,t)dt.$$

Convergences (3.11) imply that the sequence U_ε satisfies

$$(3.14)\quad \begin{cases} U_\varepsilon \rightharpoonup \int_0^T \psi(t) u(x,t) dt & \text{weakly in } H_0^1(\Omega) \text{ and strongly in } L^2(\Omega) \\ U_\varepsilon = 0 & \text{on } S_\varepsilon \end{cases}$$

and that

$$(3.15)\quad \begin{cases} \int_0^T \psi''(t)\tilde{u}_\varepsilon(x,t)dt \to \int_0^T \psi''(t)u(x,t)dt & \text{strongly in } L^2(\Omega) \\ \int_0^T \psi''(t)\tilde{u}_\varepsilon(x,t)dt = 0 & \text{on } S_\varepsilon. \end{cases}$$

It is now easy to pass to the limit in each term of (3.13), using hypothesis (2.1)(iii) and (iv) (note that $\langle \gamma_\varepsilon, \varphi U_\varepsilon \rangle_\Omega = 0$). Using Fubini's and Deny's Theorems (see (2.9)), we obtain

$$(3.16)\qquad \begin{cases} \int_0^T \psi'' \left(\int_\Omega u\varphi \, dx\right) dt + \int_0^T \psi \left(\int_\Omega \varphi u \, d\mu\right) dt + \int_0^T \psi \left(\int_\Omega \nabla u \nabla\varphi \, dx\right) dt \\ \qquad = \int_0^T \psi \left(\int_\Omega f\varphi \, dx\right) dt. \end{cases}$$

Since $\psi \in \mathcal{D}((0,T))$ is arbitrary, we have proved that

$$(3.17)\qquad \frac{d^2}{dt^2}\int_\Omega u(x,t)\varphi(x)dx + a(u,\varphi) = \int_\Omega f\varphi \, dx \quad \text{in } \mathcal{D}'(0,T),\ \forall\varphi \in \mathcal{D}(\Omega).$$

In view of (3.12), the density of $\mathcal{D}(\Omega)$ in V (see the Appendix below) allows one to extend (3.17) to every test function $\varphi \in V$.

We thus have proved (3.9) which (see Remark 3.1) is equivalent to (3.8a).

Third step: passing to the limit in the initial data.

From (3.11) and Proposition 2.7 we deduce that

$$(3.18)\qquad \langle\vartheta, \tilde{u}_\varepsilon(\cdot)\rangle_\Omega \to \langle\vartheta, u(\cdot)\rangle_\Omega \quad \text{strongly in } C^0([0,T])$$

for any $\vartheta \in H^{-1}(\Omega)$. Since $u_\varepsilon(0) = \tilde{u}_\varepsilon^0$ tends to u^0 weakly in $H_0^1(\Omega)$ (see (3.6)), we obtain

$$(3.19)\qquad u(0) = u^0.$$

In order to prove that $u'(0) = u^1$ we will apply Proposition 2.9 to $v_\varepsilon = u'_\varepsilon$. Let us first check that

$$(3.20)\qquad P_\varepsilon u''_\varepsilon \rightharpoonup u'' \quad \text{weakly in } L^1(0,T;W^{-1,q}(\Omega)).$$

Indeed observe that

$$u''_\varepsilon = \Delta u_\varepsilon + f_\varepsilon \quad \text{in } Q_\varepsilon = \Omega_\varepsilon \times (0,T)$$

and thus

$$P_\varepsilon u''_\varepsilon = P_\varepsilon \Delta u_\varepsilon + P_\varepsilon f_\varepsilon.$$

In view of (3.11a), we have $\|\Delta u_\varepsilon\|_{L^\infty(0,T;H^{-1}(\Omega_\varepsilon))} \le C$ and thus by Proposition 2.8

$$\|P_\varepsilon \Delta u_\varepsilon\|_{L^\infty(0,T;W^{-1,q}(\Omega))} \le C_q \quad \text{for any } q \in (1, n/(n-1)).$$

On the other hand, $P_\varepsilon f_\varepsilon = w_\varepsilon \tilde{f}_\varepsilon$ is relatively compact in the weak topology of $L^1(0,T;L^2(\Omega))$ in view of (3.6) and (2.1)(i). This follows from Dunford's

Theorem (see [8, Theorem 1, p. 101]) since $L^2(\Omega)$ enjoys the Radon-Nikodym property (see [8, Corollary 13, p. 76]), since $w_\varepsilon \tilde{f}_\varepsilon$ is bounded in $L^1(0,T;L^2(\Omega))$, since $\int_E w_\varepsilon \tilde{f}_\varepsilon \,\mathrm{d}t$ is relatively compact in the weak topology of $L^2(\Omega)$ for any measurable subset E of $[0,T]$, and finally since the function $t \longmapsto \|w_\varepsilon \tilde{f}_\varepsilon(t)\|_{L^2(\Omega)}$ is uniformly integrable on $[0,T]$ (indeed, the functions $t \longmapsto \|\tilde{f}_\varepsilon(t)\|_{L^2(\Omega)}$ are uniformly integrable on $[0,T]$ since the sequence $\tilde{f}_\varepsilon$ is relatively compact in $L^1(0,T;L^2(\Omega))$, see [8, Theorem 4, p. 104]).

Consequently $P_\varepsilon u_\varepsilon'' = P_\varepsilon \Delta u_\varepsilon + P_\varepsilon f_\varepsilon$ is relatively compact in the weak topology of $L^1(0,T;W^{-1,q}(\Omega))$.

Finally, combining (2.1) and (3.11b), it easy to prove that

$$P_\varepsilon u_\varepsilon' = w_\varepsilon \tilde{u}_\varepsilon' \rightharpoonup u' \quad \text{weakly } * \text{ in } L^\infty(0,T;L^2(\Omega)). \tag{3.21}$$

Since $(P_\varepsilon u_\varepsilon')' = P_\varepsilon u_\varepsilon''$ because of the definition of P_ε, this implies that (3.20) is satisfied.

Combining (3.11b) and (3.20), Proposition 2.9 ensures that

$$\int_\Omega \tilde{u}_\varepsilon'(x,\cdot)\varphi(x)\mathrm{d}x \to \int_\Omega u'(x,\cdot)\varphi(x)\mathrm{d}x \quad \text{strongly in } C^0([0,T]) \tag{3.22}$$

for any $\varphi \in L^2(\Omega)$. Since $\tilde{u}_\varepsilon'(0) = \tilde{u}_\varepsilon^1$ tends to u^1 weakly in $L^2(\Omega)$ (see (3.6)), we deduce

$$u'(0) = u^1.$$

The limit $u = u(x,t)$ therefore satisfies the initial conditions (3.8b).

Fourth step: end of the proof.

In the second and third step we have proved that, up to the extraction of a subsequence (still denoted by ε), the sequence u_ε satisfies (3.11) where the limit u belongs to $L^\infty(0,T;V) \cap W^{1,\infty}(0,T;L^2(\Omega))$ and satisfies (3.8a)-(3.8b).

The uniqueness of the solution of (3.8a)-(3.8b) in $L^\infty(0,T;V)$ $\cap W^{1,\infty}(0,T;L^2(\Omega))$ (see Remark 3.1) allows us to deduce that the whole sequence u_ε satisfies (3.7) and that the limit u satisfies (3.8c).

This completes the proof of Theorem 3.1. □

We have also the following pointwise (in time) convergence result and lower semicontinuity property of the energy.

THEOREM 3.2. *Assume that the hypotheses of Theorem* 3.1 *are fulfilled. Then for any fixed* $t \in [0,T]$

$$\tilde{u}_\varepsilon(t) \rightharpoonup u(t) \quad \textit{weakly in } H_0^1(\Omega) \tag{3.23}$$

$$\tilde{u}_\varepsilon'(t) \rightharpoonup u'(t) \quad \textit{weakly in } L^2(\Omega) \tag{3.24}$$

$$(3.25)\qquad \int_\Omega |\nabla u(x,t)|^2 \mathrm{d}x + \int_\Omega |u(x,t)|^2 \mathrm{d}\mu(x) \le \liminf_{\varepsilon\to 0} \int_{\Omega_\varepsilon} |\nabla u_\varepsilon(x,t)|^2 \mathrm{d}x$$

$$(3.26)\qquad \int_\Omega |u'(x,t)|^2 \mathrm{d}x \le \liminf_{\varepsilon\to 0} \int_{\Omega_\varepsilon} |u'_\varepsilon(x,t)|^2 \mathrm{d}x$$

and

$$(3.27)\qquad E(t) \le \liminf_{\varepsilon\to 0} E_\varepsilon(t)$$

where E_ε *is defined by* (3.4) *while*

$$(3.28)\qquad E(t) = \frac{1}{2}\|u'(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla u(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\|u(t)\|^2_{L^2(\Omega;\mathrm{d}\mu)}.$$

PROOF. The convergences (3.23) and (3.24) have been proved in the third step of the proof of Theorem 3.1 (see (3.18) and (3.22)). From Theorem 2.3 we have (3.25) while (3.26) is straightforward. This immediately implies (3.27). □

In the next Section we shall show that, under special convergence assumptions on the data (which are quite stronger than (3.6)), we have

$$E_\varepsilon(t) \to E(t) \quad \text{strongly in } C^0([0,T]).$$

This convergence property of the energies will play a crucial role when proving the corrector result.

4. - Corrector for the wave equation in a domain with small holes

This Section is devoted to state and prove the corrector result when special assumptions are satisfied by the data. The proof follows along the lines of S. Brahim-Otsmane, G.A. Francfort and F. Murat [1], who adapted to the wave equation the ideas introduced by L. Tartar [18] in the elliptic case. One of the main steps of the proof is the strong convergence of the energy in $C^0([0,T])$.

Concerning the initial condition u^0_ε we shall assume that

$$(4.1)\qquad \begin{cases} u^0_\varepsilon \in H^1_0(\Omega_\varepsilon) \\ \text{there exists } g_\varepsilon \in H^{-1}(\Omega) \text{ such that} \\ \begin{cases} -\Delta u^0_\varepsilon = g_\varepsilon & \text{in } \mathcal{D}'(\Omega_\varepsilon) \\ g_\varepsilon \to g & \text{strongly in } H^{-1}(\Omega). \end{cases} \end{cases}$$

As a consequence of Theorem 2.2 we deduce that

$$\tilde{u}^0_\varepsilon \rightharpoonup u^0 \quad \text{weakly in } H^1_0(\Omega) \tag{4.2}$$

where $u^0 = u^0(x)$ is the solution of

$$\begin{cases} -\Delta u^0 + \mu u^0 = g & \text{in } \mathcal{D}'(\Omega) \\ u^0 \in V. \end{cases} \tag{4.3}$$

THEOREM 4.1. *Assume that* (2.1) *holds true and consider a sequence of data* u^0_ε, u^1_ε, f_ε *that satisfy* (4.1) *and*

$$\tilde{f}_\varepsilon \to f \quad \textit{strongly in } L^1(0,T;L^2(\Omega)) \tag{4.4}$$

$$\tilde{u}^1_\varepsilon \to u^1 \quad \textit{strongly in } L^2(\Omega)). \tag{4.5}$$

If u *denotes the unique solution of the homogenized equation* (3.8), *the sequence* u_ε *of solutions of* (3.1) *satisfies*

$$\tilde{u}'_\varepsilon \to u' \quad \textit{strongly in } C^0([0,T];L^2(\Omega)) \tag{4.6}$$

$$\tilde{u}_\varepsilon = u w_\varepsilon + R_\varepsilon \tag{4.7}$$

with

$$R_\varepsilon \to 0 \quad \textit{strongly in } C^0([0,T];W^{1,1}_0(\Omega)). \tag{4.8}$$

Moreover, if $u \in C^0(\overline{\Omega} \times [0,T])$, *then*

$$R_\varepsilon \to 0 \quad \textit{strongly in } C^0([0,T];H^1_0(\Omega)). \tag{4.9}$$

REMARK 4.1. In (4.4) and (4.5) we have assumed the strong convergence of the data and not only the weak convergence as in (3.6).

For what concerns (4.1), note that this assumption is $-\Delta u^0_\varepsilon = g_\varepsilon$ in $\mathcal{D}'(\Omega_\varepsilon)$ and not $-\Delta \tilde{u}^0_\varepsilon = g_\varepsilon$ in $\mathcal{D}'(\Omega)$. Note also that (4.1) is quite different of assuming that

$$\tilde{u}^0_\varepsilon \to u^0 \quad \text{strongly in } H^1_0(\Omega).$$

Indeed in view of Theorem 2.2, (4.1) implies that

$$\int_\Omega |\nabla \tilde{u}^0_\varepsilon|^2 \mathrm{d}x \to \int_\Omega |\nabla u^0|^2 \mathrm{d}x + \int_\Omega |u^0|^2 \mathrm{d}\mu$$

which prevents in general the strong convergence of $\tilde{u}^0_\varepsilon$ to u^0. Nevertheless, this convergence of energies to the homogenized energy is exactly what is necessary in order to prove the corrector result of Theorem 4.1. This is a natural substitute to the strong convergence of $\tilde{u}_\varepsilon^0$, which is not the convenient hypothesis here. □

Before proving Theorem 4.1 we prove the convergence of the energy. Let us recall definitions (3.4) and (3.28)

$$E_\varepsilon(t) = \frac{1}{2}\|u_\varepsilon'(t)\|^2_{L^2(\Omega_\varepsilon)} + \frac{1}{2}\|\nabla u_\varepsilon(t)\|^2_{L^2(\Omega_\varepsilon)}$$

$$E(t) = \frac{1}{2}\|u'(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla u(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\|u(t)\|^2_{L^2(\Omega;\mathrm{d}\mu)}.$$

PROPOSITION 4.2. *Assume that the hypotheses of Theorem* 4.1 *hold true. Then*

$$E_\varepsilon(\cdot) \to E(\cdot) \quad \textit{strongly in } C^0([0,T]). \tag{4.10}$$

REMARK 4.2. Let us observe that combining (3.25) and (3.26) with (4.10) we have, for any fixed $t \in [0,T]$,

$$\begin{cases} \|u_\varepsilon'(t)\|^2_{L^2(\Omega_\varepsilon)} \to \|u'(t)\|^2_{L^2(\Omega)} \\ \|\nabla u_\varepsilon(t)\|^2_{L^2(\Omega_\varepsilon)} \to \|\nabla u(t)\|^2_{L^2(\Omega)} + \|u(t)\|^2_{L^2(\Omega;\mathrm{d}\mu)}. \end{cases} \tag{4.11}$$

On the other hand, from (3.24) and (4.11a), we obtain that

$$\tilde{u}_\varepsilon'(t) \to u'(t) \quad \text{strongly in } L^2(\Omega) \tag{4.12}$$

for any fixed $t \in [0,T]$; this statement is not as strong as (4.6), but is a first attempt in this direction. □

REMARK 4.3. Further to (4.10) and (4.11) one actually has

$$\begin{cases} \|\tilde{u}_\varepsilon'(\cdot)\|^2_{L^2(\Omega_\varepsilon)} \to \|u'(\cdot)\|^2_{L^2(\Omega)} & \text{strongly in } C^0([0,T]) \\ \|\nabla \tilde{u}_\varepsilon(\cdot)\|^2_{L^2(\Omega)} \to \|\nabla u(\cdot)\|^2_{L^2(\Omega)} + \|u(\cdot)\|^2_{L^2(\Omega;\mathrm{d}\mu)} & \\ & \text{strongly in } C^0([0,T]); \end{cases} \tag{4.13}$$

indeed (4.13) follows from (4.10), (4.6) and from the definitions of E_ε and E. □

PROOF OF PROPOSITION 4.2. We have the identities

$$E_\varepsilon(t) = E_\varepsilon(0) + \int_0^t \int_{\Omega_\varepsilon} f_\varepsilon(x,s) u_\varepsilon'(x,s)\mathrm{d}x\,\mathrm{d}s \tag{4.14}$$

(4.15) $$E(t) = E(0) + \int_0^t \int_\Omega f(x,s)u'(x,s)\mathrm{d}x\,\mathrm{d}s$$

with

$$E_\varepsilon(0) = \frac{1}{2}\int_{\Omega_\varepsilon} |u_\varepsilon^1|^2\mathrm{d}x + \frac{1}{2}\int_{\Omega_\varepsilon} |\nabla u_\varepsilon^0|^2\mathrm{d}x$$

$$E(0) = \frac{1}{2}\int_\Omega |u^1|^2\mathrm{d}x + \frac{1}{2}\int_\Omega |\nabla u^0|^2\mathrm{d}x + \frac{1}{2}\int_\Omega |u^0|^2\mathrm{d}\mu.$$

In view of Theorem 3.1 and hypothesis (4.4) we have for any $t \in [0,T]$

(4.16) $$\int_0^t \int_{\Omega_\varepsilon} f_\varepsilon(x,s)u_\varepsilon'\,\mathrm{d}x\,\mathrm{d}s \to \int_0^t \int_\Omega f(x,s)u'(x,s)\mathrm{d}x\,\mathrm{d}s.$$

On the other hand, assumptions (4.1) and (4.5) imply that (see (2.17) in Theorem 2.2)

(4.17) $$E_\varepsilon(0) \to E(0).$$

Therefore

(4.18) $$E_\varepsilon(t) \to E(t) \quad \text{for any } t \in [0,T].$$

Moreover, given any $t \in [0,T]$ and $h > 0$ small enough, we have

$$|E_\varepsilon(t+h) - E_\varepsilon(t)| \leq \int_t^{t+h} \int_\Omega |\tilde{f}_\varepsilon(x,s)|\,|\tilde{u}_\varepsilon'(x,s)|\mathrm{d}x\,\mathrm{d}s$$

$$\leq \|\tilde{u}_\varepsilon'\|_{L^\infty(0,T;L^2(\Omega))} \int_t^{t+h} \|\tilde{f}_\varepsilon(s)\|_{L^2(\Omega)}\mathrm{d}s.$$

Since $\tilde{u}_\varepsilon'$ is bounded in $L^\infty(0,T;L^2(\Omega))$ and since $\tilde{f}_\varepsilon$ strongly converges in $L^1(0,T;L^2(\Omega))$, this inequality implies that

(4.19) $$|E_\varepsilon(t+h) - E_\varepsilon(t)| \to 0 \quad \text{as } h \to 0 \quad \text{uniformly in } \varepsilon.$$

The statements (4.18), (4.19) and Ascoli-Arzela's Theorem imply (4.10). □

Defining

(4.20) $$e_\varepsilon(v)(t) = \frac{1}{2}\|v'(t)\|^2_{L^2(\Omega_\varepsilon)} + \frac{1}{2}\|\nabla v(t)\|^2_{L^2(\Omega_\varepsilon)}$$

for $v \in C^0([0,T]; H_0^1(\Omega_\varepsilon)) \cap C^1([0,T]; L^2(\Omega_\varepsilon))$ and

$$e(v)(t) = \frac{1}{2}\|v'(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla v(t))\|^2_{L^2(\Omega)} + \frac{1}{2}\|v(t)\|^2_{L^2(\Omega;\mathrm{d}\mu)} \tag{4.21}$$

for $v \in C^0([0,T]; V) \cap C^1([0,T]; L^2(\Omega))$, we have the following result:

PROPOSITION 4.3. *Assume that the hypotheses of Theorem* 4.1 *hold true. Then*

$$e_\varepsilon(u_\varepsilon - w_\varepsilon\varphi)(\cdot) \to e(u-\varphi)(\cdot) \quad \textit{strongly in } C^0([0,T]) \tag{4.22}$$

for every $\varphi \in \mathcal{D}(Q)$.

REMARK 4.4. If u belongs to $\mathcal{D}(Q)$, Theorem 4.1 is a direct consequence of (4.22). Indeed, (4.7) and (4.9) immediately follow from (4.22); (4.6) is also a consequence of (4.22) by virtue of decomposition (4.42) below.

When u is not in $\mathcal{D}(Q)$, Theorem 4.1 cannot be obtained so simply: in the proof of Theorem 4.1 below we will approximate u by a sequence of smooth functions φ and deduce (4.6) and (4.7)-(4.8) from (4.22). □

PROOF OF PROPOSITION 4.3. We have

$$\begin{cases} e_\varepsilon(u_\varepsilon - w_\varepsilon\varphi)(t) = e_\varepsilon(u_\varepsilon)(t) + e_\varepsilon(w_\varepsilon\varphi)(t) - \int_\Omega \tilde{u}'_\varepsilon(x,t) w_\varepsilon(x)\varphi'(x,t)\mathrm{d}x \\ \qquad\qquad - \int_\Omega \nabla\tilde{u}_\varepsilon(x,t)\nabla(w_\varepsilon(x)\varphi(x,t))\mathrm{d}x. \end{cases} \tag{4.23}$$

We will pass successively to the limit in each term of the right hand side of (4.23).

First term. Since $e_\varepsilon(u_\varepsilon)(t) = E_\varepsilon(t)$, we have from Proposition 4.2

$$e_\varepsilon(u_\varepsilon)(\cdot) \to e(u)(\cdot) \quad \text{strongly in } C^0([0,T]). \tag{4.24}$$

Second term. Using (2.1)(iii) we obtain that (differentiating in time proves that the function is bounded in $W^{1,\infty}(0,T)$),

$$\|w_\varepsilon\varphi'(\cdot)\|^2_{L^2(\Omega_\varepsilon)} = \|w_\varepsilon\varphi'(\cdot)\|^2_{L^2(\Omega)} \to \|\varphi'(\cdot)\|^2_{L^2(\Omega)} \text{ in } C^0([0,T]). \tag{4.25}$$

On the other hand,

$$\|\nabla(w_\varepsilon\varphi)(t)\|^2_{L^2(\Omega_\varepsilon)} = \|\nabla(w_\varepsilon\varphi(t))\|^2_{L^2(\Omega)}$$

$$= -\langle\Delta w_\varepsilon, w_\varepsilon\varphi^2(t)\rangle_\Omega - 2\int_\Omega \nabla w_\varepsilon\nabla\varphi(t) w_\varepsilon\varphi(t)\mathrm{d}x - \int_\Omega |w_\varepsilon|^2\varphi(t)\Delta\varphi(t)\mathrm{d}x.$$

In view of (2.1)(iii) and (iv) we can pass to the limit in each term of the right hand side to get (note that each term is bounded in $W^{1,\infty}(0,T)$)

$$(4.26)\qquad -\int_\Omega |w_\varepsilon|^2\varphi(t)\Delta\varphi(t)\mathrm{d}x \to \int_\Omega \varphi(t)\Delta\varphi(t)\mathrm{d}x \quad \text{strongly in } C^0([0,T])$$

$$(4.27)\qquad -2\int_\Omega \nabla w_\varepsilon\nabla\varphi(t)w_\varepsilon\varphi(t)\mathrm{d}x \to 0 \quad \text{strongly in } C^0([0,T])$$

$$(4.28)\qquad -\langle\Delta w_\varepsilon, w_\varepsilon\varphi^2(t)\rangle_\Omega = \langle\mu_\varepsilon, w_\varepsilon\varphi^2(t)\rangle_\Omega \to \langle\mu,\varphi^2(t)\rangle_\Omega \quad \text{strongly in } C^0([0,T]).$$

Combining (4.25)-(4.28) we deduce that

$$(4.29)\qquad e_\varepsilon(w_\varepsilon\varphi)(\,\cdot\,) \to e(\varphi)(\,\cdot\,) \quad \text{strongly in } C^0([0,T]).$$

Third term. In view of (3.22) we have

$$\int_\Omega \tilde{u}_\varepsilon'(x,\,\cdot\,)\psi(x)\mathrm{d}x \to \int_\Omega u'(x,\,\cdot\,)\psi(x)\mathrm{d}x \quad \text{strongly in } C^0([0,T])$$

for all $\psi \in L^\infty(\Omega)$, from which we deduce

$$(4.30)\qquad \int_\Omega \tilde{u}_\varepsilon'(x,\,\cdot\,)w_\varepsilon(x)\psi(x)\mathrm{d}x \to \int_\Omega u'(x,\,\cdot\,)\psi(x)\mathrm{d}x \quad \text{strongly in } C^0([0,T]),$$

since

$$\sup_{t\in[0,T]}\left|\int_\Omega \tilde{u}_\varepsilon'(x,t)(w_\varepsilon(x)-1)\psi(x)\mathrm{d}x\right| \le \|\tilde{u}_\varepsilon'\|_{L^\infty(0,T;L^2(\Omega))}\|\psi\|_{L^\infty(\Omega)}\|w_\varepsilon-1\|_{L^2(\Omega)} \to 0$$

in view of (2.1)(iii).

Approximating $\varphi'(x,t)$ in $C^0([0,T];L^2(\Omega))$ by functions of the form $\sum_{i=1}^{k}\eta_i(t)\psi_i(x)$, where the η_i are continuous functions on $[0,T]$ and the ψ_i belong to $L^\infty(\Omega)$ for all i in $\{1,\dots,k\}$, we deduce, from (4.30) and from the $L^\infty(\Omega)$ bound of w_ε (see (2.1)(i)), that

$$(4.31)\qquad \int_\Omega \tilde{u}_\varepsilon'(x,\,\cdot\,)w_\varepsilon(x)\varphi'(x,\,\cdot\,)\mathrm{d}x \to \int_\Omega u'(x,\,\cdot\,)\varphi'(x,\,\cdot\,)\mathrm{d}x \text{ strongly in } C^0([0,T]).$$

Fourth term. Let us now consider the last term of (4.23). We have

$$(4.32)\qquad \left\{\begin{aligned} &\int_\Omega \nabla\tilde{u}_\varepsilon(x,t)\nabla(w_\varepsilon(x)\varphi(x,t))\mathrm{d}x = \langle -\Delta w_\varepsilon, \tilde{u}_\varepsilon(t)\varphi(t)\rangle_\Omega \\ &-2\int_\Omega \tilde{u}_\varepsilon(x,t)\nabla w_\varepsilon(x)\nabla\varphi(x,t)\mathrm{d}x - \int_\Omega \tilde{u}_\varepsilon(x,t)w_\varepsilon(x)\Delta\varphi(x,t)\mathrm{d}x.\end{aligned}\right.$$

Consider the function

$$t \to -2\int_\Omega \tilde{u}_\varepsilon(x,t)\nabla w_\varepsilon(x)\nabla\varphi(x,t)\mathrm{d}x - \int_\Omega \tilde{u}_\varepsilon(x,t)w_\varepsilon(x)\Delta\varphi(x,t)\mathrm{d}x.$$

Since $\tilde{u}_\varepsilon$ is bounded in $W^{1,\infty}(0,T;L^2(\Omega))$ (see Theorem 3.1), the above function is bounded in $W^{1,\infty}(0,T)$, thus relatively compact in $C^0([0,T])$. This implies that

$$(4.33)\qquad \begin{cases} -2\int_\Omega \tilde{u}_\varepsilon(x,\cdot)\nabla w_\varepsilon(x)\nabla\varphi(x,\cdot)\mathrm{d}x - \int_\Omega \tilde{u}_\varepsilon(x,\cdot)w_\varepsilon(x)\Delta\varphi(x,\cdot)\mathrm{d}x \\ \to -\int_\Omega u(x,\cdot)\Delta\varphi(x,\cdot)\mathrm{d}x \\ = \int_\Omega \nabla u(x,\cdot)\nabla\varphi(x,\cdot)\mathrm{d}x \quad \text{strongly in } C^0([0,T]). \end{cases}$$

Consider now the remaining term $\langle -\Delta w_\varepsilon, \tilde{u}_\varepsilon(t)\varphi(t)\rangle_\Omega$. Since $\tilde{u}_\varepsilon$ vanishes on the holes, we have

$$\langle -\Delta w_\varepsilon, \tilde{u}_\varepsilon(t)\varphi(t)\rangle_\Omega = \langle \mu_\varepsilon, \tilde{u}_\varepsilon(t)\varphi(t)\rangle_\Omega.$$

On the other hand, since $\mu \in H^{-1}(\Omega)$, there exists a sequence ν_k in $L^2(\Omega)$ such that

$$(4.34)\qquad \nu_k \to \mu \text{ strongly in } H^{-1}(\Omega).$$

We have

$$(4.35)\qquad \begin{cases} \|\langle -\Delta w_\varepsilon, \tilde{u}_\varepsilon\varphi\rangle_\Omega - \langle \mu, u\varphi\rangle_\Omega\|_{L^\infty(0,T)} \le \|\langle \mu_\varepsilon - \mu, \tilde{u}_\varepsilon\varphi\rangle_\Omega\|_{L^\infty(0,T)} \\ \quad +\|\langle \mu - \nu_k, \varphi(\tilde{u}_\varepsilon - u)\rangle_\Omega\|_{L^\infty(0,T)} + \|\langle \nu_k, \varphi(\tilde{u}_\varepsilon - u)\rangle_\Omega\|_{L^\infty(0,T)}. \end{cases}$$

By (2.1)(iv) and (3.7) one has

$$(4.36)\qquad \begin{cases} \lim\limits_{\varepsilon\to 0} \|\langle \mu_\varepsilon - \mu, \tilde{u}_\varepsilon\varphi\rangle_\Omega\|_{L^\infty(0,T)} \\ \le \lim\limits_{\varepsilon\to 0}(\|\tilde{u}_\varepsilon\varphi\|_{L^\infty(0,T;H_0^1(\Omega))}\|\mu_\varepsilon - \mu\|_{H^{-1}(\Omega)}) = 0. \end{cases}$$

On the other hand, (4.34) yields

$$(4.37)\qquad \begin{cases} \lim\limits_{k\to\infty} \limsup\limits_{\varepsilon\to 0} \|\langle \mu - \nu_k, \varphi(\tilde{u}_\varepsilon - u)\rangle_\Omega\|_{L^\infty(0,T)} \\ \le \lim\limits_{k\to\infty} \limsup\limits_{\varepsilon\to 0} \|\varphi(\tilde{u}_\varepsilon - u)\|_{L^\infty(0,T;H_0^1(\Omega))}\|\mu - \nu_k\|_{H^{-1}(\Omega)} = 0. \end{cases}$$

Finally, from (3.7) and Proposition 2.5, we have

$$\tilde{u}_\varepsilon \to u \quad \text{strongly in } C^0([0,T];L^2(\Omega)).$$

Therefore, for k fixed,

$$(4.38)\qquad \begin{cases} \lim\limits_{\varepsilon\to 0} \|\langle \nu_k, \varphi(\tilde{u}_\varepsilon - u)\rangle_\Omega\|_{L^\infty(0,T)} \\ \le \lim\limits_{\varepsilon\to 0}(\|\nu_k\|_{L^2(\Omega)}\|\varphi\|_{L^\infty(\Omega)}\|\tilde{u}_\varepsilon - u\|_{C^0([0,T];L^2(\Omega))}) = 0. \end{cases}$$

Combining (4.35)-(4.38) we deduce that

$$(4.39) \qquad \langle -\Delta w_\varepsilon, \tilde{u}_\varepsilon \varphi\rangle_\Omega \to \langle \mu, u\varphi\rangle_\Omega \quad \text{strongly in } C^0([0,T]).$$

From (4.23), (4.24), (4.29), (4.31), (4.32), (4.33) and (4.39) we get (4.22). The proof of Proposition 4.3 is complete. □

PROOF OF THEOREM 4.1. From Theorem 3.1 we know that

$$u \in C^0([0,T];V) \cap C^1([0,T];L^2(\Omega)).$$

Let us consider a sequence φ_k in $\mathcal{D}(Q)$ such that

$$(4.40) \qquad \varphi_k \to u \text{ strongly in } C^0([0,T];V) \cap C^1([0,T];L^2(\Omega)) \text{ as } k \to 0.$$

From Proposition 4.3 we have

$$\limsup_{\varepsilon\to 0}\{\|(\tilde{u}_\varepsilon - w_\varepsilon\varphi_k)'\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\nabla(\tilde{u}_\varepsilon - w_\varepsilon\varphi_k)\|^2_{L^\infty(0,T;L^2(\Omega))}\}$$
$$\leq 2\|e(u-\varphi_k)\|_{L^\infty(0,T)}$$

and thus

$$(4.41) \qquad \begin{cases} \lim\limits_{k\to\infty} \limsup\limits_{\varepsilon\to 0}\{\|\tilde{u}_\varepsilon - w_\varepsilon\varphi_k)'\|_{L^\infty(0,T;L^2(\Omega))} \\ \qquad +\|\nabla(\tilde{u}_\varepsilon - w_\varepsilon\varphi_k)\|_{L^\infty(0,T;L^2(\Omega))}\} = 0. \end{cases}$$

We now observe that

$$(4.42) \qquad \begin{cases} \|\tilde{u}'_\varepsilon - u'\|_{L^\infty(0,T;L^2(\Omega))} \leq \|\tilde{u}'_\varepsilon - w_\varepsilon\varphi'_k\|_{L^\infty(0,T;L^2(\Omega))} \\ +\|(w_\varepsilon - 1)\varphi'_k\|_{L^\infty(0,T;L^2(\Omega))} + \|\varphi'_k - u'\|_{L^\infty(0,T;L^2(\Omega))}. \end{cases}$$

Combining (4.40), (4.41), (4.42) and hypothesis (2.1), we easily deduce that

$$\tilde{u}'_\varepsilon \to u' \quad \text{strongly in } C^0([0,T];L^2(\Omega)).$$

Therefore (4.6) is proved.

On the other hand, we have

$$(4.43) \qquad \begin{cases} \|\nabla(\tilde{u}_\varepsilon - w_\varepsilon u)\|_{L^\infty(0,T;L^1(\Omega))} \\ \quad \leq \|\nabla(\tilde{u}_\varepsilon - w_\varepsilon\varphi_k)\|_{L^\infty(0,T;L^1(\Omega))} + \|\nabla(w_\varepsilon(\varphi_k - u))\|_{L^\infty(0,T;L^1(\Omega))} \\ \quad \leq C\|\nabla(\tilde{u}_\varepsilon - w_\varepsilon\varphi_k)\|_{L^\infty(0,T;L^2(\Omega))} \\ \qquad +\|(\varphi_k - u)\nabla w_\varepsilon\|_{L^\infty(0,T;L^1(\Omega))} + \|w_\varepsilon\nabla(\varphi_k - u)\|_{L^\infty(0,T;L^1(\Omega))} \\ \quad \leq C\|\nabla(\tilde{u}_\varepsilon - w_\varepsilon\varphi_k)\|_{L^\infty(0,T;L^2(\Omega))} \\ \qquad +\|\nabla w_\varepsilon\|_{L^2(\Omega)}\|\varphi_k - u\|_{C^0([0,T];L^2(\Omega))} \\ \qquad +\|w_\varepsilon\|_{L^\infty(\Omega)}\|\varphi_k - u\|_{C^0([0,T];H^1_0(\Omega))}. \end{cases}$$

By (2.1), (4.22), (4.40) and (4.43), we conclude that

$$\nabla R_\varepsilon = \nabla(\tilde{u}_\varepsilon - w_\varepsilon \varphi) \to 0 \quad \text{strongly in } C^0([0,T];L^1(\Omega)).$$

Thus (4.8) is proved.

Let us finally consider the case where $u \in C^0(\overline{\Omega} \times [0,T])$. In such case, the approximating sequence φ_k may be chosen to satisfy, further to (4.40), the hypothesis

$$\varphi_k \to u \quad \text{strongly in } C^0(\overline{\Omega} \times [0,T]). \tag{4.44}$$

In this case we can estimate $\nabla(\tilde{u}_\varepsilon - w_\varepsilon u)$ in $L^\infty(0,T;L^2(\Omega))$ and not only in $L^\infty(0,T;L^1(\Omega))$ as in (4.43): indeed, we have

$$\begin{aligned} &\|\nabla(w_\varepsilon(\varphi_k - u))\|_{L^\infty(0,T;L^2(\Omega))} \\ &\qquad \le \|(\varphi_k - u)\nabla w_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} + \|w_\varepsilon \nabla(\varphi_k - u)\|_{L^\infty(0,T;L^2(\Omega))} \\ &\qquad \le \|\varphi_k - u\|_{C^0(\overline{\Omega}\times[0,T])}\|\nabla w_\varepsilon\|_{L^2(\Omega)} \\ &\qquad\quad + \|w_\varepsilon\|_{L^\infty(\Omega)}\|\varphi_k - u\|_{L^\infty(0,T;H_0^1(\Omega))}. \end{aligned}$$

Similarly to (4.43), this implies

$$\nabla R_\varepsilon = \nabla(\tilde{u}_\varepsilon - w_\varepsilon u) \to 0 \quad \text{strongly in } C^0([0,T];L^2(\Omega))$$

which gives the desired result (4.9).

The proof of Theorem 4.1 is now complete. □

5. - The case of holes smaller than the critical size

In this Section we consider the particular case where the holes are smaller than the critical size. This corresponds to the assumption that the functions w_ε of hypothesis (2.1) strongly converge in $H^1(\Omega)$, which implies $\mu = 0$. In this case all the results of Sections 3 and 4 hold true, but the corrector result of Theorem 4.1 can be improved by replacing w_ε by 1 in the statement.

Let us assume that the holes S_ε are such that

$$\begin{cases} \text{there exists a sequence of test functions } w_\varepsilon \text{ satisfying} \\ \text{(i)} \quad w_\varepsilon \in H^1(\Omega), \|w_\varepsilon\|_{L^\infty(\Omega)} \le M_0 \\ \text{(ii)} \quad w_\varepsilon = 0 \quad \text{on } S_\varepsilon \\ \text{(iii)} \quad w_\varepsilon \to 1 \quad \text{strongly in } H^1(\Omega). \end{cases} \tag{5.1}$$

REMARK 5.1. Once again the main assumption is not made directly on the form and size of the holes but in terms of the family of test functions w_ε.

The main difference between assumptions (2.1) and (5.1) is that in (5.1)(iii) we assume the strong convergence of w_ε. In this case (2.1)(iv) is obviously satisfied with $\gamma_\varepsilon = 0$, $\mu_\varepsilon = -\Delta w_\varepsilon$ and $\mu = 0$. As pointed out in Remark 2.2, in the model case (Example 2.1), assumption (5.1) signifies that the size of the holes is smaller than the critical one given by (2.4), i.e. that (2.7) holds.

Hypothesis (5.1) may also be understood in terms of the capacity of S_ε with respect to Ω. More precisely, let us denote by $\mathrm{Cap}(A, B)$ the capacity of the closed se $A \subset B$ with respect to the open set B, i.e.

$$\mathrm{Cap}(A,B) = \inf_{\substack{v \in \mathcal{D}(B) \\ v=1 \text{ on } A}} \int_B |\nabla v(x)|^2 \mathrm{d}x.$$

It is easy to see that hypothesis (5.1) follows from the hypothesis

$$\mathrm{Cap}(S_\varepsilon, \Omega) \to 0 \qquad \text{as } \varepsilon \to 0.$$

Indeed in this case the function w_ε can be constructed from the capacitary potential of S_ε by setting $w_\varepsilon = 1 - p_\varepsilon$, where $p_\varepsilon \in H_0^1(\Omega)$, $p_\varepsilon = 1$ on S_ε, is the unique function which achieves the minimum in $\mathrm{Cap}(S_\varepsilon, \Omega)$.

Note finally that if we assume that (2.1) holds true with $\mu = 0$, the use of $v_\varepsilon = \varphi w_\varepsilon$ with $\varphi \in \mathcal{D}(\Omega)$ in (2.1)(iv) implies that

$$w_\varepsilon \to 1 \qquad \text{strongly in } H^1_{\mathrm{loc}}(\Omega).$$

The assumption $\mu = 0$ in (2.1) is thus equivalent to a "local version" of (5.1), where $H^1(\Omega)$ is replaced by $H^1_{\mathrm{loc}}(\Omega)$.

Let us mention two simple examples where (5.1) is satisfied.

EXAMPLE 1. As already pointed out in Example 2.1 and Remark 2.2, (5.1) is satisfied when Ω is periodically perforated by holes S_ε of form S, the size of which satisfies (2.7).

EXAMPLE 2. Another situation where (5.1) is satisfied is the case where S_ε is the union of a finite (and fixed) number N of vanishing holes, i.e. $S_\varepsilon = \bigcup_{i=1}^{N} S_\varepsilon^i$ with S_ε^i closed sets such that $S_\varepsilon^i \subset K$ for some K such that $\overline{K} \subset \Omega$ and $\mathrm{diam}(S_\varepsilon^i) \to 0$ as $\varepsilon \to 0$. □

Under assumption (5.1) all the results of Sections 3 and 4 obviously hold true, but strong convergence of the data now implies strong convergence of the solutions: indeed the corrector result of Theorem 4.1 holds true if w_ε is replaced by 1.

THEOREM 5.1. *Assume that* (5.1) *holds true and consider a sequence of data that satisfy* (3.6). *The sequence* u_ε *of solutions of* (3.1) *then satisfies*

$$\begin{cases} \tilde{u}_\varepsilon \rightharpoonup u & \text{weakly} * \text{ in } L^\infty(0,T;H_0^1(\Omega)) \\ \tilde{u}'_\varepsilon \rightharpoonup u' & \text{weakly} * \text{ in } L^\infty(0,T;L^2(\Omega)) \end{cases}$$

and

$$\begin{cases} \tilde{u}_\varepsilon(t) \rightharpoonup u(t) & \text{weakly in } H_0^1(\Omega) \\ \tilde{u}'_\varepsilon(t) \rightharpoonup u'(t) & \text{weakly in } L^2(\Omega) \end{cases}$$

for all $t \in [0,T]$, *where the limit* u *is the unique solution of*

$$\begin{cases} u'' - \Delta u = f \quad \text{in } Q = \Omega \times (0,T) \\ u = 0 \quad \text{on } \Sigma = \partial\Omega \times (0,T) \\ u(0) = u^0, \ u'(0) = u^1 \quad \text{in } \Omega \\ u \in C^0([0,T];H_0^1(\Omega)) \cap C^1([0,T];L^2(\Omega)). \end{cases} \tag{5.2}$$

Moreover, if the data satisfy the stronger assumption

$$\begin{cases} \tilde{f}_\varepsilon \to f & \text{strongly in } L^1(0,T;L^2(\Omega)) \\ \tilde{u}^0_\varepsilon \to u^0 & \text{strongly in } H_0^1(\Omega) \\ \tilde{u}^1_\varepsilon \to u^1 & \text{strongly in } L^2(\Omega) \end{cases} \tag{5.3}$$

then

$$\begin{cases} \tilde{u}_\varepsilon \to u & \text{strongly in } C^0([0,T];H_0^1(\Omega)) \\ \tilde{u}'_\varepsilon \to u' & \text{strongly in } C^0([0,T];L^2(\Omega)). \end{cases} \tag{5.4}$$

REMARK 5.2. When hypothesis (2.1) is replaced by hypothesis (5.1), assumption (5.3) is equivalent to the hypotheses of Theorem 4.1. Indeed, when (4.1) and (5.1) hold true, Theorem 2.2 implies the strong convergence of $\tilde{u}^0_\varepsilon$ to u^0 in $H_0^1(\Omega)$ because μ is 0 in (2.17). Conversely $g_\varepsilon = -\Delta\tilde{u}^0_\varepsilon$ and $g = -\Delta u^0$ clearly sastisfy (4.1). □

PROOF OF THEOREM 5.1. The first part of Theorem 5.1 is a mere rewriting of Theorems 3.1 and 3.2. On the other hand, hypothesis (5.3) implies that the hypotheses of Theorem 4.1 on the data u^0_ε, u^1_ε and f_ε are satisfied. Therefore (5.4b) is nothing but (4.6).

To prove (5.4a) we proceed as in the proof of Theorem 4.1. Let $\varphi_k \in \mathcal{D}(Q)$ be a sequence verifying (4.40). We have

$$\begin{cases} \|\nabla(\tilde{u}_\varepsilon - u)\|_{L^\infty(0,T;L^2(\Omega))} \leq \|\nabla(\tilde{u}_\varepsilon - w_\varepsilon\varphi_k)\|_{L^\infty(0,T;L^2(\Omega))} \\ +\|\nabla((1-w_\varepsilon)\varphi_k)\|_{L^\infty(0,T;L^2(\Omega))} + \|\nabla(\varphi_k - u)\|_{L^\infty(0,T;L^2(\Omega))}. \end{cases} \tag{5.5}$$

Arguing as in the proof of Theorem 4.1 and using now the strong convergence of w_ε to 1 in the second term of (5.5) (this is the essential novelty here) we easily obtain (5.4a). □

6. - Homogenization and corrector for non-smooth data

This Section is devoted to the study of the homogenization and corrector for the wave equation (3.1) with non-smooth data. We consider here the case where

$$u_\varepsilon^0 \in L^2(\Omega_\varepsilon), \; u_\varepsilon^1 \in H^{-1}(\Omega_\varepsilon), \; f_\varepsilon \in L^1(0,T;L^2(\Omega_\varepsilon)). \tag{6.1}$$

This functional framework naturally appears in the study of the exact controllability problems for the wave equation in domains with small holes (see D. Cioranescu and P. Donato [3], D. Cioranescu, P. Donato and E. Zuazua [4], [5]).

When the data only satisfy (6.1), the solution u_ε of (3.1) has to be defined by transposition: $u_\varepsilon = u_\varepsilon(x,t)$ is said to be a solution of (3.1) in the transposition sense if

$$\begin{cases} u_\varepsilon \in L^\infty(0,T;L^2(\Omega_\varepsilon)) \\ \int_0^T \int_{\Omega_\varepsilon} u_\varepsilon(x,t) g_\varepsilon(x,t) \mathrm{d}x \, \mathrm{d}t = -\int_{\Omega_\varepsilon} u_\varepsilon^0(x)\theta_\varepsilon'(x,0)\mathrm{d}x + \langle u_\varepsilon^1, \theta_\varepsilon(0)\rangle_{\Omega_\varepsilon} \\ \qquad\qquad + \int_0^T \int_{\Omega_\varepsilon} f_\varepsilon(x,t)\theta_\varepsilon(x,t)\mathrm{d}x \, \mathrm{d}t \end{cases} \tag{6.2}$$

for all $g_\varepsilon \in L^1(0,T;L^2(\Omega_\varepsilon))$, where $\theta_\varepsilon = \theta_\varepsilon(x,t)$ is the unique solution of

$$\begin{cases} \theta_\varepsilon'' - \Delta\theta_\varepsilon = g_\varepsilon & \text{in } Q_\varepsilon \\ \theta_\varepsilon = 0 & \text{on } \Sigma_\varepsilon \\ \theta_\varepsilon(T) = \theta_\varepsilon'(T) = 0 & \text{in } \Omega_\varepsilon. \end{cases} \tag{6.3}$$

Note that (6.3) admits a unique solution

$$\theta_\varepsilon \in C^0([0,T];H_0^1(\Omega_\varepsilon)) \cap C^1([0,T];L^2(\Omega_\varepsilon)). \tag{6.4}$$

Therefore each term in (6.2) has a meaning.

Following along the lines of J.-L. Lions [14, Chapitre I] it is easy to see that (6.2) admits a unique solution $u_\varepsilon \in L^\infty(0,T;L^2(\Omega_\varepsilon))$; moreover this solution satisfies

$$u_\varepsilon \in C^0([0,T];L^2(\Omega_\varepsilon)) \cap C^1([0,T];H^{-1}(\Omega_\varepsilon)).$$

REMARK 6.1. Another (equivalent) way of defining the solution of the wave equation (3.1), when the data only satisfy (6.1), is to consider the equation satisfied by the primitive of u_ε with respect to time. This idea will be used below in the proof of Theorem 6.2. □

LEMMA 6.1. *Consider a sequence u_ε of solutions of* (6.2) *associated to data satisfying* (6.1) *and assume that*

$$\text{(6.5)} \qquad \begin{cases} \|f_\varepsilon\|_{L^1(0,T;L^2(\Omega_\varepsilon))} \leq C \\ \|u_\varepsilon^0\|_{L^2(\Omega_\varepsilon)} \leq C \\ \|u_\varepsilon^1\|_{H^{-1}(\Omega_\varepsilon)} \leq C \end{cases}$$

Then

$$\text{(6.6)} \qquad \|u_\varepsilon\|_{L^\infty(0,T;L^2(\Omega_\varepsilon))} \leq C.$$

PROOF. In view of (6.2) we have

$$\text{(6.7)} \qquad \begin{cases} \left|\int_0^T \int_{\Omega_\varepsilon} u_\varepsilon g_\varepsilon \, \mathrm{d}x \, \mathrm{d}t\right| \leq \|u_\varepsilon^0\|_{L^2(\Omega_\varepsilon)} \|\theta_\varepsilon'(0)\|_{L^2(\Omega_\varepsilon)} \\ \qquad + \|u_\varepsilon^1\|_{H^{-1}(\Omega_\varepsilon)} \|\theta_\varepsilon(0)\|_{H_0^1(\Omega_\varepsilon)} \\ \qquad + \|f_\varepsilon\|_{L^1(0,T;L^2(\Omega_\varepsilon))} \|\theta_\varepsilon\|_{L^\infty(0,T;L^2(\Omega_\varepsilon))} \\ \qquad \leq C\{\|\theta_\varepsilon\|_{C^0([0,T];H_0^1(\Omega_\varepsilon))} + \|\theta_\varepsilon'\|_{C^0([0,T];L^2(\Omega_\varepsilon))}\}. \end{cases}$$

By using for θ_ε the a priori estimates obtained in the proof of Theorem 3.1 (see (3.10)), we deduce from (6.7) that for all $g_\varepsilon \in L^1(0,T;L^2(\Omega_\varepsilon))$ one has

$$\text{(6.8)} \qquad \left|\int_0^T \int_{\Omega_\varepsilon} u_\varepsilon g_\varepsilon \, \mathrm{d}x \, \mathrm{d}t\right| \leq C\|g_\varepsilon\|_{L^1(0,T;L^2(\Omega_\varepsilon))}$$

where the constant C does not depend on ε. This proves (6.6). □

We have the following homogenization result.

THEOREM 6.2. *Assume that* (2.1) *holds true and consider a sequence of data which satisfy*

$$\text{(6.9)} \qquad f_\varepsilon \in L^1(0,T;L^2(\Omega_\varepsilon)) \quad \textit{and} \quad \tilde{f}_\varepsilon \rightharpoonup f \quad \textit{weakly in } L^1(0,T;L^2(\Omega))$$

$$\text{(6.10)} \qquad u_\varepsilon^0 \in L^2(\Omega_\varepsilon) \quad \textit{and} \quad \tilde{u}_\varepsilon^0 \rightharpoonup u^0 \quad \textit{weakly in } L^2(\Omega)$$

$$\text{(6.11)} \qquad \begin{cases} u_\varepsilon^1 \in H^{-1}(\Omega_\varepsilon), & \|u_\varepsilon^1\|_{H^{-1}(\Omega_\varepsilon)} \leq C \\ P_\varepsilon u_\varepsilon^1 \rightharpoonup u^1 & \textit{weakly in } W^{-1,q}(\Omega) \textit{ with } 1 < q < n/(n-1), \end{cases}$$

where P_ε is the quasi-extension operator defined in Proposition 2.8.

The solution u_ε *of* (6.2) *then satisfies*

$$\tilde{u}_\varepsilon \rightharpoonup u \quad \textit{weakly} * \textit{ in } L^\infty(0,T;L^2(\Omega)) \tag{6.12}$$

where $u = u(x,t)$ *is the unique solution (in the transposition sense) of*

$$\begin{cases} u'' - \Delta u + \mu u = f & \textit{in } Q \\ u = 0 & \textit{on } \Sigma \\ u(0) = u^0, \;\; u'(0) = u^1 & \textit{in } \Omega \\ u \in C^0([0,T];L^2(\Omega)) \cap C^1([0,T];V'). \end{cases} \tag{6.13}$$

REMARK 6.2. System (6.13) has to be understood in the transposition sense:

$$\begin{cases} u \in L^\infty(0,T;L^2(\Omega)) \\ \int_0^T \int_\Omega u(x,t)g(x,t)\mathrm{d}x\,\mathrm{d}t = -\int_\Omega u^0(x)\theta'(x,0)\mathrm{d}x + \langle u^1, \theta(0)\rangle_{V',V} \\ \qquad\qquad + \int_0^T \int_\Omega f(x,t)\theta(x,t)\mathrm{d}x\,\mathrm{d}t \end{cases} \tag{6.14}$$

for all $g \in L^1(0,T;L^2(\Omega))$, where $\theta = \theta(x,t)$ is the unique solution of

$$\begin{cases} \theta'' - \Delta\theta + \mu\theta = g & \text{in } Q \\ \theta = 0 & \text{on } \Sigma \\ \theta(T) = \theta'(T) = 0 & \text{in } \Omega \\ \theta \in C^0([0,T];V) \cap C^1([0,T];L^2(\Omega)). \end{cases} \tag{6.15}$$

Another (equivalent) way of defining the solution u of (6.13) is to consider the equation satisfied by the primitive of u with respect to time (see below the proof of Theorem 6.2). □

REMARK 6.3. In view of (2.53) (Remark 2.10) we know that u^1 defined in (6.11) actually belongs to V'. Thus each term makes sense in (6.14). □

PROOF OF THEOREM 6.2. The main idea of the proof of Theorem 6.2 is to integrate the solution u_ε with respect to time, in order to work with a wave equation with usual (smooth) data where the results of the previous Sections apply. This argument was introduced by J.-L. Lions in [15] and was used in the context of exact controllability of the wave equation in perforated domains in D. Cioranescu and P. Donato [3].

Define

$$y_\varepsilon(x,t) = \int_0^t u_\varepsilon(x,s)\mathrm{d}s + z_\varepsilon(x) \tag{6.16}$$

where z_ε is the solution of the elliptic problem

(6.17) $$\begin{cases} -\Delta z_\varepsilon = -u^1_\varepsilon & \text{in } \mathcal{D}'(\Omega_\varepsilon) \\ z_\varepsilon \in H^1_0(\Omega_\varepsilon). \end{cases}$$

Define also

(6.18) $$h_\varepsilon(x,t) = \int_0^t f_\varepsilon(x,s)\mathrm{d}s.$$

The function y_ε is the unique solution of the following wave equation

(6.19) $$\begin{cases} y''_\varepsilon - \Delta y_\varepsilon = h_\varepsilon & \text{in } Q_\varepsilon \\ y_\varepsilon = 0 & \text{on } \Sigma_\varepsilon \\ y_\varepsilon(0) = z_\varepsilon, \ \ y'_\varepsilon(0) = u^0_\varepsilon & \text{in } \Omega_\varepsilon \end{cases}$$

and u_ε is nothing but $u_\varepsilon = y'_\varepsilon$.

From (6.11) we deduce (see Remark 2.10) that z_ε is bounded in $H^1_0(\Omega_\varepsilon)$ and that

$$\tilde{z}_\varepsilon \rightharpoonup z \quad \text{weakly in } H^1_0(\Omega)$$

where z is the solution of

$$\begin{cases} -\Delta z + \mu z = -u^1 \\ z \in V \end{cases}$$

with u^1 defined by (6.11).

On the other hand, in view of (6.9), we have

$$\tilde{h}_\varepsilon \rightharpoonup h = \int_0^t f(x,s)\mathrm{d}s \quad \text{weakly in } L^1(0,T;L^2(\Omega)).$$

Applying Theorem 3.1, we deduce that

$$\begin{cases} \tilde{y}_\varepsilon \rightharpoonup y & \text{weakly } * \text{ in } L^\infty(0,T;H^1_0(\Omega)) \\ \tilde{y}'_\varepsilon \rightharpoonup y' & \text{weakly } * \text{ in } L^\infty(0,T;L^2(\Omega)) \end{cases}$$

where $y = y(x,t)$ is the unique solution of

(6.20) $$\begin{cases} y'' - \Delta y + \mu y = h & \text{in } Q \\ y = 0 & \text{on } \Sigma \\ y(0) = z, \ \ y'(0) = u^0 & \text{in } \Omega \\ y \in C^0([0,T];V) \cap C^1([0,T];L^2(\Omega)). \end{cases}$$

Since h belongs to $C^0([0,T];L^2(\Omega))$ and since $-\Delta+\mu$ is an isomorphism from V into V', $y''=\Delta y-\mu y+h$ belongs to $C^0([0,T];V')$. Defining u by $u=y'$, we deduce from (6.20) that

$$u'(0)=y''(0)=\Delta z-\mu z=u^1.$$

Thus u is the solution of (6.13). Since $u_\varepsilon=y'_\varepsilon$, we have in particular

$$\tilde{u}_\varepsilon=\tilde{y}'_\varepsilon \rightharpoonup y'=u \quad \text{weakly } * \text{ in } L^\infty(0,T;L^2(\Omega)),$$

and Theorem 6.2 is proved. □

If assumptions (6.9)-(6.11) of Theorem 6.1 on the data f_ε and u^0_ε are replaced by stronger ones, the strong convergence of u_ε follows; indeed we have the following result, which is in some sense the analogue of Theorem 4.1.

THEOREM 6.3. *Assume that* (2.1) *holds true and consider a sequence of data which satisfy*

(6.21) $$f_\varepsilon\in L^1(0,T;L^2(\Omega_\varepsilon)) \quad \textit{and} \quad \tilde{f}_\varepsilon\to f \quad \textit{strongly in } L^1(0,T;L^2(\Omega))$$

(6.22) $$u^0_\varepsilon\in L^2(\Omega_\varepsilon) \quad \textit{and} \quad \tilde{u}^0_\varepsilon\to u^0 \quad \textit{strongly in } L^2(\Omega)$$

(6.23) $$u^1_\varepsilon\in H^{-1}(\Omega) \quad \textit{and} \quad u^1_\varepsilon\to u^1 \quad \textit{strongly in } H^{-1}(\Omega).$$

Then

(6.24) $$\tilde{u}_\varepsilon\to u \quad \textit{strongly in } C^0([0,T];L^2(\Omega))$$

where $u=u(x,t)$ *is the solution of* (6.13).

PROOF. Proceed as in the proof of Theorem 6.2 and observe that the sequence of solutions y_ε of (6.19) now satisfies the hypotheses of Theorem 4.1. Thus, in particular

$$\tilde{u}_\varepsilon=\tilde{y}'_\varepsilon\to y'=u \quad \text{strongly in } C^0([0,T];L^2(\Omega))$$

which is the desired result. □

REMARK 6.4. There is no contradiction between hypotheses (6.11) and (6.23). Indeed when $u^1_\varepsilon\in H^{-1}(\Omega)$ and u^1_ε tends strongly to $u^1\in H^{-1}(\Omega)$, it follows from the definition (2.48) of the operator $P_\varepsilon=R^*_\varepsilon$ that

$$P_\varepsilon u^1_\varepsilon \rightharpoonup u^1 \quad \text{weakly in} \quad W^{-1,q}(\Omega) \quad \text{with} \quad 1<q<n/(n-1).$$

Note also that assumption (6.9) of Theorem 6.2 (respectively assumption (6.21) in the statement of Theorem 6.3) can be replaced by the weaker one

$$(6.25)\quad h_\varepsilon(x,t) = \int_0^t \tilde{f}_\varepsilon(x,s)\mathrm{d}s \rightharpoonup h(x,t) = \int_0^t f(x,s)\mathrm{d}s \quad \text{weakly in } L^1(0,T;L^2(\Omega))$$

(respectively strongly in $L^1(0,T;L^2(\Omega))$). □

We now give a convergence result concerning the case of non-smooth data (see (6.1)) and holes smaller than the critical size (i.e. satisfying (5.1)). If we further assume that u_ε^1 belongs to $H^{-1}(\Omega)$ (and not only to $H^{-1}(\Omega_\varepsilon)$), we can here replace hypothesis (6.11) by a simpler one which formally corresponds to the choice $w_\varepsilon = 1$.

THEOREM 6.4. *Assume that* (5.1) *holds true and consider a sequence of data which satisfy*

$$(6.26)\qquad f_\varepsilon \in L^1(0,T;L^2(\Omega_\varepsilon)) \quad \textit{and} \quad \tilde{f}_\varepsilon \rightharpoonup f \quad \textit{weakly in } L^1(0,T;L^2(\Omega))$$

$$(6.27)\qquad u_\varepsilon^0 \in L^2(\Omega_\varepsilon) \quad \textit{and} \quad \tilde{u}_\varepsilon^0 \rightharpoonup u^0 \quad \textit{weakly in } L^2(\Omega)$$

$$(6.28)\qquad u_\varepsilon^1 \in H^{-1}(\Omega) \quad \textit{and} \quad \tilde{u}_\varepsilon^1 \rightharpoonup u^1 \quad \textit{weakly in } H^{-1}(\Omega).$$

The solution u_ε of (6.2) *then satisfies*

$$(6.29)\qquad \tilde{u}_\varepsilon \rightharpoonup u \quad \textit{weakly} * \textit{ in } L^\infty(0,T;L^2(\Omega))$$

where $u = u(x,t)$ is the unique solution (in the transposition sense) of

$$(6.30)\qquad \begin{cases} u'' - \Delta u = f & \textit{in } Q \\ u = 0 & \textit{on } \Sigma \\ u(0) = u^0,\ u'(0) = u^1 & \textit{in } \Omega \\ u \in C^0([0,T];L^2(\Omega)) \cap C^1([0,T];H^{-1}(\Omega)). \end{cases}$$

PROOF. To prove Theorem 6.4, it is sufficient to repeat the proof of Theorem 6.2, just observing that when (5.1) and (6.29) hold true, the solution z_ε of (6.17) now satisfies

$$\tilde{z}_\varepsilon \rightharpoonup z \quad \text{weakly in } H_0^1(\Omega)$$

where z is the solution of

$$\begin{cases} -\Delta z = -u^1 & \text{in } \mathcal{D}'(\Omega) \\ z \in H_0^1(\Omega). \end{cases}$$

□

REMARK 6.5. In the setting of Theorem 6.4, Theorem 6.3 applies without any modification: if (5.1), (6.21), (6.22) and (6.23) hold true, then $\tilde{u}_\varepsilon$ tends strongly in $C^0([0,T];L^2(\Omega))$ to the solution u of (6.30) (see (6.24)). □

Appendix: $\mathcal{D}(\Omega)$ is dense in V

For the sake of completeness, we present in this Appendix the proof of the following result.

THEOREM A.1. *Let Ω be an open bounded set of $\mathbb{R}^n$ and let $\mu \in H^{-1}(\Omega)$ be a positive and finite Radon measure on Ω. Then $\mathcal{D}(\Omega)$ is dense in the space $V = H_0^1(\Omega) \cap L^2(\Omega; \mathrm{d}\mu)$ endowed with its natural norm.*

PROOF. We proceed in three steps.

First step. We first prove that $V \cap L^\infty(\Omega) \cap L^\infty(\Omega; \mathrm{d}\mu)$ is dense in V.

For $v \in V$ and for any $k \geq 0$ define

$$T_k v = \begin{cases} k & \text{if } v \geq k \\ v & \text{if } |v| \leq k \\ -k & \text{if } v \leq -k. \end{cases}$$

It is well known that $T_k v \in H_0^1(\Omega) \cap L^\infty(\Omega)$ for every $k \geq 0$ and that

$$T_k v \to v \quad \text{strongly in } H_0^1(\Omega) \quad \text{as } k \to \infty. \tag{A.1}$$

On the other hand (see e.g. H. Lewy and G. Stampacchia [13, Appendix]) we know that $|T_k(x)| \leq k$ for all $x \in \Omega$, except on a set of zero capacity. Since μ is a Radon measure and belongs to $H^{-1}(\Omega)$, M. Grun-Rehomme [10, Lemme 3] asserts that every set of zero capacity is μ-measurable and is of zero μ-measure. We have therefore $|T_k v(x)| \leq k$ for all $x \in \Omega$, except on a set of μ-measure zero and thus

$$T_k v \in L^\infty(\Omega; \mathrm{d}\mu) \subset L^2(\Omega; \mathrm{d}\mu).$$

Hence $T_k v$ belongs to V.

From (A.1) we deduce that for a suitable subsequence (still denoted by k), $T_k v(x)$ tends to $v(x)$ for all $x \in \Omega$, except on a set of zero capacity (see e.g. J. Frehse [9, Theorem 2.3]). Applying M. Grun-Rehomme [10, Lemme 3] once again, we deduce that, for the same subsequence, $T_k v(x)$ tends to $v(x)$ for all $x \in \Omega$ except on a set of μ-measure zero. Finally the same argument as above ensures that $|T_k v(x)| \le |v(x)|$ except on a set of μ-measure zero. Therefore, applying Lebesgue's Theorem, we conclude that

$$T_k v \to v \quad \text{strongly in } L^2(\Omega; d\mu) \text{ as } k \to \infty.$$

We have thus shown that $V \cap L^\infty(\Omega) \cap L^\infty(\Omega; d\mu)$ is dense in V.

Second step. We now prove that $V \cap C_c^0(\Omega)$ is dense in V, where $C_c^0(\Omega)$ is the space of continuous functions with compact support in Ω.

Consider $v \in V \cap L^\infty(\Omega) \cap L^\infty(\Omega; d\mu)$ and let v_k be a sequence such that

$$\begin{cases} v_k \in \mathcal{D}(\Omega) & \text{for each } k \text{ fixed} \\ v_k \to v & \text{strongly in } H_0^1(\Omega) \text{ as } k \to \infty. \end{cases}$$

Define

$$M = \|v\|_{L^\infty(\Omega)} = \|v\|_{L^\infty(\Omega; d\mu)}$$

and consider the sequence

$$w_k = T_{M+1} v_k.$$

The function w_k belongs to $C_c^0(\Omega) \cap V$.

Since the map T_{M+1} is continuous from $H_0^1(\Omega)$ into itself, we have, in view of the definition of M

$$\text{(A.2)} \qquad w_k = T_{M+1} v_k \to T_{M+1} v = v \quad \text{strongly in } H_0^1(\Omega) \text{ as } k \to \infty.$$

Extracting a suitable subsequence we deduce (use again J. Frehse [9] and M. Grun-Rehomme [10]) that $w_k(x)$ converges to $v(x)$ for all $x \in \Omega$ except on a set of μ-measure zero. Since $|w_k(x)| \le M + 1$ except on a set of μ-measure zero, Lebesgue's Theorem implies that

$$\text{(A.3)} \qquad w_k \to v \quad \text{strongly in } L^2(\Omega; d\mu) \text{ as } k \to \infty.$$

We have thus proved that, when $v \in V \cap L^\infty(\Omega) \cap L^\infty(\Omega; d\mu)$,

$$\begin{cases} w_k \in V \cap C_c^0(\Omega) \\ w_k \to v \text{ in } V \text{ as } k \to \infty \end{cases}$$

which implies, in view of the first step, that $V \cap C_c^0(\Omega)$ is dense in V.

Third step. In order to prove Theorem A.1, it is thus sufficient to approximate in V any function of $V \cap C_c^0(\Omega)$ by functions of $\mathcal{D}(\Omega)$.

Consider $v \in V \cap C_c^0(\Omega)$ and define

$$v_\varepsilon = \rho_\varepsilon * v$$

with $\rho_\varepsilon(x) = \frac{1}{\varepsilon^n}\rho\left(\frac{x}{\varepsilon}\right)$, where $\rho \in \mathcal{D}(\mathbb{R}^n)$ is a non-negative function with support contained in the unit ball of $\mathbb{R}^n$ which satisfies $\int_{\mathbb{R}^n} \rho(x)\mathrm{d}x = 1$.

For $\varepsilon > 0$ small enough, v_ε belongs to $\mathcal{D}(\Omega)$. It is well known that

$$v_\varepsilon \to v \quad \text{strongly in } H_0^1(\Omega) \cap C(\overline{\Omega}) \quad \text{as } \varepsilon \to 0.$$

Since μ is a finite Radon measure, the embedding $C(\overline{\Omega}) \subset L^2(\Omega; \mathrm{d}\mu)$ is continuous and therefore

$$v_\varepsilon \to v \quad \text{strongly in } L^2(\Omega; \mathrm{d}\mu) \quad \text{as } \varepsilon \to 0.$$

The proof of Theorem A.1 is now complete. □

Acknowledgments

The work of Patrizia Donato was partially supported by National Research Project (40%, 1989) of Ministero della Pubblica Istruzione (Italy). The work of Enrike Zuazua was partially supported by the Centre National de la Recherche Scientifique and by Project PB86-0112-C02 of the Dirección General de Investigación Cientifica y Técnica (MEC-España) (Spain). This work was done while Patrizia Donato and Enrike Zuazua enjoyed the hospitality of the Laboratoire d'Analyse Numérique de l'Université Pierre et Marie Curie.

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