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A Few Results on a Class of Degenerate Parabolic Equations

D. BLANCHARD - G.A. FRANCFORT

Introduction

This paper is devoted to the study of a nonlinear possibly degenerate parabolic equation. Specifically an equation of the type

$$
(1) \qquad \begin{cases} \dfrac{\partial b(u)}{\partial t} - \operatorname{div} \mathrm{D}\varphi(\operatorname{grad} u) = f & \text{in } \Omega \times (0,T), \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ b(u)|_{t=0} = b(u_0) & \text{in } \Omega, \end{cases}
$$

is considered on a bounded domain Ω of $\mathbb{R}^N$, with a monotone real valued energy b and a convex coercive potential φ.

Equation of the type (1) were firstly studied by Lions [13], Raviart [19] or Bamberger [2] under the assumption that b and φ exhibit power type nonlinearities of power α and p respectively, with

$$
(2) \qquad \frac{1}{\alpha} > \frac{1}{p} - \frac{1}{N}.
$$

Grange and Mignot [12] addressed a similar problem in an abstract setting with embedding restrictions that reduce to (2) in the case of power type nonlinearities.

In a previous paper [5] we were concerned with the case of a locally Lipschitz monotone function b with arbitrary growth at infinity, together with possible horizontal *plateaus*; the strict monotonicity of b was not assumed. Further the potential φ was taken to be a function of x and ξ (vector of $\mathbb{R}^N$), convex and C^1 in ξ and uniformly bounded below and above by a power $|\xi|^q$ of $|\xi|$ $(q > 1)$. Existence of a solution was then established under appropriate restrictions on u_0 whenever f is an element of $W^{1,1}(0,T;W^{-1,q'}(\Omega))$, $\dfrac{1}{q} + \dfrac{1}{q'} = 1$ (cf. Theorem 2 of [5]).

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Alt and Luckhaus [1] have obtained an existence result for certain systems that generalize (1). When applied to a single equation like (1), their result yields a solution for any f in $L^{q'}(0,T;W^{-1,q'}(\Omega))$ provided that b be continuous (some amount of discontinuity is even allowed in specific cases) and that $\mathrm{D}\varphi$ be strongly monotone. At the expense of a simple approximation process, a careful study of our previous paper would show that the existence theorem (Theorem 2 of [5]) still holds under the only assumption of continuity of b. Notice that the strong monotonicity of $\mathrm{D}\varphi$ is never assumed in [5]. The additional regularity in time of the forcing term f ($W^{1,1}$ versus $L^{q'}$) is however an essential obstacle that we propose to partially remove here.

The method developed by Alt and Luckhaus in [1] heavily relies on the monotone character of b. In particular the existence proof hinges on the following lemma (cf. Lemma 1.9 p. 322 of [1]).

LEMMA. *Let u_n be a weakly converging sequence of $L^q(0,T;W_0^{1,q}(\Omega))$ and $b(u_n(t))$ be uniformly equiintegrable in $L^1(\Omega)$ on $[0,T]$. If there further exists a constant C such that*

$$\frac{1}{h}\int_0^{T-h}\int_\Omega [b(u_n(t+h)) - b(u_n(t))]\ [u_n(t+h) - u_n(t)]\,\mathrm{d}x\,\mathrm{d}t \le C,$$

for any h in $[0,T]$, then $b(u_n)$ is a Cauchy sequence in $L^1((0,T)\times\Omega)$ (and it converges to $b(u)$).

A fully developed proof of this powerful lemma is fairly technical.

After a first section devoted to a brief recall of the theorem obtained in [5] and to a listing of the various hypotheses needed in the present study, our goal in the second section of the paper is to derive the existence result for a forcing term f in $L^{q'}(0,T;W^{-1,q'}(\Omega))$ and for a locally Lipschitz b through a different and faster truncation method which makes little use of the monotonicity properties of b (cf. Theorem 1). The drawback of the proposed derivation lies in the seemingly unavoidable locally Lipschitz character of b. The generalization to a continuous b is only performed when additional space regularity is met by the initial condition and by the forcing term (cf. Theorem 1).

Theorem 1 supposes no regularity on φ and is thus valid under the only assumptions of convexity and coercivity of $\varphi(x,\xi)$ with respect to ξ. Such a generalization is performed through a Yosida type regularization for lower semi-continuous convex functionals with q growth ($q > 1$). This is the object of Lemma 1.

The third section of the paper is concerned with various extensions of the existence result of Section 2 in which the internal energy b may be taken to be discontinuous. Furthermore the presence of infinite bareers on b at finite values of the field is investigated. In this setting f must in essence be local, except when b does not grow too quickly. Theorem 2 sums up the obtained results.

The fourth section is devoted to the case of a forcing term f in

$L^1((0,T)\times\Omega)$. The existence of a solution in such a framework is to our knowledge an open problem. We briefly present two more restrictive cases for which existence can be obtained. If $D\varphi(x,\xi)$ is linear in ξ, a comparison method communicated to us by Benilan [4] yields the existence result (cf. Theorem 3). If $D\varphi(x,\xi)$ is strongly monotone and $b(t)$ is an homeomorphism with minimal growth at infinity in $|t|^\gamma$, $\gamma > \sup\left\{0, \dfrac{N(2-q)}{q-1}\right\}$, an existence result can be derived by adapting a method devised by Boccardo and Gallouët [6] for the case of a linear internal energy functional (cf. Theorem 4).

A physically minded reader may challenge the very usefulness of considering an internal energy functional that could remain unchanged for different values of the field and thus appear as some kind of inverse phase change problem. Let us emphasize however that such a phenomenon is perfectly compatible with the rules of thermodynamics. Fluid models that exhibit a discontinuity of the pressure field as a function of the density at constant temperature have been derived by e.g. Milton and Fisher (cf. [15], [16]).

1. - Assumptions and basic theorem

Throughout the paper Ω denotes a bounded domain of $\mathbb{R}^N$ ($N\geq 1$) with Lipschitz boundary $\partial\Omega$, whereas q, q^*, T, α, β are five strictly positive real numbers satisfying

$$\frac{2N}{N+2} < q < +\infty, \qquad q^* = \begin{cases} \dfrac{Nq}{N-q} & \text{if } q < N, \\ +\infty & \text{if } q > N, \end{cases} \qquad \alpha \leq \beta. \tag{3}$$

Finally $a(x)$ is an element of $L^1(\Omega)$.

The space $W_0^{1,q}(\Omega)$ is the usual Sobolev space of functions of $L^q(\Omega)$ with weak derivatives in $L^q(\Omega)$ and null trace on $\partial\Omega$; the space $W^{-1,q'}(\Omega)$ is the dual space of $W_0^{1,q}(\Omega)$ and q' is the conjugate exponent of q, i.e., $\dfrac{1}{q}+\dfrac{1}{q'}=1$.

The internal energy b is a real valued function of the real variable with the following properties:

$$\begin{cases} b \text{ is monotone increasing,} \\ b(0)=0. \end{cases} \tag{4}$$

REMARK 1. Throughout the paper $\psi(t)$ (respectively $\psi_n(t)$, $\overline{\psi}_n(t)$) will denote the primitive of $b(t)$ (respectively $b_n(t)$, $\overline{b}_n(t)$) with value 0 at $t=0$

and $\psi^*(t)$ (respectively $\psi_n^*(t)$, $\overline{\psi}_n^*(t)$) the convex conjugate of $\psi(t)$ (respectively $\psi_n(t)$, $\overline{\psi}_n(t)$), i.e., for any t in $\mathbb{R}$,

$$\begin{cases} \psi^*(t) = \sup\limits_{s\in\mathbb{R}}\{ts - \psi(s)\}, \\ \psi(t) + \psi^*(b(t)) = tb(t). \end{cases}$$

REMARK 2. In the context of Remark 1 the following observation was made by Alt and Luckhaus (cf. [1], Remark 1.2, p. 314):

$$|b(t)| \leq \delta\psi^*(b(t)) + \sup_{|\sigma|\leq\frac{1}{\delta}} |b(\sigma)|,$$

for any t in $\mathbb{R}$ and any δ in $\mathbb{R}_+ - \{0\}$.

The potential $\varphi(x,\xi)$ is a scalar valued function defined on $\Omega\times\mathbb{R}^N$ with the following properties:

$$(5)\qquad \begin{cases} \varphi(x,\xi) \text{ is a convex normal integrand on } \Omega\times\mathbb{R}^N, \\ \alpha|\xi|^q \leq \varphi(x,\xi) \leq a(x) + \beta|\xi|^q, \text{ a.e. on } \Omega \text{ and for every } \xi \text{ of } \mathbb{R}^N, \\ \varphi(0) = 0. \end{cases}$$

If φ satisfies (5) and is also C^1 in ξ, for almost every x of Ω, it will be referred to as a C^1 *admissible potential* and a sequence φ_n satisfying (5) with constant α and β and a function $a(x)$ independent of n will be referred to as a sequence of *uniformly admissible potentials.*

The following theorem can be found in Blanchard and Francfort [5] (cf. Theorems 1 and 2, p. 1034-1035 and the conclusion of the accompanying Erratum, p. 761):

THEOREM 0. *Assume that assumptions* (3), (4), (5) *hold true, that b is locally Lipschitz, that φ is C^1-admissible and that*

$$(6)\qquad u_0 \in W_0^{1,q}(\Omega),\ b(u_0) \in L^1_{\mathrm{loc}}(\Omega)\cap W^{-1,q'}(\Omega),$$

$$(7)\qquad f \in W^{1,1}(0,T;W^{-1,q'}(\Omega)).$$

Then the problem

$$\begin{cases} \dfrac{\partial b(u)}{\partial t} - \operatorname{div} \mathrm{D}\varphi(x, \operatorname{grad} u) = f & \text{in } \Omega\times(0,T), \\ u = 0 & \text{on } \partial\Omega\times(0,T), \\ b(u)|_{t=0} = b(u_0) & \text{in } \Omega, \end{cases}$$

admits a solution

$$\begin{cases} u \in L^\infty(0,T;W_0^{1,q}(\Omega)), \\ b(u) \in C^0([0,T];L^1(\Omega)) \cap W^{1,\infty}(0,T;W^{-1,q'}(\Omega)). \end{cases}$$

Furthermore if u_{01} *and* u_{02} *satisfy* (6) *while* f_1 *and* f_2 *satisfy* (7) *and if* $b(u_{01}) - b(u_{02})$ *and* $f_1 - f_2$ *are positive (in a distributional sense), then there exists a solution* u_1 *(respectively* u_2*) associated to* u_{01}, f_1 *(respectively* u_{02}, f_2*) such that* $b(u_1) - b(u_2)$ *is almost everywhere positive on* $\Omega \times (0,T)$. *Finally, if* $b(u_0)$ *and* f *also belong to* $L^2(\Omega)$ *and* $W^{1,1}(0,T;L^2(\Omega))$ *respectively, then* $b(u)$ *belongs to* $L^\infty(0,T;L^2(\Omega))$.

REMARK 3. An extension of this theorem to the case of continuous b is readily obtained through a simple approximation process. Specifically the primitive $\psi(t)$ of the function $b(t)$ is replaced by its Yosida approximation $\psi_n(t)$ defined as

$$\psi_n(t) = \inf_s \left\{ \frac{n}{2}(t-s)^2 + \psi(s) \right\},$$

and Theorem 0 is applied to the derivative $b_n(t)$ of the $C^{1,1}$ convex function $\psi_n(t)$. As n tends to infinity one passes to the limit through a proof identical to that of Theorem 2 in [5] (cf. [5], p. 1051-1056) since, for almost every x of Ω,

$$\begin{aligned} 0 \le b_n(u_0(x))u_0(x) &= |b_n(u_0(x))|\ |u_0(x)| \le |b(u_0(x))|\ |u_0(x)| \\ &= b(u_0(x))u_0(x) \in L^1(\Omega), \end{aligned}$$

and thus the same a priori estimates hold true.

A few basic properties of the Yosida approximation $\psi_n(t)$ of a convex lower semi-continuous proper $\mathbb{R} \cup \{+\infty\}$ - valued function $\psi(t)$ of the real variable will be used in the sequel, namely

* ψ_n is an increasing sequence of $C^{1,1}$ convex functions, converging to ψ pointwise,
* for every t in $\mathbb{R}$, $|\psi_n'(t)|$ is bounded above by any element $z(t)$ of the subdifferential $\partial\psi(t)$ of ψ at t (cf. e.g. Barbu [3], Corollary 2.2, p. 58),
* $b_n(t) = \psi_n'(t)$ converges to $b_0(t) = \inf\{z | z \in \partial\psi(t)\}$ for any t in the domain of ψ,
* if b_0 is locally Lipschitz, then $|b_n'(t)|$ is uniformly bounded above on compact subsets of $\mathbb{R}$.

REMARK 4. If, in the context of Theorem 0, there exists a strictly positive real constant γ such that for every t in $\mathbb{R}$

$$b'(t) \ge \gamma,$$

then the solution u exhibited in Theorem 0 has the following additional regularity property:

$$\frac{\partial u}{\partial t} \in L^2(0,T;L^2(\Omega)),$$

as can be verified by simple inspection of the proofs of Theorems 1 and 2 of [5].

Before concluding this preliminary section we wish to draw the reader's attention to the restriction $q > \dfrac{2N}{N+2}$ imposed in (3). This restriction which implies the following compact embeddings

$$W_0^{1,q}(\Omega) \subset L^2(\Omega) \subset W^{-1,q'}(\Omega),$$

was used in our previous paper so as to prove existence of a solution for a forcing term in $W^{1,1}(0,T;L^2(\Omega))$. The existence Theorem 0 was then deduced from the former weaker existence theorem. Theorem 0 can in fact be derived without the help of the previous theorem in which case the embedding restriction may be removed and all the results of the present study hold true for any q lying in $(1,\infty)$. Confronted with such a dilemna we decided to keep the restriction $q > \dfrac{2N}{N+2}$ since Theorem 0 was only proved *stricto sensu* for such q's but we made every effort *not to* use the embedding property through the present study. Thus the reader who is willing to accept Theorem 0 without the above mentioned restriction on q will be correct in extending all subsequent results to the range $1 < q < +\infty$.

2. - Existence result with forcing term without time regularity

This section is devoted to the proof of the existence result anounced in the introduction for the case of a convex lower semi-continuous potential φ and of a forcing term in $L^{q'}(0,T;W^{-1,q'}(\Omega))$.

We propose to establish the

THEOREM 1. *Under the assumptions* (3), (4), (5), *if*

$$u_0 \in W_0^{1,q}(\Omega),\ \ b(u_0) \in L^1_{\mathrm{loc}}(\Omega) \cap W^{-1,q'}(\Omega), \tag{8}$$

and if

(a) $f \in L^{q'}(0,T;W^{-1,q'}(\Omega))$, *and b is locally Lipschitz,*

or

(b) $f \in L^{q'}(0,T;W^{-1,q'}(\Omega)) \cap L^1(0,T;L^p(\Omega)),\ \ b(u_0) \in L^p(\Omega)$ *with* $\dfrac{1}{p} + \dfrac{1}{q^*} < 1$, *and b is continuous,* *the problem*

$$
\begin{cases}
\dfrac{\partial b(u)}{\partial t} - \operatorname{div} Y = f & \text{in } \Omega \times (0,T), \\
Y \in \partial\varphi(x, \operatorname{grad} u) & \text{in } \Omega \times (0,T), \\
u = 0 & \text{on } \partial\Omega \times (0,T), \\
b(u)|_{t=0} = b(u_0) & \text{in } \Omega,
\end{cases}
$$

admits a solution

$$
\begin{cases}
u \in L^q(0,T; W_0^{1,q}(\Omega)), \ Y \in L^{q'}(0,T; [L^{q'}(\Omega)]^N), \\
b(u) \in L^\infty(0,T; L^1(\Omega)) \cap W^{1,q'}(0,T; W^{-1,q'}(\Omega)).
\end{cases}
$$

In case (b), $b(u)$ *also belongs to* $L^\infty(0,T;L^p(\Omega))$. *Furthermore if in either case* $b(u_0)$ *and* f *also belong to* $L^r(\Omega)$ *and* $L^1(0,T;L^r(\Omega))$ *respectively* $(1 < r < +\infty)$ *then* $b(u)$ *belongs to* $L^\infty(0,T;L^r(\Omega))$. *Finally if* u_{01} *and* u_{02} *satisfy* (8) *while* f_1 *and* f_2 *satisfy* (a) *or* (b) *and if* $b(u_{01}) - b(u_{02})$ *and* $f_1 - f_2$ *are positive (in a distributional sense) then there exists a solution* u_1 *(respectively* u_2*) associated to* u_{01}, f_1, *(respectively* u_{02}, f_2*) such that* $u_1 - u_2$ *is almost everywhere positive on* $\Omega \times (0,T)$.

REMARK 5. Note that the comparison part of Theorem 1 is stronger than that of the previously established theorem since it concerns $u_1 - u_2$ rather than $b(u_1) - b(u_2)$; thus Theorem 0 was not optimal in that respect.

Similarly, Theorem 0 could have been proved with the weaker hypotheses (5) on φ by reproducing in that context the sixth step of the proof of Theorem 1.

The difference between Theorem 1 and Theorem 0 is two-fold: the forcing term f has no time regularity and the potential φ has no regularity with respect to the field variable. These two difficulties are quite distinct and they will be dealt with separately.

PROOF OF THEOREM 1. The proof of Theorem 1 is divided into seven steps. The first step consists in devising a smooth approximation φ_n of φ (and approximations b_n of b and f_n of f) so as to be in a position to apply Theorem 0 and obtain a solution u_n. To this effect a regularization lemma (Lemma 1) is proved for φ in the spirit of the classical Yosida approximation (cf. e.g. Brézis [8], pp. 38- 39). A priori estimates are obtained on u_n and $b_n(u_n)$ in the second step. The third step is devoted to the derivation of an $L^\infty(0,T;L^r(\Omega))$ bound if $b(u_0)$ and f belong to $L^r(\Omega)$ and $L^1(0,T;L^r(\Omega))$ respectively. The fourth step is devoted to the identification of the limit of $b_n(u_n)$ in case (a). It is performed with the help of a type of argument communicated to us by Murat [18] which leads to the pointwise convergence of $b_n(u_n)$. This is the object of Lemma 2. The fifth step is devoted to the identification of the limit of $b_n(u_n)$ in case (b). The sixth step identifies the limit of the fluxes $\mathrm{D}\varphi_n(x, \operatorname{grad} u_n)$ in both cases. The final step proves the comparison result.

STEP 1 - *Regularization of* φ *and* b. The following lemma holds true:

LEMMA 1. *Let φ satisfy* (5), *then the sequences φ_n defined for all $\xi \in \mathbb{R}^N$ and almost every x of Ω as*

$$\varphi_n(x,\xi) = \inf_{\eta \in \mathbb{R}^N} \left\{ \frac{n}{q} |\xi - \eta|^q + \varphi(x,\eta) \right\}$$

is a sequence of C^1-uniformly admissible potential (in the sense of (5)) *such that $\varphi_n(x,\xi)$ converges monotonically to $\varphi(x,\xi)$ for every ξ in $\mathbb{R}^N$ and almost every x in Ω.*

PROOF OF LEMMA 1. Let us consider a fixed element x of Ω for which $\varphi(x,\xi)$ is continuous in ξ. Checking that for every ξ in $\mathbb{R}^N$, $\varphi_n(x,\xi)$ is an increasing sequence that converges to $\varphi(x,\xi)$ and that there exists a strictly positive real number α' (independent of x and ξ) such that $\varphi(x,\xi) \geq \alpha' |\xi|^q$ is an easily performed task.

Since $\varphi(x,\xi)$ is convex and finite (thus continuous) in ξ it is the convex conjugate of its convex conjugate function $\varphi^*(x,\eta)$ and $\varphi_n(x,\xi)$ reads as

$$\varphi_n(x,\xi) = \inf_{\eta \in \mathbb{R}^N} \left\{ \frac{n}{q} |\xi - \eta|^q + (\varphi^*)^*(x,\eta) \right\}.$$

As such $\varphi_n(x,\xi)$ identifies with the convex conjugate $\psi_n^*(x,\xi)$ of

$$\psi_n(x,\eta) = \frac{n}{q'} |\eta|^{q'} + \varphi^*(x,\eta),$$

where q' is the conjugate exponent of q (cf. e.g. Moreau [17]). The strict convexity of $\psi_n(x,\eta)$ in η implies that the subdifferential of $\psi_n^*(x,\xi)$ is reduced to a single element, $\varphi_n'(x,\xi)$ which in view of the continuous character of $\varphi_n(x,\xi) = \psi_n^*(x,\xi)$ in ξ is also the Gateaux derivative of $\varphi_n(x,\xi)$ (cf. Moreau [17], pp. 65-66). The Frechet differentiability of $\varphi_n(x,\xi)$ will be achieved if $\varphi_n'(x,\xi)$ is proved to be continuous in ξ. To this effect an arbitrary converging sequence ξ_p of $\mathbb{R}^N$ is considered. Its limit is denoted by ξ_0. Since $\varphi_n(x,\xi)$ has q growth at infinity in ξ and is convex, its subdifferential has $q-1$ growth at infinity, thus $\varphi_n'(x,\xi_p)$ is bounded uniformly in p. A subsequence p' of p is such that $\varphi_n'(x,\xi_{p'})$ converges to z_n as p' tends to infinity. The convexity of $\varphi_n(x,\xi)$ implies that, for all η's in $\mathbb{R}^N$,

$$\varphi_n(x,\xi_{p'} + \eta) - \varphi_n(x,\xi_{p'}) \geq \eta \cdot \varphi_n'(x,\xi_{p'}),$$

and the continuous character of φ_n finally yields

$$\varphi_n(x,\xi_0 + \eta) - \varphi_n(x,\xi_0) \geq \eta \cdot z_n.$$

Thus z_n belongs to the subdifferential of $\varphi_n(x,\xi)$ at the point ξ_0, i.e., $z_n = \varphi_n'(x,\xi_0)$, which proves the continuity of $\varphi_n'(x,\xi)$ in ξ.

We have proved thus far that $\varphi_n(x,\xi)$ is for almost every x of Ω a C^1-convex function in ξ, uniformly bounded below by $\alpha'|\xi|^q$ and bounded above by $a(x)+\beta|\xi|^q$ (cf. (5)). The measurable character of $\varphi_n(x,\xi)$ in x for every ξ of $\mathbb{R}^N$ is a direct consequence of the continuity of $\varphi(x,\xi)$ with respect to ξ which permits to view $\varphi_n(x,\xi)$ as a countable infimum, namely,

$$\varphi_n(x,\xi)=\inf_{\eta\in\mathbb{Q}^N}\left\{\frac{n}{q}|\xi-\eta|^q+\varphi(x,\eta)\right\}.$$

The proof of Lemma 1 is complete.

Set

$$T_n(t)=\begin{cases} t & \text{if } |t|\le n,\\ n\,\mathrm{sg}(t) & \text{if } |t|\ge n.\end{cases} \tag{9}$$

Recalling Remark 3, we denote by $b_n(t)$ the derivative of the Yosida approximation of $\psi_n(t)$, consider

$$\bar{b}_n(t)=b_n(t)+\frac{t}{n^2},$$

and apply Theorem 0 to

$$\begin{cases} \dfrac{\partial \bar{b}_n(u_n)}{\partial t}-\operatorname{div} \mathrm{D}\varphi_n(x,\,\operatorname{grad} u_n)=f_n & \text{in } \Omega\times(0,T),\\ u_n=0 & \text{on } \partial\Omega\times(0,T),\\ \bar{b}_n(u_n)|_{t=0}=\bar{b}_n(T_n(u_0)) & \text{in } \Omega,\end{cases} \tag{10}$$

where f_n is a smooth approximation of f in $L^{q'}(0,T;W^{-1,q'}(\Omega))$ in case (a) and in $L^1(0,T;L^p(\Omega))$ in case (b).

A solution u_n in $L^\infty(0,T;W_0^{1,q}(\Omega))$ with $\bar{b}_n(u_n)$ in

$$C^0([0,T];L^1(\Omega))\cap W^{1,\infty}(0,T;W^{-1,q'}(\Omega))$$

is obtained.

REMARK 6. By virtue of Remark 4 and because of the Lipschitz character of $\bar{b}_n$, both u_n and $\bar{b}_n(u_n)$ belong to $W^{1,1}(0,T;L^1(\Omega))$ and

$$u_n|_{t=0}=T_n(u_0).$$

STEP 2 - *A priori estimates.* Upon multiplication of the first equation of (10) by u_n, integration over $\Omega\times(0,t)$ of the resulting expression, appropriate integration by parts and application of Alt-Luckhaus lemma (cf. Appendix) we obtain for almost every t of $(0,T)$

$$
(11)\qquad
\begin{aligned}
&\int_\Omega \overline{\psi}^*_n(\bar{b}_n(u_n(t)))\mathrm{d}x + \int_0^t\int_\Omega \mathrm{D}\varphi_n(x,\,\mathrm{grad}\,u_n(s))\cdot \mathrm{grad}\,u_n(s)\mathrm{d}x\,\mathrm{d}s \\
&= \int_0^t \langle f_n(s), u_n(s)\rangle_{W^{-1,q'}(\Omega);W_0^{1,q}(\Omega)}\,\mathrm{d}s + \int_\Omega \overline{\psi}^*_n(\bar{b}_n(T_n(u_0)))\mathrm{d}x,
\end{aligned}
$$

in the notation of Remark 1. But, according to Remarks 1 and 3

$$
(12)\qquad 0 \le \overline{\psi}^*_n(\bar{b}_n(T_n(u_0(x)))) \le \bar{b}_n(T_n(u_0(x)))T_n(u_0(x)) \le b(u_0(x))u_0(x) + 1,
$$

for almost any x of Ω. In view of (6) Brézis-Browder theorem ([9], Theorem 1) implies that $b(u_0)u_0$ belongs to $L^1(\Omega)$ thus $\overline{\psi}^*_n(\bar{b}_n(T_n(u_0)))$ belongs to $L^1(\Omega)$ and is bounded in $L^1(\Omega)$ independently of n. The coercive character of φ_n (cf. Lemma 1 and (5)) then yields the following uniform estimates in the parameter n:

$$
(13)\qquad \begin{cases} \overline{\psi}^*_n(\bar{b}_n(u_n)) & \text{is bounded in } L^\infty(0,T;L^1(\Omega)), \\ u_n & \text{is bounded in } L^q(0,T;W_0^{1,q}(\Omega)). \end{cases}
$$

By virtue of Remark 2, the bound (13) on $\overline{\psi}^*_n(\bar{b}_n(u_n))$ implies that

$$
(14)\qquad \begin{cases} \bar{b}_n(u_n) \text{ is bounded in } L^\infty(0,T;L^1(\Omega)) \text{ independently of } n, \\ \bar{b}_n(u_n) \text{ is uniformly equintegrable in } L^1(\Omega) \text{ on } [0,T]. \end{cases}
$$

Finally because φ_n is a sequence of C^1-uniformly admissible potentials

$$
(15)\qquad \mathrm{D}\varphi_n(x,\,\mathrm{grad}\,u_n) \text{ is bounded in } L^{q'}(0,T;[L^{q'}(\Omega)]^N) \text{ independently of } n,
$$

and by virtue of the equation

$$
(16)\qquad \frac{\partial \bar{b}_n(u_n)}{\partial t} \text{ is bounded in } L^{q'}(0,T;W^{-1,q'}(\Omega)) \text{ independently of } n.
$$

Appropriate extractions of weakly converging subsequences (still indexed by n) in (13)–(16) lead to the following statements of weak convergence when n tends to infinity:

$$
(17)\qquad \begin{cases} u_n \rightharpoonup u & \text{weakly in } L^q(0,T;W_0^{1,q}(\Omega)), \\ \bar{b}_n(u_n) \rightharpoonup \chi & \text{weakly in} \\ & L^\infty(0,T;L^1(\Omega)) \cap W^{1,q'}(0,T;W^{-1,q'}(\Omega)), \\ \mathrm{D}\varphi_n(x,\,\mathrm{grad}\,u_n) \rightharpoonup Y & \text{weakly in } L^{q'}(0,T;[L^{q'}(\Omega)]^N). \end{cases}
$$

Finally passing to the weak limit in the first and third equations of (10) yields

$$\begin{cases} \dfrac{\partial \chi}{\partial t} - \operatorname{div} Y = f & \text{in } \Omega \times (0,T), \\ \chi|_{t=0} = b(u_0) & \text{in } \Omega. \end{cases} \tag{18}$$

STEP 3 - $L^\infty(0,T;L^r(\Omega))$ *bound.* If f belongs to $L^1(0,T;L^r(\Omega))$ the sequence f_n can be chosen so as to converge in $L^1(0,T;L^r(\Omega))$ in both cases. Introduce

$$F(t) = \begin{cases} t & \text{if } |t| \le 1, \\ |t|^{r-2}t & \text{if } |t| \ge 1. \end{cases}$$

Multiplication of the first equation of (10) by $F(\bar{b}_n(u_n))$, integration of the resulting expression over $\Omega \times (0,t)$, $t \le T$, and appropriate integration by parts would yield, for every t in $[0,T]$,

$$\int_\Omega G(\bar{b}_n(u_n(t)))\mathrm{d}x$$

$$+ \int_0^t \int_\Omega \bar{b}_n'(u_n(s))F'(\bar{b}_n(u_n(s)))\mathrm{D}\varphi_n(x, \operatorname{grad} u_n(s)) \cdot \operatorname{grad} u_n(s)\mathrm{d}x\,\mathrm{d}s$$

$$= \int_0^t \int_\Omega f_n(s)F(\bar{b}_n(u_n(s)))\mathrm{d}x\,\mathrm{d}s + \int_\Omega G(\bar{b}_n(T_n(u_0)))\mathrm{d}x,$$

where $G(t)$ is the primitive of $F(t)$ with value 0 at 0. Implicit use has been made in the derivation of the above equality of Remark 6 together with the chain rule for the composition of a Lipschitz function with a $W_0^{1,q}(\Omega)$ function (cf. e.g. Boccardo and Murat [7], Theorem 4.3 or Marcus and Mizel [14], Corollary 1.3, p. 353).

Note that

$$\begin{cases} |F(t)| \le |t|^{r-1} + 1, \\ \dfrac{|t|^r}{r} + \dfrac{1}{2} \ge G(t) \ge \dfrac{|t|^r}{r} - 1. \end{cases}$$

Since there exists a constant C, depending only on r, such that, almost everywhere on Ω,

$$G(\bar{b}_n(T_n(u_0(x)))) \le \frac{1}{r}\left|\bar{b}_n(T_n(u_0(x)))\right|^r + \frac{1}{2}$$

$$\le C\left(|b_n(T_n(u_0(x)))|^r + \frac{1}{n^r}\right) + \frac{1}{2} \le C(|b(u_0(x))|^r + 1),$$

and since by hypothesis $b(u_0) \in L^r(\Omega)$ the monotone character of F and $\bar{b}_n$, Hölder's inequality applied to the first term in the right hand side of the equality and the elementary estimates on $F(t)$ and $G(t)$ yield, for every t in $[0,T]$,

$$\frac{1}{r}\|\bar{b}_n(u_n(t))\|^r_{L^r(\Omega)} \le C + \|f_n\|_{L^1(0,T;L^r(\Omega))}\left(\sup_{t\in[0,T]}\|\bar{b}_n(u_n(t))\|^r_{L^r(\Omega)}\right)^{\frac{r-1}{r}}, \tag{19}$$

where C is a constant independent of n. Thus,

(20) $\bar{b}_n(u_n)$ is bounded in $L^\infty(0,T;L^r(\Omega))$ independently of n.

The above presented argument is not entirely rigorous because $\bar{b}_n(u_n)$ is not *a priori* known to belong to $L^\infty(0,T;L^r(\Omega))$. A complete proof would involve a truncation of F at an arbitrary height R and an appropriate rewriting of (19). Estimate (20) would be obtained upon letting R tend to infinity in inequality (19).

STEP 4 - *Case* (a). This step is devoted to the identification of χ in case (a). To this effect the pointwise convergence of $b(u_n)$ to χ is proved by a method suggested to us by F. Murat [18]. Specifically the following lemma is proved:

LEMMA 2. *Let h_n be a sequence of Lipschitz monotone real valued functions with $h_n(0)=0$ such that h_n' is uniformly bounded on compact subsets of $\mathbb{R}$. Let s be a real number lying in $(1,\infty)$ and assume that v_n is a sequence of elements of $L^s(0,T;W_0^{1,s}(\Omega))\cap W^{1,1}(0,T;L^1(\Omega))$ such that, as n tends to infinity,*

(21) *v_n is bounded in $L^s(0,T;W_0^{1,s}(\Omega))$,*

(22) *$\dfrac{\partial h_n(v_n)}{\partial t}$ is bounded in $L^{s'}(0,T;W^{-1,s'}(\Omega))$.*

Then a subsequence of $h_n(v_n)$ (still denoted by $h_n(v_n)$) converges almost pointwise on $\Omega\times(0,T)$ to a measurable $\overline{\mathbb{R}}$-valued function ($\overline{\mathbb{R}}=\mathbb{R}\cup\{-\infty,+\infty\}$).

This lemma is applied to $s=q$, $v_n=u_n$, $h_n=\bar{b}_n$ and it yields the almost pointwise convergence of $\bar{b}_n(u_n)$ to χ. The weak convergence of $\bar{b}_n(u_n)$ to χ in $L^1((0,T)\times\Omega)$ (cf. (17)) and a straightforward application of Egoroff's theorem imply the strong convergence of $\bar{b}_n(u_n)$ to χ in $L^1((0,T)\times\Omega)$. The identification between χ and $b(u)$ is an immediate consequence of the monotone character of $\bar{b}_n$ and b and of the continuous character of b (cf. e.g. Blanchard and Francfort [5], p. 1054, or Section 4, proof of Theorem 3 in this paper).

We now return to the

PROOF OF LEMMA 2. Let us define the real valued function $H_k(t)$ as a monotone C^∞ function such that

$$H_k(t)=\begin{cases} t & \text{if } |t|\le k/2,\\ k\,\mathrm{sg}(t) & \text{if } |t|\ge k.\end{cases} \tag{23}$$

Consider the Lipschitz monotone function $S_k^n(t)$ defined as

$$S_k^n(t) = \int_0^t H_k'(s) h_n'(s) \mathrm{d}s.$$

The subsequence $h_n(v_n)$, $S_k^n(v_n)$ lie in $L^s(0,T;W_0^{1,s}(\Omega)) \cap W^{1,1}(0,T;L^1(\Omega))$ because of the Lipschitz character of $h_n(t)$ and $S_k^n(t)$. The derivation formulae for the composition of a $W_0^{1,s}(\Omega)$ function by a Lipschitz function are applicable. Thus, almost everywhere in $\Omega \times (0,T)$,

$$\operatorname{grad} S_k^n(v_n) = H_k'(v_n) h_n'(v_n) \operatorname{grad} v_n, \tag{24}$$

$$\frac{\partial S_k^n(v_n)}{\partial t} = H_k'(v_n) h_n'(v_n) \frac{\partial v_n}{\partial t} = H_k'(v_n) \frac{\partial h_n(v_n)}{\partial t}. \tag{25}$$

Since H_k' has compact support and h_n' is uniformly bounded on the support of H_k', (21), (24) imply that

(26) $S_k^n(v_n)$ is bounded in $L^s(0,T;W_0^{1,s}(\Omega))$ independently of n.

The bound (22) yields the existence of a bounded sequence g_n in $L^{s'}(0,T;[L^{s'}(\Omega)]^N)$ such that

$$\frac{\partial h_n(v_n)}{\partial t} = \operatorname{div} g_n,$$

and (25) reads, almost everywhere in $\Omega \times (0,T)$, as

$$\frac{\partial S_k^n(v_n)}{\partial t} = H_k'(v_n) \operatorname{div} g_n = \operatorname{div}(H_k'(v_n) g_n) - H_k''(v_n)\ (\operatorname{grad} v_n \cdot g_n). \tag{27}$$

In (27) we have made implicit use once again of the derivation formula for the composition of v_n with the Lipschitz function $H_k'(t)$. Thus (27) implies that

(28) $\dfrac{\partial S_k^n(v_n)}{\partial t}$ is bounded in $L^{s'}(0,T;W^{-1,s'}(\Omega)) + L^1((0,T) \times \Omega)$.

Because of the bound (26), (28), an Aubin type lemma (cf. e.g. Simon [20], Corollary 4) implies that

$$S_k^n(v_n) \text{ lies in a compact set of } L^s(0,T;L^1(\Omega)),$$

which in turn yields the almost pointwise converge of a subsequence of $S_k^n(v_n)$ (still indexed by n). Note that the monotone character of h_n has not been used up to this point.

Recalling the definition (10) of the truncation function and invoking the monotone character of h_n leads to the following identity:

$$T_{h_n(k/2)}(S_k^n(t)) = T_{h_n(k/2)}(h_n(t)),$$

for any n, k or t. Hence for any strictly positive real number k

$$\text{(29)} \qquad T_{h_n(k)}(h_n(v_n)) \text{ converges almost everywhere on } \Omega \times (0,T).$$

The uniform bound of h_n' on compact sets implies the pointwise convergence of h_n to a locally Lipschitz function h which together with (29) immediately implies the almost pointwise convergence of $T_{h(k)}(h_n(v_n))$ on $\Omega \times (0,T)$. A simple argument which uses the convergence of h_n to h would yield that, for any strictly positive real number k, $T_k(h_n(v_n))$ converges almost everywhere on $\Omega \times (0,T)$. But if a sequence of measurable functions is such that all of its truncates converge almost pointwise, the sequence itself converges almost pointwise to a measurable $\overline{\mathbb{R}}$-valued function, which concludes the proof of Lemma 2.

REMARK 7. The uniform bound on the derivative of the Lipschitz functions h_n on compact subsets of $\mathbb{R}$ is necessary in order to obtain (26). This feature precludes the consideration of a continuous b instead of a locally Lipschitz one as already mentioned in the introduction. Let us emphasize, as in the introduction, that Alt-Luckhaus lemma (Lemma 1.9 p. 322 of [1]) would permit to remove the restriction that b be locally Lipschitz in part (a) of Theorem 1.

STEP 5 - *Case* (b). The identification of χ in case (b) is much simpler than in case (a). Because $b(u_0)$ belong to $L^p(\Omega)$ and f to $L^1(0,T;L^p(\Omega))$, (20) applied with $r = p$ permits to conclude that

$$\text{(30)} \qquad \overline{b}_n(u_n) \text{ is bounded in } L^\infty(0,T;L^p(\Omega)),$$

independently of n. Since the restriction on the range of possible p's implies that $L^p(\Omega)$ is compactly embedded in $W^{-1,q'}(\Omega)$ and since (cf. (16)) $\dfrac{\partial \overline{b}_n(u_n)}{\partial t}$ is bounded in $L^{q'}(0,T;W^{-1,q'}(\Omega))$ an Aubin type lemma may again be invoked. Thus

$$\overline{b}_n(u_n) \to \chi \text{ strongly in } C^0([0,T];W^{-1,q'}(\Omega)),$$

as n tends to infinity. Because of the first convergence in (17), we obtain the following statement of convergence, as n tends to infinity,

$$\text{(31)} \qquad \int_0^T \int_\Omega \overline{b}_n(u_n) u_n \, dx \, dt \to \int_0^T \int_\Omega \chi u \, dx \, dt.$$

The extremality relation reads as

$$\bar{b}_n(u_n)u_n = \overline{\psi}_n(u_n) + \overline{\psi}_n^*(\bar{b}_n(u_n)), \tag{32}$$

almost everywhere on $\Omega \times (0,T)$. Integrating (32) over $(0,T) \times \Omega$ yields

$$\int_0^T\int_\Omega \bar{b}_n(u_n)u_n \,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_\Omega \overline{\psi}_n(u_n)\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega \overline{\psi}_n^*(\bar{b}_n(u_n))\mathrm{d}x\,\mathrm{d}t.$$

The definitions of $\bar{b}_n$, b_n, together with a straightforward application of Remark 1 imply that

$$\begin{aligned}
&\overline{\psi}_n(t) \geq \psi_n(t),\\
&\overline{\psi}_n^*(\bar{b}_n(t)) = \psi_n^*(b_n(t)) + t^2/2n^2 \geq \psi_n^*(b_n(t)),
\end{aligned}$$

for every t in $\mathbb{R}$. Then

$$\int_0^T\int_\Omega \bar{b}_n(u_n)u_n \,\mathrm{d}x\,\mathrm{d}t \geq \int_0^T\int_\Omega \psi_n(u_n)\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega \psi_n^*(b_n(u_n))\mathrm{d}x\,\mathrm{d}t.$$

Since ψ_n is an increasing sequence of convex functions, ψ_n^* is a decreasing sequence of convex functions converging pointwise to ψ^*. Thus, for a fixed n_0 and $n \geq n_0$,

$$\int_0^T\int_\Omega \bar{b}_n(u_n)u_n \,\mathrm{d}x\,\mathrm{d}t \geq \int_0^T\int_\Omega \psi_{n_0}(u_n)\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega \psi^*(b_n(u_n))\mathrm{d}x\,\mathrm{d}t.$$

The weak lower semi-continuous character of ψ_{n_0} and (17) imply that

$$\underline{\lim}\int_0^T\int_\Omega \psi_{n_0}(u_n)\mathrm{d}x\,\mathrm{d}t \geq \int_0^T\int_\Omega \psi_{n_0}(u)\mathrm{d}x\,\mathrm{d}t, \tag{33}$$

as n tends to infinity.

Since u_n is in particular bounded in $L^q(\Omega \times (0,T))$, $\frac{u_n}{n^2}$ converges strongly to 0 in $L^1(\Omega \times (0,T))$ and the weak lower semi-continuous character of ψ^* implies that

$$\underline{\lim}\int_0^T\int_\Omega \psi^*(b_n(u_n))\mathrm{d}x\,\mathrm{d}t \geq \int_0^T\int_\Omega \psi^*(\chi)\mathrm{d}x\,\mathrm{d}t, \tag{34}$$

as n tends to infinity. Recalling (31), (33), (34), we obtain

$$\int_0^T\int_\Omega \chi u \,\mathrm{d}x\,\mathrm{d}t \geq \int_0^T\int_\Omega \psi_{n_0}(u)\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega \psi^*(\chi)\mathrm{d}x\,\mathrm{d}t.$$

Letting n_0 tend to infinity in the above inequality permits to conclude that

$$\int_0^T\int_\Omega \chi u \,\mathrm{d}x\,\mathrm{d}t \geq \int_0^T\int_\Omega (\psi(u) + \psi^*(\chi))\mathrm{d}x\,\mathrm{d}t. \tag{35}$$

Since, for almost any (x,t) in $\Omega \times (0,T)$,

$$\chi(x,t)u(x,t) \leq \psi(u(x,t)) + \psi^*(\chi(x,t)),$$

inequality (35) implies almost pointwise equality between the left and right hand sides of the above inequality from which it is immediately deduced that

$$\chi = b(u) \text{ almost everywhere in } \Omega \times (0,T). \tag{36}$$

REMARK 8. The continuous character of b has not been used in the identification (36) of χ which thus holds true under the only hypotheses that (30) hold and that ψ be a convex lower semi-continuous proper $\mathbb{R} \cup \{+\infty\}$-valued function with $\psi(0) = 0$. Note however that the continuous character of b has been used in passing to the limit in the initial condition $\overline{b}_n(T_n(u_0))$. We will see in Section 3 for which conditions it is possible to do away with the continuity of b. The identification of the limit of the sequence of approximation $b_n(u_n)$ will be performed exactly as in step 5 of the proof of Theorem 1, but for the absence of the term in t/n^2 in the approximation $\overline{b}_n$ of b.

STEP 6 - *Identification of Y*. This step is devoted to the identification of Y. To this effect equation (11) is integrated in time over $(0,T)$.

It yields

$$\begin{aligned}
&\int_0^T\int_0^t \mathrm{D}\varphi_n(x, \operatorname{grad} u_n(s)) \cdot \operatorname{grad} u_n(s)\mathrm{d}x\,\mathrm{d}s \\
&= \int_0^T\int_0^t \langle f_n(s),\ u_n(s)\rangle_{W^{-1,q'}(\Omega);W_0^{1,q}(\Omega)} \,\mathrm{d}s\,\mathrm{d}t \\
&+ T\int_\Omega \overline{\psi}_n^*(\overline{b}_n(T_n(u_0(x))))\mathrm{d}x - \int_0^T\int_\Omega \overline{\psi}_n^*(\overline{b}_n(u_n(t)))\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{37}$$

Our goal is to bound from above the lim-sup of the left hand side of equality (37). By virtue of the first convergence in (17) the first term on the right hand side of (37) converges to

$$\int_0^T \int_0^t \langle f(s),\ u(s)\rangle_{W^{-1,q'}(\Omega);W_0^{1,q}(\Omega)} \,\mathrm{d}s\,\mathrm{d}t,$$

as n tends to infinity.

Because $\overline{\psi}_n$, $\overline{b}_n$ converge pointwise to ψ, b on $\mathbb{R}$

$$\overline{\psi}_n^*(\overline{b}_n(T_n(u_0(x)))) = \overline{b}_n(T_n(u_0(x)))T_n(u_0(x)) - \overline{\psi}_n(T_n(u_0(x)))$$

converges to

$$\psi^*(b(u_0(x))) = b(u_0(x))u_0(x) - \psi(u_0(x)),$$

for almost every x of Ω. Estimate (12) and the dominated convergence theorem enable us to pass to the limit in the second term in the right hand side (37). We obtain

$$T\int_\Omega \psi^*(b(u_0(x)))\mathrm{d}x.$$

Finally in the spirit of the argument leading to (34), the second convergence in (17) in both cases (or the weak convergence deduced from (30) in case (b)) implies that

$$\underline{\lim}\int_0^T\int_\Omega \overline{\psi}_n^*(\overline{b}_n(u_n(t)))\mathrm{d}x\,\mathrm{d}t \geq \int_0^T\int_\Omega \psi^*(b(u(t)))\mathrm{d}x\,\mathrm{d}t. \tag{38}$$

Note that the identification between χ and $b(u)$ has been implicitly used in (38).

We have thus proved that, as n tends to infinity,

$$\begin{aligned}&\overline{\lim}\int_0^T\int_0^t\int_\Omega \mathrm{D}\varphi_n(x,\ \operatorname{grad} u_n(s))\cdot\operatorname{grad} u_n(s)\mathrm{d}x\,\mathrm{d}s\\ &\leq \int_0^T\int_0^t \langle f(s),\ u(s)\rangle_{W^{-1,q'}(\Omega);W_0^{1,q}(\Omega)}\mathrm{d}s\,\mathrm{d}t\\ &\quad + T\int_\Omega \psi^*(b(u_0(x)))\mathrm{d}x - \int_0^T\int_\Omega \psi^*(b(u(t)))\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{39}$$

A renewed application of Alt-Luckhaus lemma to the first equation of (18) multiplied by u and integrated over $\Omega\times(0,t)$, then over $(0,T)$, permits to identify the right hand side of inequality (39) and to conclude that

$$
\begin{aligned}
&\overline{\lim} \int_0^T \int_0^t \int_\Omega \mathrm{D}\varphi_n(x,\ \operatorname{grad} u_n(s)) \cdot \operatorname{grad} u_n(s) \mathrm{d}x\, \mathrm{d}s\, \mathrm{d}t \\
&\leq \int_0^T \int_0^t \int_\Omega Y(s) \cdot \operatorname{grad} u(x) \mathrm{d}x\, \mathrm{d}s\, \mathrm{d}t.
\end{aligned} \tag{40}
$$

The identification of Y is then performed with the help of the extremality relation for φ_n, namely

$$
\varphi_n(x,\ \operatorname{grad} u_n) + \varphi_n^*(x,\ \mathrm{D}\varphi_n(\operatorname{grad} u_n)) = \mathrm{D}\varphi_n(x,\ \operatorname{grad} u_n) \cdot \operatorname{grad} u_n, \tag{41}
$$

almost everywhere on $\Omega \times (0, T)$. Integrating (41) over $(0, t) \times \Omega$, then over $(0, T)$ and using the increasing character of φ_n and the decreasing character of φ_n^* (cf. Lemma 1) leads to

$$
\begin{aligned}
&\int_0^T \int_0^t \int_\Omega \mathrm{D}\varphi_n(x,\ \operatorname{grad} u_n(s)) \cdot \operatorname{grad} u_n(s) \mathrm{d}x\, \mathrm{d}s \mathrm{d}t \\
&\geq \int_0^T \int_0^t \int_\Omega \varphi_p(x,\ \operatorname{grad} u_n(s)) \mathrm{d}x\, \mathrm{d}s\, \mathrm{d}t \\
&+ \int_0^T \int_0^t \int_\Omega \varphi^*(\mathrm{D}\varphi_n(x,\ \operatorname{grad} u_n(x))) \mathrm{d}x\, \mathrm{d}s\, \mathrm{d}t,
\end{aligned} \tag{42}
$$

for every n greater than p.

We pass to the lim-sup in n in (42), using (40) and the lower semi-continuous properties of φ_p and φ^*. We then pass to the limit in p of the resulting expression with the help of the monotone convergence theorem and finally obtain

$$
\begin{aligned}
\int_0^T \int_0^t \int_\Omega Y(s) \cdot \operatorname{grad} u(s) \mathrm{d}x\, \mathrm{d}s\, \mathrm{d}t \geq &\int_0^T \int_0^t \int_\Omega \varphi(x,\ \operatorname{grad} u(s)) \mathrm{d}x\, \mathrm{d}s\, \mathrm{d}t \\
&+ \int_0^T \int_0^t \int_\Omega \varphi^*(Y(s)) \mathrm{d}x\, \mathrm{d}s\, \mathrm{d}t.
\end{aligned} \tag{43}
$$

The identification of $Y(x, t)$ as an element of the subdifferential $\partial\varphi(x,\ \operatorname{grad} u)$ of $\varphi(x, \cdot)$ at the point $\operatorname{grad} u(x, t)$ for almost every (x, t) in $\Omega \times (0, T)$ is performed with the help of (43) exactly as in the case of χ at the end of Step 5.

The proof of the existence part of Theorem 1 is complete.

STEP 7 - *Comparison.* Since $\bar{b}_n$ is a monotone homeomorphism on $\mathbb{R}$, the comparison part of Theorem 0 applied to u_{1n} and u_{2n} — the relevant solutions of (10) associated with u_{01}, f_1^n and u_{02}, f_2^n respectively — implies that, for almost any (x,t) in $\Omega \times (0,T)$,

$$u_{1n}(x,t) \geq u_{2n}(x,t). \tag{44}$$

Note that we have implicitly chosen f_1^n and f_2^n such that $f_1^n - f_2^n$ remains positive almost everywhere in $\Omega \times (0,T)$, whereas $T_n(u_{01}) - T_n(u_{02})$ is automatically positive almost everywhere in Ω. The comparison result is obtained by passing to the weak limit in (44) as n tends to infinity.

REMARK 9. If in the context of Theorem 1 the potential φ is assumed to be strongly monotone, i.e., if for every ξ and η in $\mathbb{R}^N$ and for almost every x in Ω

$$(y(x,\xi) - y(x,\eta)) \cdot (\xi - \eta) \geq \alpha|\xi - \eta|^q,$$

where $y(x,\xi)$ is any element in the subdifferential of $\varphi(x,\cdot)$ at the point ξ, then $(T-t)\mathrm{D}\varphi_n(x, \operatorname{grad} u_n)$ is easily seen to converge strongly to $(T-t)Y$ in $L^{q'}(0,T;[L^{q'}(\Omega)]^N)$ as n tends to infinity: firstly the first equation of (10) is multiplied by $u_n - u$ and integrated over $(0,t) \times \Omega$; then, after appropriate integration by parts of the resulting equality, the term

$$-\int_0^t \int_\Omega Y(s) \cdot \operatorname{grad}(u_n - u)(s) \mathrm{d}x\, \mathrm{d}s$$

is added to both sides of the equality and Alt-Luckhaus lemma (cf. Appendix) is applied; finally strong monotonicity is used and the result is obtained by passing to the limit as n tends to infinity.

3. - Infinite bareers and discontinuous internal energies

This section is concerned with the extension of Theorem 1 to the case of a discontinuous internal energy. The discontinuity may in particular be a jump or even an infinite bareer for a finite value u_0 of the variable u.

Physical settings for which the internal energy is a discontinuous function of the field are numerous and well documented mathematically under the "phase-change" label. Setting $b(u) = +\infty$ for $u \geq u_0 \geq 0$ is however a seldom encountered hypothesis, although its physical interpretation is clear: the field variable u is constrained to remain always smaller than u_0. The interested reader may refer to the work of Frémond [10], [11] on constrained internal variables.

We propose to prove the following

THEOREM 2. *Under the assumptions* (3), (5), *if* ψ *is a convex lower semi-continuous* $\mathbb{R} \cup \{+\infty\}$*- valued energy potential with* $\psi(0) = 0$, *if for almost any* x *in* Ω, $\chi_0(x) \in \partial\psi(u_0(x))$ *where*

$$\begin{cases} u_0 \in W_0^{1,q}(\Omega), \qquad \chi_0 \in L^1_{\mathrm{loc}}(\Omega) \cap W^{-1,q'}(\Omega), \\ f \in L^{q'}(0,T; W^{-1,q'}(\Omega)), \end{cases}$$

and if either

(a) $$\chi_0 \in L^p(\Omega), \ \ f \in L^1(0,T; L^p(\Omega)) \ \textit{with} \ 1/p + 1/q^* < 1,$$

and

$$\operatorname{div}(|\operatorname{grad} u_0|^{q-2} \operatorname{grad} u_0) \in L^p(\Omega),$$

or

(b) $$\psi(t) \leq C(|t|^{p'} + 1) \ \textit{with} \ p' < q^*,$$

then

$$\begin{cases} \dfrac{\partial \chi}{\partial t} - \operatorname{div} Y = f & \textit{in } \Omega \times (0,T), \\ \chi \in \partial\psi(u) & \textit{in } \Omega \times (0,T), \\ Y \in \partial\varphi(x, \operatorname{grad} u) & \textit{in } \Omega \times (0,T), \\ u = 0 & \textit{on } \partial\Omega \times (0,T), \\ \chi|_{t=0} = \chi_0 & \textit{in } \Omega, \end{cases}$$

admits a solution

$$\begin{cases} u \in L^q(0,T; W_0^{1,q}(\Omega)), \ \ Y \in L^{q'}(0,T; [L^{q'}(\Omega)]^N), \\ \chi \in L^\infty(0,T; L^p(\Omega)) \cap W^{1,q'}(0,T; W^{-1,q'}(\Omega)), \end{cases}$$

and a comparison result similar to that of Theorem 1 holds true.

REMARK 10. As mentioned in the introduction there is price to pay for the occurence of discontinuities in the internal energy, namely the locality hypothesis on f in case (a) or the growth hypothesis on ψ in case (b). Furthermore note that in case (b) the growth condition implicitly implies that χ_0 belongs to $L^p(\Omega)$, $\dfrac{1}{p} + \dfrac{1}{p'} = 1$.

PROOF OF THEOREM 2. Recalling Remark 3, $\psi(t)$ is replaced by its Yosida approximation $\psi_n(t)$, with derivative $b_n(t)$. The proof of Theorem 2 is then divided into two steps. Firstly an adequate approximation $(u_0^n, b_n(u_0^n))$ of (u_0, χ_0) is devised; this is the object of the first step of the proof. A sequence of approximating solutions with $b_n(u_0^n)$ as initial condition is considered upon applying Theorem 1 and the limit process is performed in the second step of the proof.

STEP 1. A sequence g_n of smooth approximations of

$$g \equiv -\mathrm{div}(|\mathrm{grad}\, u_0|^{q-2}\, \mathrm{grad}\, u_0) + \chi_0$$

is considered. In case (b) g_n converges to g in $W^{-1,q'}(\Omega)$ whereas in case (a) the convergence takes place in $L^p(\Omega)$. The approximation u_0^n of u_0 is defined as the solution of

$$(45) \qquad \begin{cases} -\mathrm{div}(|\mathrm{grad}\, u_0^n|^{q-2}\, \mathrm{grad}\, u_0^n) + b_n(u_0^n) = g_n & \text{in } \Omega, \\ u_0^n = 0 & \text{on } \partial\Omega. \end{cases}$$

Note that (45) is known to have a unique solution u_0^n in $W_0^{1,q}(\Omega)$ (cf. Brézis-Browder [9], Theorem 7 or Webb [22]).

Multiplication of the first equation of (45) by u_0^n, integration over Ω of the resulting expression and appropriate integration by parts lead, with the help of the extremality condition, to the following *a priori* estimates:

$$(46) \qquad \begin{cases} u_0^n & \text{is bounded in } W_0^{1,q}(\Omega), \\ \psi_n^*(b_n(u_0^n)) & \text{is bounded in } L^1(\Omega), \end{cases}$$

independently of n.

In case (b), the growth estimate on $\psi(t)$ implies the same growth estimate on $\psi_n(t)$ since $\psi_n(t) \leq \psi(t)$; thus the conjugate function $\psi_n^*(t)$ satisfies, for t in $\mathbb{R}$ and every n,

$$(47) \qquad \psi_n^*(t) \geq \frac{1}{p}\left(\frac{p-1}{Cp}\right)^{p-1} |t|^p - C,$$

where $\dfrac{1}{p} = 1 - \dfrac{1}{p'}$, and the second estimate in (46) permits to conclude that

$$b_n(u_n^0) \text{ is bounded in } L^p(\Omega),$$

independently of n.

In case (a), Alt-Luckhaus remark (cf. Remark 2) does not apply since $b(t)$ may not be everywhere defined. The sequence g_n is however bounded in $L^p(\Omega)$, which permits the multiplication of the first equation of (45) by $|b_n(u_0^n)|^{p-2} b_n(u_0^n)$ — or rather by a Lipschitz function of $b_n(u_0^n)$ with $p-1$ growth (cf. Step 3 in Section 2). Upon performing the usual steps one concludes that

$$b_n(u_n^0) \text{ is bounded in } L^p(\Omega),$$

independently of n. Thus in both cases a subsequence of u_n^0, $b_n(u_0^n)$ (still indexed by n) is such that, as n tends to infinity,

$$\begin{cases} u_0^n \rightharpoonup \tilde{u}_0 & \text{weakly in } W_0^{1,q}(\Omega), \\ b_n(u_0^n) \rightharpoonup \tilde{b}_0 & \text{weakly in } L^p(\Omega). \end{cases}$$

Since $1/p+1/q^* < 1$, the space $W_0^{1,q}(\Omega)$ is compactly embedded in $L^{p'}(\Omega)$; thus

$$\int_\Omega u_0^n b_n(u_0^n)\mathrm{d}x \to \int_\Omega \tilde{u}_0\tilde{b}_0\,\mathrm{d}x,$$

as n tends to infinity. The extremality relation on ψ_n permits to conclude, exactly as in the fifth step of the proof of Theorem 1, that

$$\tilde{b}_0(x) \in \partial\psi(\tilde{u}_0(x)) \text{ almost everywhere in } \Omega.$$

The weak limit Y_0 of (a subsequence of) $|\mathrm{grad}\, u_0^n|^{q-2}\,\mathrm{grad}\, u_0^n$ satisfies

$$-\mathrm{div}\, Y_0 + \tilde{b}_0 = g \quad \text{in } \Omega. \tag{48}$$

Its value is computed by proving that

$$\overline{\lim_{n\to+\infty}} \int_\Omega |\mathrm{grad}\, u_0^n|^q\,\mathrm{d}x \le \int_\Omega Y_0\,\mathrm{grad}\,\tilde{u}_0\,\mathrm{d}x. \tag{49}$$

Inequality (49) results from the multiplication of the first equality of (45) by u_0^n which yields, by virtue of Brézis-Browder's theorem,

$$\int_\Omega |\mathrm{grad}\, u_0^n|^q\,\mathrm{d}x = \langle g_n, u_0^n\rangle_{W^{-1,q'}(\Omega);W_0^{1,q}(\Omega)} - \int_\Omega b_n(u_0^n)u_0^n\,\mathrm{d}x$$
$$= \langle g_n, u_0^n\rangle_{W^{-1,q'}(\Omega);W_0^{1,q}(\Omega)} - \int_\Omega \psi_n(u_0^n)\mathrm{d}x - \int_\Omega \psi_n^*(b_n(u_0^n))\mathrm{d}x.$$

Taking the lim-sup of the above equality yields (49) since

$$\begin{aligned} &\underline{\lim}\left\{\int_\Omega \psi_n(u_0^n)\mathrm{d}x + \int_\Omega \psi_n^*(b_n(u_0^n))\mathrm{d}x\right\} \ge \int_\Omega \psi(\tilde{u}_0)\mathrm{d}x + \int_\Omega \psi^*(\tilde{b}_0)\mathrm{d}x \\ &= \int_\Omega \tilde{b}_0\tilde{u}_0\,\mathrm{d}x = \langle g, u_0\rangle_{W^{-1,q'}(\Omega),W_0^{1,q}(\Omega)} - \int_\Omega Y_0\,\mathrm{grad}\,\tilde{u}_0\,\mathrm{d}x, \end{aligned} \tag{50}$$

as can be seen by appropriately freezing the index of ψ_n and by making use of the decreasing character's of ψ_n^*. Note that the last equality in (50) results from the multiplication of (48) by $\tilde{u}_0$.

An extremality argument of the type used at the end of steps 5 or 6 in the proof of Theorem 1 permits to conclude, in view of (49), that

$$Y_0 = |\operatorname{grad} \tilde{u}_0|^{q-2} \operatorname{grad} \tilde{u}_0.$$

We have proved thus far that

$$\left\{\begin{array}{ll} -\operatorname{div}(|\operatorname{grad} \tilde{u}_0|^{q-2} \operatorname{grad} \tilde{u}_0) + \tilde{b}_0 = g & \text{in } \Omega, \\ \tilde{b}_0 \in \partial\psi(\tilde{u}_0) & \text{almost everywhere in } \Omega, \\ \tilde{u}_0 = 0 & \text{on } \partial\Omega, \end{array}\right. \tag{51}$$

whereas

$$\left\{\begin{array}{ll} -\operatorname{div}(|\operatorname{grad} u_0|^{q-2} \operatorname{grad} u_0) + \chi_0 = g & \text{in } \Omega, \\ \chi_0 \in \partial\psi(u_0) & \text{almost everywhere in } \Omega, \\ u_0 = 0 & \text{on } \partial\Omega. \end{array}\right. \tag{52}$$

Multiplication of the difference between the first equation of (51) and that of (52) by $(\tilde{u}_0 - u_0)$ and appropriate integration by parts over Ω immediately implies that $u_0 = \tilde{u}_0$ and that $\chi_0 = \tilde{b}_0$. We have thus shown the existence of a sequence u_0^n in $W_0^{1,q}(\Omega)$ such that as n tends to infinity

$$\left\{\begin{array}{ll} u_0^n \rightharpoonup u_0 & \text{weakly in } W_0^{1,q}(\Omega), \\ b_n(u_0^n) \rightharpoonup \chi_0 & \text{weakly in } L^p(\Omega) \text{ (and strongly in } W^{-1,q'}(\Omega)). \end{array}\right. \tag{53}$$

STEP 2. Theorem 1 is applied to

$$\left\{\begin{array}{ll} \dfrac{\partial b_n(u_n)}{\partial t} - \operatorname{div} Y_n = f & \text{in } \Omega \times (0,T), \\ Y_n \in \partial\varphi(x, \operatorname{grad} u_n) & \text{in } \Omega \times (0,T), \\ u_n = 0 & \text{on } \partial\Omega \times (0,T), \\ b_n(u_n)|_{t=0} = b_n(u_0^n) & \text{in } \Omega, \end{array}\right. \tag{54}$$

yielding a solution in $L^q(0,T;W_0^{1,q}(\Omega))$ with Y_n in $L^{q'}(0,T;[L^{q'}(\Omega)]^N)$ and $b_n(u_n)$ in $L^\infty(0,T;L^1(\Omega)) \cap W^{1,q'}(0,T;W^{-1,q'}(\Omega))$. Further, since $b_n(u_0^n)$ belongs to $L^p(\Omega)$, $b_n(u_n)$ belongs to $L^\infty(0,T;L^p(\Omega))$ in case (a).

In both cases, multiplication of the first equality of (54) by u_n, appropriate integration by parts and consideration of (53) would yield, as in the second step of the proof of Theorem 1, the following estimates and statements of weak convergences as n tends to infinity:

$$\left\{\begin{array}{ll} \psi_n^*(b_n(u_n)) & \text{is bounded in } L^\infty(0,T;L^1(\Omega)), \\ \dfrac{\partial b_n(u_n)}{\partial t} & \text{is bounded in } L^{q'}(0,T;W^{-1,q'}(\Omega)), \end{array}\right. \tag{55}$$

$$\left\{\begin{array}{ll} u_n \rightharpoonup u & \text{weakly in } L^q(0,T;W_0^{1,q}(\Omega)), \\ Y_n \rightharpoonup Y & \text{weakly in } L^{q'}(0,T;[L^{q'}(\Omega)]^N). \end{array}\right. \tag{56}$$

The fourth step of the proof of that theorem is however doomed to failure in the present setting because of a lack of uniform estimate on b_n' (cf. Remark 7).

The derivation of an appropriate estimate on $b_n(u_n)$ is performed in case (a) in a manner identical to that which led to the estimate (20) in the third step of the proof of Theorem 1. Thus, in case (a),

$$b_n(u_n) \rightharpoonup \chi \text{ weak-}* \text{ in } L^\infty(0,T;L^p(\Omega)),$$

as n tends to infinity.

In case (b), inequality (47) is recalled and the first estimate in (55) permits to conclude that $b_n(u_n)$ is bounded in $L^\infty(0,T;L^p(\Omega))$.

Thus, in both cases, a subsequence of $b_n(u_n)$ (still indexed by n) is such that

$$b_n(u_n) \rightharpoonup \chi \text{ weak-}* \text{ in } L^\infty(0,T;L^p(\Omega)), \tag{57}$$

as n tends to infinity with

$$\frac{1}{1-1/q^*} < p \leq +\infty. \tag{58}$$

Note that in case (b) the restriction on the range of p' leads to (58).

Furthermore, by virtue of the second convergence in (53) and upon passing to the distributional limit in the first equation of (54), Y and χ satisfy

$$\left\{\begin{array}{ll} \dfrac{\partial \chi}{\partial t} - \operatorname{div} Y = f & \text{in } \Omega \times (0,T), \\ \chi|_{t=0} = \chi_0 & \text{in } \Omega. \end{array}\right. \tag{59}$$

The lower bound (58) on p implies that $L^p(\Omega)$ is compactly embedded in $W^{-1,q'}(\Omega)$ thus Aubin's lemma together with (57) and the second estimate in (55) leads to the following statement of strong convergence:

$$b_n(u_n) \rightarrow \chi \text{ strongly in } C^0([0,T];W^{-1,q'}(\Omega)),$$

as n tends to infinity. The identification of χ is then performed with the help of the extremality condition for ψ_n.

The procedure is quasi-identical to that developed for the identification of χ in Step 5 of the proof of Theorem 1 (cf. Remark 8). It will not be repeated here. Thus

$$\chi \in \partial\psi(u),$$

almost everywhere in $\Omega \times (0,T)$.

As far as the identification of Y is concerned we refer the reader to the sixth step in the proof of Theorem 1. The task is simpler here since only one fixed potential, namely φ, needs to be considered. There is however a small difference with the previous situation in the handling of the term

$$T \int_\Omega \psi_n^*(b_n(u_0^n))\mathrm{d}x,$$

whose lim-sup must be bounded above by

$$T \int_\Omega \psi^*(\chi_0)\mathrm{d}x.$$

To this effect, we recall that

$$\int_\Omega \psi_n^*(b_n(u_0^n))\mathrm{d}x = \int_\Omega b_n(u_0^n)u_0^n\,\mathrm{d}x - \int_\Omega \psi_n(u_0^n)\mathrm{d}x,$$

and pass to the lim-sup in each term of the right-hand side. The first term passes to the limit (cf. e.g. (53)), yielding

$$\int_\Omega \chi_0 u_0\,\mathrm{d}x.$$

It remains to find a lower bound to

$$\underline{\lim} \int_\Omega \psi_n(u_0^n)\mathrm{d}x.$$

Since ψ_n is an increasing sequence of functions, for any n_0,

$$\underline{\lim} \int_\Omega \psi_n(u_0^n)\mathrm{d}x \geq \underline{\lim} \int_\Omega \psi_{n_0}(u_0^n)\mathrm{d}x \geq \int_\Omega \psi_{n_0}(u_0)\mathrm{d}x, \tag{60}$$

for $n \geq n_0$. We have used the lower-semi continuous character of ψ_{n_0} and Fatou's lemma in the second inequality of (60).

Thus, as n tends to infinity,

$$\overline{\lim} \int_\Omega \psi_n^*(b_n(u_0^n))\mathrm{d}x \leq - \int_\Omega \psi_{n_0}(u_0)\mathrm{d}x + \int_\Omega \chi_0 u_0\,\mathrm{d}x.$$

But $\psi(u_0)$ belongs to $L^1(\Omega)$ since $\psi(u_0) \leq \chi_0 u_0$ almost everywhere and $\chi_0 u_0$ lies in $L^1(\Omega)$. Since ψ_{n_0} converges to ψ, the monotone convergence theorem, then the extremality condition for ψ lead to

$$\overline{\lim} \int_\Omega \psi_n^*(b_n(u_0^n))\mathrm{d}x \leq \int_\Omega \psi^*(\chi_0)\mathrm{d}x,$$

which was the sought result.

Thus, almost everywhere in $\Omega \times (0, T)$,

$$Y \in \partial\varphi(x, \operatorname{grad} u),$$

and the proof of the existence part of Theorem 2 is complete. The comparison result is obtained as before by passing to the weak limit in the comparison result for the approximating sequences.

4. - Existence results with forcing term in $L^1((0,T) \times \Omega)$

When the forcing term f is merely integrable over $\Omega \times (0, T)$, a new kind of estimate has to be divised because multiplication of the first equation of (1) by either u, $\dfrac{\partial u}{\partial t}$ or $b(u)$ is in general impossible for want of L^∞ - estimate on those fields.

If f_n is a smooth approximation of f and u_n is the solution to (1)—with f_n as forcing term—which is produced through direct application of case (b) of Theorem 1, then strong convergence of $b(u_n)$ in $L^\infty(0, T; L^1(\Omega))$ is obtained by multiplication of the difference

$$\frac{\partial}{\partial t}(b(u_n) - b(u_m)) - \operatorname{div}(\mathrm{D}\varphi(x, \operatorname{grad} u_n) - \mathrm{D}\varphi(x, \operatorname{grad} u_m)) = f_n - f_m$$

by

$$\operatorname{sg}(u_n - u_m) = \operatorname{sg}(b(u_n) - b(u_m)),$$

integration of the resulting expression over $\Omega \times (0, t)$ $(0 \leq t \leq T)$ and appropriate integration by parts. This "hand-waving" arguments is easily rendered rigorous upon approximating the sign function by the Lipschitz function $\operatorname{sg}_\eta(t)$ defined as

$$\operatorname{sg}_\eta(t) = \begin{cases} \dfrac{1}{\eta} t & \text{if } |t| \leq \eta, \\ 1 & \text{if } |t| > \eta, \end{cases}$$

considering the coercive and Lipschitz approximation $\bar{b}_p$ of b and using the weak lower semi-continuous character of $\displaystyle\int_\Omega |\cdot| \mathrm{d}x$ with respect to $L^\infty(0, T; L^p(\Omega))$ norm (n and m remain fixed during this limit process).

This very good and easy estimate on $b(u_n)$ is however of little use as long as no estimate on u_n is available. In the case where $D\varphi(x,\xi)$ is linear in ξ the comparison part of Theorem 1 will permit to obtain a weak L^1-estimate on u_n (cf. proof of Theorem 3). When b is an homeomorphism with minimal growth, a weak $L^r(0,T;W_0^{1,r}(\Omega))$ estimate, with r small, will be derived through application of an estimation technique due to Boccardo and Gallouët [6] (cf. proof of Theorem 4). In the latter case the strongly monotone character of $D\varphi(x,\xi)$ will be essential in passing to the limit on the fluxes $D\varphi(x, \operatorname{grad} u_n)$.

We now address the first setting for which $\varphi(x,\xi)$ is of the form $A(x)\xi\cdot\xi$, where A is an element of $L^\infty(\Omega,\mathbb{R}^{N^2})$ satisfying, for almost every x of Ω and every ξ in $\mathbb{R}^N$,

$$
(61)\qquad \begin{cases} A_{ij}(x)=A_{ji}(x), \\ \alpha|\xi|^2 \le A_{ij}(x)\xi_i\xi_j \le \beta|\xi|^2. \end{cases}
$$

The idea of the proof of the following theorem was communicated to us by Benilan [4].

THEOREM 3. *Assume that assumptions* (4) *and* (61) *hold true with a continuous b,*

$$
u_0 \in H_0^1(\Omega),\ \ b(u_0) \in L^1_{\mathrm{loc}}(\Omega)\cap H^{-1}(\Omega),
$$

and

$$
f \in L^1((0,T)\times\Omega).
$$

Then the problem

$$
(62)\qquad \begin{cases} \dfrac{\partial b(u)}{\partial t} - \operatorname{div}(A(x)\operatorname{grad} u) = f & \text{in } \Omega\times(0,T), \\ u=0 & \text{in } \partial\Omega\times(0,T), \\ b(u)|_{t=0} = b(u_0) & \text{in } \Omega, \end{cases}
$$

admits a solution $u \in L^1(0,T;L^1(\Omega))$ *with* $\int_0^t u(s)\mathrm{d}s \in L^\infty(0,T;W_0^{1,r}(\Omega))$ *for* $r < \dfrac{N}{N-1}$, $b(u) \in L^\infty(0,T;L^1(\Omega))$ *and comparison holds true in the sense of Theorem 1.*

REMARK 11. Note that by virtue of Brézis-Browder's theorem ([9], Theorem 1), together with Remark 2, $b(u_0)$ actually belongs to $L^1(\Omega)$.

PROOF OF THEOREM 3. Recall the truncation operator T_n (cf. (9)) and apply case (b) of Theorem 1 to

$$
(63)\qquad \begin{cases} \dfrac{\partial b(u_n)}{\partial t} - \operatorname{div}(A(x)\operatorname{grad} u_n) = T_n f & \text{in } \Omega\times(0,T), \\ u_n=0 & \text{in } \partial\Omega\times(0,T), \\ b(u_n)_{t=0} = b(T_n(u_0)) & \text{in } \Omega. \end{cases}
$$

Theorem 1 yields a solution u_n in $L^2(0,T;H_0^1(\Omega))$ with $b(u_n)$ in $L^\infty(0,T;L^s(\Omega))\cap W^{1,2}(0,T;H^{-1}(\Omega))$ $(s<+\infty)$.

Further the remarks at the beginning of this section and the easily achieved strong convergence of $b(T_n(u_0))$ in $L^1(\Omega)$ prove that

$$b(u_n)\to\chi \text{ strongly in } L^\infty(0,T;L^1(\Omega)), \tag{64}$$

as n tends to infinity.

We now seek an appropriate estimate on u_n. To this effect we consider $\overline{v}_n$ (respectively $\underline{v}_n$) solution of

$$\begin{cases} \dfrac{\partial b(\overline{v}_n)}{\partial t}-\operatorname{div}(A(x)\operatorname{grad}\overline{v}_n)=(T_n(f))^+ & \text{in } \Omega\times(0,T),\\ \overline{v}_n=0 & \text{on } \partial\Omega\times(0,T),\\ b(\overline{v}_n)|_{t=0}=b((T_n(u_0))^+) & \text{in } \Omega\times(0,T), \end{cases} \tag{65}$$

(respectively $\underline{v}_n$, $-(T_n(f))^-$, $-(T_n(u_0))^-$). The comparison result of Theorem 1 yields, almost everywhere on $\Omega\times(0,T)$,

$$\underline{v}_n\le u_n\le\overline{v}_n. \tag{66}$$

Since $(T_n(t))^+$ and $(T_n(t))^-$ are two monotonically increasing sequences, $\overline{v}_n$ is a monotonically increasing sequence while $\underline{v}_n$ is a monotonically decreasing sequence.

The linearity of the fluxes has not been used thus far. It comes into play through the integration of the first equation of (63) (or (65)) over $(0,t)$, $0\le t\le T$; we obtain

$$-\operatorname{div}\left(A(x)\operatorname{grad}\left(\int_0^t u_n(s)\mathrm{d}s\right)\right)=\int_0^t f_n(s)\mathrm{d}s-b(u_n(t))+b(T_n(u_0)).$$

Similar equations hold for $\overline{v}_n$ and $\underline{v}_n$. By virtue of (64), the right hand side of the above equation is a Cauchy sequence in $L^\infty(0,T;L^1(\Omega))$ and De Giorgi's theorem on linear elliptic equations with non-smooth coefficients (cf. e.g. Stampacchia [21]) implies that

$$\begin{gathered}\int_0^t u_n(s)\mathrm{d}s\ \left(\text{respectively } \int_0^t \overline{v}_n(s)\mathrm{d}s,\ \int_0^t \underline{v}_n(s)\mathrm{d}s\right) \text{ is a Cauchy}\\ \text{sequence in } L^\infty(0,T;W_0^{1,r}(\Omega)),\ r<\frac{N}{N-1}.\end{gathered} \tag{67}$$

The monotone convergence theorem permits to conclude that as n tends to infinity

$$\begin{gathered}\overline{v}_n \text{ (respectively } \underline{v}_n\text{) converges monotonically and strongly}\\ \text{in } L^1(\Omega\times(0,T)) \text{ to } \overline{v} \text{ (respectively } \underline{v}\text{)},\end{gathered} \tag{68}$$

whereas the lack of monotonicity of the sequence u_n prevents us from reaching a similar conclusion about u_n. Gathering (66) and (68) leads us to

$$\underline{v} \leq u_n \leq \overline{v} \text{ almost everywhere on } \Omega \times (0,T),$$

from which a weak $L^1(\Omega \times (0,T))$ estimate is immediately deduced. Hence

$$u_n \rightharpoonup u \quad \text{weakly in } L^1(\Omega \times (0,T)),$$

and

$$\int_0^t u_n(s)\mathrm{d}s \to \int_0^t u(s)\mathrm{d}s \quad \text{strongly in } L^\infty(0,T;W_0^{1,r}(\Omega)),\ r < \frac{N}{N-1},$$

as n tends to infinity.

The identification of χ is now straightforward. Recall the definition (9) of the truncation and let φ be an arbitrary positive element of $C_0^\infty(\Omega \times (0,T))$ and w be an arbitrary element of $L^1(\Omega \times (0,T))$. The monotone character of b yields for every strictly positive real number R

$$\int_0^T\int_\Omega \varphi T_R(b(u_n) - b(w))\ (u_n - w)\mathrm{d}x\,\mathrm{d}t \geq 0. \tag{69}$$

Since u_n converges weakly to u in $L^1(\Omega \times (0,T))$ while $T_R(b(u_n) - b(w))$ converges weak-$*$ in $L^\infty(\Omega \times (0,T))$ and almost everywhere to $T_R(\chi - b(w))$, a straightforward application of Egoroff's theorem permits to pass to the limit in (69). Thus

$$\int_0^T\int_\Omega \varphi T_R(\chi - b(w))\ (u - w)\mathrm{d}x\,\mathrm{d}t \geq 0.$$

Since φ and R are arbitrary, $(\chi - b(w))\ (u-w)$ is found to be almost everywhere positive on $\Omega \times (0,T)$ and the continuous character of b yields

$$\chi = b(u) \text{ almost everywhere on } \Omega \times (0,T).$$

Finally the comparison result is obtained as before by passing to the weak limit in the comparison result for the approximating sequences. The proof of Theorem 3 is complete.

THEOREM 4. *If assumptions* (3), (4), (5) *hold true, with a continuous* b, *if* b *is an homeomorphism such that* $|b(t)| \geq C(|t|^\gamma - 1)$ *where* $\gamma > \sup\left\{0, \dfrac{N(2-q)}{q-1}\right\}$, *and if further* φ *is* C^1*-admissible and* $\mathrm{D}\varphi(x,\xi)$ *is* *strongly monotone (in the sense of Remark 9), then for every* u_0, f *such that*

$$u_0 \in W_0^{1,q}(\Omega),\ \ b(u_0) \in L^1_{\mathrm{loc}}(\Omega) \cap W^{-1,q'}(\Omega),$$
$$f \in L^1(\Omega \times (0,T)),$$

there exists a solution u *to*

$$\begin{cases} \dfrac{\partial b(u)}{\partial t} - \operatorname{div} \mathrm{D}\varphi(x, \operatorname{grad} u) = f & \text{in } \Omega \times (0,T), \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ b(u)_{t=0} = b(u_0) & \text{in } \Omega, \end{cases}$$

with

$$u \text{ in } L^r(0,T; W_0^{1,r}(\Omega)),\ \ \mathrm{D}\varphi(x, \operatorname{grad} u) \text{ in } L^{r/q-1}(0,T;[L^{r/q-1}(\Omega)]^N)$$
$$\text{and } b(u) \text{ in } L^\infty(0,T;L^1(\Omega)) \cap W^{1,r/q-1}(0,T;W^{-1,r/q-1}(\Omega))$$

where $r < q - \dfrac{N}{\gamma + N}$ *and comparison holds true in the sense of Theorem 1.*

REMARK 12. Note that since $q - N/(\gamma+N) > q-1$ as soon as $\gamma > 0$, r may be chosen such that $r/(q-1) > 1$. Actually a careful examination of the proof of Lemma 3 below would demonstrate that an $L^{r/q-1}(0,T;[L^{r/q-1}(\Omega)]^N)$ estimate on $\mathrm{D}\varphi(x, \operatorname{grad} u)$ (or rather on an approximating sequence $\mathrm{D}\varphi(x, \operatorname{grad} u_n)$) can be derived without any restriction on the range of possible strictly positive γ's. Thus the restriction on γ is only needed to lend a local meaning to the quantity $\operatorname{grad} u$.

The proof of Theorem 4 is close to that of a similar result in which the internal energy b is taken to be linear. The latter result is due to Boccardo and Gallouët (cf. [6], Theorem 4). We will thus merely sketch the proof of Theorem 4 and refer the interested reader to the previously mentioned paper of Boccardo and Gallouët. The proof is essentially two-fold. Firstly an estimate on $\operatorname{grad} u$ is being sought. This is performed by application of a modified Boccardo-Gallouët type estimate (cf. Lemma 3); as such it is independent of the strongly monotone character of $\mathrm{D}\varphi$. Then strong monotonicity is used to identify the limit flux.

We now recall the estimate obtained by Boccardo and Gallouët (cf. [6]) and adapt it to our setting.

LEMMA 3. *Under assumptions* (3), (4), (5) *with a continuous* b *satisfying*

$$|b(t)| \geq C(|t|^\gamma - 1) \text{ where } \gamma > \sup\left\{0, \frac{N(2-q)}{q-1}\right\},$$

if

$$\begin{cases} v_n^0 \in W_0^{1,q}(\Omega), \quad b(v_n^0) \in L^p(\Omega), \\ f_n \in L^{q'}(0,T;W^{-1,q'}(\Omega)) \cap L^1(0,T;L^p(\Omega)) \ (1/p + 1/q^* < 1), \end{cases} \tag{70}$$

and if further,

$$\left\{\begin{array}{ll} b(v_n^0) & \textit{is bounded in } L^1(\Omega), \\ f_n & \textit{is bounded in } L^1(0,T;L^1(\Omega)), \end{array}\right. \tag{71}$$

independently of n, then there exists a solution v_n in $L^q(0,T;W_0^{1,q}(\Omega))$ of

$$\left\{\begin{array}{ll} \dfrac{\partial b(v_n)}{\partial t} - \operatorname{div} Y_n = f_n & \textit{in } \Omega\times(0,T), \\ Y_n \in \partial\varphi(x, \operatorname{grad} v_n) & \textit{in } \Omega\times(0,T), \\ v_n = 0 & \textit{on } \partial\Omega\times(0,T), \\ b(v_n)|_{t=0} = b(v_n^0) & \textit{in } \Omega, \end{array}\right. \tag{72}$$

such that

$$v_n \textit{ is bounded in } L^r(0,T;W_0^{1,r}(\Omega)) \tag{73}$$

independently of n, for every $r < q - N/(\gamma + N)$.

SKETCH OF THE PROOF OF LEMMA 3. In view of (70), part (b) of Theorem 1 applies to (72) yielding a solution v_n in $L^q(0,T;W_0^{1,q}(\Omega))$. Further, in the spirit of the opening remarks, multiplication of the first equation of (72) by $\operatorname{sg}(v_n)$ and the bounds (71) easily imply that

$$b(v_n) \text{ is bounded in } L^\infty(0,T;L^1(\Omega)), \tag{74}$$

independently of n. Following the proof of Theorem 1 of [6] we propose to multiply the first equation of (72) by $T_p(v_n)$ (cf. (9)) and by $\theta_p(v_n)$ where

$$\theta_p(t) = \left\{\begin{array}{ll} 0 & \text{if } |t| \le p, \\ (|t| - p)\operatorname{sg}(t) & \text{if } p \le |t| \le p+1, \\ \operatorname{sg}(t) & \text{if } |t| \ge p+1. \end{array}\right.$$

The exact test fields that will be of use are coercive approximations of $T_p(v_n)$ and $\theta_p(v_n)$; specifically the term εv_n is added to both fields and n remains fixed during the approximation process. The resulting expression are integrated over $\Omega\times(0,T)$. The flux term is appropriately integrated by parts and the coercivity of φ is used, together with the bound (71) on f_n. The energy term needs to be handled with more care. Considering for example the multiplication by $T_p(v_n) + \varepsilon v_n$, we obtain

$$\int_0^T \int_\Omega \left\langle \frac{\partial b(v_n)}{\partial t}, T_p(v_n) + \varepsilon v_n \right\rangle_{W^{-1,q'}(\Omega);W_0^{1,q}(\Omega)} (t)\,\mathrm{d}x\,\mathrm{d}t.$$

Since $T_p(t)+\varepsilon t$ is invertible, the above quantity reads as

$$(75)\qquad \int_0^T\int_\Omega \left\langle \frac{\partial b((T_p+\varepsilon)^{-1}(w_n^\varepsilon))}{\partial t},\ w_n^\varepsilon \right\rangle_{W^{-1,q'}(\Omega);W_0^{1,q}(\Omega)} (t)\mathrm{d}x\,\mathrm{d}t,$$

where

$$w_n^\varepsilon = T_p(v_n)+\varepsilon v_n.$$

Upon denoting by $Z_p^\varepsilon(t)$ the primitive of $b\circ(T_p+\varepsilon)^{-1}$ with value 0 at 0, expression (75) becomes after direct application of Alt-Luckhaus Lemma

$$(76)\qquad \int_\Omega (Z_p^\varepsilon)^*(b(v_n(T)))\mathrm{d}x - \int_\Omega (Z_p^\varepsilon)^*(b(v_n^0))\mathrm{d}x.$$

The first term of (76) is positive since $(Z_p^\varepsilon(t))^*$ is a convex function with minimum at $t=0$. The extremality relation on $(Z_p^\varepsilon)^*$ yields

$$\int_\Omega (Z_p^\varepsilon)^*(b(v_n^0))\mathrm{d}x \le \int_\Omega b(v_n^0)(T_p(v_n^0)+\varepsilon v_n^0)\mathrm{d}x$$

$$\le p\|b(v_n^0)\|_{L^1(\Omega)} + \varepsilon\int_\Omega b(v_n^0)v_n^0\mathrm{d}x \le Cp \quad (\varepsilon << 1),$$

where the last inequality is a consequence of (71). A similar argument could be applied to the case of the multiplication of the first equation of (72) by $\theta_p(v_n)$. Thus in both cases only the term resulting from the flux needs to be considered; the present analysis is reduced to that of [6]. Appropriate summations over p yield — as in [6] — for every (r,s) such that

$$(77)\qquad \begin{gathered} 1<r<\inf(q,N),\\ s>r/q, \end{gathered}$$

$$(78)\qquad \int_0^T\int_\Omega |\operatorname{grad} v_n|^r\,\mathrm{d}x\,\mathrm{d}t \le C\left[p^{r/q}+\varepsilon(p)\left(\int_0^T\int_\Omega |v_n|^{qs/q-r}\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{q-r}{q}}\right].$$

In (78) $\varepsilon(p)$ is such that it goes to zero as p goes to infinity while C is a generic positive constant. The proof of estimate (73) then reduces to a careful analysis of the term $\int_0^T\int_\Omega |v_n|^{qs/q-r}\,\mathrm{d}x\,\mathrm{d}t$. The bound (74) on $b(v_n)$ implies that

$$(79)\qquad \int_\Omega |v_n(t)|^\gamma\,\mathrm{d}x \text{ is bounded in } L^\infty(0,T) \text{ independently of } n,$$

and Hölder's inequality together with estimate (79) yields

$$\int_0^T\int_\Omega |v_n|^{qs/q-r}\,\mathrm{d}x\,\mathrm{d}t \le C\int_0^T\left(\int_\Omega |v_n|^{r^*}\,\mathrm{d}x\right)^{r/r^*}\mathrm{d}t$$

where r^* is the Sobolev exponent of r $\left(r^* = \dfrac{rN}{N-r}\right)$ provided that

$$s = \frac{r(\gamma+N)(q-r)}{qN}.$$

In that case estimate (78) immediately implies estimate (73). Restriction (77) on the range of permissible s's translates into $\dfrac{(\gamma+N)(q-r)}{N} > 1$, from which the restrictions on γ and r are immediately deduced.

We are now in a position to address the proof of Theorem 4 upon applying Lemma 3 to a relevant approximation sequence of u.

SKETCH OF THE PROOF OF THEOREM 4. Let f_n be a smooth approximation of f. Case (b) of Theorem 1 yields a solution u_n, $b(u_n)$ of

$$\text{(80)}\qquad \begin{cases} \dfrac{\partial b(u_n)}{\partial t} - \operatorname{div} \mathrm{D}\varphi(x, \operatorname{grad} u_n) = f_n & \text{in } \Omega\times(0,T),\\ u_n = 0 & \text{on } \partial\Omega\times(0,T),\\ b(u_n)|_{t=0} = b(T_n(u_0)) & \text{in } \Omega, \end{cases}$$

with

$$\begin{cases} u_n & \text{in } L^q(0,T;W_0^{1,q}(\Omega)),\\ b(u_n) & \text{in } L^\infty(0,T;L^s(\Omega))\cap W^{1,q'}(0,T;W^{-1,q'}(\Omega)), \end{cases}$$

($s < +\infty$). Since it is easily seen that, as n tends to infinity,

$$\text{(81)}\qquad b(T_n(u_0)) \text{ converges to } b(u_0) \text{ strongly in } L^1(\Omega),$$

the remarks at the beginning of this section lead to

$$\text{(82)}\qquad b(u_n) \to \chi \text{ strongly in } L^\infty(0,T;L^1(\Omega)).$$

Because b is assumed to be an homeomorphism, (82) immediately implies that a subsequence of u_n (still indexed by n) converges almost everywhere in $\Omega\times(0,T)$ to a measurable function u. Thus

$$\chi = b(u).$$

In view of (81), (82), Lemma 3 applies to $v_n = u_n$; it implies that, for

$r < q - \dfrac{N}{\gamma + N}$

(83) $$u_n \rightharpoonup u \text{ weakly in } L^r(0,T;W_0^{1,r}(\Omega)),$$

as n tends to infinity. There again we have identified u_n with one of its subsequences.

The limit flux is identified as in Boccardo-Gallouët [6] by proving that, as n tends to infinity,

(84) $$\operatorname{grad} u_n \to \operatorname{grad} u \text{ strongly in } L^1(\Omega \times (0,T)).$$

In order to prove (84) the first equations in (80) for n and m are substracted from each other then multiplied by $\eta\, \mathrm{sg}_\eta(u_n - u_m)$ and the resulting expressions are integrated over $\Omega \times (0,T)$. Note once again that, in all rigor, this operation should be performed on relevant approximations of u_n and u_m. The contribution of the term $\eta \dfrac{\partial}{\partial t}(b(u_n) - b(u_m))$ will vanish as η tends to zero in view of (81), (82). The contribution of the flux term will be

$$\int_{D_{n,m,\eta}} (\mathrm{D}\varphi(x, \operatorname{grad} u_n) - \mathrm{D}\varphi(x, \operatorname{grad} u_m))\ (\operatorname{grad} u_n - \operatorname{grad} u_m)\mathrm{d}x\, \mathrm{d}s$$

where

$$D_{n,m,\eta} = \{(x,t) \in \Omega \times (0,T) |\ |u_n(x,t) - u_m(x,t)| \le \eta\}.$$

From here on the argument is identical to that of [6], p. 156-157. It relies on the strongly monotone character of the graph of $\mathrm{D}\varphi$ together with the measure convergence of u_n to u.

The coercivity of φ together with (83) implies the existence of a subsequence of $\mathrm{D}\varphi(x, \operatorname{grad} u_n)$ (still indexed by n) such that, as n tends to infinity,

$$\mathrm{D}\varphi(x, \operatorname{grad} u_n) \rightharpoonup Y \text{ weakly in } L^{r/q-1}(0,T;[L^{r/q-1}(\Omega)]^N).$$

The identification of Y is immediate because the continuous character of $\mathrm{D}\varphi(x,\xi)$ with respect to ξ and (84) imply that a subsequence of $\mathrm{D}\varphi(x, \operatorname{grad} u_n)$ converges almost pointwise to $\mathrm{D}\varphi(x, \operatorname{grad} u)$ in $\Omega \times (0,T)$. Note that the continuous character of $\mathrm{D}\varphi$ is appealed to, for the first time, at this point of the proof.

Finally the comparison result is obtained by passing to the weak limit in the comparison result for the approximating sequences.

REMARK 13. A careful examination of the convergence of the approximated energies (i.e., $b(u_n)$) would show that in Theorem 1 case (b), as well as in Theorem 3 and 4, $b(u)$ possesses the following regularity:

$$b(u) \in C^0([0,T];L^1(\Omega)).$$

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Appendix

The following lemma is a more restrictive version of a lemma due to Alt and Luckhaus (cf. [1], Lemma 1.5, p. 315).

LEMMA. *Let Ω be a bounded domain of $\mathbb{R}^N$ and $\tilde{\psi}(t)$ be a C^1 convex function on $\mathbb{R}$ with $\tilde{b}(t)$ as derivative ($\tilde{\psi}(0) = 0$). Let $\tilde{\psi}^*$ denote its convex conjugate. Assume that*

$$\begin{cases} u \in L^s(0,T; W_0^{1,s}(\Omega)), \ 1 < s < +\infty, \\ \tilde{b}(u) \in L^\infty(0,T; L^1(\Omega)), \\ \dfrac{\partial \tilde{b}(u)}{\partial t} \in L^{s'}(0,T; W^{-1,s'}(\Omega)), \quad \dfrac{1}{s} + \dfrac{1}{s'} = 1. \end{cases}$$

Assume further that there exists an element u_0 in $W_0^{1,s}(\Omega)$ such that

$$\tilde{b}(u_0) \in L^1_{\mathrm{loc}}(\Omega) \cap W^{-1,s'}(\Omega),$$

and

$$\tilde{b}(u)|_{t=0} = \tilde{b}(u_0).$$

Then

$$\psi^*(\tilde{b}(u)) \in L^\infty(0,T; L^1(\Omega)),$$

and, for almost any t in $(0,T)$,

$$\int_\Omega \psi^*(\tilde{b}(u(t)) \mathrm{d}x - \int_\Omega \psi^*(\tilde{b}(u_0)) \mathrm{d}x = \int_0^t \left\langle \frac{\partial \tilde{b}(u(s))}{\partial t}, u(s) \right\rangle_{W^{-1,s'}(\Omega); W_0^{1,s}(\Omega)} \mathrm{d}s.$$

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