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Homogeneous Cauchy-Riemann Structures

ANDREAS KRÜGER

Introduction

Assume we are given a real Lie group $\mathbf{G}$, a closed connected subgroup $\mathbf{H}$ of $\mathbf{G}$, their Lie algebras $\boldsymbol{g}$ and $\boldsymbol{h}$, and the corresponding homogeneous manifold $\pi : \mathbf{G} \to \mathbf{G}/\mathbf{H} = \mathbf{M}$. Then any $\mathbf{G}$-homogeneous almost complex structure on $\mathbf{M}$ is determined by the *classifying* vector subspace $\boldsymbol{q} = \pi_*^{-1}(\overline{H\mathbf{M}_{\pi(e)}})$ of $\boldsymbol{g}^{\mathbb{C}} = \boldsymbol{g} \otimes_{\mathbb{R}} \mathbb{C}$, where $H\mathbf{M} \subset T\mathbf{M} \otimes_{\mathbb{R}} \mathbb{C}$ is the bundle of holomorphic tangent vectors, i.e., the $+\mathbf{i}$-eigenspace of the almost complex structure. By a theorem of Frölicher (see [11], § 20), that structure is integrable, i.e., $\mathbf{M}$ is a complex manifold, iff $\boldsymbol{q}$ is a sub*algebra* of $\boldsymbol{g}^{\mathbb{C}}$. This theorem makes it possible to answer function theoretic questions using algebraic methods. We generalize it to homogeneous almost Cauchy-Riemann structures, and use this generalization to classify all such structures on spheres.

The organization of the present paper is as follows: In Chapter 1, we give some preliminaries. Chapter 2 then generalizes Frölicher's result to (almost) Cauchy-Riemann structures in the most straightforward manner: one only needs to change *complex* to *Cauchy-Riemann*. The proof given also shows how to compute the Levi form or algebra of a given $\mathbf{G}$-homogeneous almost Cauchy-Riemann structure in terms of the Lie algebra $\boldsymbol{g}$ of $\mathbf{G}$ (our Theorem 2.4), although such calculations are not carried out in this paper.

In Chapter 3, we show that $\mathbf{G}$-equivariant Cauchy-Riemann diffeomorphisms between $\mathbf{G}$-homogeneous almost Cauchy-Riemann manifolds correspond to automorphisms of $\boldsymbol{g}$ which map the classifying spaces onto each other. As it turns out, the assumption that the diffeomorphisms are $\mathbf{G}$-equivariant can be relaxed slightly.

Finally, in Chapter 4, we give a classification of all homogeneous almost CR structures on spheres, hypersurface or not. One implication of the work done here is a new proof of a theorem by Feldmueller and Lehmann [10], namely, that any (integrable) homogeneous CR hypersurface structure on $\mathbf{S}^{2k+1}$, $k \geq 2$, is CR diffeomorphic to the usual one which $\mathbf{S}^{2k+1}$ inherits from $\mathbb{C}^{k+1}$. Besides this, the only nontrivial CR structures on spheres are the usual complex structure

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of the Riemannian sphere, and an infinite family of structures on $\mathbf{S}^{4k+3}$ of type $(4k+3,1)$, for any $k \geq 0$. If $k=0$, this family has been known since the days of Ellie Cartan. Indeed, the general case can be constructed from this special one via the Hopf fibration $\mathbf{S}^3 \hookrightarrow \mathbf{S}^{4k+3} \to \mathbf{P}^k_{\mathbb{H}}$. We also obtain some non-integrable almost CR structures, namely, the usual almost complex structure on $\mathbf{S}^6$, a family of almost CR hypersurface structures on $\mathbf{S}^5$ and on $\mathbf{S}^{4k+3}$, $k \geq 1$, and finally, an exceptional structure of type $(4k+3, 2k)$ on $\mathbf{S}^{4k+3}$, $k \geq 1$.

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1. - Preliminaries

In this chapter, we sum up basic definitions and results used throughout this paper. A good introduction into this material can be found in [13]. The theorems given here have been extracted from [1]. Corollary (1.6) is an immediate consequence of [3], Lemma 1.1.

DEFINITION 1.1. (Almost CR manifolds and CR maps). An *almost* CR *structure* $(R\mathbf{M}, J)$ *of type* (n,l) on a real analytic manifold $\mathbf{M}$ of dimension n is a real analytic rank $2l$ subbundle $R\mathbf{M}$ of the tangent bundle $T\mathbf{M}$ of $\mathbf{M}$, together with a real analytic anti-involutive (i.e., $J^2 = -\mathrm{id}$) bundle automorphism $J : R\mathbf{M} \to R\mathbf{M}$. An *almost* CR *manifold of type* (n,l) is a real analytic manifold $\mathbf{M}$ of dimension n, endowed with an almost CR structure of type (n,l). Given two almost CR manifolds $(\mathbf{M}_1, (R\mathbf{M}_1, J_1))$ and $(\mathbf{M}_2, (R\mathbf{M}_2, J_2))$, a real analytic map $f : \mathbf{M}_1 \to \mathbf{M}_2$ is called a CR *map* iff its differential $f_* : T\mathbf{M}_1 \to T\mathbf{M}_2$ restricts to a bundle morphism $f_* : R\mathbf{M}_1 \to R\mathbf{M}_2$ such that $f_* \circ J_1 = J_2 \circ f_*$. Furthermore, f is called a CR *diffeomorphism* iff it is a diffeomorphism and a CR map and its inverse is a CR map as well. A covering map $\tilde{\mathbf{M}} \to \mathbf{M}$, where $\tilde{\mathbf{M}}$ and $\mathbf{M}$ are almost CR manifolds, is called a CR *covering* iff it restricts to local CR diffeomorphisms.

The *conjugate* of a given CR structure $(R\mathbf{M}, J)$ is $(R\mathbf{M}, -J)$. An almost CR manifold of type $(2n, n)$ is commonly called an *almost complex manifold.* Any complex manifold $\mathbf{M}$ is an example of this, $J : T\mathbf{M} \to T\mathbf{M}$ being identified with multiplication with $\mathbf{i}$. The other extreme is $R\mathbf{M} = \{0\}$, i.e., an almost CR manifold and structure of type $(n, 0)$, which are called *trivial.* An almost CR structure of type $(2n+1, n)$ is called an *almost* CR *hypersurface structure* and the corresponding almost CR manifold an *almost* CR *hypersurface.*

DEFINITION 1.2. (CR vector fields, homogeneous almost CR manifolds). A vector field $X \in \Gamma(\mathbf{M}, T\mathbf{M})$ is called a CR *vector field* iff the local one-parameter groups of transformations it induces consist of CR maps. The set of all such vector fields is denotes by $\Gamma_{\mathrm{CR}}(\mathbf{M}, T\mathbf{M})$.

Given the real Lie group $\mathbf{G}$, an almost CR manifold $\mathbf{M}$ and its structure are called $\mathbf{G}$-*homogeneous* iff $\mathbf{G}$ acts transitively on $\mathbf{M}$ from the left via CR maps. Defining the Lie algebra $\boldsymbol{g}$ of $\mathbf{G}$ in terms of right-invariant vector fields as usual, we get a natural homomorphism $\boldsymbol{g} \to \Gamma_{\mathrm{CR}}(\mathbf{M}, T\mathbf{M})$ of Lie algebras. Finally, an almost CR manifold or structure is called *homogeneous* iff it is $\mathbf{G}$-homogeneous for some real Lie group $\mathbf{G}$.

DEFINITION 1.3. (CR manifolds, integrability, complexification). An almost CR structure $(R\mathbf{M}, J)$ of type (n, l) on a real-analytic manifold $\mathbf{M}$ which is a closed submanifold of some complex manifold $\mathbf{X}$ via the embedding $\varphi : \mathbf{M} \hookrightarrow \mathbf{X}$ is called *inherited from* $\mathbf{X}$ iff φ becomes a CR map such that

$$R\mathbf{M}_m = \varphi_*^{-1}(\{v \in T\mathbf{X}_{\varphi(m)} : v \in \varphi_*(T\mathbf{M}_m) \quad \text{and} \quad \mathbf{i}v \in \varphi_*(T\mathbf{M}_m)\}), \qquad \text{for all } m \in \mathbf{M}.$$

An almost CR structure which is inherited from some complex manifold is called an *integrable* almost CR structure, or, alternatively, a CR *structure*, and the resulting almost CR manifold a CR *manifold*. An almost CR hypersurface with integrable structure is called a CR *hypersurface*. If a CR structure is inherited from a complex manifold $\mathbf{X}$ of complex dimension $n - l$, i.e., the smallest complex vector subspace of $T\mathbf{X}_{\varphi(m)}$ containing $\varphi_*(T\mathbf{M}_m)$ is all of $T\mathbf{X}_{\varphi(m)}$, for all $m \in \mathbf{M}$, then $\mathbf{X}$ is called a *complexification* of $\mathbf{M}$.

THEOREM 1.4. (Existence of complexifications). *Any* CR *manifold* $\mathbf{M}$ *admits a complexification* $\mathbf{M}^{\mathbb{C}}$. *The germ of* $\mathbf{M}^{\mathbb{C}}$ *near* $\mathbf{M}$ *is unique. Moreover, if* $\mathbf{N}$ *is another* CR *manifold and* $f : \mathbf{M} \to \mathbf{N}$ *a* CR *map, there exist complexifications* $\mathbf{M}^{\mathbb{C}}$ *and* $\mathbf{N}^{\mathbb{C}}$ *of* $\mathbf{M}$ *and* $\mathbf{N}$ *and a holomorphic map* $f^{\mathbb{C}} : \mathbf{M}^{\mathbb{C}} \to \mathbf{N}^{\mathbb{C}}$ *extending* f. *The germ of* $f^{\mathbb{C}}$ *near* $\mathbf{M}$ *is unique.*

REMARK 1.5. It should be noted that all manifolds and almost CR structures considered in this paper are assumed to be real analytic. The exceptions are vector fields, which in general are assumed to be smooth only. In particular, for a given subbundle $S\mathbf{M}$ of the tangent bundle $T\mathbf{M}$ of a manifold $\mathbf{M}$, we let $\Gamma(\mathbf{M}, S\mathbf{M})$ denote the Lie algebra of all C^∞-sections of $S\mathbf{M}$.

COROLLARY 1.6. *Given a* CR *manifold* $\mathbf{M}$ *and a finite-dimensional Lie algebra* $\boldsymbol{g} \subset \Gamma_{\mathrm{CR}}(\mathbf{M}, T\mathbf{M})$, *there exists a complexification* $\mathbf{M}^{\mathbb{C}}$ *of* $\mathbf{M}$ *such that, for any* $X \in \boldsymbol{g}$, *there is exactly one element of* $\Gamma_{\mathrm{CR}}(\mathbf{M}^{\mathbb{C}}, T\mathbf{M}^{\mathbb{C}})$ *extending* X.

THEOREM 1.7. *Consider an almost* CR *manifold* $(\mathbf{M}, (R\mathbf{M}, J))$. *Let* $H\mathbf{M}$ *denote the* $\mathbf{i}$-*eigenspace of* $J^{\mathbb{C}} : R\mathbf{M} \otimes_{\mathbb{R}} \mathbb{C} \to R\mathbf{M} \otimes_{\mathbb{R}} \mathbb{C}$, *which is a subbundle* *of* $T\mathbf{M} \otimes_{\mathbb{R}} \mathbb{C}$. *Then* $\mathbf{M}$ *is integrable iff* $H\mathbf{M}$ *is* involutive, *i.e., iff* $\Gamma(\mathbf{M}, H\mathbf{M})$ *is a Lie subalgebra of* $\Gamma(\mathbf{M}, T\mathbf{M} \otimes_{\mathbb{R}} \mathbb{C})$.

2. - The Classifying Algebra

THEOREM AND DEFINITION 2.1. (Classifying space and algebra). *Consider the homogeneous space* $\pi : \mathbf{G} \to \mathbf{G}/\mathbf{H} = \mathbf{M}$, *where* $\mathbf{H}$ *is a closed subgroup of the real Lie group* $\mathbf{G}$. *Let* $\boldsymbol{g}$ *and* $\boldsymbol{h}$ *denote the Lie algebras of* $\mathbf{G}$ *and* $\mathbf{H}$, *and* $\boldsymbol{g}^{\mathbb{C}} = \boldsymbol{g} \otimes_{\mathbb{R}} \mathbb{C}$ *and* $\boldsymbol{h}^{\mathbb{C}} = \boldsymbol{h} \otimes_{\mathbb{R}} \mathbb{C}$ *their complexifications. The derivative of* π *at* e *yields a map* $\pi_* : \boldsymbol{g}^{\mathbb{C}} \to \boldsymbol{g}^{\mathbb{C}}/\boldsymbol{h}^{\mathbb{C}} = T\mathbf{M}_p \otimes_{\mathbb{R}} \mathbb{C}$, *where* $p = \pi(e)$ *is the basepoint of* $\mathbf{M}$.

Given a $\mathbf{G}$*-homogeneous almost* CR *structure* $(R\mathbf{M}, J)$ *on* $\mathbf{M}$, *we consider the subbundle*

$$H\mathbf{M} = \{v \in R\mathbf{M} \otimes_{\mathbb{R}} \mathbb{C} : J(v) = \mathbf{i}v\}$$

of $T\mathbf{M} \otimes_{\mathbb{R}} \mathbb{C}$. *Then the almost* CR *structure is uniquely defined by* $\boldsymbol{q} = \pi_*^{-1}(\overline{H\mathbf{M}_p}) \subset \boldsymbol{g}^{\mathbb{C}}$, *which satisfies the conditions* $\boldsymbol{q} \cap \overline{\boldsymbol{q}} = \boldsymbol{h}^{\mathbb{C}}$ *and* $\mathrm{Ad}(\mathbf{H})(\boldsymbol{q}) \subset \boldsymbol{q}$.

Conversely, any complex vector subspace $\boldsymbol{q}$ *of* $\boldsymbol{g}^{\mathbb{C}}$ *which satisfies those two conditions can be derived from some* $\mathbf{G}$*-homogeneous almost* CR *structure on* $\mathbf{M}$ *in this fashion. We thus say that* $\boldsymbol{q}$ classifies *that almost* CR *structure or manifold.*

If $\boldsymbol{q}$ *classifies a given structure, then* $\overline{\boldsymbol{q}}$ *classifies the conjugate structure. The type of the structure classified by a given* $\boldsymbol{q}$ *is* $(\dim_{\mathbb{R}} \boldsymbol{g} - \dim_{\mathbb{R}} \boldsymbol{h}, \dim_{\mathbb{C}} \boldsymbol{q} - \dim_{\mathbb{C}} \boldsymbol{h}^{\mathbb{C}})$. *In particular, the trivial structure is classified by* $\boldsymbol{q} = \boldsymbol{h}^{\mathbb{C}}$, *whereas a structure is almost complex iff* $\boldsymbol{q} + \overline{\boldsymbol{q}} = \boldsymbol{g}^{\mathbb{C}}$.

Furthermore, a given almost CR *structure is integrable iff it is classified by a subalgebra of* $\boldsymbol{g}^{\mathbb{C}}$, *which is then called the* classifying algebra *of that* CR *structure or manifold.*

Finally, assume that $\mathbf{M} = \mathbf{G}/\mathbf{H}$, *classified by* $\boldsymbol{q}$, *and* $\mathbf{N} = \mathbf{G}/\mathbf{L}$, *classified by* $\boldsymbol{r}$, *are two* $\mathbf{G}$*-homogeneous almost* CR *manifolds. Let* $\boldsymbol{h}$ *and* $\boldsymbol{l}$ *denote the Lie algebras of* $\mathbf{H}$ *and* $\mathbf{L}$. *Then a* $\mathbf{G}$*-equivariant map* $f : \mathbf{M} \to \mathbf{N}$ *is a* CR *map iff* $f_*(\boldsymbol{q}/\boldsymbol{h}^{\mathbb{C}}) \subset \boldsymbol{r}/\boldsymbol{l}^{\mathbb{C}}$, *where* $f_* : \boldsymbol{g}^{\mathbb{C}}/\boldsymbol{h}^{\mathbb{C}} \to \boldsymbol{g}^{\mathbb{C}}/\boldsymbol{l}^{\mathbb{C}}$ *denotes the derivative of* f *at the basepoint of* $\mathbf{M}$.

PROOF. This theorem is a consequence of (1.7) and the theorem on the invariant fundamental bilinear form, which is to follow.

The following two lemmas are well known (compare [13]) and given only for reference.

LEMMA AND DEFINITION 2.2. (The fundamental bilinear form). *Given two subbundles* $E\mathbf{M}$ *and* $F\mathbf{M}$ *of the tangent bundle* $T\mathbf{M}$ *of the* C^∞*-manifold* $\mathbf{M}$, *let* $\varphi : T\mathbf{M} \to T\mathbf{M}/(E\mathbf{M} + F\mathbf{M})$ *denote the projection. Then there is a natural bilinear bundle map* $B : E\mathbf{M} \times F\mathbf{M} \to T\mathbf{M}/(E\mathbf{M} + F\mathbf{M})$ *such that, for all*

$X \in \Gamma(\mathbf{M}, E\mathbf{M})$ and $Y \in \Gamma(\mathbf{M}, F\mathbf{M})$, we have

$$B(X(m), Y(m)) = \varphi([X, Y](m)), \qquad \textit{for all } m \in \mathbf{M}.$$

In particular, if $E\mathbf{M} = F\mathbf{M}$, then $B = 0$ iff $E\mathbf{M}$ is involutive.

PROOF. Using the above equation as a definition, we only need to show that is does not depend on the particular choice of X and Y, given $X(m)$ and $Y(m)$. This can be concluded from the following lemma:

LEMMA 2.3. *Let $F\mathbf{M}$ be a subbundle of the tangent bundle $T\mathbf{M}$ of the C^∞-manifold $\mathbf{M}$. If $Y \in \Gamma(\mathbf{M}, F\mathbf{M})$ and $m \in \mathbf{M}$ are such that $Y(m) = 0$, then, for any $X \in \Gamma(\mathbf{M}, T\mathbf{M})$, we have $[X, Y](m) \in F\mathbf{M}_m$.*

PROOF. Choose $Y_1, \ldots, Y_r \in \Gamma(\mathbf{M}, F\mathbf{M})$ such that $(Y_1(m), \ldots, Y_r(m))$ is a basis of $F\mathbf{M}_m$. Then there are C^∞-functions $a_1, \ldots, a_r$ on $\mathbf{M}$ vanishing at m such that, near m, we have $Y = \sum_{i=1}^{r} a_i Y_i$. We get

$$[X, Y] = \sum_{i=1}^{r} (X(a_i) Y_i + a_i [X, Y_i]),$$

which evaluates at m to yield the desired result.

REMARK 2.4. Let $\mathbf{H}$ be a closed subgroup of a given real Lie group $\mathbf{G}$, $\boldsymbol{h}$ and $\boldsymbol{g}$ their Lie algebras, and $\pi : \mathbf{G} \to \mathbf{G}/\mathbf{H} = \mathbf{M}$ the corresponding homogeneous space with basepoint $p = \pi(e) \in \mathbf{M}$. The derivative π_* of π yields a map $\boldsymbol{g} \to T\mathbf{M}_p$ which will be denoted by π_* also. Then any $\mathbf{G}$-invariant subbundle $V\mathbf{M}$ of $T\mathbf{M}$ corresponds to a vector subspace $\boldsymbol{v} = \pi_*^{-1}(V\mathbf{M}_p)$ of $\boldsymbol{g}$, satisfying $\boldsymbol{h} \subset \boldsymbol{v} = \mathrm{Ad}(\mathbf{H})(\boldsymbol{v})$, which determines $V\mathbf{M}$ uniquely. Conversely, any such space can be obtained from some invariant bundle.

THEOREM 2.5. (Invariant fundamental form). *Given the assumptions as in the preceding remark, let $V\mathbf{M}$ and $W\mathbf{M}$ be two $\mathbf{G}$-invariant subbundles of $T\mathbf{M}$ corresponding to $\boldsymbol{v}, \boldsymbol{w} \subset \boldsymbol{g}$, and let $I\mathbf{M}$ be the $\mathbf{G}$-invariant subbundle of $T\mathbf{M}$ corresponding to $[\boldsymbol{v}, \boldsymbol{w}] + \boldsymbol{v} + \boldsymbol{w}$. Then the image of the bilinear map $B : V\mathbf{M} \times W\mathbf{M} \to T\mathbf{M}/(V\mathbf{M} + W\mathbf{M})$ as given in (2.2) is $\varphi(I\mathbf{M})$, where $\varphi : T\mathbf{M} \to T\mathbf{M}/(V\mathbf{M} + W\mathbf{M})$ denotes the natural projection, and B satisfies the fundamental equation*

$$B(\pi_*(v), \pi_*(w)) = -\varphi(\pi_*([v, w])), \qquad \textit{for all } v \in \boldsymbol{v},\ w \in \boldsymbol{w}.$$

PROOF. As B and hence its image are $\mathbf{G}$-invariant, we only need to prove the fundamental equation. To do so, use the fact that $\mathrm{Ad}(\mathbf{H})$ stabilizes both $\boldsymbol{v}$ and $\boldsymbol{w}$, from which we conclude $[\boldsymbol{h}, \boldsymbol{v}] \subset \boldsymbol{v}$ and $[\boldsymbol{h}, \boldsymbol{w}] \subset \boldsymbol{w}$. One consequence of this is $\varphi \circ \pi_*[v, w] = \varphi \circ \pi_*[v', w']$, for all v', $w' \in \boldsymbol{g}$ such that $\pi_*(v - v') = \pi_*(w - w') = 0$. Secondly, let $V\mathbf{G}$, $W\mathbf{G}$, and $H\mathbf{G}$ denote the left-invariant subbundles of $T\mathbf{G}$

such that $VG_e = v$, $WG_e = w$, and $HG_e = h$. Then $\Gamma(\mathbf{G}, H\mathbf{G})$ normalizes both $\Gamma(\mathbf{G}, V\mathbf{G})$ and $\Gamma(\mathbf{G}, W\mathbf{G})$. It should be noted that $H\mathbf{G}$ is exactly the kernel of π_*.

Consider left-invariant vector fields X_v, X_w on $\mathbf{G}$ such that $X_v(e) = v$, $X_w(e) = w$. In this paper, we use the definition of the Lie algebra structure on $\boldsymbol{g} = T\mathbf{G}_e$ in terms of right-invariant vector fields, so $[X_v, X_w](e) = -[v, w]$.

Now choose a neighbourhood U of p in $\mathbf{M}$ such that there exists a C^∞-map $\sigma : U \to \mathbf{G}$ such that $\sigma(p) = e$ and $\pi \circ \sigma = \mathrm{id}_U$, and consider $\tilde{Y}_v, \tilde{Y}_w \in \Gamma(\sigma(U), T\mathbf{G})$ given by

$$\tilde{Y}_v(\sigma(u)) = \sigma_* \circ \pi_*\big(X_v(\sigma(u))\big) \quad \text{and} \quad \tilde{Y}_w(\sigma(u)) = \sigma_* \circ \pi_*\big(X_w(\sigma(u))\big), \qquad \text{for all } u \in U.$$

By making U smaller if necessary, we may assume that there exist $Y_v \in \Gamma(\mathbf{G}, V\mathbf{G})$, $Y_w \in \Gamma(\mathbf{G}, W\mathbf{G})$ extending $\tilde{Y}_v$, $\tilde{Y}_w$ which satisfy $X_v - Y_v$, $X_w - Y_w \in \Gamma(\mathbf{G}, H\mathbf{G})$. By what has been said earlier, this implies $\pi_*[Y_v, Y_w] = = \pi_*[X_v, X_w]$.

By construction, there exist $Z_v \in \Gamma(\mathbf{M}, V\mathbf{M})$, $Z_w \in \Gamma(\mathbf{M}, W\mathbf{M})$ which are related to Y_v, Y_w via σ. Clearly, $Z_v(p) = \pi_*(v)$ and $Z_w(p) = \pi_*(w)$. So we get

$$\begin{aligned} B(\pi_*(v), \pi_*(w)) &= \varphi\big([Z_v, Z_w](p)\big) = \varphi \circ \pi_* \circ \sigma_*\big([Z_v, Z_w](p)\big) \\ &= \varphi \circ \pi_*\big([Y_v, Y_w](e)\big) = -\varphi \circ \pi_*[v, w]. \end{aligned}$$

DEFINITION 2.6. (Complexification of Lie groups). A *complexification* of a given real Lie group $\mathbf{G}$ is a complex Lie group $\mathbf{G}^{\mathbb{C}}$ which is a complexification of $\mathbf{G}$, viewed as a trivial CR manifold, such that the injection $\mathbf{G} \hookrightarrow \mathbf{G}^{\mathbb{C}}$ becomes a homomorphism of (real) Lie groups. Its derivative can then be used to identify the Lie algebra of $\mathbf{G}^{\mathbb{C}}$ with the complexification $\boldsymbol{g}^{\mathbb{C}}$ of the Lie algebra $\boldsymbol{g}$ of $\mathbf{G}$.

Given a $\mathbf{G}$-homogeneous CR manifold $\mathbf{M}$, we can ask whether there exists a complexification $\mathbf{M}^{\mathbb{C}}$ of $\mathbf{M}$ such that the $\mathbf{G}$-action on $\mathbf{M}$ can be extended to a $\mathbf{G}^{\mathbb{C}}$-action on $\mathbf{M}^{\mathbb{C}}$. If this is the case, we call $\mathbf{M}^{\mathbb{C}}$ a $\mathbf{G}^{\mathbb{C}}$*-complexification* of $\mathbf{M}$.

COROLLARY 2.7. (Classifying is isotropy algebra). *If $\mathbf{G}^{\mathbb{C}}$ is a complexification of the real Lie group $\mathbf{G}$, and $\mathbf{M}$ a $\mathbf{G}$-homogeneous CR manifold classified by $\boldsymbol{q}$ which admits the $\mathbf{G}^{\mathbb{C}}$-complexification $\mathbf{M}^{\mathbb{C}}$, then the Lie algebra of the isotropy group of the basepoint $p \in \mathbf{M} \subset \mathbf{M}^{\mathbb{C}}$ in $\mathbf{G}^{\mathbb{C}}$ is $\boldsymbol{q}$.*

PROOF. By (1.5), there exists a complexification $\mathbf{M}^{\mathbb{C}}$ of $\mathbf{M}$ such that, for any $x \in \boldsymbol{g}$, the corresponding CR vectorfield on $\mathbf{M}$ can be extended to a holomorphic vector field on $\mathbf{M}^{\mathbb{C}}$. Employing our usual notation, we get the commutative diagram

$$\begin{tikzcd} \boldsymbol{g} \arrow[r] \arrow[d] & \Gamma_{CR}(\mathbf{M}, T\mathbf{M}) \arrow[r, "\mathrm{eval}"] & T\mathbf{M}_p \arrow[d] \\ \boldsymbol{g}^{\mathbb{C}} \arrow[r] & \Gamma_{CR}(\mathbf{M}^{\mathbb{C}}, T(\mathbf{M}^{\mathbb{C}})) \arrow[r, "\mathrm{eval}"] & T(\mathbf{M}^{\mathbb{C}})_p . \end{tikzcd}$$

The composite map $\boldsymbol{g} \to T\mathbf{M}_p$ given by the first row is π_*, its kernel $\boldsymbol{h}$. The kernel $\boldsymbol{n}$ of the map $\boldsymbol{g}^{\mathbb{C}} \to T(\mathbf{M}^{\mathbb{C}})_p$ given by the second row is clearly a Lie algebra as well. Complexifying the first row, we get the commutative diagram

$$\begin{tikzcd} & & \{0\} \arrow[d] \\ & & \overline{H\mathbf{M}_p} \arrow[d] \\ \boldsymbol{g}^{\mathbb{C}} \arrow[r] \arrow[d, "="] & \Gamma_{CR}(\mathbf{M}, T\mathbf{M}) \otimes_{\mathbb{R}} \mathbb{C} \arrow[r] & T\mathbf{M}_p \otimes_{\mathbb{R}} \mathbb{C} \arrow[d] \\ \boldsymbol{g}^{\mathbb{C}} \arrow[r] & \Gamma_{CR}(\mathbf{M}^{\mathbb{C}}, T(\mathbf{M}^{\mathbb{C}})) \arrow[r] & T(\mathbf{M}^{\mathbb{C}})_p, \arrow[d] \\ & & \{0\} \end{tikzcd}$$

the column being exact. This shows $\boldsymbol{q} = \boldsymbol{n}$.

THEOREM 2.8. ($\mathbf{G}^{\mathbb{C}}$-complexifications up to CR coverings). *Given a real Lie group* $\mathbf{G}$ *with Lie algebra* $\boldsymbol{g}$, *a closed and connected subgroup* $\mathbf{H}$ *of* $\mathbf{G}$ *with Lie algebra* $\boldsymbol{h}$, *a* $\mathbf{G}$*-homogeneous* CR *structure on* $\mathbf{M} = \mathbf{G}/\mathbf{H}$ *which is classified by* $\boldsymbol{q} \subset \boldsymbol{g}^{\mathbb{C}}$, *and a complexification* $\mathbf{G}^{\mathbb{C}}$ *of* $\mathbf{G}$, *let* $\mathbf{Q}$ *denote the connected subgroup of* $\mathbf{G}$ *with Lie algebra* $\boldsymbol{q}$.

There is a $\mathbf{G}$*-invariant CR covering* $\mathbf{M} \to \mathbf{M}_1$ *such that the* CR *manifold* $\mathbf{M}_1$ *admits a* $\mathbf{G}^{\mathbb{C}}$*-complexification* $\mathbf{M}_1^{\mathbb{C}}$ *iff* $\mathbf{Q}$ *is topologically closed in* $\mathbf{G}^{\mathbb{C}}$. *If this is the case,* $\mathbf{H}$ *is clearly the connected component of* $\mathbf{G} \cap \mathbf{Q}$ *at* e, *and we get the commutative diagram of* $\mathbf{G}$*-equivariant* CR *maps*

$$\begin{tikzcd} \mathbf{M} \arrow[r] \arrow[d, "="] & \mathbf{G}/(\mathbf{G} \cap \mathbf{Q}) \arrow[r, hook] \arrow[d] & \mathbf{G}^{\mathbb{C}}/\mathbf{Q} \arrow[d] \\ \mathbf{M} \arrow[r] & \mathbf{M}_1 \arrow[r, hook] & \mathbf{M}_1^{\mathbb{C}} . \end{tikzcd}$$

Here $\mathbf{G}/(\mathbf{G}\cap\mathbf{Q}) \hookrightarrow \mathbf{G}^{\mathbb{C}}/\mathbf{Q}$ *and* $\mathbf{M}_1 \hookrightarrow \mathbf{M}_1^{\mathbb{C}}$ *are complexification maps, all others* CR *coverings.*

3. - Equivalence

Traditionally, two $\mathbf{G}$-homogeneous CR manifolds have often been considered the same iff they are CR diffeomorphic. In the chapter following this one, we will show that no two of the CR structures on spheres, examined there, are the same in this sense, although we cannot answer the corresponding question for the nonintegrable almost CR structures (see (4.3)). However, it is more consistent with the point of view of this paper also to take into consideration the structure of $\mathbf{M}$ as a $\mathbf{G}$-homogeneous space. In this sense, two CR diffeomorphic $\mathbf{G}$-homogeneous CR manifolds are not the same unless the CR diffeomorphism preserves the group of CR automorphisms given by $\mathbf{G}$, either elementwise or as a whole. So we get two different notions of *equivalence*: A *strong equivalence* is a CR diffeomorphism which is $\mathbf{G}$-equivariant. If we can make a CR diffeomorphism $\mathbf{G}$-equivariant by changing the $\mathbf{G}$-action on either one of the two manifolds using an automorphism of $\mathbf{G}$, we call it a *weak equivalence*. This latter notion corresponds to the one used by Sasaki in his papers [30-31]. Our classification of the nonintegrable structures will be up to these equivalences.

DEFINITION 3.1. (Strong and weak equivalence). Given a fixed real Lie group $\mathbf{G}$ and two $\mathbf{G}$-homogeneous CR manifolds $\mathbf{M}$ and $\mathbf{N}$, a pair (f,ρ) is called a *weak equivalence* iff $f:\mathbf{M}\to\mathbf{N}$ is a CR diffeomorphism and $\rho:\mathbf{G}\to\mathbf{G}$ an automorphism of $\mathbf{G}$ such that $f(gm)=\rho(g)f(m)$, for all $g\in\mathbf{G}$ and $m\in\mathbf{M}$. A map $f:\mathbf{M}\to\mathbf{N}$ is called a *strong equivalence* iff $(f,\mathrm{id}_{\mathbf{G}})$ is a weak equivalence.

We will frequently consider the special case where the underlying real-analytic manifolds and the $\mathbf{G}$-actions are identical, i.e., $\mathbf{M}$ and $\mathbf{N}$ differ by their CR structures only. We then call these CR structures *weakly* (*strongly*) *equivalent* iff a weak (strong) equivalence exists.

EXAMPLE 3.2. Given any $n\geq 0$, the $\mathbf{SU}(n+1)$-homogeneous CR hypersurface structure which $\mathbf{S}^{2n+1}$ inherits from $\mathbb{C}^{n+1}$ is weakly equivalent to its conjugate, $(v\to\overline{v}, A\to\overline{A})$ being an equivalence. As we shall see in (4.12), these two structures are not strongly equivalent.

REMARK 3.3. (Functorial property of weak equivalence). The map $g_0\to(m\to g_0m,\ g\to g_0gg_0^{-1})$ defines a group homomorphism from $\mathbf{G}$ to the group of weak equivalences of a $\mathbf{G}$-homogeneous almost CR manifold $\mathbf{M}$. A straightforward computation shows that the image of this homomorphism is normal.

THEOREM 3.4. (Equivalence on Lie algebra level). *Let* $\mathbf{G}$ *be a real Lie group,* $\mathbf{H} \subset \mathbf{G}$ *a closed subgroup,* $\boldsymbol{g}$ *and* $\boldsymbol{h}$ *their Lie algebras, and* $\pi : \mathbf{G} \to \mathbf{G}/\mathbf{H} = \mathbf{M}$ *the corresponding homogeneous space. Then two* $\mathbf{G}$*-homogeneous almost* CR *structures on* $\mathbf{M}$ *classified by* $\boldsymbol{q}_1, \boldsymbol{q}_2 \subset \boldsymbol{g}^{\mathbb{C}}$ *are weakly equivalent iff there exists an automorphism* φ *of* $\mathbf{G}$ *stabilizing* $\mathbf{H}$ *such that* $\varphi_*(\boldsymbol{q}_1) = \boldsymbol{q}_2$. *They are strongly equivalent iff* φ *can be chosen to be inner. In particular, if all automorphisms of* $\mathbf{G}$ *are inner, two* $\mathbf{G}$*-homogeneous* CR *structures are strongly equivalent iff they are weakly equivalent.*

PROOF. Let $\mathbf{M}_1$ (respectively $\mathbf{M}_2$) denote the CR manifold classified by $\boldsymbol{q}_1$ (respectively $\boldsymbol{q}_2$). Given an equivalence (f, ρ) between them, choose $g_0 \in \mathbf{G}$ such that $g_0 f(p) = p$, where $p = \pi(e)$ is the basepoint. Define $\varphi(g) = g_0\rho(g)g_0^{-1}$, for all $g \in \mathbf{G}$. We get $\pi(\varphi(g)) = g_0 f(gp)$, from which we deduce $\varphi(\mathbf{H}) = \mathbf{H}$. Now the map $\omega : m \to g_0 f(m)$ is a CR map $\mathbf{M}_1 \to \mathbf{M}_2$, so its derivative satisfies $\omega_*(\boldsymbol{q}_1/\boldsymbol{h}^{\mathbb{C}}) = \boldsymbol{q}_2/\boldsymbol{h}^{\mathbb{C}}$. Combined with $\pi \circ \varphi = \omega \circ \pi$, this yields $\varphi_*(\boldsymbol{q}_1) = \boldsymbol{q}_2$.

Conversely, assume φ is given. To deal with both cases at once, let $g_0 \in \mathbf{G}$ be e if φ is outer and such that $g_0\varphi(g)g_0^{-1} = g$, for all $g \in \mathbf{G}$, if φ is inner. Define an equivalence (f, ρ) by

$$f(gp) = g_0\varphi(g)p \quad \text{and} \quad \rho(g) = g_0\varphi(g)g_0^{-1}, \qquad \text{for all } g \in \mathbf{G}.$$

We can use Theorem 2.1 to show that $f : \mathbf{M}_1 \to \mathbf{M}_2$ is a CR diffeomorphism by considering the $\mathbf{G}$-action $*$ on $\mathbf{M}_2$ given by $g{*}m = \rho(g)m$, for all $g \in \mathbf{G}$, $m \in \mathbf{M}_2$.

4. - Homogeneous almost CR Structures on Spheres

We first give a summary of the results of our classification of all homogeneous almost CR structures on spheres. At this point, no information about the groups and actions has been included. Such information can be found later in this chapter.

THEOREM 4.1. *Given a nontrivial homogeneous almost* CR *manifold* $\mathbf{M}$ *which is, as an analytic manifold, a sphere* $\mathbf{S}^n$, *there exists a* CR *diffeomorphism between* $\mathbf{M}$ *and one of the following*:

(a) *The well-known "classical" almost* CR *spheres, namely*:

  (i) *the Riemannian Sphere,*

  (ii) *the usual almost complex manifold* $\mathbf{S}^6$, *and*

  (iii) *the hypersurface* $\mathbf{S}^{2k+1}$ *in* $\mathbb{C}^{k+1}$, $k \geq 1$.

(b) *A family of almost* CR *hypersurface structures on* $\mathbf{S}^5$ *parametrized by the closed interval* $[0, 1]$. *The only integrable one among these is the structure which* $\mathbf{S}^5$ *inherits from* $\mathbb{C}^3$, *which corresponds to the parameter value* 1.

(c) *A family of* CR *hypersurface structures on* $\mathbf{S}^3$, *which can be parametrized by* $(0, 1]$. *The parameter value* 1 *corresponds to the structure which* $\mathbf{S}^3$ *inherits from* $\mathbb{C}^2$. *All other values correspond to structures which have been pulled up from generic* $\mathbf{SO}(3)$*-orbits in the complex quadric* $\{(z_1, z_2, z_3) \in \mathbb{C}^3 : z_1^2 + z_2^2 + z_3^2 = 1\}$ *(from whence they inherit their* CR *hypersurface structures).*

(d) *Several families of almost* CR *structures on* $\mathbf{S}^{4k+3}$, *for* $k \geq 1$, *which are closely related to the Hopf fibration* $\mathbf{S}^3 \hookrightarrow \mathbf{S}^{4k+3} \to \mathbf{P}^k_{\mathbb{H}}$. *The general philosophy is that we can endow the fibre* $\mathbf{S}^3$ *of a choosen basepoint* $p \in \mathbf{S}^{4k+3}$ *with any homogeneous* CR *structure, and* $\mathbf{P}^k_{\mathbb{H}}$ *with either the trivial or the usual almost complex structure. We pull back this structure on* $\mathbf{P}^k_{\mathbb{H}}$ *to the orthogonal complement of* $T\mathbf{S}^3_p$ *in* $T\mathbf{S}^{4k+3}_p$ *and extend homogeneously. With respect to the homogeneous almost* CR *structure on* $\mathbf{S}^{4k+3}$ *constructed in this fashion, both* $\mathbf{S}^3 \hookrightarrow \mathbf{S}^{4k+3}$ *and* $\mathbf{S}^{4k+3} \to \mathbf{P}^k_{\mathbb{H}}$ *become* CR *maps. More precisely, we can choose*

   (i) *the trivial structure on* $\mathbf{P}^k_{\mathbb{H}}$ *and any of the* CR *hypersurface structures given above for* $\mathbf{S}^3$, *which yields a family of* CR *structures of type* $(4k + 3, 1)$ *parametrized by* $(0, 1]$,

   (ii) *the almost complex structure on* $\mathbf{P}^k_{\mathbb{H}}$ *and the trivial one on* $\mathbf{S}^3$, *which yields a single non-integrable almost* CR *structure on* $\mathbf{S}^{4k+3}$ *of type* $(4k + 3, 2k)$,

   (iii) *the almost complex structure on* $\mathbf{P}^k_{\mathbb{H}}$ *and any homogeneous* CR *hypersurface structure on* $\mathbf{S}^3$. *However, there is a complication in this case: an equivalence of two* CR *structures on* $\mathbf{S}^3$ *can, in general, not be extended to* $\mathbf{S}^{4k+3}$ *without changing the structure of* $\mathbf{P}^k_{\mathbb{H}}$. *So, instead of* $(0, 1]$, *we get a certain parameter space of three dimensions which is described more precisely later. However, it turns out that the only integrable, among these almost* CR *hypersurface structures, is the one* $\mathbf{S}^{4k+3}$ *inherits from the complex vector space* $\mathbb{H}^{k+1}$.

The following consequence of this classification had been proved in [10], using a different approach.

COROLLARY 4.2. *Up to* CR *diffeomorphism, the only homogeneous* CR *hypersurface structure, which* $\mathbf{S}^{2k+1}$, *for* $k \geq 2$, *admits, is the one it inherits from* $\mathbb{C}^{k+1}$.

OPEN QUESTION 4.3. Theorem (4.1) leaves open the possibility that the given parameter spaces are larger than necessary. Indeed, the more technical classification which it summarizes is up to weak equivalence (in the sense of the previous chapter). It is conceivable that two structures are not weakly equivalent, but do admit a CR diffeomorphism between them.

However, in the case of $\mathbf{S}^3$ it is well known that two homogeneous CR hypersurface structures corresponding to different parameter values do not admit CR diffeomorphisms between them, see [9]. As it will be shown later in this chapter, this can be generalized easily to cover the homogeneous CR structures of type $(4k+3,1)$ on $\mathbf{S}^{4k+3}$.

This implies that the families of integrable CR structures given above are not unneccessarily large. However, the corresponding question for the families of non-integrable almost CR structures on $\mathbf{S}^5$ and $\mathbf{S}^{4k+3}$ remains open.

After this summary of our results, we now give a more detailed version and its proof. Both rest on the following theorem, which summarizes a classification of Lie groups acting transitively on spheres which has been carried out in the 40s and 50s by Borel, Montgomery, Samelson, and Poncet, see [6], [7], [23], [24], [28].

THEOREM 4.4. (Homogeneous spheres). *Choose $n \geq 2$, let $\mathbf{G}$ be a real Lie group which acts transitively and effectively on the n-sphere $\mathbf{S}^n$. Then there exists a connected, compact and simple subgroup $\mathbf{K}$ of $\mathbf{G}$ which still acts transitively on $\mathbf{S}^n$. More precisely, the following is a complete list of the cases which can occur for $\mathbf{S}^n = \mathbf{K}/\mathbf{L}$:*

(A) $n \neq 3$, $\mathbf{S}^n = \mathbf{SO}(n+1)/\mathbf{SO}(n)$;

(B) $n = 2k+1$, $\mathbf{S}^{2k+1} = \mathbf{SU}(k+1)/\mathbf{SU}(k)$;

(C) $n = 4k+3$, $\mathbf{S}^{4k+3} = \mathbf{Sp}(k+1)/\mathbf{Sp}(k)$;

(D) $\mathbf{S}^6 = \mathbf{G}_2/\mathbf{SU}(3)$;

(E) $\mathbf{S}^7 = \mathbf{Spin}(7)/\mathbf{G}_2$;

(F) $\mathbf{S}^{15} = \mathbf{Spin}(9)/\mathbf{Spin}(7)$.

Here, a Lie group is called *simple* iff all normal proper subgroups are discrete. It turns out that $\mathbf{SO}(4) = (\mathbf{SU}(2) \times \mathbf{SU}(2))/\{\pm 1\}$ is not simple, which is why $n = 3$ has to be excluded in (A).

PROOF. That we can restrict our attention to compact groups can be derived from [23]. A concise account of this classification can be found in either [15] or [26]. For some information on (B)–(F), see [12], on (D), also [14].

THEOREM 4.5. (Classification). *For each of the six cases* (A)–(F) *of the preceeding theorem, the following presents a list of all strong equivalence classes of nontrivial homogeneous almost* CR *structures which does not contain the same strong equivalence class twice.*

(A) $\mathbf{S}^n = \mathbf{SO}(n+1)/\mathbf{SO}(n)$:

$(n = 2)$: *The Riemannian sphere.*

$(n \geq 3)$: *No nontrivial $\mathbf{SO}(n+1)$-homogeneous almost* CR *structure on $\mathbf{S}^n$ exists.*

(B) $\mathbf{S}^{2k+1} = \mathbf{SU}(k+1)/\mathbf{SU}(k)$:

$(k = 1)$: *This is the same as $\mathbf{S}^3 = \mathbf{Sp}(1)/\mathbf{Sp}(0)$ and will be dealt with under* (C).

($k = 2$): *All nontrivial* $\mathbf{SU}(3)$*-homogeneous almost* CR *structure on* $\mathbf{S}^5$ *are hypersurface structures. The strong equivalence classes can be parametrized by the closed interval* $[-1, 1]$. *The endpoints* ± 1 *correspond to the only integrable structures, namely the one inherited from* $\mathbb{C}^3$ *and its conjugate. Two structures corresponding to distinct parameter values* s *and* t *are weakly equivalent iff* $s = -t$, *which is the case iff they are conjugate.*

($k \geq 3$): *The hypersurface structure which* $\mathbf{S}^{2k+1}$ *inherits from* $\mathbb{C}^{k+1}$ *and its conjugate, which are weakly equivalent.*

(C) $\mathbf{S}^{4k+3} = \mathbf{Sp}(k+1)/\mathbf{Sp}(k)$:

($k = 0$): *A family of* CR *hypersurface structures parametrized by the half-open interval* $(0, 1]$. *Any two structures which are weakly equivalent are strongly equivalent. Each structure is strongly equivalent to its conjugate. The parameter value* 1 *corresponds to the structure which* $\mathbf{S}^3$ *inherits from* $\mathbb{C}^2$. *Any other parameter value corresponds to a* CR *hypersurface structure which has been pulled up to* $\mathbf{S}^3$ *from a generic orbit (topologically* $\mathbf{P}^3_{\mathbb{R}}$*) of* $\mathbf{Sp}(1)/\mathbb{Z}_2 = \mathbf{SO}(3)$ *in the complex quadric* $\{(z_1, z_2, z_3) \in \mathbb{C}^3 : z_1^2 + z_2^2 + z_3^2 = 1\}$.

($k \geq 1$): *Any two structures which are weakly equivalent are strongly equivalent.*

(i) *An infinite family of equivalence classes of* CR *structures of type* $(4k+3, 1)$ *parametrized by the half-open interval* $(0, 1]$, *each structure being equivalent to its conjugate,*

(ii) *a single equivalence class of non-integrable almost* CR *structures of type* $(4k+3, 2k)$, *and*

(iii) *a family of equivalence classes of almost* CR *hypersurface structures parametrized by the set of all points* $(x, y, z) \in \mathbb{R}^3$ *which are not contained in the line* $(0, \mathbb{R}, 0)$ *and satisfy either* $x > 0$ *or* $x = y = z^2 - 1 = 0$. *Conjugation corresponds to* $(x, y, z) \rightarrow (x, -y, z)$. *The structure which* $\mathbf{S}^{4k+3}$ *inherits from* $\mathbb{H}^{k+1}$ *(viewed as a complex vector space) spans the only strong equivalence class containing (integrable)* CR *structures. It corresponds to the parameter* $(0, 0, -1)$.

(D) $\mathbf{S}^6 = \mathbf{G}_2/\mathbf{SU}(3)$: *The usual almost complex structure on* $\mathbf{S}^6$, *which is strongly equivalent to its conjugate.*

(E) $\mathbf{S}^7 = \mathbf{Spin}(7)/\mathbf{G}_2$: *No nontrivial homogeneous almost* CR *structure.*

(F) $\mathbf{S}^{15} = \mathbf{Spin}(9)/\mathbf{Spin}(7)$: *No nontrivial homogeneous almost* CR *structure.*

PROOF. The proof of this theorem will be given in a case-by-case study which will take up the rest of this chapter.

REMARK 4.6. Given a real Lie group $\mathbf{G}$ and a $\mathbf{G}$-homogeneous almost CR structure $(R\mathbf{M}, J)$ on $\mathbf{M}$, where $\pi : \mathbf{G} \to \mathbf{G}/\mathbf{H} = \mathbf{M}$ is a homogeneous space with basepoint $p = \pi(e)$, then $R\mathbf{M}_p$ is an even-dimensional $\mathbf{H}$-invariant subspace of $T\mathbf{M}_p$ and J_p a $\mathbf{H}$-equivariant map $R\mathbf{M}_p \to R\mathbf{M}_p$. Conversely, we can extend any such data $(R\mathbf{M}_p, J_p)$ to a unique $\mathbf{G}$-homogeneous almost CR structure on $\mathbf{M}$.

PROOF 4.7 OF (A), $n = 2$: $(\mathbf{S}^2 = \mathbf{SO}(3)/\mathbf{SO}(2))$. Note that $\boldsymbol{so}(2)$ is a maximal torus of $\boldsymbol{so}(3)$. Let $\{\alpha, -\alpha\}$ denote the two roots of $\boldsymbol{sl}(2,\mathbb{C}) = \boldsymbol{so}(3)^{\mathbb{C}}$ with respect to this torus. There are only two candidates for classifying spaces of nontrivial CR structures, which lead to the Riemannian structure and its conjugate. As the Weyl group contains an inner automorphism which maps one to the other, these two are strongly equivalent. An equivalence can be given directly by the map $z \to -\overline{z}^{-1}$.

PROOF 4.8 OF (A), $n \geq 3$: $(\mathbf{S}^n = \mathbf{SO}(n+1)/\mathbf{SO}(n))$. (It does not hurt to include the case $n = 3$ here.) Identifying $\mathbf{SO}(n)$ with the isotropy group of the unit vector $e_{n+1} \in \mathbf{S}^n$, we consider the action of $\mathbf{SO}(n)$ on the tangent space $T\mathbf{S}^n_{e_{n+1}}$. This action can be identified with the usual action of $\mathbf{SO}(n)$ on $\mathbb{R}^n$. As this action is irreducible, the only nontrivial $\mathbf{SO}(n)$-invariant possibility for $R\mathbf{S}^n_{e_{n+1}}$ is $T\mathbf{S}^n_{e_{n+1}} \approx \mathbb{R}^n$ itself. So $J_{e_{n+1}}$ can be viewed as an anti-involutive endomorphism of $\mathbb{R}^n$ centralizing $\mathbf{SO}(n)$. This implies that the subgroup of $\mathbf{GL}(n,\mathbb{R})$ generated by $J_{e_{n+1}}$ and $\mathbf{SO}(n)$ is compact, hence, after possible change of inner product, $J_{e_{n+1}} \in \mathbf{O}(n)$. Clearly, $\det J_{e_{n+1}} = 1$. But the center of $\mathbf{SO}(n)$ does not contain an anti- involution, which yields a contradiction.

For the following considerations, two well-known algebraic results are needed.

SCHUR'S LEMMA 4.9. *Let G be a Lie group which acts linearily and irreducibly on a finite-dimensional real vector space V. Then the set* $\mathrm{End}_{\mathbf{G}}(V)$ *of all* $\mathbf{G}$*-equivariant* $\mathbb{R}$*-linear maps $V \to V$ is a finite-dimensional associative division algebra over* $\mathbb{R}$.

FROBENIUS' THEOREM 4.10. *The only finite-dimensional associative division algebras over* $\mathbb{R}$ *are* $\mathbb{R}$, $\mathbb{C}$, *and* $\mathbb{H}$.

PROOF 4.11 OF (B), $k = 2$: $(\mathbf{S}^5 = \mathbf{SU}(3)/\mathbf{SU}(2))$. The only nontrivial $\mathbf{SU}(2)$-invariant vector subspaces of $T\mathbf{S}^5_{e_3}$ are $\mathbb{C}e_1 \oplus \mathbb{C}e_2$ and $\mathbb{R}\mathbf{i}e_3$, which forces $R\mathbf{S}^5_{e_3} = \mathbb{C}e_1 \oplus \mathbb{C}e_2$ for any nontrivial $\mathbf{SU}(3)$-homogeneous almost CR structure $(R\mathbf{S}^5, J)$ on $\mathbf{S}^5$.

Pulled up to $\boldsymbol{sl}(3,\mathbb{C}) = \boldsymbol{su}(3)^{\mathbb{C}}$, the complexification of $\mathbb{C}e_1 \oplus \mathbb{C}e_2$ corresponds to

$$\boldsymbol{sl}(3,\mathbb{C}) \cap \begin{pmatrix} * & * & * \\ * & * & * \\ * & * & 0 \end{pmatrix}.$$

Using root theory or direct calculations, one verifies that the only two possibilities for the classifying space $\boldsymbol{q}$ to become an algebra are

$$\boldsymbol{q} = \boldsymbol{sl}(3,\mathbb{C}) \cap \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ * & * & 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{q} = \boldsymbol{sl}(3,\mathbb{C}) \cap \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & 0 \end{pmatrix}.$$

The first corresponds to the usual structure which $\mathbf{S}^5$ inherits from $\mathbb{C}^3$, the second to its conjugate. So these are only nontrivial (integrable) CR structures.

To find all almost CR structures, identify $R\mathbf{S}^5 = \mathbb{C}e_1 \oplus \mathbb{C}e_2$ with the space $\mathbb{H}$ of quaternions. Then

$$\begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \in \mathbf{SU}(2)$$

becomes multiplication with $a + b\mathbf{j}$ on the right. By Schur's Lemma and Frobenius' Theorem, any $\mathbb{R}$-linear map $\mathbb{H} \to \mathbb{H}$ which is equivariant with respect to this $\mathbf{SU}(2)$-action can be written as multiplication with some quaternion on the left. In particular, the equivariant anti-involutions correspond to the points of the set $\mathbf{S}^2$ of purely imaginary unit quaternions. Clearly, the conjugate of the structure given by $u \in \mathbf{S}^2$ is given by $-u$.

Now any inner automorphism of $\mathbf{SU}(3)$ stabilizing $\mathbf{SU}(2)$ acts on $\mathbb{H}\oplus\mathbb{R}\mathbf{i}e_3 = \boldsymbol{su}(3)/\boldsymbol{su}(2)$, preserving the two summands. The restrictions to $\mathbb{H}$ of all such automorphisms form a group $\mathbf{S}^1 \times \mathbf{SU}(2)$, where $\mathbf{SU}(2)$ acts as above and $\mathbf{S}^1 \subset \mathbb{C} \subset \mathbb{H}$ by multiplication on the left. So $z \in \mathbf{S}^1$ induces a map $u \to z^{-1}uz$ on the sphere of anti-involutions, i.e., we can rotate that sphere around the axis connecting $\mathbf{i}$ and $-\mathbf{i}$. So $\{t\mathbf{i} + \sqrt{1-t^2}\mathbf{j} : t \in [-1,1]\}$ contains exactly one element from each strong equivalence class.

Finally, the outer automorphism $A \to \overline{A}$ of $\mathbf{SU}(3)$ can be restricted to $\mathbb{H}$, where it yields $v + w\mathbf{j} \to \bar{v} + \bar{w}\mathbf{j}$. The corresponding map of the sphere of anti-involutions reverses the $\mathbf{i}$-coordinate. As this automorphism, together with all inner ones, generates the full group of automorphisms of $\mathbf{SU}(3)$, we see that $t\mathbf{i} + \sqrt{1-t^2}\mathbf{j}$ and $s\mathbf{i} + \sqrt{1-s^2}\mathbf{j}$ correspond to weakly equivalent structures iff $\{\pm s\} = \{\pm t\}$.

PROOF 4.12 OF (B), $k \geq 3$: ($\mathbf{S}^{2k+1} = \mathbf{SU}(k+1)/\mathbf{SU}(k)$). The only nontrivial $\mathbf{SU}(k)$-stable vector subspaces of $T\mathbf{S}^{2k+1}_{e_{k+1}}$ are $\mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_k$ and $\mathbb{R}\mathbf{i}e_{k+1}$, forcing $R\mathbf{S}^{2k+1}_{e_{k+1}} = \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_k$.

Consider the algebra $\boldsymbol{a}$ of all $\mathbb{R}$-linear endomorphisms of $\mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_k$ which are equivariant with respect to the action of $\mathbf{SU}(k)$. The set $Z \subset \mathbb{C}$ of all k^{th} roots of unity is contained in the center of $\mathbf{SU}(k)$, hence also in the center of $\boldsymbol{a}$, as is the real vector space $\mathbb{C}$ spanned by Z. Schur's Lemma and Frobenius' Theorem now imply $\boldsymbol{a} = \mathbb{C}$, so the only $\mathbf{SU}(k)$-invariant anti-involutions on $\mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_k$ are $\pm\mathbf{i}$. They correspond to the CR hypersurface structure which $\mathbf{S}^{2k+1}$ inherits from $\mathbb{C}^{k+1}$, and its conjugate. A weak equivalence between these structures is given by $(v \to \bar{v}, A \to \overline{A})$. Any inner automorphism of $\mathbf{SU}(k+1)$

which stabilizes $\mathbf{SU}(k)$ comes from a matrix which stabilizes $\mathbb{C}e_{k+1}$ as well, so that inner automorphism is an inner automorphism of $\mathbf{SU}(k)$ and hence also of the classifying algebra $\boldsymbol{q}$ of either of the two structures under consideration. This shows that these structures are not strongly equivalent.

PROOF 4.13 OF (C), $k = 0$: ($\mathbf{S}^3 = \mathbf{SU}(2) = \mathbf{Sp}(1)$). The material presented here can also be found in [9], and the structures thus obtained have also been studied elsewhere, see, for example, [3].

Any complex vector subspace $\boldsymbol{q}$ of $\boldsymbol{sl}(2,\mathbb{C}) = \boldsymbol{su}(2)^{\mathbb{C}}$ is invariant under the (trivial) isotropy group. Assuming that $\boldsymbol{q}$ is not trivial itself, $\boldsymbol{q}\cap\overline{\boldsymbol{q}} = \{0\}$ implies $\dim_{\mathbb{C}}\boldsymbol{q} = 1$ and hence that $\boldsymbol{q}$ is an algebra, so the structure classified by $\boldsymbol{q}$ is automatically integrable. The classifying algebras are parametrized by $\mathbf{P}^2_{\mathbb{C}}\backslash\mathbf{P}^2_{\mathbb{R}}$ after identifying $\boldsymbol{sl}(2,\mathbb{C})$ with $\mathbb{C}^3$ via the basis

$$\left(x_0 = \begin{pmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{pmatrix},\ y_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\ z_0 = \begin{pmatrix} 0 & -\mathbf{i} \\ -\mathbf{i} & 0 \end{pmatrix}\right)$$

of $\boldsymbol{su}(2)$. As all automorphisms of $\mathbf{SU}(2)$ are inner, the notions of strong and weak equivalence coincide. The equivalence classes correspond to the orbits of $\mathbf{SU}(2)$ on $\mathbf{P}^2_{\mathbb{C}}\backslash\mathbf{P}^2_{\mathbb{R}}$ via the restriction of the adjoint action of $\mathbf{SL}(2,\mathbb{C})$, which yields the covering map $\mathbf{SU}(2) \to \mathbf{SO}(3)$.

Given $v, w \in \mathbb{R}^3$, then $[v + \mathbf{i}w] \in \mathbf{P}^2_{\mathbb{C}}\backslash\mathbf{P}^2_{\mathbb{R}}$ iff v and w are linearly independent. If this is the case, we can multiply with some complex number to achieve $v \perp w$ Using the action of $\mathbf{SO}(3)$, we see that any orbit in $\mathbf{P}^2_{\mathbb{C}}\backslash\mathbf{P}^2_{\mathbb{R}}$ contains a point $[x_0 + \mathbf{i}ty_0]$, where $t > 0$. The image of $[x_0 + \mathbf{i}ty_0]$ under the element of $\mathbf{SO}(3)$ which rotates x_0 to y_0 fixing z_0 is $[y_0 - \mathbf{i}tx_0] = [x_0 + \mathbf{i}t^{-1}y_0]$. So each orbit contains one point of $\{[x_0 + \mathbf{i}ty_0] : t \in (0,1]\}$. Assume there are some $s, t \in (0,1]$ and some $A \in \mathbf{SO}(3)$ such that $[A(x_0 + \mathbf{i}ty_0)] = [x_0 + \mathbf{i}sy_0]$. This implies that A stabilizes $\mathbb{C}x_0 \oplus \mathbb{C}y_0$ and hence restricts to an element of $\mathbf{O}(\mathbb{R}x_0 \oplus \mathbb{R}y_0)$. So there exist $\alpha, \beta \in \mathbb{R}, \alpha^2 + \beta^2 = 1$, such that $A(x_0) = \alpha x_0 + \beta y_0$ and $A(y_0) = \pm(-\beta x_0 + \alpha y_0)$. A straightforward calculation yields

$$[A(x_0 + \mathbf{i}ty_0)] = \left[x_0 + \frac{\pm\mathbf{i}t + (1 - t^2)\alpha\beta}{\alpha^2 + t^2\beta^2} y_0\right],$$

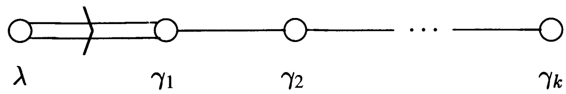
which equals $[x_0 + \mathbf{i}sy_0]$ iff $s = t$. This shows that no orbit can contain two such points.

The CR hypersurface structure which $\mathbf{S}^3$ inherits from $\mathbb{C}^2$ has $\begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix}$ as its classifying algebra, which consists of nilpotent elements and thus, using $\det(x_0 + \mathbf{i}ty_0) = 1 - t^2$, corresponds to the parameter value $t = 1$.

Because of $\det(ux_0 + vy_0 + wz_0) = u^2 + v^2 + w^2$, for all $u, v, w \in \mathbb{C}$, we can identify the complex quadric with the set Q of matrices in $\boldsymbol{sl}(2,\mathbb{C})$ which have determinant 1. All of these have Jordan normal form x_0, so $\mathbf{SL}(2,\mathbb{C})$ acts transitively on Q. The projection $\mathbb{C}^3 \to \mathbf{P}^2_{\mathbb{C}}$ restricts to a $2:1$-cover $Q \to D$, where D is a dense open set in $\mathbf{P}^2_{\mathbb{C}}$. Given $p \in D$, the isotropy algebra $\boldsymbol{q}$ of p

in $\mathbf{SL}(2,\mathbb{C})$, viewed as a point of $\mathbf{P}^2_{\mathbb{C}}$, coincides with p. So the CR structure on $\mathbf{S}^3$ parametrized by $t \in (0,1)$ is the one lifted from the $\mathbf{SU}(2)$-orbit through $[x_0 + \mathbf{i}ty_0]$ in D, or from the corresponding orbit in Q.

PROOF 4.14 OF (C), $k \geq 1$: ($\mathbf{S}^{4k+3} = \mathbf{Sp}(k+1)/\mathbf{Sp}(k)$). Choose a maximal torus of $\mathbf{Sp}(k+1)$ stabilizing $\mathbf{Sp}(k)$ and a system



of simple roots of $\mathbf{Sp}(k+1)$ such that $\lambda, \gamma_1, \ldots, \gamma_{k-1}$ form a system of simple roots of $\mathbf{Sp}(k)$. Then the long root $\alpha = \lambda + 2\gamma_1 + \cdots + 2\gamma_k$ is the only positive root perpendicular to any root of $\mathbf{Sp}(k)$. The corresponding copy $\mathbf{Z}$ of $\mathbf{SU}(2)$ is the centralizer of $\mathbf{Sp}(k)$ and naturally isomorphic to $N(\mathbf{Sp}(k))/\mathbf{Sp}(k)$, and hence also to the group of equivalences, acting on the set of all $\mathbf{Sp}(k+1)$-homogeneous almost CR structures on $\mathbf{S}^{4k+3}$, the orbits being the equivalence classes. (That these groups are connected uses the fact that an automorphism of $\mathbf{Sp}(k+1)$ is given by its restrictions to $\mathbf{Sp}(k)$ and $\mathbf{Z}$, and that all automorphisms of these two are inner).

Let us now consider the $\mathbf{Sp}(k)$-invariant splitting

$$T\mathbf{S}^{4k+3}_{e_{k+1}} = (\mathbb{H}e_1 \oplus \cdots \oplus \mathbb{H}e_k) \oplus (\mathbb{R}\mathbf{i}e_{k+1} \oplus \mathbb{R}\mathbf{j}e_{k+1} \oplus \mathbb{R}\mathbf{k}e_{k+1}),$$

which restricts to the $\mathbf{Sp}(k)$-invariant space $R\mathbf{S}^{4k+3}_{e_{k+1}}$, leaving three nontrivial cases.

If $R\mathbf{S}^{4k+3}_{e_{k+1}} \subset \mathbb{R}\mathbf{i}e_{k+1} \oplus \mathbb{R}\mathbf{j}e_{k+1} \oplus \mathbb{R}\mathbf{k}e_{k+1}$, the classifying space $\boldsymbol{q}$ satisfies $\boldsymbol{q} = \boldsymbol{sp}(k)^{\mathbb{C}} \oplus (\boldsymbol{q} \cap z^{\mathbb{C}})$, where z is the Lie algebra of $\mathbf{Z}$. Any automorphism of $\mathbf{Sp}(k+1)$ fixing $\boldsymbol{sp}(k)$ restricts to an automorphism of $\mathbf{Z}$ and vice versa, so we are essentially in the situation of case $k = 0$ and get the same parameter space.

If $R\mathbf{S}^{4k+3}_{e_{k+1}} \subset \mathbb{H}e_1 \oplus \cdots \oplus \mathbb{H}e_k$, we must have equality, as $\mathbf{Sp}(k)$ acts irreducibly on the right-hand side. Using Schur's Lemma and Frobenius' Theorem, we see that $J_{e_{k+1}}$ must agree with left multiplication by some purely imaginary quaternion. The structures given by the special cases $J_{e_{k+1}} = \mathbf{i}$ and $J_{e_{k+1}} = -\mathbf{i}$ are strongly equivalent via left multiplication with $\mathbf{j}$, viewed as an equivalence $\mathbf{S}^{4k+3} \to \mathbf{S}^{4k+3}$. That any other structure is equivalent to one of these can be concluded from the fact that maximal tori in $\mathbf{S}^3 \subset \mathbb{H}$ are conjugated.

Finally, consider the case $R\mathbf{S}^{4k+3}_{e_{k+1}} = W_1 \oplus W_2$, where $W_1 = \mathbb{H}e_1 \oplus \cdots \oplus \mathbb{H}e_k$ and $W_2 \subset \mathbb{R}\mathbf{i}e_{k+1} \oplus \mathbb{R}\mathbf{j}e_{k+1} \oplus \mathbb{R}\mathbf{k}e_{k+1}$ are both not trivial. Then there are anti-involutions J_1 of W_1 and J_2 of W_2 such that $Je_{k+1} = J_1 \oplus J_2$. Using the preceding argument, we may assume that J_1 coincides with multiplication from the left by $\mathbf{i}$. By what we know about the normalizer of $\mathbf{Sp}(k)$, the only maps which yield equivalences which fix J_1 are, on the level of $T\mathbf{S}^{4k+3}_{e_{k+1}}$, multiplications from the left with elements of $\mathbf{S}^1 \subset \mathbb{C} \subset \mathbb{H}$. After identifying W_2 with $\mathbb{R}^3$ via the basis $(\mathbf{i}e_{k+1}, \mathbf{j}e_{k+1}, \mathbf{k}e_{k+1})$ of W_2, those multiplications correspond to rotations around the e_1-axis in $\mathbb{R}^3$. Any choice of J_2 is uniquely defined by its $+\mathbf{i}$-eigenspace, viewed as a point $[v+\mathbf{i}w] \in \mathbf{P}^2_{\mathbb{C}} \backslash \mathbf{P}^2_{\mathbb{R}}$, where $v, w \in \mathbb{R}^3$ are linearly independent. As we may rotate around the e_1-axis, we may assume that e_2 is contained in $\mathbb{R}v \oplus \mathbb{R}w$. Multiplying with some nonzero complex number then yields $[v+\mathbf{i}w] = [e_2+\mathbf{i}(x,y,z)]$, for some $(x,y,z) \in \mathbb{R}^3 \backslash \mathbb{R}e_2$. We can multiply with the complex number -1 and then rotate back, which yields $[e_2+\mathbf{i}(-x,y,z)]$. So, without loss of generality, $x \geq 0$. If $x > 0$, then the orbit through $[e_2+\mathbf{i}(x,y,z)]$ contains only one such point. Otherwise, we can multiply with some nonzero complex number to find $[e_2+\mathbf{i}(0,y,z)] = [v'+\mathbf{i}w']$, where $v', w' \in \mathbb{R}e_2 \oplus \mathbb{R}e_3$ are perpendicular. Rotating again, we are left with the two possibilities $[e_2 \pm \mathbf{i}e_3]$. Straightforward computation shows that these two points are indeed contained in distinct orbits.

Now we need to show that, given $\mathbb{H}e_1 \oplus \cdots \oplus \mathbb{H}e_k \subset R\mathbf{S}^{4k+3}_{e_{k+1}}$ and some fixed choice of a restriction of $J_{e_{k+1}}$ to $\mathbb{H}e_1 \oplus \cdots \oplus \mathbb{H}e_k$, there is only one integrable structure. One such choice for the restriction of $J_{e_{k+1}}$ can be fixed by the decree that all root spaces which have a γ_k-component of $+1$ belong to the classifying space. There is indeed only one possibility to extend this to a classifying algebra.

Straightforward (though somewhat involved) computations show what has been said about conjugation and that the structure which $\mathbf{S}^{4k+3}$ inherits from $\mathbb{H}^{k+1}$ corresponds to the parameter $(0,0,-1)$.

LEMMA 4.15. *Two* $\mathbf{Sp}(k+1)$*-homogeneous (integrable)* CR *structures on* $\mathbf{S}^{4k+3}$ *are strongly equivalent iff there is a* CR *diffeomorphism between them.*

PROOF. In the case $k = 0$, it has been known for a long time that the parameter space given above corresponds to a family of homogeneous CR structures on $\mathbf{S}^3$ such that no two distinct such structures are CR diffeomorphic, see, for example, [9].

In the case $k \geq 1$, we only need to consider CR structures of type $(4k+3, 1)$. We use $B : R\mathbf{S}^{4k+3} \times R\mathbf{S}^{4k+3} \to T\mathbf{S}^{4k+3}/R\mathbf{S}^{4k+3}$ and

$$\varphi : T\mathbf{S}^{4k+3} \to T\mathbf{S}^{4k+3}/R\mathbf{S}^{4k+3}$$

as in (2.2) to construct

$$X\mathbf{S}^{4k+3} = \varphi^{-1}(B(R\mathbf{S}^{4k+3}, R\mathbf{S}^{4k+3})),$$

which, by (2.4) and (2.5), is a $\mathbf{Sp}(k+1)$-invariant involutive subbundle of $T\mathbf{S}^{4k+3}$. The integral manifold $\mathbf{X}$ of $X\mathbf{S}^{4k+3}$ through e_{k+1} is an orbit of the centralizer $\mathbf{Z}$ of $\mathbf{Sp}(k)$ in $\mathbf{Sp}(k+1)$. This is true for either one of the two structures.

If $\psi : \mathbf{S}^{4k+3} \to \mathbf{S}^{4k+3}$ is the CR diffeomorphism between them, we can use the $\mathbf{Sp}(k+1)$-action to achieve $\psi(e_{k+1}) = e_{k+1}$. As we have expressed $\mathbf{X}$ as an integral manifold in terms of the CR structures only, without refering to the groups, this implies $\psi(\mathbf{X}) = \mathbf{X}$. But the given CR structures on $\mathbf{S}^{4k+3}$ are completely determined by their "restrictions" (using the obvious generalization of (1.3)) to $\mathbf{X} \approx \mathbf{S}^3$. So we have reduced the general case to $k = 0$.

PROOF 4.16 OF (D): ($\mathbf{S}^6 = \mathbf{G}_2/\mathbf{SU}(3)$). The group $\mathbf{G}_2$ of automorphisms of the normed division algebra $\mathbb{O}$ of Cayley numbers acts transitively on the sphere $\mathbf{S}^6$ of purely imaginary unit Cayley numbers, the isotropy group being isomorphic to $\mathbf{SU}(3)$. There is a natural $\mathbf{G}_2$-invariant nonintegrable almost complex structure on $\mathbf{S}^6$, the fibre of J at $p \in \mathbf{S}^6$ simply being given by left multiplication with p.

In the root system of $\mathbf{G}_2$ with respect to some maximal torus of $\mathbf{SU}(3)$, the roots of $\mathbf{SU}(3)$ are the long ones. Distribute the short ones into two disjoint families ψ_1 and ψ_2 of three roots each such that any two distinct roots from the same family span an angle of $2\pi/3$. Then $\boldsymbol{su}(2)^{\mathbb{C}} \oplus \bigoplus_{\alpha\in\psi_i}(\boldsymbol{g}_2)^{\mathbb{C}}_{\alpha}$, for $i \in \{1,2\}$, are the only possible classifying spaces of nontrivial $\mathbf{G}_2$-homogeneous almost CR structures on $\mathbf{S}^6$. As there exists an element w of the Weyl group of $\mathbf{G}_2$ such that $w(\psi_1) = \psi_2$, the two structures are strongly equivalent.

PROOF 4.17 OF (E): ($\mathbf{S}^7 = \mathbf{Spin}(7)/\mathbf{G}_2$). The action of $\mathbf{Spin}(7)$ on $\mathbf{S}^7$ is effective and comes from a linear action of $\mathbf{Spin}(7)$ on $\mathbb{R}^8$, so $\mathbf{G}_2$ acts nontrivially on $T\mathbf{S}^7_p$, where $p \in \mathbf{S}^7$ denotes the basepoint. Representation theory shows that the smallest non-trivial linear representation of $\mathbf{G}_2$ is on a space of dimension 7, so there is no even-dimensional $\mathbf{G}_2$-invariant subspace of $T\mathbf{S}^7_p$ other than $\{0\}$.

PROOF 4.18 OF (F): ($\mathbf{S}^{15} = \mathbf{Spin}(9)/\mathbf{Spin}(7)$). Let $p \in \mathbf{S}^{15}$ denote the basepoint. It turns out that $T\mathbf{S}^{15}_p$ splits into two irreducible $\mathbf{Spin}(7)$-components, one of dimension 7 and one of dimension 8. The second one is the only nontrivial candidate for $R\mathbf{S}^{15}$, but a corresponding anti-involution would yield an irreducible complex representation of the Lie algebra $\boldsymbol{so}(7)$ of $\mathbf{Spin}(7)$, which would be of complex dimension 4. Such a representation does not exist.

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