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Massimo Cicognani

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The Geometric Optics for a Class of Hyperbolic Second Order Operators with Hölder Continuous Coefficients with Respect to Time

MASSIMO CICOGNANI

0. - Introduction

Several authors have considered the Cauchy problem for operators with hyperbolic principal part and coefficients which are Hölder continuous with respect to time and in a Gevrey class with respect to $x \in \mathbb{R}^n$.

F. Colombini, E. De Giorgi and S. Spagnolo [4], F. Colombini, E. Jannelli and S. Spagnolo [5], T. Nishitani [14], E. Jannelli [8], Y. Ohya and S. Tarama [16] have proved results of well-posedness of the Cauchy problem for operators of this type in some Gevrey classes of functions and ultradistributions. The propagation of Gevrey singularities of the solution has not been studied till now and this paper is devoted to this topic. The results we describe here have been partially announced in [3]. We consider second order strictly hyperbolic operators (i.e., with real distinct characteristic roots) to better point out the influence of the coefficients on the behaviour of the solution.

One can check, by means of simple examples, that refractions of Gevrey singularities, with respect to x, of the solution appear even if the characteristic roots are distinct, while it is well known that this does not happen when the coefficients of a strictly hyperbolic operator are in the same Gevrey class both with respect to time and x variable[1].

Let us present one of these examples considering the Cauchy problem

$$\begin{aligned}
&(\partial_t^2 - \partial_t\partial_x - b(t)\partial_t)u(t,x) = 0, \qquad t > 0\\
&u(0,x) = 0\\
&\partial_t u(0,x) = \delta(x - x_0) = \text{ the Dirac measure concentrated at } x_0.
\end{aligned}$$

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[1] See [12] and [17].

If Γ is the cone $\{(t,x);\ t \geq 0,\ x_0 - t \leq x \leq x_0\}$ the solution is given by

$$u(t,x) = \begin{cases} \exp\left(\int_0^{x_0-x} b(s)\mathrm{d}s\right) & , \quad (t,x) \in \Gamma \\ 0 & , \quad (t,x) \notin \Gamma. \end{cases}$$

Therefore if the instant t_1 is singular for the coefficient $b(t)$ then a singularity of $u(t,\cdot)$ branches at the point $(t_1, x_0 - t_1)$ and propagates along the half line $x = x_0 - t_1,\ t \geq t_1$.

If the singular support of $b(t)$ is the time axis $t \geq 0$ then the solution is singular with respect to x in the whole cone Γ.

On the other hand let us replace $\delta(x - x_0)$ by an initial datum $g(x)$ with wave front set $\{(x_0, \xi); \xi > 0\}$ and choose $b(t)$ to have the wave front set of $\exp\left(\int_0^t b(s)\mathrm{d}s\right)$ equal to $\{(t,\tau);\ t \geq 0,\ \tau > 0\}$. Then the solution $u(t,x)$ is singular with respect to x only at the points of the boundary of Γ exactly as in the case of a regular coefficient even if the singular support of $b(t)$ is the time axis $t \geq 0$.

Let us consider the Cauchy problem

$$\text{(C.P.)} \qquad \begin{cases} P(t,x,\mathrm{D}_t,\mathrm{D}_x)u(t,x) = 0 & (t,x) \in [0,T] \times \mathbb{R}^n \\ \mathrm{D}_t^j u(0,x) = g_j(x) & x \in \mathbb{R}^n,\ j = 0,1, \end{cases}$$

$T > 0$, for a strictly hyperbolic operator

$$P(t,x,\mathrm{D}_t,\mathrm{D}_x) = \mathrm{D}_t^2 + \sum_{h=1}^n a_h(t,x)\mathrm{D}_{x_h}\mathrm{D}_t + \sum_{j,k=1}^n b_{j,k}(t,x)\mathrm{D}_{x_j}\mathrm{D}_{x_k}$$
$$+ c(t,x)\mathrm{D}_t + \sum_{\ell=1}^n d_\ell(t,x)\mathrm{D}_{x_\ell} + e(t,x)$$

$$\mathrm{D}_t = -i\partial/\partial t,\ \ x = (x_1, \cdots, x_n),$$
$$\mathrm{D}_{x_h} = -i\partial/\partial_{x_h},\ \ h = 1, \cdots, n,$$
$$\mathrm{D}_x = (\mathrm{D}_{x_1}, \cdots, \mathrm{D}_{x_n}),$$

and assume that a_h and $b_{j,k}$ are Hölder continuous of exponent χ, $0 < \chi < 1$, from $[0,T]$ with values in the Gevrey class of functions $G_b^{(\sigma)}(\mathbb{R}^n)$ of type $\sigma \in]1, 1/(1-\chi)[$ on $\mathbb{R}^n$ [2]. Lower order terms coefficients are assumed to be continuous from $[0,T]$ with values in $G_b^{(\sigma)}(\mathbb{R}^n)$.

It follows from the results of [4], [8] and [14] that the Cauchy problem (C.P.) is well-posed for initial data $g_j(x)$, $j = 0,1$, in the classes of Gevrey functions and ultradistributions of type σ on $\mathbb{R}^n$, and the condition

[2] See section 1 for notations.

$1 < \sigma < 1/(1-\chi)$ is optimal. The problem (C.P.) can be reduced to the problem

$$(\mathrm{C.P.})_S \qquad \begin{cases} L(t,x,\mathrm{D}_t,\mathrm{D}_x)U(t,x) = 0 & (t,x) \in [0,T] \times \mathbb{R}^n \\ U(0,x) = G(x) & x \in \mathbb{R}^n \end{cases}$$

for a strictly hyperbolic system

$$L(t,x,\mathrm{D}_t,\mathrm{D}_x) = \mathrm{D}_t - \begin{vmatrix} \lambda_1(t,x,\mathrm{D}_x) & 0 \\ 0 & \lambda_2(t,x,\mathrm{D}_x) \end{vmatrix} + (c_{j,k}(t,x,\mathrm{D}_x))_{j,k=1,2}$$

where λ_j and $c_{j,k}$ are pseudo differential operators of order 1 and 1-χ respectively of the type studied in [6] and [7]. If the characteristic roots λ_1 and λ_2 satisfy one of the following alternatives

$$\text{(A1)} \qquad \lambda_j(t,x,\xi) = \alpha(t)\mu_j(x,\xi),\ \ j = 1,2, \{\mu_1,\mu_2\} = \beta(\mu_1 - \mu_2)$$

where $\{\mu_1,\mu_2\} = \nabla_x\mu_1 \cdot \nabla_\xi\mu_2 - \nabla_x\mu_2 \cdot \nabla_\xi\mu_1$ and β is a real constant;

$$\text{(A2)} \qquad \lambda_j(t,x,\xi) = \alpha_j(t)\mu(x,\xi),\ \ j = 1,2,$$

and the coefficients of P are analytic as functions of x, then we construct a parametrix for the problem $(\mathrm{C.P.})_S$ represented as a matrix of Fourier integral operators

$$(*) \qquad \begin{aligned} E(t,s) = &\begin{vmatrix} e^1_{\phi_1(t,s)}(t,s,x,\mathrm{D}_x) & 0 \\ 0 & e^2_{\phi_2(t,s)}(t,s,x,\mathrm{D}_x) \end{vmatrix} \\ &+ \left(\int_s^t P^{k,\ell}_{\Phi_{k,j(k)}}(t,t_1,s,x,\mathrm{D}_x)\mathrm{d}t_1\right)_{k,\ell=1,2}, j(1) = 2,\ \ j(2) = 1. \end{aligned}$$

Phase functions ϕ_1 and ϕ_2 derive from λ_1 and λ_2 respectively by solving eikonal equations while $\Phi_{1,2}$ and $\Phi_{2,1}$ are their products (see section 3). From the particular form of the characteristic roots we obtain a commutation law for products of phase-function and this permits us to determine the amplitudes $e^j, p^{k,\ell}$, $j,k,\ell = 1,2$, as asymptotic sums in the classes of symbols of infinite order defined by L. Zanghirati [20], L. Cattabriga and L. Zanghirati [2], by the method of transport equations studied in [2].

Fundamental solutions of the form $(*)$ have been obtained in C^∞ framework by J.C. Nosmas [15], Y. Morimoto [13], K. Taniguchi [18], for Cauchy problems related to hyperbolic operators with involutive characteristic roots and C^∞ coefficientes with respect to (t,x).

By means of the parametrix $E(t,s)$ we càn study the propagation of Gevrey singularities of the solution $U(t,x)$ of problem $(\mathrm{C.P.})_S$; in particular the form $(*)$ of $E(t,s)$ is connected with the fact that refractions may happen. By integrating by parts the terms $\int_s^t P^{k,\ell}_{\Phi_{k,j(k)}}(t,t_1,s)\mathrm{d}t_1$ and using the strict hyperbolicity of L, we can see that refractions of singularities may occur only at instants t_0 corresponding to Gevrey singularities of the coefficients of P with respect to time variable, as in the simple example we have presented.

The plan of the first part of the work is as follows: section 1 contains our main notations and definitions; in section 2 we recall the results of [2] and [20] on Fourier integral operators with amplitudes of infinite order we shall use here. Section 3 is devoted to phase functions and their products and there we obtain the commutation law that will be the key assumption in section 5. In section 4 we reduce the problem (C.P.) to the system from $(\text{C.P.})_S$. In section 5 we construct the parametrix $E(t,x)$ for the problem $(\text{C.P.})_S$ following [2] and we use it in section 6 to study the propagation of Gevrey singularities of the solution.

1. - Main notations

For $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ we set $\mathrm{D}_x = (\mathrm{D}_{x_1}, \cdots, \mathrm{D}_{x_n})$, $\mathrm{D}_{x_j} = -i\partial/\partial x_j$, $j = 1, \cdots, n$, and for $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{Z}_+^n$, $\mathbb{Z}_+$ the set of all non negative integers, let $\mathrm{D}_x^\alpha = \mathrm{D}_{x_1}^{\alpha_1} \cdots \mathrm{D}_{x_n}^{\alpha_n}$, $|\alpha| = \sum_{j=1}^{n} \alpha_j$, $\alpha! = \alpha_1! \cdots \alpha_n!$. For $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$ we shall also write $\langle x, \xi\rangle = x \cdot \xi = \sum_{j=1}^{n} x_j \xi_j$ and $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$.

Let $\Omega \subset \mathbb{R}^n$ be an open set. For $\sigma \geq 1$, $A > 0$ we denote by $G_b^{(\sigma)}(\Omega; A)$ the Banach space of all complex valued functions $u \in C^\infty(\Omega)$ such that:

$$\|u\|_{\Omega,A}^{\sigma} = \sup_{\substack{x\in\Omega \\ \alpha\in\mathbb{Z}_+^n}} A^{-|\alpha|}\alpha!^{-\sigma}|\mathrm{D}_x^\alpha u(x)| < +\infty \tag{1.1}$$

and set

$$G_b^{(\sigma)}(\Omega) = \varinjlim_{A\to+\infty} G_b^{(\sigma)}(\Omega; A)$$

$$G^{(\sigma)}(\Omega) = \varprojlim_{\Omega'\subset\subset\Omega} G_b^{(\sigma)}(\Omega')$$

$$G_0^{(\sigma)}(\Omega) = \varinjlim_{\Omega'\subset\subset\Omega} \varinjlim_{A\to+\infty} G_b^{(\sigma)}(\Omega'; A) \cap C_0^\infty(\Omega')$$

where Ω' denotes relatively compact open subsets of Ω. $G^{(1)}(\Omega)$ is the space of all analytic functions in Ω.

For $\sigma > 1$ the dual spaces of $G^{(\sigma)}(\Omega)$ and $G_0^{(\sigma)}(\Omega)$, called spaces of ultradistributions of Gevrey type σ, will be denoted by $G^{(\sigma)'}(\Omega)$ and $G_0^{(\sigma)'}(\Omega)$

respectively. As is well known the former can be identified with the subspace of ultradistribution of $G_0^{(\sigma)'}(\Omega)$ with compact support. For $u \in G^{(\sigma)'}(\Omega)$ the Fourier transform $\tilde{u}$ of u is defined by $\tilde{u}(\xi) = u(e^{-i\langle\cdot,\xi\rangle})$. We shall also denote by $WF_{(\sigma)}(u)$ the σ-wave front set of $u \in G_0^{(\sigma)'}(\Omega)$ defined as the complement in $\Omega \times \mathbb{R}^n \backslash \{0\}$ of the set of all (x_0, ξ_0) such that there exist a conic neighbourhood Γ of ξ_0 in $\mathbb{R}^n \backslash \{0\}$ and a function $\phi \in G_0^{(\sigma)}(\Omega)$ with $\phi(x_0) \neq 0$ such that

$$|(\tilde{\phi u})(\xi)| \leq C \ \exp(-h\langle\xi\rangle^{1/\sigma}), \ \ \xi \in \Gamma \tag{1.2}$$

for some positive C and h. The projection of $WF_{(\sigma)}(u)$ on Ω is equal to the σ-singular support of u denoted by sing-supp$_{(\sigma)}(u)$ and defined as the complement in Ω of the set of all x_0 having a neighbourhood Ω' such that $u \in G^{(\sigma)}(\Omega')$.

For a given X subset of the cotangent bundle $T^*(\mathbb{R}^n)$ of $\mathbb{R}^n$, $0 \notin X$, X^{con} will denote the conic hull of X in $T^*(\mathbb{R}^n)\backslash\{0\}$.

If V is a topological vector space, $\mathcal{A} \subset \mathbb{R}^k$, $m \in \mathbb{Z}_+$, $M^m(\mathcal{A}, V)$ shall denote the set of all functions defined in $\mathcal{A}$ with values in V which are continuous and bounded together with their derivatives up to order m. We shall also write $M^\infty(\mathcal{A}, V) = M(\mathcal{A}, V) = \bigcap_{m\geq 0} M^m(\mathcal{A}, V)$ and sometimes $M_z^m(\mathcal{A}, V)$ and $M_z(\mathcal{A}, V)$ respectively if $z \in \mathcal{A}$ denotes the independent variable.

2. - Symbols of infinite order and Fourier integral operators in Gevrey classes

The following classes of infinite order symbols of Gevrey type have been introduced in [20] and [2]. We refer to these works for the proofs of all results listed in this section.

DEFINITION 2.1. Let $\sigma > 1$, $\mu \in [1, \sigma]$, $A > 0$, $B_0 > 0$, $B \geq 0$ be real constants and let us set $T^n(C) = \{(x,\xi) \in T^*(\mathbb{R}^n); \ |\xi| \geq C\}$. We denote by $S_b^{\infty,\sigma,\mu}(A, B_0, B)$ the space of all complex valued functions $a(x,\xi) \in C^\infty(\mathbb{R}^{2n})$ such that for every $\varepsilon > 0$

$$\begin{aligned} \|a\|_\varepsilon^{A,B_0,B} = \sup_{\alpha,\beta\in\mathbb{Z}_+^n} \ \sup_{(x,\xi)\in T^n(B|\alpha|^\sigma+B_0)} & A^{-|\alpha|-|\beta|}\alpha!^{-\mu}\beta!^{-\sigma}\langle\xi\rangle^{|\alpha|} \\ & \cdot \exp(-\epsilon\langle\xi\rangle^{1/\sigma})|D_\xi^\alpha D_x^\beta a(x,\xi)| < +\infty \end{aligned} \tag{2.1}$$

and set

$$S_b^{\infty,\sigma,\mu} = \varinjlim_{A,B_0,B\to+\infty} S_b^{\infty,\sigma,\mu}(A, B_0, B) \tag{2.2}$$

$$\tilde{S}_b^{\infty,\sigma,\mu} = \varinjlim_{A,B_0\to+\infty} S_b^{\infty,\sigma,\mu}(A, B_0, 0). \tag{2.3}$$

The asymptotic sums in $S_b^{\infty,\sigma,\mu}$ are defined by:

DEFINITION 2.2. We say that $a(x,\xi) \in S_b^{\infty,\sigma,\mu}$ has an asymptotic expansion $a(x,\xi) \sim \sum_{j\geq 0} a_j(x,\xi)$ if there exist constant $A > 0$, $B \geq 0$, $B_0 > 0$ such that for every $\varepsilon > 0$

$$\begin{aligned}(2.4)\qquad &\sup_{j\in\mathbb{Z}_+}\ \sup_{\alpha,\beta\in\mathbb{Z}_+^n}\ \sup_{(x,\xi)\in T^n((|\alpha|+j)^\sigma+B_0)} A^{-|\alpha|-|\beta|-j}\alpha!^{-\mu}(j!\beta!)^{-\sigma}\\ &\cdot\langle\xi\rangle^{|\alpha|+j}\exp(-\varepsilon\langle\xi\rangle^{1/\sigma})|D_\xi^\alpha D_x^\beta a_j(x,\xi)| < +\infty\end{aligned}$$

and

$$\begin{aligned}(2.5)\qquad &\sup_{s\in\mathbb{Z}_+}\ \sup_{\alpha,\beta\in\mathbb{Z}_+^n}\ \sup_{(x,\xi)\in T^n(B(|\alpha|+j)^\sigma+B_0)} A^{-|\alpha|-|\beta|-s}\alpha!^{-\mu}(s!\beta!)^{-\sigma}\\ &\cdot\langle\xi\rangle^{|\alpha|+s}\exp(-\varepsilon\langle\xi\rangle^{1/\sigma})|D_\xi^\alpha D_x^\beta \sum_{j<s}(a(x,\xi)-a_j(x,\xi))| < +\infty.\end{aligned}$$

THEOREM 2.3. ([20]). *For every sequence* $\{a_j(x,\xi)\}_{j\geq 0} \subset S_b^{\infty,\sigma,\mu}$ *satisfying* (2.4) *there exists* $a(x,\xi) \in S_b^{\infty,\sigma,\mu}$ *such that* $a(x,\xi) \sim \sum_{j\geq 0} a_j(x,\xi)$ *in the sense of definition* 2.2.

We shall also need symbols of finite order ($\subset S_b^{\infty,\sigma,\mu}$) of the type studied in [6] and [7].

DEFINITION 2.4. For $\sigma \geq 1$, $\mu \in [1,\sigma]$, $A > 0$, $B \geq 0$, $B_0 > 0$, $m \in R$ we denote by $S_b^{m,\sigma,\mu}(A, B_0, B)$ the Banach space of all complex valued functions $a(x,\xi) \in C^\infty(\mathbb{R}^{2n})$ with the norm

$$\begin{aligned}(2.6)\qquad &\|a\|_m^{A,B_0,B} = \sup_{\alpha,\beta\in\mathbb{Z}_+^n}\ \sup_{(x,\xi)\in T^n(B|\alpha|^\sigma+B_0)} A^{-|\alpha|-|\beta|}\alpha!^{-\mu}\beta!^{-\sigma}\langle\xi\rangle^{-m+|\alpha|}\\ &\cdot|D_\xi^\alpha D_x^\beta a(x,\xi)| < +\infty\end{aligned}$$

and define

$$(2.7)\qquad S_b^{m,\sigma,\mu} = \varinjlim_{A,B_0,B\to+\infty} S_b^{m,\sigma,\mu}(A,B_0,B)$$

$$(2.8)\qquad \tilde{S}_b^{m,\sigma,\mu} = \varinjlim_{A,B_0\to+\infty} S_b^{m,\sigma,\mu}(A,B_0,0).$$

DEFINITION 2.5. Let $a \in S_b^{\infty,\sigma,\mu}$. We say that a is an amplitude of infinite order of type (σ,μ) if $a \in M^0(Y, G^{(\sigma)}(\mathbb{R}_x^n))$ for every relatively compact open set $Y \subset \mathbb{R}_\xi^n$. The set of all amplitudes of infinite order of type (σ,μ) will be denoted by $\mathcal{A}_b^{\infty,\sigma,\mu}$. In the set same way we define the set $\tilde{\mathcal{A}}_b^{\infty,\sigma,\mu}$ and the sets of finite order amplitudes $\mathcal{A}_b^{m,\sigma,\mu}$, $\tilde{\mathcal{A}}_b^{m,\sigma,\mu}$ replacing $S_b^{\infty,\sigma,\mu}$ by $\tilde{S}_b^{\infty,\sigma,\mu}$, $S_b^{m,\sigma,\mu}$, $\tilde{S}_b^{m,\sigma,\mu}$ respectively.

DEFINITION 2.6. Let I be an open set in $\mathbb{R}^k$ and $\sigma_1 \geq 1$. We say that $a(z, x, \xi)$ belongs to $G^{(\sigma_1)}(I;\ \mathcal{A}_b^{\infty,\sigma,\mu})$ if $a(z,\cdot,\cdot) \in \mathcal{A}_b^{\infty,\sigma,\mu}$ for every $z \in I$ and for every $I' \subset I$ there exist $A > 0$, $B_0 > 0$, $B \geq 0$ such that

$$\sup_{\gamma\in\mathbb{Z}_+^k}\ \sup_{z\in I'} A^{-\gamma}\gamma!^{-\sigma_1}\|\mathrm{D}_z^\gamma a(z,\cdot,\cdot)\|_\varepsilon^{A,B_0,B} < +\infty \tag{2.9}$$

for every $\varepsilon > 0$. Here I' denotes a relatively compact open subset of I. In a similar way we definte the classes

$$G^{(\sigma_1)}(I;\ \tilde{\mathcal{A}}_b^{\infty,\sigma,\mu}),\ G^{(\sigma_1)}(I;\ \mathcal{A}_b^{m,\sigma,\mu}),\ G^{(\sigma_1)}(I;\ \tilde{\mathcal{A}}_b^{m,\sigma,\mu})$$

replacing $\|\mathrm{D}_z^\gamma a(z,\cdot,\cdot)\|_\varepsilon^{A,B_0,B}$ with $\|\mathrm{D}_z^\gamma a(z,\cdot,\cdot)\|_\varepsilon^{A,B_0,0}$, $\|\mathrm{D}_z^\gamma a(z,\cdot,\cdot)\|_m^{A,B_0,B}$, $\|\mathrm{D}_z^\gamma a(z,\cdot,\cdot)\|_m^{A,B_0,0}$, respectively.

DEFINITION 2.7. Let $\phi(x,\xi) \in \tilde{\mathcal{A}}_b^{1,\sigma,\mu}$ be real valued. We say that ϕ belongs to a class of phase-functions $\mathcal{P}^{\sigma,\mu}(\tau)$, $0 < \tau < 1$, if there exists $B_0 > 0$ such that for $J(x,\xi) = \phi(x,\xi) - \langle x, \xi\rangle$ the estimate

$$\sum_{|\alpha+\beta|\leq 2}\ \sup_{(x,\xi)\in T^n(B_0)} |\mathrm{D}_\xi^\alpha \mathrm{D}_x^\beta J(x,\xi)|\langle\xi\rangle^{|\alpha|-1} \leq \tau \tag{2.10}$$

holds.

The above class of phase-functions has been introduced by K. Taniguchi in [17]. Next we define the spaces of Fourier integral operators we shall use here.

DEFINITION 2.8. Let $a(x,\xi) \in \mathcal{A}_b^{\infty,\sigma,\mu}$ and $\phi(x,\xi) \in \mathcal{P}^{\sigma,\mu}(\tau)$. We define the Fourier integral operator $a_\phi(x,\mathrm{D}_x)$ on $G_0^{(\sigma)}(\mathbb{R}^n)$ with amplitude a and phase-function ϕ by

$$a_\phi(x,\mathrm{D}_x)u(x) = \int e^{i\phi(x,\xi)}a(x,\xi)\tilde{u}(\xi)\ \text{đ}\xi,\ \ u \in G_0^{(\sigma)}(\mathbb{R}^n), \tag{2.11}$$

$$\text{đ}\xi = (2\pi)^{-n}\mathrm{d}\xi.$$

If $\phi(x,\xi) = \langle x,\xi\rangle$ we shall write $a(x,\mathrm{D}_x)$ instead of $a_\phi(x,\mathrm{D}_x)$ and we shall call $a(x,\mathrm{D}_x)$ a pseudo-differential operator with symbol a.

THEOREM 2.9. ([2]). *Let* $a(x,\xi) \in \mathcal{A}_b^{\infty,\sigma,\mu}$ *and* $\phi(x,\xi) \in \mathcal{P}^{\sigma,\mu}(\tau)$. *Then* (2.11) *defines a continuous linear map from* $G_0^{(\sigma)}(\mathbb{R}^n)$ *to* $G^{(\sigma)}(\mathbb{R}^n)$ *which extends to a continuous linear map from* $G^{(\sigma)'}(\mathbb{R}^n)$ *to* $G_0^{(\sigma)'}(\mathbb{R}^n)$. *If* $\tau < 1/2$ *and*

$\phi(x,\theta\xi) = \theta\phi(x,\xi)$ for $\theta > 0$ and $|\xi| > B_0$, then for every $u \in G^{(\sigma)'}(\mathbb{R}^n)$ we have

$$WF_{(\sigma)}(a_\phi(x,\mathrm{D}_x)u(x)) \subset \{T_\phi(y,\eta);\ (y,\eta) \in WF_{(\sigma)}(u),\ |\eta| \ \text{large}\}^{\mathrm{con}} \tag{2.12}$$

where $T_\phi : T^*(\mathbb{R}^n) \to T^*(\mathbb{R}^n)$, called the transformation generated by ϕ, is defined by

$$T_\phi(y,\eta) = (x,\xi) \text{ if and only if } y = \nabla_\xi\phi(x,\eta) \text{ and } \xi = \nabla_x\phi(x,\eta). \tag{2.13}$$

DEFINITION 2.10. Let $\Omega \subset \mathbb{R}^n$ be an open set. A continuous linear map from $G_0^{(\sigma)}(\Omega)$ to $G^{(\sigma)}(\Omega)$ which extends to a continuous linear map from $G^{(\sigma)'}(\Omega)$ to $G^{(\sigma)}(\Omega)$ is said to be a σ-regularizing operator on Ω. The space of σ-regularizing operators on Ω will be denoted by $\mathcal{R}^\sigma(\Omega)$.

From the results of Komatsu [10] every σ-regularizing operator on Ω has a kernel in $G^{(\sigma)}(\Omega \times \Omega)$.

THEOREM 2.11. ([2]). *Let $a(x,\xi) \in \mathcal{A}_b^{\infty,\sigma,\mu}$, $a(x,\xi) \sim 0$ according to defintion 2.2. Then $a_\phi(x,\mathrm{D}_x) \in \mathcal{R}^\sigma(\Omega)$ for every $\phi \in \mathcal{P}^{\sigma,\mu}(\tau)$ and every open set $\Omega \subset \mathbb{R}^n$.*

The following result on composition of operators of type (2.11) will permit us to follow the transport equations method in section 5.

THEOREM 2.12. ([2]). *Let $p^1(x,\mathrm{D}_x)$ be a pseudo differential operator with symbol $p^1(x,\xi) \in \tilde{\mathcal{A}}_b^{\infty,\sigma,1}$ and $p_\phi^2(x,\mathrm{D}_x)$ a Fourier integral operator with amplitude $p^2(x,\xi) \in \mathcal{A}_b^{\infty,\sigma,\mu}$ and phase-function $\phi(x,\xi) \in \mathcal{P}^{\sigma,\mu}(\tau)$. Let Ω be open and convex in $\mathbb{R}^n$ and $h \in G_0^{(\sigma)}(\mathbb{R}^n)$, $h \equiv 1$ in a neighbourhood of Ω. Then*

$$p^1(x,\mathrm{D}_x)h(x)p_\phi^2(x,\mathrm{D}_x)u(x) = q_\phi(x,\mathrm{D}_x)u(x) + Ru(x)$$

for every $u \in G^{(\sigma)'}(\Omega)$, where $R \in \mathcal{R}^\sigma(\Omega)$ and the amplitude $q(x,\xi) \in \mathcal{A}_b^{\infty,\sigma,\mu}$ has an asymptotic expansion $q(x,\xi) \sim \sum_{j\geq 0} q_j(x,\xi)$ given by

$$\begin{aligned} q_j(x,\xi) &= \sum_{|\alpha|=j} \alpha!^{-1}\mathrm{D}_y^\alpha[\mathrm{D}_\xi^\alpha p^1(x,\tilde{\nabla}_x\phi(x,y,\xi))p^2(y,\xi)]_{|y=x}, \\ \tilde{\nabla}_x\phi(x,y,\xi) &= \int_0^1 \nabla_x\phi(y+\theta(x-y),\xi)\mathrm{d}\theta. \end{aligned} \tag{2.14}$$

3. - Products of phase functions

During the whole section $\lambda_j(t,x,\xi)$, $j=1,2$, will denote two real valued symbols of class $M^0([0,T];\ \tilde{\mathcal{A}}_b^{1,\sigma,1})\cap G^{(\sigma)}(I;\ \tilde{\mathcal{A}}_b^{1,\sigma,1})$, I an open set in $[0,T]$. We shall assume $\lambda_j(t,x,\theta\xi)=\theta\lambda_j(t,x,\xi)$ for $\theta>0$ and $|\xi|>B_0$. Let us consider the solutions $\phi_j(t,s;x,\xi)$ of the eikonal equations

$$(3.1)\qquad \begin{cases} \partial_t\phi_j(t,s;x,\xi)=\lambda_j(t,x,\nabla_x\phi_j(t,s;x,\xi)) \\ \phi_j(s,s;x,\xi)=\langle x,\xi\rangle \end{cases}$$

$j=1,2$. Then we have:

PROPOSITION 3.1. *There exists T_0, $0<T_0\le T$ such that the solution ϕ_j of* (3.1) *exists uniquely in* $[0,T_0]^2$ *and* $\phi_j\in M^1([0,T_0]^2;\ \tilde{\mathcal{A}}_b^{1,\sigma,\sigma})\cap G^{(\sigma)}(I^2;\ \tilde{\mathcal{A}}_b^{1,\sigma,\sigma})$. *Furthermore, the following properties hold:*

$$(3.2)\qquad \partial_s\phi_j(t,s;x,\xi)=-\lambda_j(s,\nabla_\xi\phi_j(t,s;x,\xi),\xi),$$

$$(3.3)\qquad \phi_j(t,s;x,\theta\xi)=\theta\phi_j(t,s;x,\xi)\ \textit{for}\ \theta>0\ \textit{and}\ |\xi|>B_0,$$

$$(3.4)\qquad \phi_j(t,s;\cdot,\cdot)\in\mathcal{P}^{\sigma,\sigma}(c|t-s|)\ \textit{for}\ c>0\ \textit{independent of}\ (t,s),\ j=1,2.$$

PROOF. Existence and uniqueness of the solution together with properties (3.2), (3.3) and (3.4) have been proved in [17] and [19]. Equations (3.1) and (3.2) yield $\phi_j\in G^{(\sigma)}(I^2;\ \tilde{\mathcal{A}}_b^{1,\sigma\sigma})$ since $\lambda_j\in G^{(\sigma)}(I;\ \tilde{\mathcal{A}}_b^{1,\sigma,\sigma})$.

DEFINITION 3.2. Let $j\in\{1,2\}$. We say that a curve

$$\{t,q_j(t,s;y,\eta),\ \ p_j(t,s;y,\eta)\}\subset[0,T]\times T^*(\mathbb{R}^n)$$

is the bicharacteristic curve with respect to λ_j through (s,y,η) if (q_j,p_j) satisfies the equation

$$(3.5)\qquad \begin{cases} \mathrm{d}q_j/\mathrm{d}t=-\nabla_\xi\lambda_j(t,q_j,p_j),\ \ \mathrm{d}p_j/\mathrm{d}t=\nabla_x\lambda_j(t,q_j,p_j) \\ (q_j,p_j)_{|t=s}=(y,\eta). \end{cases}$$

We denote by $\mathcal{C}_j(t,s)$ the transformation

$$(3.6)\qquad T^*(\mathbb{R}^n)\backslash\{0\}\ni(y,\eta)\to\mathcal{C}_j(t,s;y,\eta)=(q_j,p_j)(t,s;y,\eta)\in T^*(\mathbb{R}^n)\backslash\{0\}.$$

It follows from our assumptions on λ_j that

$$(3.7)\qquad q_j,|\eta|^{-1}p_j\in M^1([0,T_0]^2;\ \tilde{\mathcal{A}}_b^{0,\sigma,\sigma})\cap G^{(\sigma)}(I^2;\ \tilde{\mathcal{A}}_b^{0,\sigma,\sigma})$$

and are positively homogeneous of degree zero for $|\eta| > B_0$;

$$|q_j(t,s;y,\eta) - y| + \langle\eta\rangle^{-1}|p_j(t,s;y,\eta) - \eta| \leq c_0|t-s| \tag{3.8}$$

for a suitable constant $c_0 > 0$ and every $(t,s;y,\eta) \in [0,T]^2 \times T^*(\mathbb{R}^n)$. Furthermore we have $T_{\phi_j(t,s)} = \mathcal{C}_j(t,s)$ if $\phi_j(t,s)$ is the solution of (3.1) and $T_{\phi_j(t,s)}$, $\mathcal{C}_j(t,s)$ are the transformations defined by (2.13) and (3.6) respectively[3].

DEFINITION 3.3. Let ϕ_j, $j = 1,2$, be the solution of (3.1). We define the product $\phi_i(t,t_1)\#\phi_k(t_1,s) = \Phi_{i,k}(t,t_1,s)$, $i,k \in \{1,2\}$, as the solution of the equation

$$\begin{aligned} &\partial_t\Phi_{i,k}(t,t_1,s;x,\xi) = \lambda_i(t,x,\nabla_x\Phi_{i,k}(t,t_1,s;x,\xi)) \\ &\Phi_{i,k}(t_1,t_1,s;x,\xi) = \phi_k(t_1,s;x,\xi). \end{aligned} \tag{3.9}$$

Equations (3.1) and (3.9) can be solved by similar arguments.

THEOREM 3.4. *The solution* $\phi_{i,k}(t,t_1,s)$ *of* (3.9) *exists uniquely in* $[0,T_0]^3$, *with a smaller* T_0 *if necessary, and*

$$\Phi_{i,k} \in M^1([0,T_0]^3;\ \tilde{\mathcal{A}}_b^{1,\sigma,\sigma}) \cap G^{(\sigma)}(I^3;\ \tilde{\mathcal{A}}_b^{1,\sigma,\sigma}).$$

Furthermore we have:

$$\Phi_{i,k}(t,t_1,s;x,\theta\xi) = \theta\Phi_{i,k}(t,t_1,s;x,\xi) \qquad \textit{for } \theta > 0 \textit{ and } |\xi| > B_0, \tag{3.10}$$

$$\Phi_{i,k}(t,t_1,s;\cdot,\cdot) \in \mathcal{P}^{\sigma,\sigma}(c|t-t_1| + c|t_1-s|) \tag{3.11}$$

for $c > 0$ *independent of* (t,t_1,s),

$$T_{\Phi_{i,k}(t,t_1,s)} = \mathcal{C}_i(t,t_1)\cdot\mathcal{C}_k(t_1,s). \tag{3.12}$$

PROOF. Let $(X,\Xi)(t_1,s;y,\eta) = \mathcal{C}_k(t_1,s;y,\eta)$ and consider the solution (q,p) of

$$\begin{aligned} &\mathrm{d}/\mathrm{d}t\ q(t,t_1,s;y,\eta) = -\nabla_\xi\lambda_i(t,q,p) \\ &\mathrm{d}/\mathrm{d}t\ p(t,t_1,s;y,\eta) = \nabla_x\lambda_i(t,q,p) \\ &(q,p)_{|t=t_1} = (X,\Xi)(t_1,s;y,\eta). \end{aligned} \tag{3.13}$$

q, $|\eta|^{-1}p$ belong to $M^1([0,T_0]^3;\ \tilde{\mathcal{A}}_b^{0,\sigma,\sigma}) \cap G^{(\sigma)}(I^3;\ \tilde{\mathcal{A}}_b^{0,\sigma,\sigma})$, are positively homogeneous of degree zero for large $|\eta|$ and satisfy

$$|q(t,t_1,s;y,\eta) - X| + \langle\eta\rangle^{-1}|p(t,t_1,s;y,\eta) - \Xi| < c_0|t-t_1|.$$

[3] See [11], [17].

If we denote by

$$y = Y(t, t_1, s; x, \eta)$$

the inverse function of

$$x = q(t, t_1, s; y, \eta)$$

then we obtain the solution of (3.9) by

$$\Phi_{i,k}(t, t_1, s; x, \eta) = v(t, t_1, s; Y(t, t_1, s; x, \eta), \eta)$$

where

$$\begin{aligned} v(t, t_1, s; y, \eta) &= \phi_k(t_1, s; X(t_1, s; y, \eta), \eta) \\ &+ \int_{t_1}^{t} (p \cdot \nabla_\xi \lambda_i)(\tau, q(\tau, t_1, s), p(\tau, t_1, s)) d\tau. \end{aligned}$$

Thus all assertions of the theorem except (3.12) follow from Proposition 3.1.

By using equation (3.9) we can represent the solution $(q, p)(t, t_1; s; y, \eta)$ of (3.13) by

$$y = \nabla_\xi \Phi_{i,k}(t, t_1, s; q, \eta), \;\; p = \nabla_x \Phi_{i,k}(t, t_1, s; q, \eta).$$

Hence we have $T_{\Phi_{i,k}(t,t_1,s)}(y, \eta) = (q, p)(t, t_1, s; y, \eta) = \mathcal{C}_i(t, t_1) \circ \mathcal{C}_k(t_1, s)(y, \eta)$, completing the proof.

DEFINITION 3.5. For fixed $(x, \eta) \in \mathbb{R}^{2n}$ set

$$y = y(t, t_1, s; x, \eta) = \nabla_\xi \Phi_{i,k}(t, t_1, s; x, \eta).$$

$(X, \Xi)(t, t_1, s; x, \eta) = \mathcal{C}_k(t_1, s; y, \eta)$ is called the critical point of $\Phi_{i,k}(t, t_1, s; x, \eta)$.

The following statements hold:

PROPOSITION 3.6.

(i) $\Phi_{i,k}(t, t, s) = \phi_k(t, s)$, $\Phi_{i,k}(t, s, s) = \phi_i(t, s)$

(ii) $\Phi_{j,j}(t, t_1, s) = \phi_j(t, s)$

(iii) $X, \langle\eta\rangle^{-1}\Xi \in M^1([0, T]^3;\ \tilde{\mathcal{A}}_b^{0,\sigma,\sigma}) \cap G^{(\sigma)}(I^3;\ \tilde{\mathcal{A}}_b^{0,\sigma,\sigma})$

(iv) $\delta_1 < \langle\eta\rangle^{-1}|\Xi| < \delta_2$ *for suitable* $\delta_1, \delta_2 > 0$ *and every* $(t, t_1, s; x, \eta)$ *with* $|\eta| > B_0$

(v) (X, Ξ) *satisfies the equation*

$$\begin{aligned} X &= \nabla_\xi \phi_i(t, t_1; x, \Xi) \\ \Xi &= \nabla_x \phi_k(t_1, s; X, \eta) \end{aligned}$$

(vi) $\nabla_x \Phi_{i,k}(t, t_1, s; x, \eta) = \nabla_x \phi_i(t, t_1; x, \Xi)$

$\nabla_\xi \Phi_{i,k}(t, t_1, s; x, \eta) = \nabla_\xi \phi_k(t_1, s; X, \eta)$

(vii) $\Phi_{i,k}(t, t_1, s; x, \eta) = \phi_i(t, t_1; x, \Xi)\langle X, \Xi\rangle + \phi_k(t_1, s, X, \eta)$

(viii) $\partial_{t_1} \Phi_{i,k}(t, t_1, s) = \lambda_k(t_1, X, \Xi) - \lambda_i(t_1, X, \Xi)$

$\partial_s \Phi_{i,k}(t, t_1, s) = -\lambda_k(t, x, \nabla_x \Phi_{i,k})$.

PROOF. Statements (i) and (ii) follow directly from definition of $\Phi_{i,k}$ while (iii), (v) and (vi) are consequences of Theorem 3.4. We get (iv) by the inequality $|\Xi\eta| \leq c_0|t_1 - s|\langle\eta\rangle$ that holds for $|\eta| > B_0$. To prove (vii) we verify that $\psi(t, t_1, s) = \phi_i(t, t_1, s; X, \Xi) - \langle X, \Xi\rangle + \phi_k(t_1, s; X, \eta)$ is the solution of equation (3.9) by using Proposition 3.1, the above statements (v), (vi) and the equalities $\phi_i(t_1, t_1, x, \Xi) = x \cdot \Xi$, $X(t_1, t_1, s; x, \eta) = x$. Differentiating the equality (vii) we obtain (viii) and the proof is complete.

REMARK. Since we need only products of two phase-functions and do not use compositions between Fourier integral operators, we have given a definition of $\Phi_{i,k}$ different from that one of [11] and [17]. By means of statement (vii) in Proposition 3.6 we see that our definition is equivalent indeed to that one of [11] and [17].

Next we want to prove commutation laws for products of phase-functions.

THEOREM 3.7. *Assume that* λ_1 *and* λ_2 *satisfy one of the following alternatives:*

(3.14)(i) $\lambda_j(t, x, \xi) = \alpha(t)\mu_j(x, \xi)$, $j = 1, 2$, *with* $\alpha(t)$ *a continuous and positive function on* $[0, T]$ *and* $\{\mu_1, \mu_2\} = \beta(\mu_1 - \mu_2)$

where $\{\mu_1, \mu_2\} = \nabla_x\mu_1 \cdot \nabla_\xi\mu_2 - \nabla_x\mu_2 \cdot \nabla_\xi\mu_1$ *are the Poisson brackets of* μ_1 *and* μ_2, β *a real constant;*

(3.14)(ii) $\lambda_j(t, x, \xi) = \alpha_j(t)\mu(x, \xi)$, $\alpha_1(t)$ *and* $\alpha_2(t)$ *continuous functions on* $[0, T]$, $\alpha_1(t) \neq \alpha_2(t)$ *for every* $t \in [0, T]$.

Then there exists $\theta(t, t_1, s) \in C^1([0, T]^3)$ *such that* $s \leq \theta \leq t$ *for every* (t, t_1, s), $\theta(t, t, s) = s$, $\theta(t, s, s) = t$, $\partial_{t_1}\theta < 0$, $\theta(t, \theta, s) = t_1$ *and*

$$\Phi_{k,i}(t, \theta, s) = \Phi_{i,k}(t, t_1, s). \tag{3.15}$$

PROOF. If (3.14)(i) is satisfied we set

$$A(t) = \int_0^t \alpha(\tau)\mathrm{d}\tau, \qquad B(t) = \int_0^t \alpha(\tau)\exp(\beta A(\tau))\mathrm{d}\tau \tag{3.16}$$

and

$$\theta = B^{-1}(B(t) - B(t_1) + B(s)). \tag{3.17}$$

We have to prove (3.15) only. Let (X, Ξ) be the critical point of $\Phi_{k,i}(t, \theta, s)$ and $(q, p)(\tau)$ the bicharacteristic curve for λ_k with value (X, Ξ) at $\tau = \theta$. Then we have

$$\mathrm{d}/\mathrm{d}\tau(\mu_i - \mu_k)(q(\tau), p(\tau)) = \beta\alpha(\tau)(\mu_i - \mu_k)(q(\tau), p(\tau)) \tag{3.18}$$

and this yelds

$$(\mu_i - \mu_k)(X, \Xi) = (\mu_i - \mu_k)(x, \nabla_x\Phi_{k,i}(t, \theta, s))\exp(\beta(A(\theta) - A(t))). \tag{3.19}$$

Thus by Proposition 3.6 we obtain

$$\begin{aligned}\partial_t\Phi_{k,i}(t, \theta, s) &= \lambda_k(t, x, \nabla_x\Phi_{k,i}(t, \theta, s)) + (\lambda_i - \lambda_k)(\theta, X, \Xi)\partial_t\theta \\ &= \lambda_i(t, x, \nabla_x\Phi_{k,i}(t, \theta, s))\end{aligned}$$

and $\Phi_{k,i}(t, \theta, s)_{|t=t_1} = \Phi_{k,i}(t_1, s, s) = \phi_k(t_1, s)$, proving (3.15).

If (3.14)(ii) is satisfied we change the defintion of the function $B(t)$ in (3.16) to $B(t) = \int\limits_0^t(\alpha_1 - \alpha_2)(\tau)\mathrm{d}\tau$ and set again $\theta = B^{-1}(B(t) - B(t_1) + B(s))$. Then we repeat the foregoing arguments to obtain property (3.15) also in this case.

The following corollaries will be used in sections 5 and 6.

Corollary 3.8. *Assume that property* (3.15) *is satisfied with*

$$\theta(t, t_1, s) = \theta = B^{-1}(B(t) - B(t_1) + B(s))$$

and set $b(t) = \mathrm{d}/\mathrm{d}tB(t)$. *Then for* $p(t, t_1, s; x, \xi) \in M^0([0, T_0]^3;\ \mathcal{A}_b^{\infty,\sigma,\sigma})$ *we have*

$$\begin{aligned}&\int\limits_s^t P_{\Phi_{\imath,k}(t,t_1,s)}(t, t_1, s; x, \mathrm{D}_x)\mathrm{d}t_1 \\ &\qquad = \int\limits_s^t P_{\Phi_{k,\imath}(t,t_1,s)}(t, \theta, s; x, \mathrm{D}_x)b(t_1)/b(\theta)\mathrm{d}t_1\end{aligned} \tag{3.20}$$

Proof. Equality (3.20) easily follows from property (3.15) and equality $t_1 = \theta(t, \theta, s)$.

Corollary 3.9. *Assume the property* (3.15) *is satisfied with*

$$\theta(t, t_1, s) = \theta = B^{-1}(B(t) - B(t_1) + B(s)),$$

$B(0) = 0$, *and for* $(y, \eta) \in T^*(\mathbb{R}^n)\backslash\{0\}$, *set*

$$(x, \xi) = \mathcal{C}_{h_0}(t, t_1)\mathcal{C}_{h_1}(t_1, t_2) \cdots \mathcal{C}_{h_\nu}(t_\nu, 0)(y, \eta),$$

with $t = t_0 \geq t_1 \geq \cdots \geq t_\nu \geq t_{\nu+1} = 0$, $h_k \in \{1, 2\}$, $h_j \neq h_{j+1}$, $k = 0, \cdots, \nu$, $j = 0, \cdots, \nu - 1$. *Then we have*

$$(x, \xi) = \mathcal{C}_{h_0}(t, \theta_\nu)\mathcal{C}_{h_1}(\theta_\nu, 0)(y, \eta) \tag{3.21}$$

for

$$\theta_\nu = B^{-1}\left(\sum_{j=0}^{\nu-1}(-1)^j B(t_{\nu-j})\right). \tag{3.22}$$

PROOF. Property (3.21) is obtained by $T_{\Phi_{\iota,k}(t,\tau,s)} = \mathcal{C}_i(t,\tau) \circ \mathcal{C}_k(\tau, s)$, commutation law (3.15) and $\mathcal{C}_k(t, \tau) \circ \mathcal{C}_k(\tau, s) = \mathcal{C}_k(t, s)$.

Now we apply integration by parts to operators of the form $\int_s^t P_{\Phi_{\iota,k}}(t, t_1, s; x, \mathrm{D}_x)\mathrm{d}t_1$.

THEOREM 3.10. *Let* $p(t, t_1, s; x, \xi)$ *be a symbol of class*

$$M^0([0,T]^3;\ \mathcal{A}_b^{\infty,\sigma,\sigma}) \cap G_{t_1}^{(\sigma)}(\mathcal{I};\ \mathcal{A}_b^{\infty,\sigma,\sigma}).$$

Assume that

$$\inf_{[0,T]\times T^n(B_0)} \langle\xi\rangle^{-1}|\lambda_1(t, x, \xi) - \lambda_2(t, x, \xi)| \geq \delta > 0. \tag{3.23}$$

If $E(t, s)$ *denotes the operator* $\int_s^t P_{\Phi_{i,k}}(t, t_1, s; x, \mathrm{D}_x)\mathrm{d}t_1$ *then we have*

$$\begin{aligned} &WF_{(\sigma)}(E(t, s)u) \\ &\subset \{(x, \xi) = \mathcal{C}_j(t, s)(y, \eta);\ j = 1, 2, (y, \eta) \in WF_{(\sigma)}(u),\ |\eta| > B_0\}^{\mathrm{con}} \\ &\cup \{(x, \xi) = \mathcal{C}_i(t, t_1)\mathcal{C}_k(t_1, s)(y, \eta);\ s < t_1 < t,\ t_1 \notin \mathcal{I},\ (y, \eta) \\ &\in WF_{(\sigma)}(u), |\eta| > B_0\}^{\mathrm{con}}, \end{aligned} \tag{3.24}$$

for every $u \in G^{(\sigma)'}(\mathbb{R}^n)$.

PROOF. Take a $\psi(t_1) \in G_0^{(\sigma)}(R)$ and a $\varepsilon > 0$ such that $\psi(t_1) = 1$ on $\mathcal{I}_{-\varepsilon} = \{t_1 \in \mathcal{I};\ \mathrm{dist}(t_1, \partial\mathcal{I}) \geq \varepsilon\}$, supp $\psi(t_1) \subset \mathcal{I}_{-\varepsilon/2}$ and split $E(t, s)$ in

$E_1(t,s)+E_2(t,s)$ with $E_1(t,s)$ defined by

$$E_1(t,s)u(x)=\int_s^t \mathrm{d}t_1 \int đ\xi\ e^{i\Phi_{i,k}(t,t_1,s;x,\xi)}\psi(t_1)p(t,t_1,s;x,\xi)\tilde{u}(\xi)$$

for $u \in G_0^{(\sigma)}(\mathbb{R}^n)$,

and E_2 consequently. By Proposition 3.6 and property (3.23) we obtain

$$(\partial_{t_1}\Phi_{i,k}(t,t_1,s))^{-1} \in M^0([0,T]^2;\ \tilde{A}_b^{-1,\sigma,\sigma}) \cap G^{(\sigma)}(I^3;\ \tilde{A}_b^{-1,\sigma,\sigma}). \tag{3.25}$$

We can apply integration by parts to $E_1(t,s)$ as follows:

$$\int_s^t i\partial_{t_1}\Phi_{i,k}e^{i\Phi_{i,k}}(i\partial_{t_1}\Phi_{i,k})^{-1}\psi p\ \mathrm{d}t_1=[e^{i\Phi_{i,k}}(i\partial_{t_1}\Phi_{i,k})^{-1}\psi p]_{t_1=t}$$

$$+[e^{i\Phi_{i,k}}(i\partial_{t_1}\Phi_{i,k})^{-1}\psi p]_{t_1=s}-\int_s^t e^{i\Phi_{i,k}}\partial_{t_1}[(i\partial_{t_1}\Phi_{i,k})^{-1}\psi p]\mathrm{d}t_1$$

$$=e^{i\phi_k(t,s)}q_1^k(t,s)+e^{i\phi_i(t,s)}q_1^i(t,s)-\int_s^t e^{i\Phi_{i,k}}p_1(t,t_1,s)\mathrm{d}t_1.$$

Repeating inductively this process we obtain sequences of symbols

$$\{p_j(t,t_1,s;x,\xi)\}_{j\ge 0}\subset M^0([0,T]^3;\ A_b^{\infty,\sigma,\sigma})\cap G_{t_1}^{(\sigma)}([0,T];\ A_b^{\infty,\sigma,\sigma}),$$

$$\{q_j^i(t,s;x,\xi)\}_{j\ge 0},\ \{q_j^k(t,s;x,\xi\}_{j\ge 0}\subset M^0([0,T]^2;\ A_b^{\infty,\sigma,\sigma})$$

with

$$p_0=\psi p,\ \ p_j=(-1)^j\partial_{t_1}[p_{j-1}(i\partial_{t_1}\Phi_{i,k})^{-1}],\ \ j\ge 1 \tag{3.26}$$

$$q_j^i(t,s)=[(i\partial_{t_1}\Phi_{i,k})^{-1}p_{j-1}]_{t_1=s},\ \ j\ge 1 \tag{3.27}$$

$$q_j^k(t,s)=[(i\partial_{t_1}\Phi_{i,k})^{-1}p_{j-1}]_{|t_1=t},\ \ j\ge 1. \tag{3.28}$$

By property (3.25), $\{q_j^i\}_{j\ge 0}$ and $\{q_j^k\}_{j\ge 0}$ satisfy condition (2.4) and, from Theorem 2.3, there exist q^i, $q^k \in M^0([0,T]^2, A_b^{\infty,\sigma,\sigma})$ such that $q^\ell \sim \sum_{j\ge 1} q_j^\ell$, $\ell=i,k$. Furthermore we have $0\sim -p_0+\sum_{j\ge 0}(p_j-p_{j+1})$. Thus from Theorem 2.11 and equalities (3.26), (3.27), (3.28), $E_1(t,s)-q^i_{\phi_i}(t,s;x,\mathrm{D}_x)-q^k_{\phi_k}(t,s;x,\mathrm{D}_x)$ is a σ- regularizing operator with kernel in $M^0([0,T]^2;\ G^{(\sigma)}(\mathbb{R}^n\times\mathbb{R}^n))$.

This yield by Theorem 2.9 that $WF_{(\sigma)}(E_1(t,s)u)$ propagates along the bicharacteristic curves for λ_1 and λ_2. Since by Theorems 2.9 and 3.4 we have

$$WF_{(\sigma)}(E_2(t,s)u) \subset \{(x,\xi) = C_i(t,t_1)C_k(t_1,s)(y,\eta);\\ s < t_1 < t,\ t_1 \notin I_{-\varepsilon},\ (y,\eta) \in WF_{(\sigma)}(u), |\eta| > B_0\}^{\mathrm{con}},$$

we get estimate (3.24) letting $\varepsilon \to 0$.

4. - The Cauchy Problem

LEMMA 4.1. *Let $\rho(\tau) = c^{-1}\exp(-\langle\tau\rangle)$, $c = \int_{-\infty}^{+\infty}\exp(-\langle\tau\rangle)\mathrm{d}\tau$. For $\lambda(t,x,\xi) \in M^0([0,T];\ \tilde{\mathcal{A}}_b^{1,\sigma,1}) \cap G^{(\sigma)}(I;\ \tilde{\mathcal{A}}_b^{1,\sigma,1})$ we assume*

$$\|\lambda(t,x,\xi) - \lambda(s,x,\xi)\|_1^{A,B_0,0} \le C|t-s|^{\chi} \tag{4.1}$$

with $A, B_0, C > 0$, $0 < \chi < 1$, constants independent on $(t,s) \in [0,T]^2$.
Define an approximation of λ by

$$\begin{aligned}\lambda^a(t,x,\xi) &= \int \tilde{\lambda}(\tau,x,\xi)\rho((t-\tau)\langle\xi\rangle)\langle\xi\rangle\mathrm{d}\tau\\ &= \int \tilde{\lambda}(t-\tau\langle\xi\rangle^{-1},x,\xi)\rho(\tau)\mathrm{d}\tau\end{aligned} \tag{4.2}$$

where $\tilde{\lambda}(\tau,x,\xi) = \lambda(\tau,x,\xi)$ for $\tau \in [0,T], \tilde{\lambda}(\tau,x,\xi) = \lambda(0,x,\xi)$ for $\tau < 0$, $\tilde{\lambda}(\tau,x,\xi) = \lambda(T,x,\xi)$ for $\tau > T$. Then the following properties hold:

(i) $\lambda^a(t,x,\xi) \in M^0([0,T];\ \tilde{\mathcal{A}}_b^{1,\sigma,1}) \cap M^1([0,T];\ \tilde{\mathcal{A}}_b^{2-\chi,\sigma,1}) \cap G^{(\sigma)}(I;\ \tilde{\mathcal{A}}_b^{1,\sigma,\sigma})$

(ii) $\lambda - \lambda^a \in M^0([0,T];\ \tilde{\mathcal{A}}_b^{1-\chi,\sigma,1}) \cap G^{(\sigma)}(I;\ \tilde{\mathcal{A}}^{0,\sigma,\sigma})$.

PROOF. Write

$$\partial_t\lambda^a = \int [\tilde{\lambda}(\tau,x,\xi) - \lambda(t,x,\xi)](t-\tau)^{-\chi}(t-\tau)^{\chi}\rho'((t-\tau)\langle\xi\rangle)\langle\xi\rangle^2\mathrm{d}\tau,$$

$\rho'(\tau) = \mathrm{d}/\mathrm{d}\tau\rho(\tau)$, and

$$\lambda^a - \lambda = \int [\tilde{\lambda}(\tau,x,\xi) - \lambda(t,x,\xi)](t-\tau)^{-\chi}(t-\tau)^{\chi}\rho((t-\tau)\langle\xi\rangle)\langle\xi\rangle\mathrm{d}\tau.$$

By using (4.1) and the change of variable $t - \tau = \tau_1\langle\xi\rangle^{-1}$ we can prove $\lambda^a \in M^1([0,T];\ \tilde{\mathcal{A}}_b^{2-\chi,\sigma,1})$ and $\lambda^a - \lambda \in M^0([0,T];\ \tilde{\mathcal{A}}_b^{1-\chi,\sigma,1})$. To end the proof take $\psi \in G_0^{(\sigma)}(R)$ and $\varepsilon > 0$ such that $0 \leq \psi \leq 1$, $\psi = 1$ on $I_{-\varepsilon} = \{\tau \in I;\ \mathrm{dist}(\tau, \partial I) \geq \varepsilon\}$, supp $\psi \subset I_{-\varepsilon/2}$ and split λ^a in two parts

$$\lambda^a(t,x,\xi) = \int \psi(t - \tau\langle\xi\rangle^{-1})\tilde{\lambda}(t - \tau\langle\xi\rangle^{-1}, x, \xi)\rho(\tau)\mathrm{d}\tau$$
$$+ \int (1 - \psi(\tau))\tilde{\lambda}(\tau, x, \xi)\rho((t-\tau)\langle\xi\rangle)\langle\xi\rangle\mathrm{d}\tau = \overline{\lambda}^a + \overline{\overline{\lambda}}{}^a.$$

It is easy to see that $\overline{\lambda}^a \in G^{(\sigma)}([0,T];\ \tilde{\mathcal{A}}_b^{1,\sigma,\sigma})$. Since $1 - \psi(\tau) = 0$ for $\tau \in I_{-\varepsilon}$, if $t \in I_{-2\varepsilon}$ we have

$$|\mathrm{D}_t^\gamma \mathrm{D}_x^\beta \mathrm{D}_\xi^\alpha \overline{\overline{\lambda}}{}^a(t,x,\xi)| \leq \varepsilon^{-\gamma} \int |t-\tau|^\gamma |\mathrm{D}_x^\beta \mathrm{D}_\xi^\alpha [\tilde{\lambda}(\tau,x,\xi)\rho^{(\gamma)}((t-\tau)\langle\xi\rangle)]| \langle\xi\rangle^{1+\gamma}\mathrm{d}\tau$$

and we can complete the proof of (i). By similar arguments we can complete also the proof of (ii).

Let us now consider a differential operator

$$\begin{aligned} P(t,x,\mathrm{D}_t,\mathrm{D}_x) &= \mathrm{D}_t^2 + \left(\sum_{h=1}^n a_h(t,x)\mathrm{D}_{x_h}\right)\mathrm{D}_t \\ &+ \sum_{j,k=1}^n b_{j,k}(t,x)\mathrm{D}_{x_j}\mathrm{D}_{x_k} + c(t,x)\mathrm{D}_t \\ &+ \sum_{\ell=1}^n d_\ell(t,x)\mathrm{D}_{x_\ell} + e(t,x). \end{aligned} \tag{4.3}$$

Hereafter we assume that the characteristic roots $\lambda_j'(t,x,\xi)$, $j = 1,2$, of P are real and satisfy

$$\inf_{[0,T]\times T^n(B_0)} \langle\xi\rangle^{-1}|\lambda_1'(t,x,\xi) - \lambda_2'(t,x,\xi)| \geq \delta > 0. \tag{4.4.}$$

We shall denote by λ_1 and λ_2 two real valued functions such that

$$\lambda_j(t,\cdot,\cdot) \in C^\infty(\mathbb{R}^{2n}),\ \lambda_j = \lambda_j' \text{ on } T^n(B_0),\ j = 1,2,$$

and assume:

$$\lambda_j \in M^0([0,T];\ \tilde{\mathcal{A}}_b^{1,1,1}) \cap G^{(\sigma)}(I;\ \tilde{\mathcal{A}}_b^{1,1,1}) \tag{4.5}$$

$$\|\lambda_j(t,x,\xi) - \lambda_j(s,x,\xi)\|_1^{A,B_0,0} \leq C|t-s|^\chi,\ j = 1,2 \tag{4.6}$$

with $A, B_0, C > 0$, $0 < \chi < 1$, constants independent on $(t,s) \in [0,T]^2$;

$$c(t,x),\ d_\ell(t,x),\ e(t,x) \in M^0([0,T];\ G^{(1)}(\mathbb{R}^n)) \cap G_b^{(\sigma)}(I' \times \mathbb{R}^n) \tag{4.7}$$

for every relatively compact open $I' \subset I$;

$$1 < \sigma < 1/(1-\chi). \tag{4.8}$$

THEOREM 4.2. *Assume for the operator P defined by* (4.3) *all conditions* (4.4)-(4.8). *Then the Cauchy problem*

$$\begin{cases} P(t,x;D_t,D_x)u(t,x) = 0 & \text{in } [0,T]\times\mathbb{R}^n \\ D_t^j u(0,x) = g_j(x), & j = 0,1, \end{cases} \tag{4.9}$$

is well-posed for data $g_j \in G_0^{(\sigma)'}(\mathbb{R}^n)$ and can be reduced to the system form

$$\begin{cases} LU(t,x) = 0 \\ U(0,x) = G(x) \end{cases} \tag{4.10}$$

for

$$L = D_t - \begin{vmatrix} \lambda_1(t,x,D_x) & 0 \\ 0 & \lambda_2(t,x,D_x) \end{vmatrix} + (a_{i,j}(t,x,D_x))_{i,j=1,2}, \tag{4.11}$$

$a_{i,j} \in M^0([0,T];\ \tilde{\mathcal{A}}_b^{1-\chi,1,1}) \cap G^{(\sigma)}(I;\ \tilde{\mathcal{A}}_b^{1-\chi,\sigma,\sigma})$.

REMARK. The well-posedness of the problem (4.9) in the classes $G^{(\sigma)}$ and $G_0^{(\sigma)'}$ has been proved in [4] and [14]. By the results of [4], condition (4.8) is optimal. The well-posedness of the problem (4.9) in the Gevrey classes of exponent σ could be deduced from this reduction and the results of [12] and [17]. See also [17] for a reduction to a system form.

PROOF. Let $\lambda_j^a(t,x,\xi)$ be the approximations of $\lambda_j(t,x,\xi)$, $j=1,2$, defined by equality (4.2). From Lemma 4.1 we get

$$P(t,x;D_t,D_x) = (D_t - \lambda_2^a(t,x,D_x) + c(t,x))(D_t - \lambda_1^a(t,x,D_x)) + R(t,x,D_x)$$

with a symbol $R \in M^0([0,T];\ \tilde{\mathcal{A}}_b^{2-\chi,1,1}) \cap G^{(\sigma)}([0,T];\ \tilde{\mathcal{A}}_b^{2-\chi,\sigma,\sigma})$.

For $u(t,x)$ satisfying $Pu = 0$ put $u_1 = \langle D_x\rangle u$, $u_2 = (D_t - \lambda_1^a)u$ and $U = {}^t(u_1,u_2)$. Then U satisfies $L_1U = 0$ for a system

$$L_1 = D_t - \begin{vmatrix} \lambda_1^a & \langle D_x\rangle \\ 0 & \lambda_2^a \end{vmatrix} + \begin{vmatrix} [\lambda_1^a, \langle D_x\rangle]\langle D_x\rangle^{-1} & 0 \\ R\langle D_x\rangle^{-1} & c(t,x) \end{vmatrix}.$$

To diagonalize the principal part of L_1 we set

$$M(t,x,D_x) = \begin{vmatrix} 1 & m(t,x,D_x) \\ 0 & 1 \end{vmatrix}$$

with $m(t,x,\xi) = \langle\xi\rangle(\lambda_2^a - \lambda_1^a)^{-1}(t,x,\xi)$ for $|\xi| > B_0$; we have

$$m \in M^0([0,T];\ \tilde{\mathcal{A}}_b^{0,1,1}) \cap M^1([0,T];\ \tilde{\mathcal{A}}_b^{1-\chi,1,1}) \cap G^{(\sigma)}(I;\ \tilde{\mathcal{A}}_b^{0,\sigma,\sigma})$$

and $M^{-1} = \begin{vmatrix} 1 & -m \\ 0 & 1 \end{vmatrix}$. Defining L by $L = M^{-1}L_1M$ the proof is complete taking into account the property (ii) of $\lambda - \lambda^a$ in Lemma 4.1.

5. - Transport equations

In this section we assume for the operator P defined by (4.3) all conditions (4.4)-(4.8) and one of the alternatives (3.14) to be satisfied by the characteristic roots. $\theta(t,t_1,s) = B^{-1}(B(t) - B(t_1) + B(s))$ will be the function for which commutation law (3.15) holds. Δ will denote the set

$$\Delta = \{(t,t_1,s) \in [0,T_0]^3;\ \ s \le t_1 \le t \text{ or } s \ge t_1 \ge t\}$$

and Θ the operator from $M^0(\Delta;\ \mathcal{A}_b^{\infty,\sigma,\mu})$ to $M^0(\Delta;\ \mathcal{A}_b^{\infty,\sigma,\mu})$ defined by

$$\Theta p(t,t_1,s;x,\xi) = -p(t,\theta,s;x,\xi)\partial_{t_1}\theta. \tag{5.1}$$

Note that $\sup_{t_1\in[s,t]} \|p(t,t_1,s)\|_\varepsilon^{A,B,B_0} = \sup_{t_1\in[s,t]} \|p(t,\theta,s)\|_\varepsilon^{A,B,B_0}$ for every A,B,B_0, $\varepsilon > 0$.

For the diagonal part of the system L in (4.11) we put

$$p_j(t,x,\mathrm{D}_t,\mathrm{D}_x) = \mathrm{D}_t - \lambda_j(t,x,\mathrm{D}_x) + a_{j,j}(t,x,\mathrm{D}_x),\ \ j = 1,2.$$

A parametrix for p_j has been constructed in [2] with the method of transport equations[4]; in this section we use the commutation law for products of phase-functions obtained in Theorem 3.7 and Corollary 3.8 to show how the methods of [2] can be applied to the problem (4.10).

PROPOSITION 5.1. ([2]). *Let Ω be open and convex in $\mathbb{R}^n$ and let $h \in G_0^{(\sigma)}(\mathbb{R}^n)$, $h = 1$ in a neighbourhood of Ω. There exist*

$$e^j(t,s;x,\xi) \in M^0([0,T_0];\ \mathcal{A}_b^{\infty,\sigma,\sigma}) \cap G^{(\sigma)}(I;\ \mathcal{A}_b^{\infty,\sigma,\sigma})$$

independent of Ω and h such that:

$$p_j(t,x;\mathrm{D}_t,\mathrm{D}_x)h(x)e^j_{\phi_j}(t,s;x,\mathrm{D}_x)u = \mathbb{R}_\Omega u \qquad \textit{for every } u \in G^{(\sigma)'}(\Omega), \tag{5.2}$$

[4] See [17] for a construction with the method of multi-products of Fourier integral operators.

with $\mathbb{R}_\Omega \in M^0([0,T_0]^2;\ \mathcal{R}^{(\sigma)}(\Omega))$;

$$e^j_{\phi_j}(s,s) = 1. \tag{5.3}$$

Take now $p^{k,\ell}(t,t_1,s;x,\xi)$ of class $M^0(\Delta;\ \mathcal{A}_b^{\infty,\sigma,\sigma}) \cap M^1_t([0,T];\ \mathcal{A}_b^{\infty,\sigma,\sigma})$, $k,\ell \in \{1,2\}$, to be determinated later on as asymptotic expansions

$$p^{k,\ell} \sim \sum_{j\geq 0} p_j^{k,\ell}(t,t_1,s;x,\xi), \qquad \mathrm{D}_t p^{k,\ell} \sim \sum_{j\geq 0} \mathrm{D}_t p_j^{k,\ell}(t,t_1,s;x,\xi),$$

and set

$$\mathcal{E}(t,s) = \begin{vmatrix} e^1_{\phi_1}(t,s) & 0 \\ 0 & e^2_{\phi_2}(t,s) \end{vmatrix} + \left(\int_s^t p^{k,\ell}_{\Phi_{k,j(k)}}(t,t_1,s)\mathrm{d}t_1 \right)_{k,\ell=1,2} \tag{5.4}$$

with $e^j(t,s;x,\xi)$, $j=1,2$, the symbols described in Proposition 5.1, $j(1)=2$, $j(2)=1$. Denote by $\tilde{p}_j^{k,\ell}(t,t_1,s)$ the symbols defined by

$$\tilde{p}_j^{k,\ell}(t,t_1,s;y_k(t,t_1,s;x,\xi),\xi) = p_j^{k,\ell}(t,t_1,s;x,\xi),$$

$k,\ell = 1,2,\ \ j \geq 0$,

$$y_k(t,t_1,s;x,\xi) = \nabla_\xi \Phi_{k,j(k)}(t,t_1,s;x,\xi).$$

For the operator L of (4.11), h and Ω as in Proposition 5.1, from Theorem 2.12 and Corollary 3.8 we have:

$$\begin{aligned} Lh\mathcal{E}(t,s) = {} & \begin{vmatrix} 0 & \tilde{e}^2_{\phi_2}(t,s) \\ \tilde{e}_{\phi_1}(t,s) & 0 \end{vmatrix} i(p^{k,\ell}_{\phi_{j(k)}}(t,t,s))_{k,\ell=1,2} \\ & + \left(\int_s^t q^{k,\ell}_{\Phi_{k,j(k)}}(t,t_1,s)\mathrm{d}t_1 \right)_{k,\ell=1,2} \\ & + (r^{k,\ell}(t,s))_{k,\ell=1,2} \qquad \text{on } G^{(\sigma)'}(\Omega), \end{aligned} \tag{5.4$_L$}$$

where $r^{k,\ell} \in M^0([0,T]^2;\ \mathcal{R}^{(\sigma)}(\Omega))$,

$$\tilde{e}^k_{\phi_k}(t,s) = a_{j(k),k}(t)he^k_{\phi_k}(t,s) \bmod M^0([0,T]^2;\ \mathcal{R}^{(\sigma)}(\Omega)),$$

$$q^{k,\ell}(t,t_1,s;x,\xi) = \tilde{q}^{k,\ell}(t,t_1,s;y_k(t,t_1,s;x,\xi),\xi)$$

and $\tilde{q}^{k,\ell}$ are the asymptotic sums $\tilde{q}^{k,\ell} \sim \sum_{j\geq 0} \tilde{q}_j^{k,\ell}$ defined by:

$$(\tilde{q}_0^{k,\ell}(t,t_1,s;y,\xi))_{k,\ell=1,2} = (\mathrm{D}_t - \Lambda)(\tilde{p}_0^{k,\ell}(t,t_1,s;y,\xi))_{k,\ell=1,2} \tag{5.5$_0$}$$

$$(5.5)_j \qquad (\tilde{q}_j^{k,\ell}(t,t_1,s;y,\xi))_{k,\ell=1,2} = (\mathrm{D}_t - \Lambda)(\tilde{p}_j^{k,\ell}(t,t_1,s;y,\xi))_{k,\ell=1,2} - (\tilde{s}_j^{k,\ell}(t,t_1,s;y,\xi)), \qquad j \geq 1,$$

$$\Lambda = \begin{vmatrix} \tilde{\alpha}_{11}(t,t_1,s) & \tilde{\alpha}_{1,2}(t,t_1,s)\Theta \\ \tilde{\alpha}_{2,1}(t,t_1,s)\Theta & \tilde{\alpha}_{2,2}(t,t_1,s) \end{vmatrix}, \qquad \Theta \text{ defined by equality (5.1)},$$

$$\tilde{\alpha}_{k,\ell}(t,t_1,s;y,\xi) = \alpha_{k,\ell}(t,t_1,s;x_k(t,t_1,s;y,\xi),\xi),$$

$x_k(t,t_1,s;y,\xi)$ the inverse function of $y = y_k(t,t_1,s;x,\xi)$,

$$\alpha_{k,k} = 1/2i \sum_{j,h=1}^{n} \partial_{\xi_j}\partial_{\xi_h}\lambda_k(t,x,\nabla_x\Phi_{k,j(k)}(t,t_1,s))\partial_{x_j}\partial_{x_h}\Phi_{k,j(k)}(t,t_1,s) + a_{k,k}(t,x,\nabla_x\Phi_{k,j(k)}(t,t_1,s)),$$

$$\alpha_{k,j(k)} = a_{k,j(k)}(t,x,\nabla_x\Phi_{k,j(k)}(t,t_1,s)),$$

$$(\tilde{s}_j^{k,\ell}(t,t_1,s))_{k,\ell=1,2} = \sum_{h=1}^{j} \sum_{|\beta|\leq h+1} \begin{vmatrix} \tilde{\rho}_{\beta,h}^{1,1} & \tilde{\rho}_{\beta,h}^{1,2}\Theta \\ \tilde{\rho}_{\beta,h}^{2,1}\Theta & \tilde{\rho}_{\beta,h}^{2,2} \end{vmatrix} (\mathrm{D}_y^\beta \tilde{p}_{j-h}^{k,\ell})_{k,\ell=1,2}$$

$$\tilde{\rho}_{\beta,h}^{k,\ell}(t,t_1,s;y,\xi) = \rho_{\beta,h}^{k,\ell}(t,t_1,s;x_k(t,t_1,s;y,\xi),\xi),$$

$$\sum_{|\beta|<h+1} \rho_{\beta,h}^{k,k}(t,t_1,s;x,\xi)\mathrm{D}_y^\beta p_{j-h}^{k,\ell}(t,t_1,s;x,\xi)$$

$$= \sum_{|\alpha|=h+1} \alpha!^{-1}\mathrm{D}_z^\alpha(\partial_\xi^\alpha\lambda_k(t,x,\tilde{\nabla}_x\Phi_{k,j(k)}(t,t_1,s;x,z,\xi)p_{j-h}^{k,\ell}(t,t_1,s;z,\xi))_{|z=x}$$

$$+ \sum_{|\alpha|=h} \alpha!^{-1}\mathrm{D}_z^\alpha(\partial_\xi^\alpha a_{k,k}(t,x,\tilde{\nabla}_x\Phi_{k,j(k)}(t,t_1,s;x,z,\xi)p_{j-k}^{k,\ell}(t,t_1,s;z,\xi))_{|z=x},$$

$$\sum_{|\beta|\leq h+1} \rho_{\beta,h}^{k,j(k)}(t,t_1,s;x,\xi)\mathrm{D}_y^\beta p_{j-h}^{k,\ell}(t,t_1,s;x,\xi)$$

$$= \sum_{|\alpha|=h} \alpha!^{-1}\mathrm{D}_z^\alpha(\partial_\xi a_{k,j(k)}(t,x,\tilde{\nabla}_x\Phi_{k,j(k)}(t,t_1,s;x,z,\xi)p_{j-h}^{k,\ell}(t,t_1,s;z,\xi))_{z=x},$$

$$\tilde{\nabla}_x\Phi_{k,j(k)}(t,t_1,s;x,z,\xi) = \int_0^1 \nabla_x\Phi_{k,j(k)}(t,t_1,s;z+\tau(x-z),\xi)\mathrm{d}\tau.$$

Thus by Theorem 2.11, equality $(5.4)_L$, $(5.5)_0$ and $(5.5)_j$, we shall have $Lh\mathcal{E}(t,s)$ equal to a σ-regularizing operator taking as $\tilde{p}_j^{k,\ell}$ the solution of the transport equations

$$(5.6)_0 \qquad \begin{aligned} &(\mathrm{D}_t - \Lambda(t,t_1,s))(\tilde{p}_0^{k,\ell}) = 0 \\ &(\tilde{p}_0^{k,\ell}(t_1,t_1,s)) = -i \begin{vmatrix} 0 & \tilde{e}^2(t_1,s) \\ \tilde{e}^1(t_1,s) & 0 \end{vmatrix} \end{aligned}$$

$$(5.6)_j \qquad \begin{aligned} &(\mathrm{D}_t - \Lambda(t,t_1,s))(\tilde{p}_j^{k,\ell}) = (\tilde{s}_j^{k,\ell}) \\ &(\tilde{p}_j^{k,\ell}(t_1,t_1,s)) = 0, \qquad j \geq 1. \end{aligned}$$

From the assumptions of this section the derivatives $\mathrm{D}_\xi^\alpha \mathrm{D}_y^\beta \tilde{p}_j^{k,\ell}$ can be estimated in the same way used for the solutions of the equations $(3.13)_0$ and $(3.13)_h$ in [2], taking there $m = 1$. The key point of this section is the use of commutation law established in Theorem 3.7 and of equality (3.20) in Corollary 3.8 to get equalities $(5.4)_L$, $(5.5)_0$ and $(5.5)_j$. We mean that, denoting by $\mathcal{H}$ the set of all operators $\mathcal{E}(t,s)$ of the form (5.4), we have $L\mathcal{H} \subset \mathcal{H}$. In fact the action of L on $\mathcal{E}(t,s)$ defined by (5.4) produces terms of the form

$$\int_s^t p_{\Phi_{1,2}}(t,t_1,s)\mathrm{d}t_1 + \int_s^t q_{\Phi_{2,1}}(t,t_1,s)\mathrm{d}t_1$$

that we reduce to $\int_s^t [p+\Theta q]_{\Phi_{1,2}}(t,t_1,s)\mathrm{d}t_1$ or to $\int_s^t [\Theta p+q]_{\Phi_{2,1}}(t,t_1,s)\mathrm{d}t_1$ with the aid of equality (3.20). Without this tool the method of transport equations developed in [2] cannot be applied to an operator in the system form (4.11) to construct a parametrix of the form (5.4).

Thus summing up we can state:

THEOREM 5.2. *Assume all conditions* (4.4)-(4.8) *and one of the alternatives* (3.14) *to be satisfied for the operator P defined by* (4.3) *and its characteristic roots. Let L be the operator obtained in Theorem* 4.2.

Then there exists an operator $\mathcal{E}(t,s)$ of the form (5.4) *such that for every open convex set $\Omega \subset \mathbb{R}^n$ and $h \in G_0^{(\sigma)}(\mathbb{R}^n)$, $h = 1$ in a neighbourhood of Ω, we have*

(5.7) *$Lh\mathcal{E}(t,s)$ is a σ-regularizing operator on Ω with kernel in* $M^0([0,T_0]^2;\ G^{(\sigma)}(\mathbb{R}^n \times \mathbb{R}^n))$,

(5.8) $\mathcal{E}(s,s) = I$.

In the next section we shall also need the following:

PROPOSITION 5.3. *Let all assumptions of Theorem* 5.2 *hold and* $\mathcal{I}'$ *be an open interval,* $\mathcal{I}' \subset \mathcal{I}$, $\mathcal{I}$ *the open set satisfying conditions* (4.5) *and* (4.7). *If* $s, t \in \mathcal{I}'$, $s < t$, *then the solutions of equations* $(5.6)_0$ *and* $(5.6)_j$, $j \geq 1$, *are of class* $G^{(\sigma)}_{t_1}([s,t];\ \mathcal{A}_b^{\infty,\sigma,\sigma})$.

PROOF. To prove this property for the solution of $(5.6)_0$ we have to consider systems of the type

$$(5.9)\qquad \begin{cases} \mathrm{D}_t p(t,t_1,s) = \hat{\alpha}_{1,1}(t,t_1,s)p(t,t_1,s) + \hat{\alpha}_{1,2}(t,t_1,s)q(t,\theta,s) \\ \mathrm{D}_t q(t,t_1,s) = \hat{\alpha}_{2,1}(t,t_1,s)p(t,\theta,s) + \hat{\alpha}_{2,2}(t,t_1,s)q(t,t_1,s) \end{cases}$$

with $\theta = B^{-1}(B(t) - B(t_1) + B(s))$. Since $s, t \in \mathcal{I}' \subset \mathcal{I}$ we may assume $p(t_1,t_1,s)$, $q(t_1,t_1,s) \in G^{(\sigma)}_{t_1}(\mathcal{I}';\ \mathcal{A}_b^{\infty,\sigma,\sigma})$, $\theta \in G^{(\sigma)}(\Delta')$,

$$\Delta' = \{(t,t_1,s) \in \Delta;\ s,t \in \mathcal{I}'\},$$

$\hat{\alpha}_{k,\ell} \in G^{(\sigma)}(\mathcal{I}'^3;\ \tilde{\mathcal{A}}_b^{1-\chi,\sigma,\sigma})$. Setting

$$\hat{p}(t,t_1,s) = p(t,t_1,s)\exp(-i\int_{t_1}^{t} \alpha_{1,1}(\tau,t_1,s)\mathrm{d}\tau),$$

$$\hat{q}(t,t_1,s) = q(t,t_1,s)\exp(-i\int_{t_1}^{t} \alpha_{2,2}(\tau,t_1,s)\mathrm{d}\tau)$$

and then $\hat{p}(t,t_1,s) = \hat{\hat{p}}(t_1,\theta,s)$, $\hat{q}(t,t_1,s) = \hat{\hat{q}}(\theta,t_1,s)$ we reduce problem (5.9) to

$$(5.10)\qquad \begin{cases} \mathrm{D}_{t_1}\hat{\hat{p}}(t,t_1,s) = \hat{\hat{\alpha}}(t,t_1,s)\hat{\hat{q}}(t,t_1,s) \\ \mathrm{D}_t\hat{\hat{q}}(t,t_1,s) = \hat{\hat{\beta}}(t,t_1,s)\hat{\hat{p}}(t,t_1,s) \end{cases}$$

with initial conditions

$$(5.11)\qquad \begin{aligned} \hat{\hat{p}}(t,s,s) &= \hat{\hat{e}}_1(t,s) \in G^{(\sigma)}(\mathcal{I}'^2;\ \mathcal{A}_b^{\infty,\sigma,\sigma}), \\ \hat{\hat{q}}(s,t_1,s) &= \hat{\hat{e}}_2(t_1,s) \in G^{(\sigma)}(\mathcal{I}'^2;\ \mathcal{A}^{\infty,\sigma,\sigma}), \end{aligned}$$

and coefficients $\hat{\hat{\alpha}}$, $\hat{\hat{\beta}} \in G^{(\sigma)}(\mathcal{I}'^3;\ \mathcal{A}_b^{1-\chi,\sigma,\sigma})$. From

$$\hat{\hat{p}}(t,t_1,s) = \hat{\hat{e}}_1(t,s) + i\int_{s}^{t_1} \hat{\hat{\alpha}}(t,\tau,s)\hat{\hat{q}}(t,\tau,s)\mathrm{d}\tau$$

and

$$\hat{q}(t,t_1,s) = \hat{e}_2(t_1,s) + i\int_s^t \hat{\beta}(\tau,t_1,s)\hat{p}(\tau,t_1,s)\mathrm{d}\tau$$

we can prove inductively on the order of the derivatives that

$$\hat{p},\ \hat{q} \in G^{(\sigma)}_{(t,t_1)}(\mathcal{I}'^2;\ \mathcal{A}_b^{\infty,\sigma,\sigma}),$$

hence the solution (p,q) of a system of type (5.9) is of class $G^{(\sigma)}_{(t,t_1)}(\mathcal{I}'^2;\ \mathcal{A}_b^{\infty,\sigma,\sigma})$. This proves the requested property of the solutions of the equation $(5.6)_0$. By the same method we can treat the solutions of equations $(5.6)_j$, $j \geq 1$, completing the proof by induction on j.

6. - Propagation of Gevrey singularities

In this section we consider the Cauchy problem (4.9) for the operator P defined by equality (4.3) and its reduction to the first order system from (4.10) for L given by (4.11), assuming all hypothesis of Theorem 5.2 to be satisfied.

Since problems (4.9) and (4.10) have the finite speed of propagation property[5] we may assume the initial data for $t=0$ to be with compact support. We need the following:

DEFINITION 6.1. Let $\lambda_j \in M^0([0,T];\ \tilde{\mathcal{A}}_b^{1,1,1}) \cap G^{(\sigma)}(\mathcal{I};\ \tilde{\mathcal{A}}_b^{1,1,1})$ be the symbols satisfying conditions (4.4), (4.5), (4.6) and (3.14). Take

$$t = t_0 \geq t_1 \geq t_2 \cdots \geq t_\nu \geq t_{\nu+1} = 0, \qquad \nu \geq 1,$$

and $h_k \in \{1,2\}$, $h_j \neq h_{j+1}$, $k = 0,\cdots,\nu$, $j = 0,\cdots,\nu-1$. A point

$$(x,\xi) = \mathcal{C}_{h_0}(t,t_1) \circ \mathcal{C}_{h_1}(t_1,t_2) \circ \mathcal{C}_{h_\nu}(t_\nu,0)(y,\eta)$$

will be called an end point at $t = t_0$ of a trajectory of step ν with initial point (y,η). If V is a closed conic set in $T^*(\mathbb{R}^n)\backslash\{0\}$ and $\mathcal{F}$ a closed set in $[0,T]$, we shall denote by $\Gamma_\nu(t_0,\mathcal{F},V)$ the conic hull in $T^*(\mathbb{R}^n)\backslash\{0\}$ of the set of all end points at $t = t_0$ of trajectories of step ν with initial point $(y,\eta) \in V$, $|\eta| > B_0$, and such that $t_j \in \mathcal{F}$, $j = 1,\cdots,\nu$.

To have uniform notations we set

$$\Gamma_0(t_0,\mathcal{F},V) = \{(x,\xi) = \mathcal{C}_i(t_0,0)(y,\eta);\ i = 1,2,(y,\eta) \in V,\ |\eta| > B_0\}^{\mathrm{con}}.$$

Corollary 3.9 can be restated in the following way:

[5] See [8].

PROPOSITION 6.2. *Let $\mathcal{F} \subset [0,T]$ be a closed set and $V \subset T^*(\mathbb{R}^n)\backslash\{0\}$ a closed conic set. Then*

$$\overline{\bigcup_{\nu\geq 1} \Gamma_\nu(t_0, \mathcal{F}, V)} = \Gamma_1\left(t_0, \overline{\bigcup_{\nu\geq 1} \mathcal{F}_\nu}, V\right) \tag{6.1}$$

with

$$\mathcal{F}_\nu = \left\{ \theta_\nu = B^{-1}\left(\sum_{j=0}^{\nu-1}(-1)^j B(t_{\nu-j})\right);\ t_0 \geq t_1 \geq \cdots \geq t_\nu \geq 0,\right. \tag{6.2}$$

$$\left. t_j \in \mathcal{F},\ j = 1, \cdots, \nu \right\}.$$

DEFINITION 6.3. Let $\mathcal{F} \subset [0, T_0]$ be a closed set. Denote by Π the set of all partitions $t_0 \geq t_1 \geq \cdots \geq t_\nu \geq 0$ and set:

$$\delta(\mathcal{F}, \pi) = \sum_{*} \sup_{\tau\in[t_k, t_{k-1}]} \operatorname{dist}(\tau, \mathcal{F}) \tag{6.3}$$

for $\pi \in \Pi$, where $\sum_*$ is performed on the indexes k such that $[t_k, t_{k-1}] \cap \mathcal{F} \neq \emptyset$; then put

$$\delta(\mathcal{F}) = \inf_{\pi\in\Pi} \delta(\mathcal{F}, \pi). \tag{6.4}$$

We say that $\mathcal{F}$ has the property (H) if $\delta(\mathcal{F}) = 0$.

REMARK. If the boundary of $\mathcal{F}$ has Lebesgue measure equal to zero, then $\mathcal{F}$ has the property (H).

We can now prove:

THEOREM 6.4. *Assume all hypothesis of Theorem* 5.2 *to be satisfied. Let us consider the Cauchy problem* (4.9) *for the operator P defined by equality* (4.3) *and its reduction to the system form* (4.10) *for L given by* (4.11) *obtained in Theorem* 4.2. *If $u(t,x)$ and $U(t,x)$ are the solution of problems* (4.9) *and* (4.10) *respectively, we have*

$$\bigcup_{j=0}^{1} WF_{(\sigma)}(D_t^j u(t_0, \cdot)) = WF_{(\sigma)}(U(t_0, \cdot)) \quad \textit{for every } t_0,\ 0 \leq t_0 \leq T_0; \tag{6.5}$$

$$WF_{(\sigma)}(U(t_0, \cdot)) \subset \Gamma_1(t_0, [0, t_0],\ WF_{(\sigma)}(G)). \tag{6.6}$$

Furthermore, if $\mathcal{F} \cap [0, t_0]$ *has the property* (H) *for* $\mathcal{F} = [0, T_0] \backslash \mathcal{I}$, *then the following estimate holds:*

$$
\begin{aligned}
WF_{(\sigma)}(U(t_0, \cdot\,)) &\subset \overline{\bigcup_{\nu \geq 0} \Gamma_\nu(t_0, \mathcal{F}, WF_\sigma(G))} \\
&= \bigcup_{\mu=0}^{1} \Gamma_\mu \left(t_0, \overline{\bigcup_{\nu \geq 1} \mathcal{F}_\nu}, \ WF_{(\sigma)}(G) \right)
\end{aligned}
\tag{6.7}
$$

with $\mathcal{F}_\nu \subset [0, T_0]$ *defined by* (6.2).

REMARK. If $\mathcal{I} = [0, T_0]$ then $\mathcal{F} = \emptyset$ and estimate (6.7) is a particular case of the results of [12] and [17] for operators with coefficients in $G_b^{(\sigma)}([0, T_0] \times \mathbb{R}_x^n)$.

PROOF. Equality (6.5) has been proved in Theorem 4.2. Since problem (4.10) is well posed in the classes $G^{(\sigma)}(\mathbb{R}^n)$ and $G_0^{(\sigma)'}(\mathbb{R}^n)$ and has the finite speed of propagation property, we may assume that the initial data $G(x)$ has compact support and we can represent the solution $U(t, x)$ by

$$
U(t, x)|_{[0, T_0] \times \Omega} = \mathcal{E}(t, 0) G(x) + V_\Omega(t, x)
$$

for every relatively compact open convex set $\Omega \subset \mathbb{R}^n$, with $\mathcal{E}(t, s)$ the operator constructed in section 5 and $V_\Omega(t, \cdot\,) \in G^{(\sigma)}(\mathbb{R}^n)$.

Thus estimate (6.6) follows from Theorems 2.9 and 3.4. If $\mathcal{F} \cap [0, t_0]$ has the property (H), for every $\varepsilon > 0$ we can take $\pi_\varepsilon \in \Pi$, $\pi_\varepsilon = t_0 \geq t_1 \geq \cdots \geq t_{\nu_\varepsilon} \geq 0$, such that

$$
\delta(\mathcal{F} \cap [0, t_0], \pi_\varepsilon) < \varepsilon. \tag{6.8}
$$

We may assume that no one of the intervals $[t_k, t_{k+2}]$, $k = 0, \cdots, \nu_\varepsilon - 2$, is completely contained in $\mathcal{I}$. From the well-posedness of problem (4.10) in the classes $G^{(\sigma)}(\mathbb{R}^n)$ and $G_0^{(\sigma)'}(\mathbb{R}^n)$ and the finite speed of propagation property, we can write

$$
U(t_0, x) = \mathcal{E}(t_0, t_1) h_1 \mathcal{E}(t_1, t_2) h_2 \cdots h_{\nu_\varepsilon} \mathcal{E}(t_{\nu_\varepsilon}, 0) G(x) + W_\Omega(t, x)
$$

for every relatively compact open convex set $\Omega \subset \mathbb{R}^n$ with suitable $h_j(x) \in G_0^{(\sigma)}$, $j = 1, \ldots, \nu_\varepsilon$, and $W_\Omega(t, \cdot\,) \in G^{(\sigma)}(\mathbb{R}^n)$.

By Theorems 2.9, 3.4 and 3.10 together with Propositions 5.3 and 6.2 and property (6.8), we thus obtain

$$
WF_{(\sigma)}(U(t_0, \cdot\,)) \subset \bigcup_{\nu=0}^{1} \Gamma_\nu(t_0, \tilde{\mathcal{F}}_\varepsilon, \ WF_{(\sigma)}(G))
$$

where $\tilde{\mathcal{F}}_\varepsilon$ is a closed set satisfying $\sup_{\tau\in\tilde{\mathcal{F}}_\varepsilon} \operatorname{dist}\left(\tau, \bigcup_{\nu\geq 1} \mathcal{F}_\nu\right) \leq c\varepsilon$ with a constant $c > 0$ depending only on the function $\theta(t, t_1, s) = B^{-1}(B(t) - B(t_1) + B(s))$. Letting $\varepsilon \to 0$ we get estimate (6.7).

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Massimo Cicognani
Dipartimento di Matematica
Via Machiavelli 35
I-44100, Ferrara