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Some Techniques for the Characterization of Intermediate Spaces

ANITA TABACCO VIGNATI

0. - Introduction

In the last few years R. Coifman, M. Cwikel, R. Rochberg, Y. Sagher and G. Weiss have developed a method of interpolation for families of Banach spaces that generalizes the complex method of A.P. Calderon (see [CCRSW 1] and [CCRSW 2]).

As a further development we presented in [T-V 2] a method of interpolation for families of quasi-Banach spaces. More precisely, we considered quasi-Banach spaces $B(\vartheta)$ associated with the points $e^{i\vartheta}$ of the boundary Γ of the open unit disk D in the complex plane \mathbb{C} . Intermediate spaces $B(z)$, for each $z \in D$, were constructed in such a way that interpolation theorems for linear operators hold.

The aim of this note is to find some results that easily yield the identification of families of quasi-Banach spaces commonly used in analysis.

A large number of interesting spaces of functions in analysis has a norm defined by a sublinear operator, generally a maximal operator. Consider, for example, the case of H^p -spaces. If S denotes the space of test functions in \mathbb{R}^n and $\varphi \in S$ with $\hat{\varphi}(0) \neq 0$, we consider the operator \tilde{M} defined by

$$\tilde{M}(x, f) = \sup_{|x-y| \leq at} |f * \varphi_t(y)|, \quad a > 0$$

for every tempered distribution f . We know that $f \in H^p(\mathbb{R}^n)$ if and only if $\tilde{M}(\cdot, f) \in L^p(\mathbb{R}^n)$ and $\|f\|_{H^p} = \|\tilde{M}(\cdot, f)\|_{L^p}$, $0 < p \leq \infty$.

Observe that \tilde{M} can be described as the composition of two operators: the first one is linear and it associates to f the convolution $f * \varphi_t$, the second one is a typical maximal operator. The aim of section 2 is to prove an interpolation theorem for operators of this kind.

In section 4 we study the relations between our interpolation method and the real interpolation method developed by J. Peetre and J.L. Lions. The theorem

that we obtain yields the identification of complex-intermediate spaces from the identification of real-intermediate ones.

As an example, in section 5, from the Fefferman-Riviere-Sagher result

$$(H^{p_0}, L^\infty)_{\vartheta, p} = H^p, \quad \frac{1}{p} = \frac{1 - \vartheta}{p_0},$$

we are able to characterize the intermediate spaces when on the boundary we have H^p spaces. As we shall see, these intermediate spaces are again H^p spaces.

Finally, I would like to thank my advisors, Richard Rochberg and Guido Weiss, for introducing me to the subject and for their valuable help during my graduate studies.

1. - The complex interpolation method

We now briefly describe the complex interpolation method for families of quasi-Banach spaces, as given in [T-V 2].

For each $e^{i\vartheta} \in \Gamma$ we assign a quasi-Banach space $(B(\vartheta), |\cdot|_\vartheta)$, and denote by $c(\vartheta)$ the constants in the quasi-triangle inequalities.

We say that the family $\{B(\vartheta)\}$ is an *interpolation family* (of quasi-Banach spaces) if each $B(\vartheta)$ is continuously embedded in a Hausdorff topological vector space \mathcal{U} , the function $\vartheta \rightarrow |b|_\vartheta$ is measurable for each $b \in \bigcap_{\vartheta} B(\vartheta)$, and $\log c(\vartheta) \in L^1(\Gamma)$.

The subset \mathcal{B} of $\bigcap_{\vartheta} B(\vartheta)$ of those elements b such that $\log^+ |b|_\vartheta \in L^1(\Gamma)$ is called the *log-intersection* of the family $\{B(\vartheta)\}$.

By $\mathcal{G} = \mathcal{G}(B(\cdot), \Gamma)$ we denote the set of functions defined by

$$\mathcal{G} = \left\{ g(z) = \sum_{n=1}^N \varphi_j(z) a_j : \varphi_j \in N^+(D), a_j \in \mathcal{B}, \right. \\ \left. j = 1, \dots, N \text{ and } \| \|g\| \|_\infty < \infty \right\}$$

where $N^+(D)$ denotes the positive Nevanlinna class (see [Dur], ch. 2), and

$$\| \|g\| \|_\infty = \text{Ess sup}_{\vartheta} |g(\vartheta)|_\vartheta.$$

For every $a \in \mathcal{B}$ and $z \in D$ we define

$$|a|_z = \text{Inf} \{ \| \|g\| \|_\infty : g \in \mathcal{G}, g(z) = a \}.$$

If N_z denotes the set of elements of \mathcal{B} such that $|a|_z = 0$, the completion $B(z)$ of $(\mathcal{B}/N_z, |\cdot|_z)$ will be called the interpolation space at z of the family $\{B(\vartheta)\}$.

It can be proven that $(B(z), |\cdot|_z)$ is a quasi-Banach space with quasi-triangle inequality constant $c(z) = \exp \int_{\Gamma} \log c(\vartheta) P_z(\vartheta) d\vartheta$, where $P_z(\vartheta)$ is the Poisson kernel “centered” at z .

2. - An interpolation theorem for log-subharmonic operators

Let \mathcal{M} be the set of measurable complex-valued functions on some measure space (Y, ν) . An operator M mapping \mathcal{M} into the class \mathcal{N} of non-negative-valued measurable functions on some other measure space (X, μ) is said to be of *maximal type* provided it satisfies:

- (a) $M(\lambda a) = |\lambda| M a \quad \forall \lambda \in \mathbb{C}, \forall a \in \mathcal{M};$
- (b) $M(a) = M(|a|) \quad \forall a \in \mathcal{M};$
- (c) $M(a)(x) \leq M(b)(x) \quad \text{if } |a(y)| \leq |b(y)|, a, b \in \mathcal{M};$
- (d) $M \left[\int_{\Gamma} f(\cdot, \vartheta) d\vartheta \right] (x) \leq \int_{\Gamma} M(f(\cdot, \vartheta))(x) d\vartheta.$

If $\{B(\vartheta)\}$ is an interpolation family, with containing space \mathcal{U} , we say that an operator $\tilde{M} : \mathcal{U} \rightarrow \mathcal{N}$ is a *log-subharmonic operator* associated to the family $\{B(\vartheta)\}$ if it can be expressed as the composition $M \cdot L$ of a linear operator L mapping \mathcal{U} into \mathcal{M} and of a maximal-type operator M .

The reason for such a name is clear if we note that

PROPOSITION 2.1. *Let \tilde{M} be a log-subharmonic operator associated to the family $\{B(\vartheta)\}$. If $f(z) = \sum_{j=1}^n \varphi_j(z) a_j \in \mathcal{G}(B(\cdot), \Gamma)$, then $\log \tilde{M}(f(z))(x)$ is a subharmonic function of z in the disk, for a.e. fixed x .*

Before proving the proposition we recall the following result, due to Radó (see [Aup]):

LEMMA 2.2. *Let ϕ be a positive function defined on a domain Ω ; then, $\log \phi$ is subharmonic in Ω if and only if the function*

$$z \rightarrow |e^{\alpha z}| \phi(z)$$

is subharmonic in Ω , for every complex number α .

We shall refer to such functions ϕ as *log-subharmonic*.

PROOF OF 2.1. By Radó’s criterion (lemma 2.2), it is enough to show that the functions $z \rightarrow |e^{\alpha z}| \tilde{M}(f(z))(x)$ are subharmonic for every complex number

α . Fix $z_0 \in D$ and let $\rho > 0$ be such that the closure of the ball of radius ρ centered at z_0 is contained in D . Since $L(f(z))$ is analytic, $\log|L(f(z))|$ is subharmonic in the disk and, thus, we have

$$\begin{aligned} |e^{\alpha z_0} \tilde{M}(f(z_0))(x) &= M(|e^{\alpha z_0}| |L(f(z_0))|)(x) \\ &\leq M \left[\frac{1}{2\pi} \int_0^{2\pi} |e^{\alpha(z_0 + \rho e^{i\vartheta})}| |L(f(z_0 + \rho e^{i\vartheta}))| d\vartheta \right] (x) \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} M \left[|e^{\alpha(z_0 + \rho e^{i\vartheta})}| |L(f(z_0 + \rho e^{i\vartheta}))| \right] (x) d\vartheta \\ &= \frac{1}{2\pi} \int_0^{2\pi} |e^{\alpha(z_0 + \rho e^{i\vartheta})}| \tilde{M}(f(z_0 + \rho e^{i\vartheta}))(x) d\vartheta. \end{aligned}$$

The H^p -spaces introduced above are just one of several cases where the norm of an element f in a given function space is defined by evaluating the Lebesgue norm of $\tilde{M}f$, \tilde{M} a log-subharmonic operator.

For all these spaces we can apply the following interpolation theorem:

THEOREM 2.3. *Let \tilde{M} be a log-subharmonic operator associated to an interpolation family of quasi-Banach spaces $\{B(\vartheta)\}$. Suppose that*

$$\|\tilde{M}a\|_{L^{p(\vartheta)}(X)} \leq \eta(\vartheta) |a|_{\vartheta}, \quad \forall a \in \mathcal{B},$$

where $0 < p(\vartheta) \leq \infty$ and $\frac{1}{p} \in L^1(\Gamma)$.

If $\log \eta \in L^1(\Gamma)$, then for all $a \in \mathcal{B}$

$$\|\tilde{M}a\|_{L^{p(z)}(X)} \leq \eta(z) |a|_z,$$

where $\frac{1}{p(z)} = \int_{\Gamma} \frac{1}{p(\vartheta)} P_z(\vartheta) d\vartheta$ and $\eta(z) = \exp \int_{\Gamma} \log \eta(\vartheta) P_z(\vartheta) d\vartheta$.

PROOF. For $a \in \mathcal{B}$, there exists $f(z) = \sum_{j=1}^n \varphi_j(z) a_j \in \mathcal{G}(B(\cdot), \Gamma)$ such that $f(z_0) = a$ and $\|f\|_{\infty} \leq |a|_{z_0} + \varepsilon$, for $z_0 \in D$ and $\varepsilon > 0$ fixed. To prove the theorem it is enough to show that the function $z \rightarrow \log \|\tilde{M}(f(z))(\cdot)\|_{L^{p(z)}(X)}$ is subharmonic in the disk. Indeed

$$\log \|\tilde{M}(f(0))(\cdot)\|_{L^{p(0)}(X)} \leq \frac{1}{2\pi} \int_0^{2\pi} \log \|\tilde{M}(f(\vartheta))(\cdot)\|_{L^{p(\vartheta)}(X)} d\vartheta$$

is equivalent to

$$\|\tilde{M}(f(z_0))(\cdot)\|_{L^{p(z_0)}(X)} \leq \exp \int_0^{2\pi} \log \|\tilde{M}(f(\vartheta))(\cdot)\|_{L^{p(\vartheta)}(X)} P_{z_0}(\vartheta) d\vartheta$$

via the Möbius transformation

$$w = \frac{z - z_0}{1 + \bar{z}_0 z}, \quad d(\arg z) = P_{z_0}(\vartheta) d\vartheta.$$

Therefore,

$$\begin{aligned} \|\tilde{M}a\|_{L^{p(z_0)}(X)} &= \|\tilde{M}(f(z_0))(\cdot)\|_{L^{p(z_0)}(X)} \\ &\leq \exp \int_0^{2\pi} \log \eta(\vartheta) |f(\vartheta)|_{\vartheta} P_{z_0}(\vartheta) d\vartheta \\ &\leq \eta(z_0) \|f\|_{\infty} \\ &\leq \eta(z_0) (|a|_{z_0} + \varepsilon). \end{aligned}$$

Thus, letting $\varepsilon \rightarrow 0$, we get

$$\|\tilde{M}a\|_{L^{p(z_0)}(X)} \leq \eta(z_0) |a|_{z_0}.$$

To show our claim, we note that since the function $p(z)$ is strictly positive on D , for any $\rho > 0$ we can find $r > 0$ such that $0 < r < p(z)$ if $z \in \overline{B_{\rho}(z_0)} = \{z : |z - z_0| \leq \rho\} \subset D$.

Moreover, since subharmonicity is a local property, it suffices to show

$$\begin{aligned} &\log \|\tilde{M}(f(z_0))(\cdot)\|_{L^{p(z_0)}(X)} \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} \log \|\tilde{M}(f(z_0 + \rho e^{i\vartheta}))(\cdot)\|_{L^{p(z_0 + \rho e^{i\vartheta})}(X)} d\vartheta \end{aligned}$$

for any such $\rho > 0$.

Define

$$\ell(z) = 1 - \frac{r}{p(z)}$$

and let g be a simple and positive function on X of the form $g(x) = \sum_{j=1}^N \alpha_j \chi_{E_j}$, with $\alpha_j > 0$ and E_j pairwise disjoint sets of finite measure. Then $g(x)^{\ell(z)}$ is a log-subharmonic function in the disk for every fixed x . Moreover, since $\tilde{M}(f(z))(x)$ is a log-subharmonic function, also $(\tilde{M}(f(z)))^{\gamma}(x)$ is log-subharmonic in D for

every fixed x . We also have that $\tilde{M}(f(z))(\cdot)$ is in $L^r_{loc}(X)$ (i.e. $(\tilde{M}(f(z)))^r(\cdot)$ is integrable on sets of finite measure) for every $z \in D$, since $\tilde{M}(f(z)) \in \bigcap_{\vartheta} L^{p(\vartheta)}(X)$.

So

$$\begin{aligned} I(z) &\equiv \int_X g(x)^{\ell(z)} (\tilde{M}(f(z)))^r(x) dx \\ &= \sum_{j=1}^N \alpha_j^{\ell(z)} \int_{E_j} (\tilde{M}(f(z)))^r(x) dx \\ &= \sum_{j=1}^N \beta_j(z) \end{aligned}$$

is well defined. We claim that $I(z)$ is a log-subharmonic function in the disk. We need only prove that every β_j is log-subharmonic since, by Radó's criterion (Lemma 2.2), a finite sum of log-subharmonic functions is also log-subharmonic.

But

$$\log \beta_j(z) = \ell(z) \log \alpha_j + \log \int_{E_j} (\tilde{M}(f(z)))^r(x) dx.$$

Thus, it remains only to show that

$$\delta_j(z) \equiv \log \int_{E_j} (\tilde{M}(f(z)))^r(x) dx$$

is subharmonic.

We know that $(\tilde{M}(f(z)))^r(x)$ is log-subharmonic; therefore, for any complex number α we have:

$$\begin{aligned} &\int_{E_j} |e^{\alpha z_0}| (\tilde{M}(f(z_0)))^r(x) dx \\ &\leq \int_{E_j} \left[\frac{1}{2\pi} \int_0^{2\pi} |e^{\alpha(z_0 + \rho e^{i\vartheta})}| (\tilde{M}(f(z_0 + \rho e^{i\vartheta})))^r(x) d\vartheta \right] dx \\ &= \frac{1}{2\pi} \int_0^{2\pi} |e^{\alpha(z_0 + \rho e^{i\vartheta})}| \left[\int_{E_j} (\tilde{M}(f(z_0 + \rho e^{i\vartheta})))^r(x) dx \right] d\vartheta. \end{aligned}$$

Thus, again by Lemma 2.2, δ_j is log-subharmonic in D . So we have

$$\begin{aligned} \log \int_X g(x)^{\ell(z_0)} (\tilde{M}(f(z_0)))^r(x) dx &= \log I(z_0) \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} \log I(z_0 + \rho e^{i\vartheta}) d\vartheta \\ &= \frac{1}{2\pi} \int_0^{2\pi} \log \left[\int_X g(x)^{\ell(z_0 + \rho e^{i\vartheta})} (\tilde{M}(f(z_0 + \rho e^{i\vartheta})))^r(x) dx \right] d\vartheta. \end{aligned}$$

We can assume $\|g\|_{L^1(X)} = 1$ and use Hölder’s inequality on the right hand side with indices $q = \frac{p(z_0 + \rho e^{i\vartheta})}{r}$ and $q' = \frac{1}{\ell(z_0 + \rho e^{i\vartheta})}$.

Thus, we have

$$\begin{aligned} \log I(z_0) &\leq \frac{1}{2\pi} \int_0^{2\pi} \log \|g\|_{L^1(X)}^{\ell(z_0 + \rho e^{i\vartheta})} \|\tilde{M}(f(z_0 + \rho e^{i\vartheta}))(\cdot)\|_{L^{p(z_0 + \rho e^{i\vartheta})}(X)}^r d\vartheta \\ &= \frac{r}{2\pi} \int_0^{2\pi} \log \|\tilde{M}(f(z_0 + \rho e^{i\vartheta}))(\cdot)\|_{L^{p(z_0 + \rho e^{i\vartheta})}(X)} d\vartheta. \end{aligned}$$

Taking the Supremum over all such g ’s, we obtain

$$\begin{aligned} \log \|\tilde{M}(f(z_0))(\cdot)\|_{L^{p(z_0)}(X)}^r &= \log \sup_{\substack{g \text{ simple} \\ \|g\|_{L^1(X)}=1}} \int_X g(x)^{\ell(z_0)} (\tilde{M}(f(z_0)))^r(x) dx \\ &\leq \frac{r}{2\pi} \int_0^{2\pi} \log \|\tilde{M}(f(z_0 + \rho e^{i\vartheta}))(\cdot)\|_{L^{p(z_0 + \rho e^{i\vartheta})}(X)} d\vartheta. \end{aligned}$$

Hence,

$$\begin{aligned} \log \|\tilde{M}(f(z_0))(\cdot)\|_{L^{p(z_0)}(X)} &\leq \int_0^{2\pi} \log \|\tilde{M}(f(z_0 + \rho e^{i\vartheta}))(\cdot)\|_{L^{p(z_0 + \rho e^{i\vartheta})}(X)} d\vartheta. \end{aligned}$$

Theorem 2.3 generalizes a result by M. Cwikel, M. Milman and Y. Sagher (see [CMS]) for couples of quasi-Banach spaces.

3. - The real method of interpolation: the K , \tilde{K} , J and \tilde{J} functionals

We start by summarizing the K -method and the J -method of interpolation. We take the definitions and properties from [BL], where these methods are treated in a systematic way.

Let $(A_j, \|\cdot\|_j)$ be a quasi-Banach space with quasi-triangle inequality constant c_j , $j = 0, 1$. We suppose $A = (A_0, A_1)$ is a compatible pair; i.e. A_0, A_1 are continuously embedded in a common Hausdorff topological vector space. For $a \in \Sigma(A) = A_0 + A_1$, we define the K -functional by letting

$$(3.1) \quad K(t, a) = \inf_{\substack{a = a_0 + a_1 \\ a_j \in A_j}} (\|a_0\|_0 + \|a_1\|_1), \quad t > 0.$$

PROPOSITION 3.2. *For any $a \in \Sigma(A)$, $K(t, a)$ is a positive, increasing and concave function of t . Moreover,*

$$(3.3) \quad K(t, a) \leq \max\left(1, \frac{t}{s}\right) K(s, a)$$

$$(3.4) \quad K(t, a + b) \leq c_0[K(c_1t/c_0, a) + K(c_1t/c_0, b)].$$

For $0 < \vartheta < 1$, $0 < q \leq \infty$ (and $0 \leq \vartheta \leq 1$, $q = \infty$) we let $(A_0, A_1)_{\vartheta, q; K} = A_{\vartheta, q; K}$ denote the space of all $a \in \Sigma(A)$ such that

$$(3.5) \quad \|a\|_{\vartheta, q; K} = \left\{ \int_0^\infty [t^{-\vartheta} K(t, a)]^q dt/t \right\}^{1/q} < \infty.$$

PROPOSITION 3.6. *For all $a \in A_{\vartheta, q; K}$ we have*

$$K(t, a) \leq \gamma_{\vartheta, q} t^\vartheta \|a\|_{\vartheta, q; K}$$

where

$$\gamma_{\vartheta, q} = [q\vartheta(1 - \vartheta)]^{1/q}.$$

A variant of the $K_{\vartheta, q}$ -functional is the discrete $K_{\vartheta, q}$ -method. We shall replace the continuous variable t by a discrete variable n ; the relation between them being $t = 2^n$. If $\lambda^{\vartheta, q}$ denotes the space of all sequences $\{\alpha_v\}_{v=-\infty}^\infty$ such that

$$\|\{\alpha_v\}\|_{\lambda^{\vartheta, q}} \equiv \left\{ \sum_v [2^{-v\vartheta} |\alpha_v|]^q \right\}^{1/q} < \infty,$$

we have

THEOREM 3.7. *For $a \in \Sigma(A)$ we put $\alpha_v = K(2^v, a)$. Then $a \in A_{\vartheta,q;K}$ if and only if $\{\alpha_v\}_{-\infty}^{\infty}$ belongs to $\lambda^{\vartheta,q}$.*

Moreover,

$$(3.8) \quad 2^{-\vartheta} \log 2 \|\{\alpha_v\}\|_{\lambda^{\vartheta,q}} \leq \|a\|_{\vartheta,q;K} \leq 2 \log 2 \|\{\alpha_v\}\|_{\lambda^{\vartheta,q}}.$$

For every $a \in \Delta(A) = A_0 \cap A_1$, let the J -functional be defined by

$$(3.9) \quad J(t, a) = \max(\|a\|_0, t\|a\|_1), \quad t > 0.$$

For $0 < \vartheta < 1$, $0 < q \leq \infty$ (and $0 \leq \vartheta \leq 1$, $q = \infty$), we define the space $(A_0, A_1)_{\vartheta,q;J} = A_{\vartheta,q;J}$ as the set of all those a in $\Sigma(A)$ that can be represented as $a = \sum_v a_v$ (convergence in $\Sigma(A)$) where $a_v \in \Delta(A)$ and

$$\|a\|_{\vartheta,q;J} = \text{Inf}_{(a_v)} \|\{J(2^v, a_v)\}\|_{\lambda^{\vartheta,q}} < \infty.$$

There is a continuous representation of the space $A_{\vartheta,q;J}$, which is analogous to the continuous representation of the space $A_{\vartheta,q;K}$, but we shall not need it.

It is possible to show that the spaces $A_{\alpha,q;K}$ and $A_{\vartheta,q;J}$ so defined are again quasi-Banach spaces with quasi-triangle inequality constants $c_0^{1-\vartheta} c_1^{\vartheta} \max(1, 2^{1/q-1})$. Moreover, the K and J methods give rise to the same space with equivalent quasi-norms, as long as $0 < \vartheta < 1$, $0 < q \leq \infty$. More precisely, we have:

THEOREM 3.10. *Let A be a compatible couple of quasi-Banach spaces and assume that $0 < \vartheta < 1$, $0 < q \leq \infty$. Then $A_{\vartheta,q;J} = A_{\vartheta,q;K} = A_{\vartheta,q}$ and*

$$(3.11) \quad \frac{1}{16c} \|a\|_{\vartheta,q;J} \leq \|a\|_{\vartheta,q;K} \leq c_{\vartheta,q} \|a\|_{\vartheta,q;J},$$

where

$$(3.12) \quad c_{\vartheta,q} = 2^{1+1/p} \log 2 \left\{ \frac{1 - 2^{-\rho}}{(1 - 2^{\rho(\vartheta-1)})(1 - 2^{-\vartheta\rho})} \right\}^{1/\rho}$$

and c, ρ are such that $c \geq \max(c_0, c_1)$, $(2c)^{\rho} = 2$ and $q/\rho \geq 1$.

PROPOSITION 3.13. *Let A be a compatible couple of quasi-Banach spaces. Then, if $q < \infty$, $\Delta(A)$ is dense in $A_{\vartheta,q}$.*

We shall introduce two new functionals, the \tilde{K} and \tilde{J} functionals. They are defined in a way similar to the K and J functionals, using the so called Gagliardo completion norms. For $a \in \Delta(A)$, let

$$(3.14) \quad |a|_0 = \lim_{t \rightarrow \infty} K(t, a)$$

$$(3.15) \quad |a|_1 = \lim_{t \rightarrow 0} \frac{1}{t} K(t, a).$$

Clearly $|a|_j \leq \|a\|_j, j = 0, 1$.

In certain particular situations also the converse inequality is true. Consider, for example, the L^p -spaces case. Let $A_0 = L^1(M), A_1 = L^\infty(M)$ where (M, dx) is any measure space. We know the explicit expression of the K -functional, $K(t, a) = \int_0^t f^*(s)ds$, where f^* is the non-increasing rearrangement of the function f .

Therefore,

$$|f|_0 = \lim_{t \rightarrow \infty} K(t, a) = \int_0^\infty f^*(s)ds = \|f\|_{L^1(M)}$$

$$|f|_1 = \lim_{t \rightarrow 0} \frac{1}{t} K(t, a) = \lim_{t \rightarrow 0} \frac{1}{t} \int_0^t f^*(s)ds = f^*(0) = \|f\|_{L^\infty(M)}.$$

In general we do not have equality; however, the new norms $|\cdot|_j, j = 0, 1$, play a rôle similar to the old ones, $\|\cdot\|_j, j = 0, 1$.

Back to our situation, for $a \in \Delta(A)$ we define the \tilde{K} -functional as

$$(3.16) \quad \tilde{K}(t, a) \equiv \inf_{\substack{a = a_0 + a_1 \\ a_j \in A_j}} (|a_0|_0 + t|a_1|_1), \quad t > 0.$$

PROPOSITION 3.17. *Let $a \in \Delta(A)$; then*

$$\tilde{K}(t, a) \leq K(t, a) \leq \max(c_0, c_1) \tilde{K}(t, a).$$

PROOF. Clearly $\tilde{K}(t, a) \leq K(t, a)$. To prove the second inequality, note that $K(t, a) \leq |a|_0$, since $K(t, a)$ is an increasing function of t . Moreover $K(t, a) \leq t|a|_1$, since $K(t, a)/t$ is a decreasing function of t , by (3.3).

So, if $a = a_0 + a_1, a_j \in \Delta(A), j = 0, 1$, using (3.3) and (3.4) we obtain

$$\begin{aligned} K(t, a) &\leq c_0[K(c_1t/c_0, a_0) + K(c_1t/c_0, a_1)] \\ &\leq \max(c_0, c_1) [K(t, a_0) + K(t, a_1)] \\ &\leq \max(c_0, c_1) [|a_0|_0 + t|a_1|_1]. \end{aligned}$$

Taking the infimum over all possible decompositions, we get

$$K(t, a) \leq \max(c_0, c_1) \tilde{K}(t, a).$$

For $a \in \Delta(A)$, let the \tilde{J} -functional be defined by

$$(3.18) \quad \tilde{J}(t, a) \equiv \max(|a|_0, t|a|_1), \quad t > 0.$$

Clearly $\tilde{J}(t, a) \leq J(t, a)$, but unlike the case of the K and \tilde{K} -functionals one does not have that J and \tilde{J} are equivalent.

It is possible, however, to prove the following result:

PROPOSITION 3.19. *Let $\varepsilon > 0$, $a \in \Delta(A)$. There exists a representation $a = \sum_{|v| \leq N+1} a_v$, with $a_v \in \Delta(A)$, such that*

$$\tilde{J}(2^v, a_v) \leq 3 \max(c_0, c_1)(1 + \varepsilon) K(2^v, a).$$

PROOF. Take $a \in \Delta(A)$ and $\varepsilon > 0$; then there exists N such that

$$\begin{aligned} |a|_0 &\leq (1 + \varepsilon) K(2^v, a) && \text{if } v \geq N \\ 2^v |a|_1 &\leq (1 + \varepsilon) K(2^v, a) && \text{if } -v \leq N. \end{aligned}$$

We can, therefore, find two sequences $\{a_{j,v}\}$, $j = 0, 1$, such that

$$\begin{aligned} a &= a_{0,v} + a_{1,v} \\ |a_{0,v}|_0 + 2^v |a_{1,v}|_1 &\leq (1 + \varepsilon) K(2^v, a) \\ a_{0,v} &= 0 && \text{if } -v \leq -N \\ a_{1,v} &= 0 && \text{if } v \geq N. \end{aligned}$$

Let $a_v = a_{0,v} - a_{0,v-1} = a_{1,v-1} - a_{1,v}$. Then $a_v = 0$ for $|v| \geq N + 1$. Thus $\sum_{|v| \leq N+1} a_v = a_{0,N} - a_{0,-N-1} = a - a_{1,N} - a_{0,-N-1} = a$. Moreover,

$$\begin{aligned} \tilde{J}(2^v, a_v) &= \max(|a_v|_0, 2^v |a_v|_1) \\ &\leq \max(c_0, c_1) \max(|a_{0,v}|_0 + |a_{0,v-1}|_0, 2^v [|a_{1,v-1}|_1 + |a_{1,v}|_1]) \\ &\leq \max(c_0, c_1) (1 + \varepsilon) [K(2^v, a) + K(2^{v-1}, a)] \\ &\leq 3 \max(c_0, c_1) (1 + \varepsilon) K(2^v, a). \end{aligned}$$

For $0 < \vartheta < 1$, $0 < q \leq \infty$ (and $0 \leq \vartheta \leq 1$, $q = \infty$) we let $(A_0, A_1)_{\vartheta,q,\tilde{K}} = A_{\vartheta,q,\tilde{K}}$ denote the completion of $\Delta(A)$ with respect to the quasi-norm $\|a\|_{\vartheta,q,\tilde{K}} = \left\{ \int_0^\infty [t^{-\vartheta} \tilde{K}(t, a)]^q dt/t \right\}^{1/q}$.

We define the space $(A_0, A_1)_{\vartheta,q,\tilde{J}} = A_{\vartheta,q,\tilde{J}}$ as the completion of $\Delta(A)$ with respect to the quasi-norm

$$\|a\|_{\vartheta,q,\tilde{J}} = \text{Inf}_{\{a_v\}} \|\{\tilde{J}(2^v, a_v)\}\|_{\lambda^{\vartheta,q}},$$

where the infimum is taken over all the representations of a of the form $a = \sum_v a_v$ with $a_v \in \Delta(A)$.

THEOREM 3.20. *Let A be a compatible couple of quasi-Banach spaces. Then, for every $a \in \Delta(A)$ we have*

$$(3.21) \quad \|a\|_{\vartheta,q;K} \leq \max(c_0, c_1) \|a\|_{\vartheta,q;\tilde{K}} \leq \max(c_0, c_1) \|a\|_{\vartheta,q;K}$$

and

$$(3.22) \quad \|a\|_{\vartheta,q;\tilde{J}} \leq 12 \max(c_0, c_1) \|a\|_{\vartheta,q;K} \leq 12 \max(c_0, c_1) c_{\vartheta,q} \|a\|_{\vartheta,q;\tilde{J}}$$

where $c_{\vartheta,q}$ is as in (3.12).

Before proving the theorem we need a couple of lemmas.

LEMMA 3.23 (see [BL]). *Let $(B, |\cdot|)$ be a quasi-normed vector space with quasi-triangle inequality constant c . Let ρ be defined by the equation $(2c)^\rho = 2$. If $a = \sum_{j=0}^{\infty} a_j$ converges in B , then*

$$|a| \leq 2^{1/\rho} \left(\sum_{j=0}^{\infty} |a_j|^\rho \right)^{1/\rho}.$$

LEMMA 3.24. *Let $a \in \Delta(A)$; then*

$$K(t, a) \leq \min \left(1, \frac{t}{s} \right) \tilde{J}(s, a).$$

PROOF. $K(t, a) \leq \min \left(|a|_0, \frac{t}{s} |a|_1 \right)$

$$\leq \min(\tilde{J}(s, a), \frac{t}{s} \tilde{J}(s, a)) = \min \left(1, \frac{t}{s} \right) \tilde{J}(s, a).$$

PROOF OF THEOREM 3.20. By Proposition 3.17, we immediately obtain 3.21. Now take $a \in \Delta(A)$; by Proposition 3.19 there exists a representation $a = \sum_{|v| \leq N} a_v$ with $a_v \in \Delta(A)$ such that

$$\tilde{J}(2^v, a_v) \leq 4 \max(c_0, c_1) K(2^v, a).$$

Thus,

$$\|\{\tilde{J}(2^v, a_v)\}\|_{\lambda^{\vartheta,q}} \leq 4 \max(c_0, c_1) \|\{K(2^v, a)\}\|_{\lambda^{\vartheta,q}}.$$

Therefore, using (3.8), we obtain

$$\begin{aligned} \|a\|_{\vartheta,q;\tilde{J}} &\leq 4 \max(c, c_1) \frac{2^\vartheta}{\log 2} \|a\|_{\vartheta,q;K} \\ &\leq 12 \max(c_0, c_1) \|a\|_{\vartheta,q;K}. \end{aligned}$$

To prove the second part of the inequality, take $a \in \Delta(A)$ and assume $a = \sum_v a_v$, with $a_v \in \Delta(A)$. We know that $K(t, a)$ is a c -norm ($c \geq \max(c_0, c_1)$). Choosing c large and ρ so that $(2c)^\rho = 2$, we have $p = q/\rho \geq 1$. Then, using Lemmas 3.23 and 3.24 we obtain

$$\begin{aligned} K(t, a) &\leq 2^{1/\rho} \{ \sum_v [K(t, a_v)]^\rho \}^{1/\rho} \\ &\leq 2^{1/\rho} \{ \sum_v [\min(1, t2^{-v}) \tilde{J}(2^v, a_v)]^\rho \}^{1/\rho}. \end{aligned}$$

So

$$\begin{aligned} K(2^\mu, a) &\leq 2^{1/\rho} \{ \sum_v [\min(1, 2^{\mu-v}) \tilde{J}(2^v, a_v)]^\rho \}^{1/\rho} \\ &= 2^{1/\rho} \{ \sum_v [\min(1, 2^v) \tilde{J}(2^{\mu-v}, a_{\mu-v})]^\rho \}^{1/\rho}. \end{aligned}$$

Thus, using (3.8) and Minkowski's inequality for series, we obtain

$$\begin{aligned} \|a\|_{\vartheta, q; K} &\leq 2 \log 2 \{ \sum_\mu [2^{-\mu\vartheta} K(2^\mu, a)]^q \}^{1/q} \\ &= 2 \log 2 \{ [\sum_\mu [2^{-\mu\vartheta} K(2^\mu, a)]^{\rho p}]^{1/p} \}^{1/\rho} \\ &\leq 2 \log 2 \{ [\sum_\mu [2^{-\mu\vartheta\rho} 2(\sum_v [\min(1, 2^v) J(2^{\mu-v}, a_{\mu-v})]^\rho)]^p]^{1/p} \}^{1/\rho} \\ &\leq 2^{1+1/\rho} \log 2 \{ \sum_v \min(1, 2^{v\rho}) [\sum_\mu (2^{-\mu\vartheta} J(2^{\mu-v}, a_{\mu-v}))^q]^{1/p} \}^{1/\rho} \\ &= 2^{1+1/\rho} \log 2 \{ \sum_v \min(1, 2^{v\rho}) [\sum_\mu (2^{-(\mu+v)\vartheta} J(2^\mu, a_\mu))^q]^{1/p} \}^{1/\rho} \\ &= 2^{1+1/\rho} \log 2 \{ \sum_v \min(1, 2^{v\rho}) 2^{-v\rho\vartheta} \}^{1/\rho} \{ \sum_\mu (2^{-\mu\vartheta} J(2^\mu, a_\mu))^q \}^{1/q}. \end{aligned}$$

Therefore, $\|a\|_{\vartheta, q; K} \leq c_{\vartheta, q} \|a\|_{\vartheta, q; \tilde{J}}$, with

$$\begin{aligned} c_{\vartheta, q} &= 2^{1+1/\rho} \log 2 \{ \sum_v \min(1, 2^{v\rho}) 2^{-v\rho\vartheta} \}^{1/\rho} \\ &= 2^{1+1/\rho} \log 2 \left\{ \sum_{v < 0} 2^{v\rho - v\vartheta\rho} + \sum_{v \geq 0} 2^{-\vartheta v\rho} \right\}^{1/\rho} \\ &= 2^{1+1/\rho} \log 2 \left\{ \sum_{v > 0} 2^{v\rho(\vartheta-1)} + (1 - 2^{-\vartheta\rho})^{-1} \right\}^{1/\rho} \\ &= 2^{1+1/\rho} \log 2 \left\{ \frac{1}{1 - 2^{\rho(\vartheta-1)}} - 1 + \frac{1}{1 - 2^{-\vartheta\rho}} \right\}^{1/\rho} \\ &= 2^{1+1/\rho} \log 2 \left\{ \frac{1 - 2^{-\rho}}{(1 - 2^{\rho(\vartheta-1)})(1 - 2^{-\vartheta\rho})} \right\}^{1/\rho}. \end{aligned}$$

4. - Complex interpolation of real interpolation spaces

Let $0 < \alpha(\vartheta) < 1$, $0 < q(\vartheta) \leq \infty$ be two measurable functions defined on Γ and let $B(\vartheta) = (B_0, B_1)_{\alpha(\vartheta), q(\vartheta)}$, where (B_0, B_1) is a compatible pair of quasi-Banach spaces and $(B_0, B_1)_{\alpha, q}$ denotes the space obtained by the real method of interpolation.

PROPOSITION 4.1. *If $\frac{1}{q} \in L^1(\Gamma)$, then $\{B(\vartheta)\}$ is an interpolation family of quasi-Banach spaces.*

PROOF. The containing space \mathcal{U} can be taken to be $B_0 + B_1$. The measurability of $\vartheta \rightarrow \|b\|_{\vartheta}$, $b \in \cap_{\vartheta} B(\vartheta)$, is a consequence of the measurability of the functions α and q and the definition of

$$\|b\|_{\vartheta} = \|b\|_{\alpha(\vartheta),q(\vartheta)} = \begin{cases} \left\{ \int_0^{\infty} [t^{-\alpha(\vartheta)} K(t, b)]^{q(\vartheta)} \frac{dt}{t} \right\}^{1/q(\vartheta)} & \text{if } q(\vartheta) < \infty \\ \text{Sup}_t t^{-\alpha(\vartheta)} K(t, b) & \text{if } q(\vartheta) = \infty. \end{cases}$$

Moreover $c(\vartheta) = c_0^{1-\alpha(\vartheta)} c_1^{\alpha(\vartheta)} \max(1, 2^{1/q(\vartheta)-1})$; thus, $\log c(\vartheta)$ is integrable, since $\frac{1}{q} \in L^1(\Gamma)$.

PROPOSITION 4.2. *Suppose*

$$(4.3) \quad \frac{1}{q} \log q, \frac{1}{q} \log \alpha, \frac{1}{q} \log(1 - \alpha) \in L^1(\Gamma);$$

then $B_0 \cap B_1 \subset \mathcal{B}$, where \mathcal{B} denotes the log-intersection of the family $\{B(\vartheta)\}$.

PROOF. Take $b \in B_0 \cap B_1$; since $\|b\|_{\Delta(B)} = \max(\|b\|_0, \|b\|_1)$, we have

$$K(t, b) = \text{Inf}_{b=b_0+b_1} (\|b_0\|_0 + t\|b_1\|_1) \leq \min(\|b\|_0, t\|b\|_1) \leq \min(1, t)\|b\|_{\Delta(B)}.$$

Thus,

$$\begin{aligned} \|b\|_{\alpha(\vartheta),q(\vartheta)} &= \left\{ \int_0^{\infty} [t^{-\alpha(\vartheta)} K(t, b)]^{q(\vartheta)} dt/t \right\}^{1/q(\vartheta)} \\ &\leq \left\{ \int_0^{\infty} [t^{-\alpha(\vartheta)} \min(1, t)\|b\|_{\Delta(B)}]^{q(\vartheta)} dt/t \right\}^{1/q(\vartheta)} \\ &= \|b\|_{\Delta(B)} \left\{ \int_0^1 t^{(1-\alpha(\vartheta))q(\vartheta)} dt/t + \int_1^{\infty} t^{-\alpha(\vartheta)q(\vartheta)} dt/t \right\}^{1/q(\vartheta)} \\ &= \|b\|_{\Delta(B)} \left\{ \frac{1}{q(\vartheta)\alpha(\vartheta)(1-\alpha(\vartheta))} \right\}^{1/q(\vartheta)} \quad \text{if } q(\vartheta) < \infty, \end{aligned}$$

or

$$\begin{aligned} \|b\|_{\alpha(\vartheta),q(\alpha)} &= \text{Sup}_t t^{-\alpha(\vartheta)} K(t, b) \leq \text{Sup}_t t^{-\alpha(\vartheta)} \min(1, t) \|b\|_{\Delta(B)} \\ &= \max \left[\text{Sup}_{0 < t \leq 1} t^{-\alpha(\vartheta)+1}, \text{Sup}_{1 < t < \infty} t^{-\alpha(\vartheta)} \right] \|b\|_{\Delta(B)} = \|b\|_{\Delta(B)} \text{ if } q(\vartheta) = \infty. \end{aligned}$$

Therefore, by (4.3), $\log^+ \|b\|_{\alpha(\vartheta),q(\vartheta)} \in L^1(\Gamma)$.

PROPOSITION 4.4. *If (4.3) holds, then $\mathcal{B} \subset (B_0, B_1)_{\alpha(z),q(z)}$, where*

$$(4.5) \quad \alpha(z) = \int_{\Gamma} \alpha(\vartheta) P_z(\vartheta) d\vartheta$$

$$(4.6) \quad \frac{1}{q(z)} = \int_{\Gamma} \frac{1}{q(\vartheta)} P_z(\vartheta) d\vartheta.$$

PROOF. Fix $z \in D$; then there exist $\vartheta, \varphi \in \Gamma$ such that $\alpha(\vartheta) \leq \alpha(z) \leq \alpha(\varphi)$. Thus, $A \equiv (B_0, B_1)_{\alpha(\vartheta),q(\vartheta)} \cap (B_0, B_1)_{\alpha(\varphi),q(\varphi)} \subset (B_0, B_1)_{\alpha(z),q(z)}$. Indeed, if $a \in A$, we have

$$\begin{aligned} &\phi_{\alpha(z),q(z)}(K(\cdot, a)) \\ &\leq \max(1, 2^{1/q(z)-1}) \left\{ \left[\int_0^1 [t^{-\alpha(z)} K(t, a)]^{q(z)} dt/t \right]^{1/q(z)} \right. \\ &\quad \left. + \left[\int_1^\infty [t^{-\alpha(z)} K(t, a)]^{q(z)} dt/t \right]^{1/q(z)} \right\} \\ &= \max(1, 2^{\frac{1}{q(z)}-1}) \left\{ \left[\int_0^1 [t^{-\alpha(\varphi)} K(t, a)]^{q(z)} t^{(\alpha(\varphi)-\alpha(z))q(z)} \frac{dt}{t} \right]^{\frac{1}{q(z)}} \right. \\ &\quad \left. + \left[\int_1^\infty [t^{-\alpha(\vartheta)} K(t, a)]^{q(z)} t^{(\alpha(\vartheta)-\alpha(z))q(z)} \frac{dt}{t} \right]^{\frac{1}{q(z)}} \right\}. \end{aligned}$$

Using Proposition 3.6 we get

$$\begin{aligned} &\phi_{\alpha(z),q(z)}(K(\cdot, a)) \\ &\leq \max(1, 2^{1/q(z)-1}). \end{aligned}$$

$$\begin{aligned}
 & \cdot \left\{ \gamma_{\alpha(\varphi), q(\varphi)} \|a\|_{\alpha(\varphi), q(\varphi)} \left[\int_0^1 t^{(\alpha(\varphi) - \alpha(z))q(z)} \frac{dt}{t} \right]^{\frac{1}{q(z)}} \right. \\
 & \left. + \gamma_{\alpha(\vartheta), q(\vartheta)} \|a\|_{\alpha(\vartheta), q(\vartheta)} \left[\int_1^\infty t^{(\alpha(\vartheta) - \alpha(z))q(z)} \frac{dt}{t} \right]^{\frac{1}{q(z)}} \right\} \\
 & = \max(1, 2^{1/q(z)-1}) \cdot \left\{ \frac{\gamma_{\alpha(\varphi), q(\varphi)} \|a\|_{\alpha(\varphi), q(\varphi)}}{[q(z)(\alpha(\varphi) - \alpha(z))]^{1/q(z)}} + \frac{\gamma_{\alpha(\vartheta), q(\vartheta)} \|a\|_{\alpha(\vartheta), q(\vartheta)}}{[q(z)(\alpha(z) - \alpha(\vartheta))]^{1/q(z)}} \right\} < \infty.
 \end{aligned}$$

Thus $B \subset \bigcap_{\vartheta} B(\vartheta) \subset A \subset (B_0, B_1)_{\alpha(z), q(z)}$.

THEOREM 4.7. *Suppose (4.3) holds; then*

$$|b|_z \leq d(z) \|b\|_{\alpha(z), q(z); K}$$

for every $b \in B_0 \cap B_1$, where $\alpha(z)$ and $q(z)$ are defined as in (4.5) and (4.6).

PROOF. Take $0 \neq b \in B_0 \cap B_1$. By Proposition 2.6, there exists a representation of b of the form $b = \sum_{|v| \leq N} b_v$, with $b_v \in \Delta(B)$, such that

$$(4.8) \quad \tilde{J}(2^v, b_v) \leq 4 \max(c_0, c_1) K(2^v, b).$$

Fix $\xi \in D$, and let $w(z)$ and $1/s(z)$ be the two unique analytic functions in D whose real parts are $\alpha(z)$ and $1/q(z)$ respectively, and such that $w(\xi)$ and $1/s(\xi)$ are real. Let $\rho(\vartheta) = \min(\rho, q(\vartheta))$, where ρ is defined by the equation $(2c)^\rho = 2$ and $c = \max(c_0, c_1)$. Note that $q(\vartheta)/\rho(\vartheta) \geq 1$ and so we can apply Theorem 3.20 to every $\|\cdot\|_{\alpha(\vartheta), q(\vartheta)}$.

Let $f_v(z) = b_v B_v(z)$, where

$$B_v(z) = 2^{v(w(z) - w(\xi))} \left\{ \frac{\tilde{J}(2^v, b_v) 2^{-v\alpha(\xi)}}{\|b\|_{\alpha(\xi), q(\xi); K}} \right\}^{\frac{s(\xi)}{s(z)} - 1} \frac{k(\xi)}{k(z)} \frac{L(\xi)}{L(z)},$$

$$k(z) = \exp \int_0^{2\pi} \log k(\vartheta) H_z(\vartheta) d\vartheta \quad \text{with } k(\vartheta) = c_{\alpha(\vartheta), q(\vartheta)},$$

$$L(z) = \exp \int_0^{2\pi} \log L(\vartheta) H_z(\vartheta) d\vartheta \quad \text{with } L(\vartheta) = \left\{ \frac{2\alpha(\xi)}{\log 2} 4 \max(c_0, c_1) \right\}^{\frac{s(\xi)}{s(\vartheta)}}.$$

Note that $k(z)$ and $L(z)$ are well defined, since (4.3) holds and $\log(1 - 2^{-t}) \approx \log(t \log 2)$ as $t \rightarrow 0$. Moreover, $B_\nu \in N^+(D)$ and $b_\nu \in \mathcal{B}$ by Proposition 4.2. Finally, define

$$g(z) = \sum_{|v| \leq N} f_\nu(z).$$

Observe that $g(\xi) = b$ and, using (3.22), (4.8), (3.8),

$$\begin{aligned} \|g(\vartheta)\|_{\alpha(\vartheta),q(\vartheta);K} &\leq k(\vartheta) \|g(\vartheta)\|_{\alpha(\vartheta),q(\vartheta);J} \\ &\leq k(\vartheta) \left\{ \sum_{|v| \leq N} [2^{-v\alpha(\vartheta)} \bar{J}(2^v, B_\nu, b_\nu)]^{q(\vartheta)} \right\}^{1/q(\vartheta)} \\ &= k(\xi) \frac{L(\xi)}{L(\vartheta)} \|b\|_{\alpha(\xi),q(\xi);K}^{1-\frac{q(\xi)}{q(\vartheta)}} \left\{ \sum_{|v| \leq N} [2^{-v\alpha(\xi)} \bar{J}(2^v, b_\nu)]^{q(\xi)} \right\}^{1/q(\vartheta)} \\ &\leq k(\xi) \frac{L(\xi)}{L(\vartheta)} \|b\|_{\alpha(\xi),q(\xi);K}^{1-\frac{q(\xi)}{q(\vartheta)}} [4 \max(c_0, c_1)]^{\frac{q(\xi)}{q(\vartheta)}} \cdot \left\{ \sum_{|v| \leq N} [2^{-v\alpha(\xi)} K(2^v, b)]^{q(\xi)} \right\}^{1/q(\vartheta)} \\ &\leq k(\xi) \frac{L(\xi)}{L(\vartheta)} [4 \max(c_0, c_1)]^{\frac{2\alpha(\xi)}{\log 2}} 2^{\frac{q(\xi)}{q(\vartheta)}} \|b\|_{\alpha(\xi),q(\xi);K} \\ &= k(\xi)L(\xi) \|b\|_{\alpha(\xi),q(\xi);K} < \infty. \end{aligned}$$

Therefore $g \in \mathcal{G}(B(\cdot), \Gamma)$ and, thus,

$$|b|_\xi = |g(\xi)|_\xi \leq \|g\|_\infty \leq d(\xi) \|b\|_{\alpha(\xi),q(\xi);K}.$$

A similar result has been proven by E. Hernandez ([Her]) in the Banach space case, and by M. Cwikel, M. Milman and Y. Sagher ([CMS]) for couples of quasi-Banach spaces.

5. - Example: H^p spaces

We shall use the results obtained in sections 2 and 4 to identify the intermediate spaces when on the boundary we have H^p spaces. We shall follow the notations of Calderon and Torchinski and we refer to [CT 1] and [CT 2] for the most important properties of these spaces. We recall that if $0 < p < 1$, $H^p(\mathbb{R}^n)$ is a quasi-Banach space and the constant in the quasi-triangle inequality is $c_p = 2^{1/p-1}$, while, if $p \geq 1$, $H^p(\mathbb{R}^n)$ is a Banach space, and coincides with the ordinary $L^p(\mathbb{R}^n)$, with equivalence of norms, if $p > 1$.

Suppose $B(\vartheta) = H^{p(\vartheta)}(\mathbb{R}^n)$; we shall see that the interpolation space at z is $B(z) = H^{p(z)}(\mathbb{R}^n)$ where:

$$(5.1) \quad \frac{1}{p(z)} = \int_{\Gamma} \frac{1}{p(\vartheta)} P_z(\vartheta) d\vartheta.$$

We shall prove this when $0 < p_0 < p(\vartheta) \leq \infty$. If we want to remove the restriction $0 < p_0 < p(\vartheta)$, we can not apply Theorem 4.7 and we have to use the definition of interpolation space directly. This was done in [T-V 1].

We first recall the following result due to C. Fefferman, N. Riviere and Y. Sagher (see [FRS]).

THEOREM 5.2. $(H^{p_0}(\mathbb{R}^n), L^\infty(\mathbb{R}^n))_{\vartheta, p} = H^p$, where $\frac{1}{p} = \frac{1-\vartheta}{p_0} + \vartheta$, $0 < \vartheta < 1$.

THEOREM 5.3. Suppose $B(\vartheta) = H^{p(\vartheta)}(\mathbb{R}^n)$, $0 < p_0 + \varepsilon < p(\vartheta) \leq \infty$. Then $B(z) = H^{p(z)}(\mathbb{R}^n)$, with equivalence of quasi-norms, where $p(z)$ is defined in (5.1).

PROOF. Fix $z \in D$. We can assume $p(z) < \infty$, since otherwise we have $p(\vartheta) = \infty$ a.e. and thus $B(w) = H^\infty(\mathbb{R}^n)$ for every $w \in D$.

By Theorem 5.2, $H^{p(\vartheta)}(\mathbb{R}^n) = (H^{p_0}, L^\infty)_{1-p_0/p(\vartheta), p(\vartheta)}$. Applying Theorem 4.7, we get $|f|_z \leq d(z) \|f\|_{1-p_0/p(z), p(z)}$, or $\|f\|_z \leq k(z) \|f\|_{H^{p(z)}}$ for every $f \in H^p \circ \cap L^\infty$.

Moreover, if $\varphi \in S$ and $\hat{\varphi}(0) \neq 0$, we can define

$$\tilde{M}(x, f) = \text{Sup}_{|x-y| \leq at} |f * \varphi_t(y)|, \quad a > 0$$

for every tempered distribution f . We know that $f \in H^p(\mathbb{R}^n)$ if and only if $\tilde{M}(\cdot, f)$ is in $L^p(\mathbb{R}^n)$, and $\|f\|_{H^p} = \|\tilde{M}(\cdot, f)\|_{L^p}$.

It is easy to see that \tilde{M} is a log-subharmonic operator associated with the family $\{B(\vartheta)\}$. Therefore, using Theorem 2.3, we get

$$\|\tilde{M}(\cdot, f)\|_{L^{p(z)}} = \|f\|_{H^{p(z)}} \leq |f|_z$$

for every $f \in \mathcal{B}$. By Proposition 4.2, this is true in particular for $f \in H^{p_0} \cap L^\infty$.

Finally we observe that the completions of $H^{p_0} \cap L^\infty$ with respect to both norms $\|\cdot\|_{H^{p(z)}}$ and $|\cdot|_z$ give us $H^{p(z)}(\mathbb{R}^n)$ by Proposition 3.13. Consequently, since the log-intersection \mathcal{B} is contained in $H^{p(z)}(\mathbb{R}^n)$ by Proposition 4.4, the spaces $H^{p(z)}(\mathbb{R}^n)$ and $B(z)$ coincide with equivalence of norms.

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