Annali della
Scuola Normale Superiore di Pisa
Classe di Scienze

Augusto Visintin

Pattern evolution

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 17, nº 2 (1990), p. 197-225

<http://www.numdam.org/item?id=ASNSP_1990_4_17_2_197_0>

Pattern Evolution

AUGUSTO VISINTIN

Introduction

The aim of this paper is to introduce a variational model of *morphogenesis*, namely to describe the evolution of Euclidean sets. In particular, the proposed model can represent the evolution of sets of finite perimeter in the sense of Caccioppoli and De Giorgi, that is the evolution of oriented *non-Cartesian surfaces* of codimension 1. It can also describe the evolution of certain boundaries of *fractional dimension*, in a sense to be specified.

The classical representation of non-Cartesian surfaces is based on the minimization of the total variation functional, perturbed by a linear term proportional to the prescribed curvature of the surface. Fundamental results on the regularity of minimizing surfaces were obtained by De Giorgi [2,3]. Apparently, that model had not yet been extended to the corresponding evolution problem, in which the curvature is a prescribed function of space and time.

One of the difficulties encountered in the evolution problem is due to lack of time regularity. Here this is overcome by introducing a *hysteresis* effect. This leads to the formulation of a very natural time discretization scheme; it is then shown that from the family of solutions of these approximate problems it is possible to extract a converging subsequence, and a limit problem is formulated. However, in the present model the desired curvature is not exactly attained by the solution, although this deviation can be made arbitrarily small, by choosing an appropriate coefficient.

This formulation can be extended by replacing the total variation with any functional of a larger class fulfilling a generalization of the classical *coarea formula*. This allows to represent the evolution of surfaces of infinite perimeter, for which a sort of measure of *fractional dimension* is finite.

This research was initially aimed to describe the evolution of two-phase systems, accounting for surface tension effects; these are represented by the classical *Gibbs-Thomson law*, which states that the mean curvature of the surface of separation between different phases is proportional to the relative temperature. As we already remarked, our model corresponds just

Pervenuto alla Redazione il 23 Novembre 1988 e in forma definitiva il 20 Novembre 1989.

to an approximate formulation of this condition. Moreover a less uncomplete description of the phenomenon would require the coupling with the dynamic of diffusion of latent heat of phase transition. This is known as the *Stefan problem with surface tension*, and still lacks a satisfactory formulation. This model of surface evolution can also be applied in problems of *image interpolation.*

A different approach to the evolution of sets of finite perimeter was considered by Brakke [1], who studied the motion of a surface driven by its mean curvature from the viewpoint of *geometric measure theory.* Another formulation of the same problem, based on the parametric representation of surfaces, was dealt with by Huisken [8,9,10]. This approach was also considered by Dziuk [4], who studied the numerical approximation of that problem.

We remark that Brakke's solution is a *varifold,* hence it can be very irregular. On the other hand, the parametric representation can degenerate in finite time. On the contrary, the solution of our model exists for any time, and degenerations, like formation of a spike, are here excluded; this is possible because the evolution of the surface can be discontinuous in time.

1. - Presentation of results

1.1 *Non-convex problems.*

Each set A included in a (bounded) enviromental set $\Omega \subset \mathbf{R}^N$ ($N \geq 1$) will be represented by its characteristics function χ_A (here defined by $\chi_A = 1$ in A, $\chi_A = -1$ in $\Omega \backslash A$, so that $|\chi_A| = 1$ in Ω). The family X of measurable characteristic functions being non-convex, some compactness properties consistent with the structure of X will be necessary for our developments.

For a moment let us consider a simple stationary problem. Let us fix a (non-convex) functional $\Psi : L^\infty(\Omega) \to [0, +\infty]$ such that

$$\text{Dom}(\Psi) := \{v \in L^\infty(\Omega) : \Psi(v) < +\infty\} \subset X. \tag{1.1}$$

Thus Ψ corresponds to an application from a family of measurable subsets of Ω into $\mathbf{R}^+$. Then for any given $u \in L^1(\Omega)$ we consider the following variational inequality:

$$\begin{cases} \text{To find } w \in L^\infty(\Omega) \text{ such that } u \in \partial\Psi(w), \text{ namely} \\ \Psi(w) - \Psi(v) \leq \int_\Omega u(w - v)\, dx \text{ for every } v \in L^\infty(\Omega). \end{cases} \tag{1.2}$$

Let us introduce some terminology: a variational inequality will be said *non-convex* if either it contains a non-convex functional, or the test functions belong to a non-convex set; otherwise it will be said *convex.* Obviously this corresponds to the distinction between convex and non-convex problems. For instance (1.2) is a *non-convex* variational inequality.

If the injection of Dom(Ψ) into $L^1(\Omega)$ is compact, and if Ψ is lower semi-continuous with respect to the topology of $L^1(\Omega)$, then (1.2) has at least one solution. This statement is an obvious consequence of the direct method of the calculus of variations applied to the functional $v \longmapsto \Psi(v) - \int_\Omega u\, v\, dx$.

For instance, these assumptions are fulfilled if, like in the classical theory of Caccioppoli and De Giorgi [2,3,7],

$$\Psi = V + I_X,$$

where

$$V(u) := \int_\Omega |\nabla u| := \sup_{\eta \in C^1_c(\Omega)^N, |\eta| \le 1} \int_\Omega u \operatorname{div} \eta \, dx \, (\le +\infty) \quad \forall u \in L^1(\Omega), \tag{1.3}$$

$$I_X(u) = 0 \text{ if } u \in X, \quad I_X(u) = +\infty \text{ if } u \notin X.$$

For such a Ψ, one can show that (1.2) is equivalent to the following system

$$\int_\Omega |\nabla w| - \int_\Omega |\nabla v| \le \int_\Omega u(w - v)\, dx \quad \forall v : \Omega \to [-1, 1] \text{ measurable}, \tag{1.4}$$

$$|w| = 1 \text{ a.e. in } \Omega. \tag{1.5}$$

The implication "(1.4), (1.5) $\Rightarrow$ (1.2)" is obvious; the converse is based on the so-called *coarea formula* of Fleming and Rishel [6; 7, p. 20]; we refer to [13,14] for the proof of this statement.

The interest of this result stays in that a *non-convex* problem, here (1.2), is reduced to the selection of the solutions of a *convex* problem, here (1.4), that fulfil a *non-convex* constraint, here (1.5). Later on we shall encounter other *selection results* of this sort.

1.2 *Hysteresis.*

The variational inequality (1.2) can have more that one solution; so this formulation cannot be used if u varies in time. Moreover in the evolution problem some compactness is needed also in time, as the constraint X is non-convex. In the model we shall propose, such a property will be provided by the introduction of a *hysteresis* effect.

Hysteresis can be easily illustrated in the case of a space-independent system. Let us fix any couple $\rho := (\rho_1, \rho_2) \in \mathbf{R}^2$, with $\rho_1 < \rho_2$, and consider the operator $f_\rho : u \longmapsto w$ outlined in fig. 1. It is easy to check that for any $u \in C^0([0,T])$ the corresponding w is in $BV([0,T])$ (space of functions

$[0,T] \to \mathbf{R}$ with bounded total variation), which has *compact* injection into $L^1(0,T)$. The operator f_ρ is causal and rate-independent: it is what will be named a *hysteresis operator.*

The operator f_ρ is not closed with respect to natural convergences; so we shall also consider its closure $\tilde{f}_\rho$, which is a multi-valued hysteresis operator. It can be represented by a system of two *convex* variational inequalities coupled with a *non-convex* constraint; cf. problem $(P1)$ of section 2.

The model of set evolution we shall propose can be regarded as the coupling of the stationary space-dependent model based on the variational inequality (1.2), with the evolution space-independent model represented by the hysteresis operator $\tilde{f}_\rho$.

1.3 *Generalized coarea formula.*

Later on we shall consider functionals of the form

$$\Psi = \Lambda + I_X, \tag{1.6}$$

with $\Lambda : L^1(\Omega) \to [0,+\infty]$ convex, lower semi-continuous in $L^1(\Omega)$ (i.e., $\Lambda = \Lambda^{**}$) and fulfilling the following *generalized coarea formula*

$$\Lambda(u) = \int_{\mathbf{R}} \Lambda(H_s(u))\, ds \,(\leq +\infty) \qquad \text{for every } u \in L^1(\Omega), \tag{1.7}$$

where we set

$$H_s(\xi) := 0 \text{ if } \xi < s, \;\; H_s(\xi) := 1 \text{ if } \xi \geq s, \qquad \text{for every } \xi,\, s \in \mathbf{R}.$$

An example of such a functional is the total variation V, cf. (1.3); in this case (1.7) coincides with the standard *coarea formula* of Fleming and Rishel [6; 7, p. 20]. Other examples of functionals fulfilling (1.7) are

$$\Lambda_r(u) := \iint_{\Omega^2} |u(x) - u(y)| \cdot |x-y|^{-(N+r)} dx\, dy \,(\leq +\infty) \tag{1.8}$$

$$\text{for every } u \in L^1(\Omega),\ \forall r \in]0,1[,$$

$$\tilde{\Lambda}_r(u) := \int_{\mathbf{R}^+} \frac{dh}{h^{1+r}} \int_{\Omega} (\operatorname*{ess\,sup}_{B_h(x)\cap\Omega} u - \operatorname*{ess\,inf}_{B_h(x)\cap\Omega} u)\, dx \,(\leq +\infty) \tag{1.9}$$

$$\forall u \in L^1(\Omega),\ \forall r \in]0,1[$$

(here $B_h(x)$ denotes the N-dimensional ball of center x and radius h).

Note that, for any $r \in]0, 1[$, Λ_r is the seminorm of the fractional Sobolev space $W^{r,1}(\Omega)$. The implications of the *generalized coarea formula* (1.7) and the properties of the functionals Λ_r and $\tilde{\Lambda}_r$ are discussed in [16]. In particular these functionals are used to introduce new classes of sets of *fractional dimension.*

1.4 *Model of set evolution.*

We shortly illustrate the basic ideas of the set evolution model here introduced. First we note that for $\Psi = V + I_X$, cf. (1.3), the stationary problem (1.2) can be rewritten as

$$\partial(V + I_X)(w) \ni u \qquad \text{in } L^\infty(\Omega). \tag{1.10}$$

Then as a rate-independent evolution model, one is tempted to consider an equation of the form

$$S\left(\frac{\partial w}{\partial t}\right) + \partial(V + I_X)(w) \ni u \qquad \text{in } L^\infty(\Omega), \text{ for } 0 < t < T, \tag{1.11}$$

where S denotes the sign graph: $S(\xi) = \{-1\}$ if $\xi < 0$, $S(0) = [-1, 1]$, $S(\xi) = \{1\}$ if $\xi > 0$. More generally, for any functional Ψ fulfilling (1.1), (1.11) could be replaced by

$$S\left(\frac{\partial w}{\partial t}\right) + \partial\Psi(w) \ni u \qquad \text{in } L^\infty(\Omega), \text{ for } 0 < t < T. \tag{1.12}$$

Obviously (1.12) corresponds to a system of the form

$$\alpha + \beta = u, \qquad \alpha \in S\left(\frac{\partial w}{\partial t}\right), \qquad \beta \in \partial\Psi(w), \tag{1.13}$$

and similarly for (1.11).

Unfortunately "technical difficulties" arise in giving an acceptable meaning to formulae (1.11), (1.12). Thus we introduce problems ($P2$) and ($P3$) of sections 3 and 4, whose interpretation is less obvious than that of (1.11) and (1.12); however problems ($P2$) and ($P3$) are natural extensions of ($P1$), which corresponds to the hysteresis behaviour outlined in fig. 1. Moreover the implicit time discretization schemes used later on to approximate problems ($P2$) and ($P3$) are reminiscent of (1.12) and (1.11), respectively.

It does not seem sound to replace $S\left(\frac{\partial w}{\partial t}\right)$ by $\frac{\partial w}{\partial t}$ in (1.11) and (1.12). In such a case, at least on the corresponding approximate problems, one could multiply these equations by $\frac{\partial w}{\partial t}$, getting $\frac{\partial w}{\partial t} \in L^2(Q)$; but this regularity would exclude any evolution of the characteristic function w. That is this modified problem would not be a model of surface evolution.

1.5 *Application.*

If $\Lambda = V$, cf. (1.3), then problem $(P3)$ of section 4 can be used to represent the quasi-stationary rate-independent evolution of oriented *non-Cartesian surfaces* of $\mathbf{R}^N$ of codimension 1, subject to a time-dependent mean curvature field u. The case of the functionals Λ_r and $\tilde{\Lambda}_r$, with $0 < r < 1$, allows to represent the evolution of more irregular sets with boundaries of *fractional dimension.*

Let us apply this model to the *phase evolution* of a solid-liquid system controlled by a time-dependent temperature field. Then it also accounts for *supercooling* and *superheating* effects and for phenomena of phase *nucleation* or *annihilation* [13,14,15].

Let H denote the mean curvature of the interface S, assumed posititve for a solid ball, and v_{S} denote the normal velocity of S, assumed positive for solidification and negative for melting. If u is continuous, then problem $(P3)$ of section 4 corresponds to the following moving boundary condition

$$-(N-1)H - u \in S(v_{\mathsf{S}}) \qquad \text{on } \mathsf{S}, \tag{1.14}$$

where S still denotes the sign graph and N is the Euclidean dimension of Ω. This equation represents a *rate-independent* evolution. On the contrary it is an open question to formulate a variational problem corresponding to the *rate-dependent* equation

$$-(N-1)H - u \in S(v_{\mathsf{S}}) + \alpha v_{\mathsf{S}} \qquad \text{on } \mathsf{S}, \tag{1.15}$$

where α denotes a time relaxation constant.

Here several constants have been normalized, however by a simple modification one can reduce himself to the case in which the right-hand side of (1.14) is multiplied by a constant $C > 0$. So for $C << 1$ one gets $|(N-1)H + u| << 1$ on S, which can be compared with the *Gibbs-Thomson* equilibrium condition:

$$-(N-1)H = u \qquad \text{on } \mathsf{S}. \tag{1.16}$$

Our model of set evolution can also be used to tackle the problem of *pattern interpolation*: given any couple of subsets of $\mathbf{R}^N$, construct a continuum of *intermediate* sets. As a model for this construction, we propose a *control problem*, governed by problem $(P3)$ as *state equation*, and with u as *control variable.*

1.6 *Plan of the paper.*

In section 2 we deal with the evolution of space-independent systems with hysteresis. First we give a precise definition of the hysteresis operators f_ρ and $\tilde{f}_\rho$; we introduce a variational formulation of $\tilde{f}_\rho$, cf. problem $(P1)$, and prove the existence of a solution. Then we consider more general hysteresis effects, corresponding to the evolution of a scalar system governed by a non-convex potential according to the so-called *delay rule*; we show that, by means of a suitable transformation, a fairly general class of hysteresis behaviours can be reduced to the elementary operators f_ρ and $\tilde{f}_\rho$.

In section 3 we formulate our model of set evolution, and prove that it has at least one solution.

In section 4 we recall the definition of the functionals fulfilling the *generalized coarea formula* (1.7), and give some examples; a more detailed presentation of this subject can be found in [16]. The use of these functionals allows to reformulate problem ($P2$) as a system of two *convex* variational inequalities coupled with a *non-convex* constraint, i.e., problem ($P3$).

Then in section 5 we outline the previously mentioned applications to the evolution of non-Cartesian surfaces and of two-phase systems. Finally we briefly discuss the problem of *pattern interpolation.*

1.7 *Bibliographical note.*

The present work is in the framework of a research on models of *surface tension* effects in two-phase systems, which now we briefly outline.

The stationary problem was first addressed in [13], where the physical aspects of metastability and nucleation were presented and the equivalence between (1.2) and the system (1.4), (1.5) was proven. In [14] this result was extended to more general non-convex potentials and to functionals fulfilling the generalization coarea formula (1.7). The implications of this formula, the properties of the functionals Λ_r and $\tilde{\Lambda}_r$, cf. (1.8) and (1.9), and the possibility of representing sets with boundaries of *fractional dimension* were then examined in [16], cf. also [17,18]. The present paper is the first one dealing with the corresponding evolution problems.

As we said above, a basic tool for the formulation of our model is the introduction of *hysteresis* effects. Mathematical models of hysteresis phenomena were extensively studied by Krasnosel'skiĭ and Pokrovskiĭ and by other Soviet mathematicians, cf. [11]. A review of the research of the present author on this subject can be found in [12].

2. - Hysteresis

2.1 *Elementary hysteresis operators.*

For any fixed couple $\rho := (\rho_1, \rho_2) \in \mathbf{R}^2$, with $\rho_1 < \rho_2$, we define the *elementary hysteresis operator*

$$f_\rho : C^0([0,T]) \times \{-1,1\} \to BV([0,T]) : (u, w^0) \longmapsto w,$$

as follows (cf. fig. 1):

$$w(0) = w^0; \tag{2.1}$$

for any $t \in]0,T]$, setting $A_t := \{\tau \in]0,t] : u(\tau) \notin [\rho_1, , \rho_2]\}$,

$$w(t) = \begin{cases} w^0 & \text{if} \quad A_t = \emptyset, \\ 1 & \text{if} \quad A_t \neq \emptyset \text{ and } u(\sup A_t) \leq \rho_1, \\ 1 & \text{if} \quad A_t \neq \emptyset \text{ and } u(\sup A_t) \geq \rho_2. \end{cases} \tag{2.2}$$

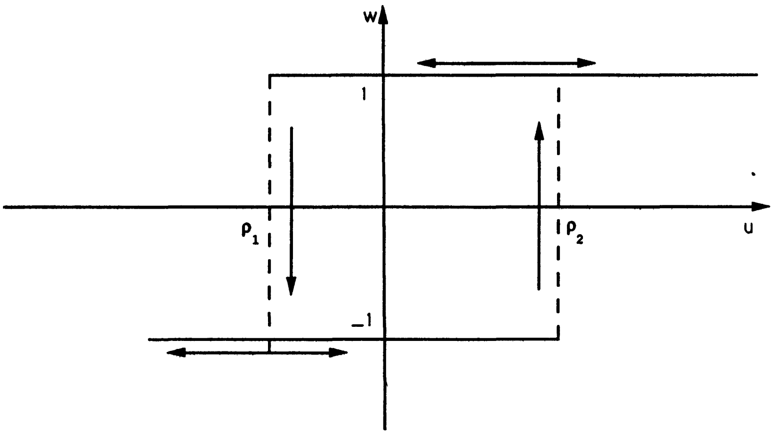

Fig. 1 - $w = f_\rho(u, w^0)$. For any $t \in [0,T[$, if $w(t) = -1$ ($w(t) = -1$, respectively), then w remains constant for $\tau > t$ as long as $u(\tau) \leq \rho_2$ ($u(\tau \geq \rho_1$, respectively). If u becomes larger than ρ_2 (smaller than ρ_1, respectively), then w jumps to 1 (to -1, respectively), and so on.

Remark that the function u has a finite number of oscillations between ρ_1 and ρ_2, if any, as it is uniformly continuous in $[0,T]$; hence the total variation of w is actually finite.

Note that not only the operator f_ρ is discontinuous, but also its graph is non-closed, in the following sense:

$$u_n \to u \text{ uniformly in } [0,T], \tag{2.3}$$

$$f_\rho(u_n, w^0) \to w \text{ weakly star in } BV(0,T) \text{ and pointwise in } [0,T], \tag{2.4}$$

do not entail

$$w = f_\rho(u, w^0). \tag{2.5}$$

As a counterexample, one can take $w^0 = -1$, $u_n(t) = \rho_2 + \frac{1}{n}$.

In order to overcome this drawback, we introduce the following *multi-valued hysteresis operator*

$$\tilde{f}_\rho : C^0([0,T]) \times \{-1,1\} \to \mathcal{P}(BV([0,T]));$$

$w \in \tilde{f}_\rho(u, w^0)$ if and only if (2.1) holds, and for any $t \in]0,T]$

$$w(t) = \begin{cases} 1 & \text{if} \quad u(t) < \rho_1, \\ 1 \text{ or } 1 & \text{if} \quad \rho_1 \leq u(t) \leq \rho_2, \\ 1 & \text{if} \quad u(t) > \rho_2, \end{cases} \tag{2.6}$$

$$\begin{cases} \text{if } u(t) \neq \rho_1, \rho_2, & \text{then } w \text{ is constant in a neighbourhood of } t, \\ \text{if } u(t) = \rho_1, & \text{then } w \text{ is non-increasing in a neighbourhood of } t, \\ \text{if } u(t) = \rho_2, & \text{then } w \text{ is non-decreasing in a neighbourhood of } t, \end{cases} \tag{2.7}$$

Note that also these conditions entail that $w \in BV([0,T])$. The operator $\tilde{f}_\rho$ is an extension of $f_\rho : f_\rho \subset \tilde{f}_\rho$, in the sense of graphs. We shall see that $\tilde{f}_\rho$ is the closure of f_ρ with respect to natural convergences.

Possibly replacing u with $a(u) := (2u - \rho_1 - \rho_2)/(\rho_2 - \rho_1)$, we can assume that $\rho_1 = -1$, $\rho_2 = 1$. This will simplify some computations.

PROPOSITION 1. *For any* $u \in C^0([0,T])$ *and any* $w^0 \in \{-1,1\}$, $w \in \tilde{f}_\rho(u, w^0)$ *if and only if* w *solves the following problem:*

PROBLEM $(P1)$ - *To find* $w \in BV([0,T])$ *such that* (2.1) *holds and*

$$|w(t)| = 1 \qquad \forall t \in]0,T], \tag{2.8}$$

$$\int_0^T [u(t) + w(t)] \cdot [w(t) - v(t)]\, dt \geq 0 \qquad \forall v :]0,T[\to [-1,1] \text{ measurable}, \tag{2.9}$$

$$\int_0^T (u - v)\, dw(t) \geq 0 \qquad \forall v : [0,T] \to [-1,1] \text{ continuous} \tag{2.10}$$

(the latter integral is in the sense of Lebesgue-Stieltjes). □

PROOF. It is easy to check that (2.6) is equivalent to (2.8), (2.9). Then, under condition (2.6), (2.7) is equivalent to (2.10). □

REMARKS. (i) If $u \in W^{1,1}(0,T)$, then (2.10) is equivalent to the following variational inequality, obtained by partial integration in time:

$$w(0)\,[u(0) - v(0)] + \int_0^T w \frac{d}{dt}(u - v)\, dt \leq w(T)\,[u(T) - v(T)] \tag{2.11}$$

$$\forall v \in W^{1,1}(0,T),\ |v| \leq 1.$$

(ii) (2.10) yields

$$\int_0^T u dw(t) \geq \int_0^T |dw(t)|; \tag{2.12}$$

this can be regarded as a *dissipation law*. The need of considering such a condition, besides the equilibrium equation (2.9), is related to *irreversibility* of hysteresis.

(iii) The graph of $\tilde{f}_\rho$ is closed in the following sense: if

$$u_n \to u \quad \text{uniformly in } [0,T], \tag{2.13}$$

$$\tilde{f}_\rho(u_n, w^0) \ni w_n \to w \tag{2.14}$$

weakly star in $BV([0,T])$ and pointwise in $[0,T]$,

then

$$w \in \tilde{f}_\rho(u, w^0); \tag{2.15}$$

indeed the corresponding equations (2.8),...,(2.10) are preserved in the limit as $n \to \infty$.

It is not difficult to check that the graph of f_ρ is dense in that of $\tilde{f}_\rho$ with respect to the convergences (2.13) and (2.14). Hence we can conclude with the following result:

PROPOSITION 2. *The operator $\tilde{f}_\rho$ is the closure of f_ρ, in the sense of the convergences* (2.13), (2.14). □

2.2 *An approximation procedure.*

We shall approximate the operator $\tilde{f}_\rho$ by means of time discretization, and show the convergence of such a scheme.

Let us fix any $m \in \mathbf{N}$ and set $k := T/m$, $w_m^0 := w^0$, $u_m^n := u(nk)$ for $n = 0, \dots, m$. Then we define, for $n = 1, \dots, m$,

$$(2.16) \qquad w_m^n = \begin{cases} 1 & \text{if} \quad u_m^n + w_m^{n-1} < 0, \\ 1 \text{ or } 1 & \text{if} \quad u_m^n + w_m^{n-1} = 0, \\ 1 & \text{if} \quad u_m^n + w_m^{n-1} > 0. \end{cases}$$

It is easy to check that there exists a function $\hat{w}_m \in BV([0,T])$ such that, denoting by u_m the linear interpolate of $u_m(nk) = u_m^n (n = 0, \dots, m)$,

$$\begin{cases} \hat{w}_m \in \tilde{f}_\rho(u_m, w^0) & \text{in } [0,T], \\ \hat{w}_m(nk) = w_m^n & n = 0, \dots, m; \end{cases}$$

so the discretization scheme (2.16) of $\tilde{f}_\rho$ is consistent.

We recall that we denote by S the sign graph. We notice that

$$(2.17) \qquad -v_2 \in S(v_1 - v_2) \qquad \forall v_1 \in [-1,1], \qquad \forall v_2 \in \{-1,1\};$$

thus in particular

$$(2.18) \qquad -w_m^{n-1} \in S(w_m^n - w_m^{n-1}) \qquad \text{for } n = 1, \dots, m.$$

Formulae (2.16) and (2.18) can be rewritten as follows

$$(2.19) \qquad z_m^n := u_m^n + w_m^{n-1} \in S^{-1}(w_m^n),$$

$$(2.20) \qquad w_m^n - w_m^{n-1} \in S^{-1}(-w_m^{n-1}),$$

that is

$$(2.21) \qquad [u_m^n + w_m^{n-1}](w_m^n - v) \geq 0 \quad \forall v \in [-1,1],$$

$$(2.22) \qquad (w_m^n - w_m^{n-1})(-w_m^{n-1} - v) \geq 0 \quad \forall v \in [1,1],\ n = 1, \dots, m;$$

these yield

$$(2.23) \qquad (w_m^n - w_m^{n-1})(u_m^n - v) \geq (w_m^n - w_m^{n-1})(u_m^n + w_m^{n-1}) \geq 0$$
$$\forall v \in [-1,1],\ n = 1, \dots, m;$$

whence

$$(w_m^n - w_m^{n-1})(u_m^n - v) \geq 0 \qquad \forall v \in [-1,1],\ n = 1,\ldots,m. \tag{2.24}$$

Also here the linear interpolate w_m of $w_m(nk) = w_n^m$ $(n = 0,\ldots,m)$ has uniformly bounded total variation, because u can oscillate between -1 and 1 only a finite number of times, if any. Therefore there exists a subsequence, still labelled by m, such that

$$w_m \to w \text{ weakly star in } BV([0,T]) \text{ and pointwise in } [0,T]. \tag{2.25}$$

Then summing (2.21) and (2.24) for $n = 1,\ldots,m$, and taking $m \to \infty$, we get (2.9) and (2.10). Also the constraint $|w_m^n| = 1$ is preserved in the limit. This concludes the proof of the convergence of the discrete scheme (2.16).

Finally we show the following property of *piecewise monotonicity*:

$$R := (u_m^n - u_m^{n-1})(w_m^n - w_m^{n-1}) \geq 0 \qquad n = 1,\ldots,m. \tag{2.26}$$

To this aim we set

$$R_1 := [(u_m^n - z_m^n) - (u_m^{n-1} - z_m^{n-1})] \cdot (w_m^n - w_m^{n-1}),$$

$$R_2 := (z_m^n - z_m^{n-1}) \cdot (w_m^n - w_m^{n-1}),$$

so that $R = R_1 + R_2$. If $w_m^n > w_m^{n-1}$ then $w_m^{n-1} = -1$, namely by (2.19)

$$u_m^n - z_m^n = -w_m^{n-1} = 1 \geq u_m^{n-1} - z_m^{n-1} (= -w_m^{n-2} = \pm 1);$$

hence $R_1 \geq 0$; the same procedure can be used if $w_m^n < w_m^{n-1}$. Still by (2.19) we have $R_2 \geq 0$; therefore $R \geq 0$.

We summarize these results in the following statement:

THEOREM 1. *Take any* $u \in C^0([0,T])$ *and* $w^0 = \pm 1$. *For any* $m \in \mathbf{N}$, *let* w_m *be the piecewise linear interpolate of* w_m^n, *defined in* (2.16). *Then, possibly extracting a subsequence,* (2.25) *holds and* $w \in \tilde{f}_\rho(u, w^0)$. *Moreover the piecewise monotonicity property* (2.26) *is fulfilled.* □

We recall that for any set A we denote by I_A its indicator function (or functional): $I_A(v) = 0$ if $v \in A$, $I_A(v) = +\infty$ if $v \notin A$.

REMARK. (2.16) is equivalent to the condition that w_m^n minimizes the (non-convex) function

$$J_m^n(v) := |v - w_m^{n-1}| + I_{\{-1,1\}}(v) - u_m^n v (\leq +\infty) \qquad \forall v \in \mathbf{R}; \tag{2.27}$$

note that by (2.17) we have

$$-w_m^{n-1} \in S(v - w_m^{n-1}) \text{ for } v = \pm 1;$$

hence

$$\begin{aligned} J_m^n(v) &= -w_m^{n-1}(v - w_m^{n-1}) + I_{\{-1,1\}}(v) - u_m^n v = \\ &= I_{\{-1,1\}}(v) - (u_m^n + w_m^{n-1})v + 1 \quad \forall v \in \mathbf{R}. \end{aligned} \tag{2.28}$$

□

2.3 *More general hysteresis behaviours.*

It is easy to check that the elementary hysteresis operator f_ρ defined by (2.1), (2.2) represents the evolution governed by the *non-convex potential*

$$\Phi_u(v) := I_{[-1,1]}(v) + \frac{1-v^2}{2} - uv := \begin{cases} \dfrac{1-v^2}{2} - uv & \text{if } |v| \leq 1, \\ +\infty & \text{if } |v| > 1, \end{cases} \tag{2.31}$$

according to the so-called *delay rule*:

(2.32) $\begin{cases} \text{for any } t > 0,\ w(t) \text{ is an either absolute or relative minimum} \\ \text{of } \Phi_{u(t)}; \text{ between two such minima, } w \text{ "chooses" that which} \\ \text{minimizes the time variation of } w \text{ in a neighbourhood of } t. \end{cases}$

Here *delay* means *hysteresis*, according to the etymology of the latter term. This behaviour can be compared with the so-called *Maxwell rule*:

$$\Phi_{u(t)}(w(t)) = \inf \Phi_{u(t)} \qquad \forall t > 0, \tag{2.33}$$

that is

$$\partial\Phi_{u(t)}(w(t)) \ni 0 \qquad \forall t > 0. \tag{2.34}$$

For Φ_u defined as in (2.31), (2.34) is equivalent to

$$\partial\Phi_{u(t)}^{**}(w(t)) \ni 0, \qquad |w(t)| = 1 \qquad \forall t > 0; \tag{2.35}$$

namely, as $\Phi_u^{**}(v) = I_{[-1,1]}(v) - uv \qquad \forall u, v \in \mathbf{R},$

$$w(t) \in S(u(t)), \qquad |w(t)| = 1 \qquad \forall t > 0. \tag{2.36}$$

Now we introduce a more general family of non-convex potentials Φ_u. First let $\lambda_1, \lambda_2 \in \mathbf{R}$, with $\lambda_1 < \lambda_2$, and $\varphi_1, \varphi_2 : \mathbf{R} \to \mathbf{R}$ be non-decreasing continuous functions such that (cf. fig. 2)

$$\begin{cases} \varphi_1' = 0 & \text{in }]\lambda_1, \lambda_2[\quad , \quad \varphi_1' > 0 \quad \text{a.e.} \quad \text{in } \mathbf{R}\backslash[\lambda_1, \lambda_2]; \\ \varphi_2' > 0 \quad \text{a.e.} & \text{in }]\lambda_1, \lambda_2[\quad , \quad \varphi_2' = 0 \qquad \text{in } \mathbf{R}\backslash[\lambda_1, \lambda_2]; \end{cases} \tag{2.37}$$

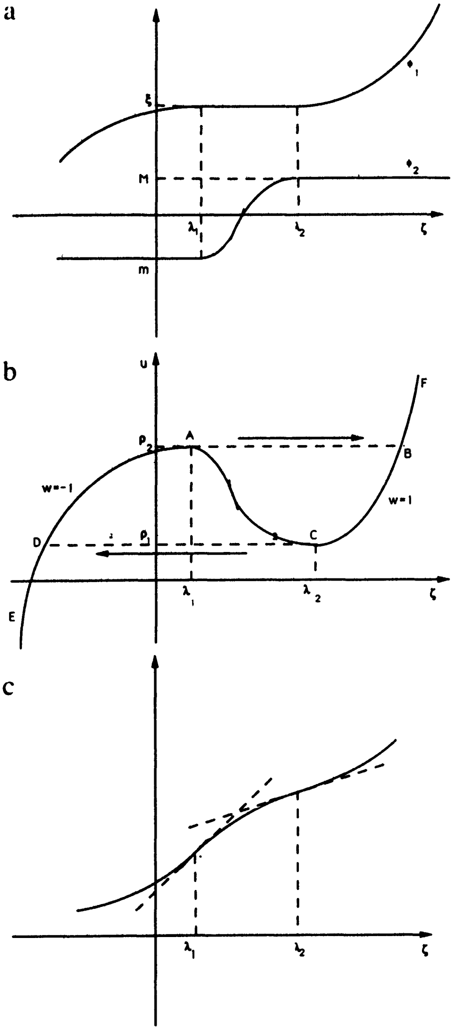

Fig. 2 - The functions φ_1, φ_2, φ, Φ_u, *are defined in* (2.37),...,(2.39)*; cf. also* (2.40), (2.41). *The hysteresis behaviour sketched in fig. 2b corresponds to the evolution of a system governed by the non-convex potential* $\Phi_u(v) = \Phi_0(v) - uv$, *according to the* delay rule (2.32).

Note the hysteresis loop *ABCD and the excluded region* $\lambda_1 \leq \varsigma \leq \lambda_2$. *Setting* $w := f_\rho(u, w^0)$, *the branches ADE and CBF correspond to* $w = -1$ *and* $w = 1$, *respectively. In fig. 2c,* Φ_0 *is locally convex for* $\varsigma \notin [\lambda_1, \lambda_2]$ *and locally concave for* $\varsigma \in]\lambda_1, \lambda_2[$.

then we set

$$\varphi(\eta) := \varphi_1(\eta) - \varphi_2(\eta) \qquad \forall \eta \in \mathbf{R}, \tag{2.38}$$

$$\Phi_u(v) := \int_0^v \varphi(\eta)\, d\eta - uv \qquad \forall u,\, v \in \mathbf{R}, \tag{2.39}$$

$$\begin{cases} \{\xi\} := \varphi_1([\lambda_1,\, \lambda_2]),\ m := \varphi_2(\lambda_1),\ M := \varphi_2(\lambda_2), \\ \rho_1 := \varphi(\lambda_2)(= \xi - M),\ \rho_2 := \varphi(\lambda_1)(= \xi - m), \end{cases} \tag{2.40}$$

$$\ell(\eta) := \frac{M - m}{2}\eta + \frac{M + m}{2} \qquad \forall \eta \in \mathbf{R}, \tag{2.41}$$

so that $\ell(1) = M$, $\ell(-1) = m$.

Note that as u ranges in $\mathbf{R}$, any either absolute or relative minimum of Φ_u is confined to $\mathbf{R}\backslash[\lambda_1,\, \lambda_2]$, so that either $\varphi_2(w) = m$ or $\varphi_2(w) = M$. It is easy to check that then the evolution defined by the delay ryle (2.32) corresponds to the hysteresis behaviour represented in fig. 2b. We will give a variational formulation of the latter relation.

First we must slightly modify the definition of f_ρ by requiring that w jumps from -1 to 1 (from 1 to -1, respectively) as soon as u reaches the value $\rho_2 = 1$ ($\rho_1 = -1$, respectively). More precisely $\check{f}_\rho$ is defined as follows. For any $u \in C^0([0,T])$ and any $w^0 \in \{-1, 1\}$, $w = \check{f}_\rho(u,\, w^0)$ if and only if (2.1) holds and, setting $B_t := \{\tau \in]0, t] : u(\tau) = \rho_1 \text{ or } \rho_2\}$.

$$w(t) = \begin{cases} w^0 & \text{if} \quad B_t = \emptyset, \\ -1 & \text{if} \quad B_t \neq \emptyset \text{ and } u(\max B_t) = \rho_1, \\ 1 & \text{if} \quad B_t \neq \emptyset \text{ and } u(\max B_t) = \rho_2. \end{cases} \tag{2.42}$$

Thus the points $(-1, 1)$ and $(1, -1)$ are excluded from the graph of $\check{f}_\rho$. We notice that this change is of little importance, as the closure of $\check{f}_\rho$ in the sense of the convergences (2.13), (2.14) coincides with that of f_ρ, i.e., $\tilde{f}_\rho$.

THEOREM 2. *For any* $u \in C^0([0,T])$ *and any* $w^0 \in \{-1, 1\}$, *there exists a measurable function* $\varsigma : [0,T] \to \mathbf{R}\backslash[\lambda_1,\, \lambda_2]$ *such that*

$$\varphi(\varsigma) = u \qquad \textit{in } [0,T], \tag{2.43}$$

$$\varphi_2(\varsigma) = \ell(f_\rho(u,\, w^0)) \qquad \textit{in } [0,T]. \tag{2.44}$$

PROOF. Let us set $w = \check{f}_\rho(u,\, w^0)$ in $[0,T]$. We shall construct a function ς such that

$$\varphi_1(\varsigma) = u + \ell(w) \qquad \text{in } [0,T], \tag{2.45}$$

$$\varphi_2(\varsigma) = \ell(w)\, (:= \frac{1}{2}[(M - m)w + M + m]) \qquad \text{in } [0,T]. \tag{2.46}$$

To this aim for any $t \in [0,T]$ we distinguish the two cases $w(t) = \pm 1$:

(i) If $w = -1$, then $\ell(w) = m$ and $u < \rho_2 = \xi - m$. Then we set $\varsigma := \varphi_1^{-1}(u+m)$; so $\varphi_1(\varsigma) = u+m$ and (2.45) holds. Moreover, as $u+m < \xi$, we have $\varsigma < \lambda_1$, whence $\varphi_2(\varsigma) = m$ and (2.46) holds.

(ii) If $w = 1$, then $\ell(w) = M$ and $u > \rho_1 = \xi - M$. Then we set $\varsigma := \varphi_1^{-1}(u+M)$; so $\varphi_1(\varsigma) = u+m$ and (2.45) holds. Moreover, as $u+m > \xi$, we have $\varsigma > \lambda_2$, whence $\varphi_2(\varsigma) = M$ and (2.46) holds. □

It is easy to see that for any $u \in C^0([0,T])$ the (discontinuous) curve $[0,T] \to \mathbf{R}^2 : t \longmapsto (\varsigma(t),\ u(t))$ defined by (2.43), (2.44) corresponds to the hysteresis behaviour sketched in fig. 2b. Hence it represents evolution governed by the potential $\Phi_u(v) := \Phi_0(v) - uv$, cf. fig. 2c, according to the delay rule (2.32).

Finally we note that the limit case $\lambda_1 = \lambda_2$ is not excluded; there φ_2 is a jump function and ς ranges in $\mathbf{R}\backslash\{\lambda_1\}$.

3. - Set Evolution

Let Ω be a bounded domain of $\mathbf{R}^N (N \geq 1)$. We set $Q := \Omega \times]0,T[$ and

$$X := \{v \in L^\infty(\Omega) : |v| = 1 \text{ a.e. in } \Omega\},$$

family of characteristic functions of measurable subsets of Ω.

Let the functional $\Psi : L^\infty(\Omega) \to [0,+\infty]$ be such that

$$\text{Dom}(\Psi) := \{v \in L^\infty(\Omega) : \Psi(v) < +\infty\} \subset X, \tag{3.1}$$

and let

$$u \in L^1(\Omega;\ W^{1,1}(0,T)), \tag{3.2}$$

$$w^0 \in \text{Dom}(\Psi). \tag{3.4}$$

We can now formulate our model of set evolution:

PROBLEM $(P2)$. *To find $w \in L^\infty(Q)$ such that $\frac{\partial w}{\partial t} \in M(Q)$ and*

$$\iint_Q (u+w)(w-v)\,dx\,dt \geq \int_0^T [\Psi(w) - \Psi(v)]\,dt \qquad \forall v \in L^\infty(Q), \tag{3.4}$$

$$\int_\Omega w^0(x)[u(x,0) - v(x,0)]\,dx + \iint_Q w\,\tfrac{\partial}{\partial t}(u - v)\,dx\,dt + \Psi(w(\cdot,T)) \tag{3.5}$$
$$\leq \Psi(w^0) + \int_\Omega w(x,T)\,[u(x,T) - v(x,T)]\,dx$$
$$\forall v \in L^1(\Omega;\, W^{1,1}(0,T)),\ |v| \leq 1 \ a.e.\ in\ Q.$$

INTERPRETATION OF PROBLEM ($P2$). Let us assume that $w \in L^\infty(\Omega) \cap L^1(\Omega;\, BV([0,T]))$. If moreover $u \in L^\infty(\Omega;\, W^{1,1}(0,T))$, then by partial integration in time, it is easy to see that (3.5) is equivalent to

$$\int_\Omega dx \int_{[0,T]} (u - v)\,dw\,(x,t) \geq \Psi(w(\cdot,T)) - \Psi(w^0) \tag{3.6}$$
$$\forall v \in L^\infty(0,T;\, C^0([0,T])),\ |v| \leq 1 \ a.e.\ in\ Q$$

(here the time integral is in the sense of Lebesgue-Stieltjes). Note that in order to write (3.6), it is sufficient to require $u \in L^\infty(\Omega;\, C^0([0,T])$.

Comparing the systems (2.8),...,(2.10) and (3.4), (3.6), we can conclude that problem ($P2$) is a natural extension of problem ($P1$) of section 2 to the space-dependent case.

Also note that (3.6) yields

$$\int_\Omega dx \int_{[0,T]} u dw(x,t) \geq \Psi(w(\cdot,T)) - \Psi(w^0) + \int_\Omega dx \int_{[0,T]} |dw(x,t)|, \tag{3.7}$$

which can be regarded as a *dissipation law*. As we already remarked about the analogous inequality (2.12), the presence of such an extra-condition is related to the *irreversibility* of the model under consideration. □

THEOREM 3. *Assume that* (3.1),...,(3.3) *hold and that*

(3.8) *Ψ is lower semi-continuous with respect to the strong topology of $L^1(\Omega)$,*

(3.9)
there exists a separable Banach space B such that
Dom(Ψ) $\subset B'$(:= *dual space of* B) $\subset L^1(\Omega)$,
the injection Dom(Ψ) $\to B'$ *is continuous,*
the injection $B' \to L^1(\Omega)$ *is compact.*

Then problem ($P2$) *has a solution such that*

$$w \in L^\infty(0,T;\, B'), \qquad \Psi(w) \in L^\infty(0,T). \tag{3.10}$$

If moreover

$$u \in L^\infty(Q), \tag{3.11}$$

then $(P2)$ *has a solution such that*

$$\Psi(w) \in BV([0,T]). \tag{3.12}$$

PROOF. Let us fix any $m \in \mathbf{N}$ and set $k := T/m$, $w_m^0(x) := w^0(x)$, $u_m^n(x) := u(x, nk)$ a.e. in Ω for $n = 1, \ldots, m$. We claim that there exist $w_m^1, \ldots, w_m^m \in \mathrm{Dom}(\Psi)$ such that

$$\int_\Omega (u_m^n + w_m^{n-1})(w_m^n - v)\,dx \geq \Psi(w_m^n) - \Psi(v) \qquad \forall v \in L^\infty(\Omega),\ n = 1, \ldots, m, \tag{3.13}$$

that is

$$u_m^n + w_m^{n-1} \in \partial\Psi(w_m^n) \qquad \text{in } L^\infty(\Omega)',\ n = 1, \ldots, m.$$

Indeed, assuming that w_m^{n-1} is known, applying the direct method of the calculus of variations and using the assumptions (3.1), (3.8), (3.9), it is easy to check that the functional

$$\mathcal{J}_m^n(v) := \Psi(v) - \int_\Omega (u_m^n + w_m^{n-1})\,v\,dx\ (\leq +\infty) \qquad \forall v \in L^\infty(\Omega) \tag{3.14}$$

has an absolute minimum w_m^n; namely (3.13) holds.

By (3.1) w_m^{n-1}, $w_m^n \in X$; hence by (2.17) we have

$$-w_m^{n-1}(x) \in S(w_m^n(x) - w_m^{n-1}(x)) \qquad \text{a.e. in } \Omega, \text{ for } n = 1, \ldots, m; \tag{3.15}$$

so taking $v = w_m^{n-1}$ in (3.13) we get

$$\|w_m^n - w_m^{n-1}\|_{L^1(\Omega)} + \Psi(w_m^n) - \Psi(w_m^{n-1}) \leq \int_\Omega u_m^n\,(w_m^n - w_m^{n-1})\,dx, \tag{3.16}$$

whence for any $\tilde{n} = 1, \ldots, m$ we have

$$\sum_{n=1}^{\tilde{n}} \|w_m^n - w_m^{n-1}\|_{L^1(\Omega)} + \Psi(w_m^{\tilde{n}}) - \Psi(w^0) \leq \sum_{n=1}^{\tilde{n}} \int_\Omega u_m^n(w_m^n - w_m^{n-1})\,dx =$$

$$= \int_\Omega (u_m^{\tilde{n}} w_n^{\tilde{n}} - u_m^1 w^0)\,dx - \sum_{n=2}^{\tilde{n}} \int_\Omega (u_m^n - u_m^{n-1}) w_m^{n-1}\,dx$$

$$\leq \text{Constant}\ \|u_m\|_{L^1(\Omega; W^{1,1}(0,T))};$$

therefore

$$\sum_{n=1}^{m} \|w_m^n - w_m^{n-1}\|_{L^1(\Omega)} + \max_{n=1,\dots,m} \Psi(w_m^n) \tag{3.17}$$

$$\le \text{Constant independent of } m.$$

Setting $M(Q) := C_c^0(Q)'$, $w_m(x,t) := w_m^n(x)$ a.e. in Ω, if $(n-1)k < t \le nk$, for $n = 1, \dots, m$, and using (3.9), the estimates (3.17) yield

$$\left\| \frac{\partial w_m}{\partial t} \right\|_{M(Q)}, \|\Psi(w_m)\|_{L^\infty(0,T)}, \|w_m\|_{L^\infty(0,T;B')} \le$$

$$\le \text{Constant independent of } m.$$

Therefore there exists a subsequence, here still labelled by m, such that

$$w_m \to w \qquad \text{weakly star in } L^\infty(Q) \cap L^\infty(0,T;B') \tag{3.19}$$

$$\frac{\partial w_m}{\partial t} \to \frac{\partial w}{\partial t} \qquad \text{weakly star in } M(Q), \tag{3.20}$$

$$w_m(\cdot,T) = w_m^m \to w(\cdot,T) \qquad \text{weakly star in } B', \text{ strongly in } L^1(\Omega). \tag{3.21}$$

By (3.9), (3.19) and (3.20), applying a standard compactness result essentially due to Aubin, we have

$$w_m \to w \qquad \text{strongly in } L^1(Q); \tag{3.22}$$

hence by (3.8) and (3.18), we get (3.10).

By (3.15) we have

$$\int_\Omega (w_m^n - w_m^{n-1})(-w_m^{n-1} - v)dx \ge 0 \qquad \forall v \in L^\infty(\Omega),\ |v| \le 1 \text{ a.e. in } \Omega; \tag{3.23}$$

for any $v \in L^1(\Omega; W^{1,1}(0,T))$ such that $|v| \le 1$ a.e. in Q, setting $v_m^n(x) := v(x, nk)$ a.e. in Ω for $n = 1, \dots, m$. (3.13) and (3.23) yield

$$\sum_{n=1}^{m} \int_\Omega (w_m^n - w_m^{n-1})(u_m^n - v_m^n)\,dx \ge \sum_{n=1}^{m} \int_\Omega (w_m^n - w_m^{n-1}) \tag{3.24}$$

$$(u_m^n + w_m^{n-1})\,dx \ge \sum_{n=1}^{m} [\Psi(w_m^n) - \Psi(w_m^{n-1})] = \Psi(w_m^m) - \Psi(w^0);$$

then by discrete partial integration we get

$$\int_\Omega w_m^0(x)\,[u_m^0(x) - v_m^0(x)]\,dx + \sum_{n=1}^{m} \int_\Omega w_m^{n-1}\,(u_m^n - u_m^{n-1} + v_m^n - v_m^{n-1})dx \tag{3.25}$$

$$+\Psi(w_m^m) \le \Psi(w^0) + \int_\Omega w_m^m(x)\,[u_m^m(x) - v_m^m(x)]\,dx.$$

Finally we rewrite (3.13) and (3.25) in terms of w_m and of the interpolate $\hat{u}_m$ of $\{u_m^n\}_{n=0,\dots,m}$, and take $m \to \infty$; this yields (3.4) and (3.5). So w is a solution of problem $(P2)$.

Now let $u \in L^\infty(Q)$; by (3.16) and (3.17) we have

$$
\begin{aligned}
\sum_{n=1}^{m} |\Psi(w_m^n) - \Psi(w_m^{n-1})| &\le \sum_{n=1}^{m} \Big| \int_\Omega u_m^n\,(w_m^n - w_m^{n-1})\,dx \Big| \\
&\le \|u\|_{L^\infty(Q)} \cdot \sum_{n=1}^{m} \|w_m^n - w_m^{n-1}\|_{L^1(\Omega)} \le \text{Constant independent of } m;
\end{aligned}
\tag{3.26}
$$

that is, denoting by $\mathcal{V}$ the total variation in $[0,T]$,

$$
\mathcal{V}(\Psi(w_m)) \le \text{Constant independent of } m.
\tag{3.27}
$$

By (3.8), $\mathcal{V} \circ \Psi$ is lower semi-continuous with respect to the strong topology of $L^1(Q)$; hence by (3.22) $\mathcal{V}(\Psi(w)) < +\infty$, and so (3.10) yields (3.12). □

REMARKS. (i) As we saw, the time-discretization scheme (3.13) is equivalent to the condition that at each time step w_m^n minimizes the functional $\hat{J}_m^n$ define in (3.14). Note that (2.17) yields

$$
-w_m^{n-1} \in S(v - w_m^{n-1}) \qquad \text{a.e. in } \Omega,\ \forall v \in X,
$$

whence, as $w_m^{n-1} \in X$,

$$
\begin{aligned}
\hat{J}_m^n(v) &= \Psi(v) - \int_\Omega u_m^n\, v\,dx + \int_\Omega [-w_m^{n-1}(v - w_m^{n-1}) + 1]\,dx = \\
&= \Psi(v) - \int_\Omega u_m^n\, v\,dx + \|v - w_m^{n-1}\|_{L^1(\Omega)} + |\Omega| \qquad \forall v \in L^\infty(\Omega).
\end{aligned}
\tag{3.28}
$$

This expression shows the relation between the evolution problem and the corresponding stationary one. Notice that the latter does not coincide with (1.2), because of the extra term $\|v - w\|_{L^1(\Omega)}$.

It can also be useful to compare the functionals J_m^n and $\hat{J}_m^n$, cf. (2.27), corresponding to the space-independent and space-dependent problems $(P1)$ and $(P2)$, respectively.

(ii) In general the solution of problem $(P2)$ is not unique. As we saw in section 2, multiple solutions can occur for $(P1)$, namely for the space-independent case; hence a fortiori for $(P2)$. □

The correspondance $(u, w^0) \longmapsto w$ defined by problem $(P2)$ determines a multi-valued operator:

$$
\tilde{F}_{\rho,\Psi} : L^1(\Omega; W^{1,1}(0,T)) \times \mathrm{Dom}(\Psi) \to \left\{ v \in L^\infty(Q) : \frac{\partial v}{\partial t} \in M(Q) \right\}
\tag{3.29}
$$

PROPOSITION 3. *Assume that* Ψ *fulfils* (3.1), (3.8), (3.9). *Then*

(i) $\tilde{F}_{\rho,\Psi}$ *is rate-independent, namely*

$$(3.30)\qquad \begin{cases} \forall u \in L^1(\Omega, W^{1,1}(0,T)),\ \forall w^0 \in \mathrm{Dom}(\Psi),\ \forall w \in \tilde{F}_{\rho,\Psi}(u, w^0) \\ \forall \textit{ increasing homeomorphism } s : [0,T] \to [0,T], \\ w \circ s \in \tilde{F}_{\rho,\Psi}(u \circ s, w); \end{cases}$$

so $\tilde{F}_{\rho,\Psi}$ *is a multi-valued hysteresis operator.*

(ii) $\tilde{F}_{\rho,\Psi}$ *is closed in the following sense: if*

$$(3.31)\qquad u_n \to u \qquad \textit{strongly in } L^1(\Omega; W^{1,1}(0,T)),$$

$$(3.32)\qquad w_n^0 \to w^0 \qquad \textit{strongly in } L^1(\Omega),$$

$$(3.33)\qquad \Psi(w_n^0) \to \Psi(w^0),$$

$$(3.34)\qquad F_{\rho,\Psi}(u_n, w_n^0) \ni w_n \to w \qquad \textit{weakly star in } L^\infty(Q),$$

$$(3.35)\qquad \frac{\partial w_n}{\partial t} \to \frac{\partial w}{\partial t} \qquad \textit{weakly star in } M(Q),$$

then

$$(3.36)\qquad w \in F_{\rho,\Psi}(u, w^0).$$

PROOF. Part (i) is obvious. In order to check part (ii), it is sufficient to notice that (3.4) and (3.5) are stable with respect to the convergences (3.31),...,(3.35). □

4. - Generalized Coarea Formula

In this section we shall mainly review some definitions and results of [14, sec. 2; 16]. First for any y, $s \in \mathbf{R}$ we set

$$H_s(y) := \begin{cases} 0 & \text{if} \quad y < s, \\ 1 & \text{if} \quad y \geq s. \end{cases}$$

DEFINITION. We denote by $GC(\Omega)$ the family of functionals $\Lambda : L^1(\Omega) \to [0,+\infty]$ which are proper, i.e. $\Lambda \not\equiv +\infty$, and which fulfil the following *generalized coarea formula*

$$\Lambda(u) = \int_{\mathbf{R}} \Lambda(H_s(u))\, dx \,(\leq +\infty) \qquad \forall u \in L^1(\Omega), \tag{4.1}$$

with the convention that the integral is set equal to $+\infty$ if the function $s \longmapsto \Lambda(H_s(u))$ is not measurable. This definition coincides with that given in [14; sec. 2], and is slightly different from that of [16]. Here it will be assumed that Λ operates on the equivalence classes of $L^1(\Omega)$, namely that

$$\forall u,\, v \in L^1(\Omega), \text{ if } u = v \text{ a.e. in } \Omega, \text{ then } \Lambda(u) = \Lambda(v). \tag{4.2}$$

Let us give some examples of functionals of $GC(\Omega)$. We set

$$V(u) := \int_\Omega |\nabla u| := \sup\left\{ \int_\Omega u \operatorname{div} \eta \, dx \,:\, \eta \in C_c^1(\Omega)^N,\, |\eta| \leq 1 \right\}. \qquad \forall u \in L^1(\Omega); \tag{4.3}$$

thus $\operatorname{Dom}(V) = BV(\Omega)$. In this case (4.1) coincides with the standard Fleming-Rishel *coarea formula* [6; 7, p. 20], and so $V \in GC(\Omega)$. We also set

$$\overline{V} := \int_{\overline{\Omega}} |\nabla u| := \sup_{\eta \in C_c^1(\mathbf{R}^N)^N,\, |\eta| \leq 1} \int_\Omega u \operatorname{div} \eta \, dx \qquad \forall u \in L^1(\Omega). \tag{4.4}$$

For any measurable function $g : \Omega^2 \to \mathbf{R}^+$, we set

$$\Lambda_g(u) := \iint_{\Omega^2} |u(x) - u(y)|\, g(x,y)\, dx\, dy \qquad \forall u \in L^1(\Omega). \tag{4.5}$$

In particular, taking $g_r(x,y) := |x-y|^{-(N+r)}$ for any $r \in]0,1[$, $\Lambda_r := \Lambda_{g_r}$ is the standard seminorm of the fractional Sobolev space $W^{r,1}(\Omega)$:

$$\|v\|_{L^1(\Omega)} + \Lambda_r(v) := \|v\|_{W^{r,1}(\Omega)}.$$

thus $\operatorname{Dom}(\Lambda_r) = W^{r,1}(\Omega)$.

For any measurable function $f : \Omega \times \mathbf{R}^+ \to \mathbf{R}^+$, we set

$$\tilde{\Lambda}_f(u) := \int_{\mathbf{R}^+} dh \int_\Omega \left(\operatorname*{ess\,sup}_{B_h(x)\cap\Omega} u - \operatorname*{ess\,inf}_{B_h(x)\cap\Omega} \right) f(x,h)\, dx \qquad \forall u \in L^1(\Omega), \tag{4.6}$$

where $B_h(x)$ is the ball of center x and radius h. In particular, for any $r \in]0,1[$, we set $f_r(x,h) := h^{-(1+r)}$ and $\tilde{\Lambda}_r := \tilde{\Lambda}_{f_r}$; also $\mathrm{Dom}(\tilde{\Lambda}_r)$ is a Banach space and

$$\text{(4.7)} \qquad \mathrm{Dom}(\tilde{\Lambda}_r) \subset \mathrm{Dom}\,(\Lambda_r)\,(= W^{r,1}(\Omega)) \qquad \forall r \in]0,1[.$$

PROPOSITION 4. [16; sec. 2]. *All the functionals V, $\overline{V}$, Λ_r and $\tilde{\Lambda}_r$ $(0 < r < 1)$ are convex and lower semi-continuous in $L^1(\Omega)$, and fulfil the generalized coarea formula* (4.1). *Moreover, undel mild regularity conditions for Ω (e.g., if Ω is of Lipschitz class), their domains have compact injections into $L^1(\Omega)$.* □

Let us set

$$\text{(4.8)} \qquad \tilde{X} := \{v \in L^1(\Omega) : |v| \le 1 \text{ a.e. in } \Omega\} = \overline{co}(X)$$

(closed convex hull of X). We recall that for any set A we denote by I_A its indicator function (or functional); namely $I_A = 0$ in A, $I_A = +\infty$ outside A. Moreover for any functional $\Phi : L^1(\Omega) \to \mathbf{R} \cup \{+\infty\}$, we denote by Φ^{**} its lower semi-continuous, convex regularized. Then we have

$$\text{(4.9)} \qquad (I_X)^{**} = I_{\tilde{X}}.$$

LEMMA 1. *Let $\Lambda \in GC(\Omega)$ be convex and lower semi-continuous with respect to the topology of $L^1(\Omega)$, i.e., $\Lambda = \Lambda^{**}$. Then for any $u \in L^1(\Omega)$,*

$$\text{(4.10)} \qquad (I_X + \Lambda)^{**}(u) = I_{\tilde{X}}(u) + \Lambda(u),$$

$$\text{(4.11)} \qquad \partial(I_X + \Lambda)(u) = \partial I_X(u) + \partial\Lambda(u) (\subset \partial I_{\tilde{X}}(u) + \partial\Lambda(u)) \qquad \textit{in } L^\infty(\Omega).$$

PROOF. This statement is a particular case of theorem 3 of [14]. □

The reason why here we introduced the class $GC(\Omega)$ stays in the following result:

THEOREM 4. *Let*

$$\text{(4.12)} \qquad \Psi = I_X + \Lambda$$

with $\Lambda \in GC(\Omega)$, convex and lower semi-continuous with respect to the topology of $L^1(\Omega)$, i.e., $\Lambda = \Lambda^{**}$. Then for any $u \in L^1(\Omega; W^{1,1}(0,T) \cap L^\infty(Q)$ and $w^0 \in X \cap \mathrm{Dom}(\Lambda)$, problem $(P2)$ of section 3 is equivalent to problem $(P3)$ below:

PROBLEM $(P3)$. *To find $w \in L^\infty(Q)$ such that $\frac{\partial w}{\partial t} \in M(Q)$ and*

$$\text{(4.13)} \qquad |w(x,t)| = 1 \text{ a.e. in } Q,$$

$$\iint\limits_{Q} (u+w)(w-v)\,dx\,dt \geq \int\limits_{0}^{T} [\Lambda(w) - \Lambda(v)]\,dt \tag{4.14}$$

$$\forall v \in L^\infty(Q),\ |v| \leq 1 \text{ a.e. in } Q,$$

$$\int\limits_{\Omega} w^0(x)[u(x,0) - v(x,0)]\,dx + \iint\limits_{Q} w\frac{\partial}{\partial t}(u-v)\,dx\,dt + \Lambda(w(\cdot,T))$$

$$\leq \Lambda(w^0) + \int\limits_{\Omega} w(x,T)[u(x,T) - v(x,T)]\,dx \tag{4.15}$$

$$\forall v \in L^1(0,T;\ W^{1,1}(0,T)),\ |v| \leq 1 \textit{ a.e. in } Q.$$

PROOF. Obviously $(P3)$ entails $(P2)$. In order to show the converse, we note that (3.4) can be rewritten in the form

$$u + w \in \partial\Psi(w) \qquad \text{in } L^\infty(\Omega), \text{ a.e. in }]0,T[.$$

This entails that $w \in X$, namely (4.13) holds. Furthermore by (4.10) we have

$$u + w \in \partial(I_X + \Lambda)^{**}(w) = \partial(I_{\bar{X}} + \Lambda)(w) \qquad \text{in } L^\infty(\Omega), \text{ a.e. in }]0,T[,$$

whence (4.14) holds, too. Finally (3.5) is obviously equivalent to (4.15). Thus $(P2)$ entails $(P3)$. □

REMARKS. (i) By (4.14) any solution of problem $(P3)$ is such that

$$\partial\Lambda(w) \neq \emptyset \text{ in } L^\infty(\Omega), \text{ a.e. in }]0,T[; \tag{4.16}$$

this can be regarded as a regularity result.

(ii) If $w \in L^1(\Omega;\ BV([0,T]))$ then (3.6) and (3.7) can be rewritten in the form

$$\int\limits_{\Omega} dx \int\limits_{[0,T]} (u-v)\,dw\,(x,t) \geq \Lambda(w(\cdot,T)) - \Lambda(w^0) \tag{4.17}$$

$$\forall v \in L^\infty(\Omega;\ W^{1,1}(0,T)),\ |v| \leq 1 \text{ a.e. in } Q,$$

(also here the integral in time is in the sense of Lebesgue-Stieltjes),

$$\int\limits_{\Omega} dx \int\limits_{[0,T]} u\,dw\,(x,t) \geq \Lambda(w(\cdot,T)) - \Lambda(w^0) + \int\limits_{\Omega} dx \int\limits_{[0,T]} |dw\,(x,t)|. \tag{4.18}$$ □

THEOREM 5. *Let* $\Lambda \in GC(\Omega)$, Λ *be convex and lower semi-continuous in* $L^1(\Omega)$ *(i.e.,* $\Lambda = \Lambda^{**}$*), and let* (3.9) *hold for* $\Psi := I_X + \Lambda$. *Then for any* $u \in L^1(\Omega; W^{1,1}(0,T)) \cap L^\infty(Q)$ *and any* $w^0 \in X \cap \mathrm{Dom}(\Lambda)$, *problem* $(P3)$ *has a solution such that*

$$\Lambda(w) \in BV([0,T]). \tag{4.19}$$

PROOF. Straightforward consequence of theorems 3 and 4. □

REMARKS. (i) After proposition 4, if either $\Lambda = V$, or $\Lambda = \overline{V}$, or $\Lambda = \Lambda_r$, or $\Lambda = \tilde{\Lambda}_r$ $(0 < r < 1)$, and if Ω is sufficiently smooth (e.g. of Lipschitz class), then theorem 5 can be applied.

(ii) A major open question is the extension of the space-dependent model of sections 3 and 4 to more general non-convex potentials; for instance for (4.12) replaced by $\Psi = \Phi_0 + \Lambda$, with Φ_0 defined as in (2.37),...,(2.39) (with $u \equiv 0$) and $\Lambda \in GC(\Omega)$, $\Lambda = \Lambda^{**}$. □

5. - Application

5.1 *Evolution of non-Cartesian surfaces.*

If $\Lambda = V$, cf. (4.3), then problem $(P3)$ of section 4 represents the quasi-stationary, rate-independent evolution of Euclidean sets of finite perimeter in the sense of Caccioppoli and De Giorgi, cf. [2,3,7]; namely the evolution of oriented non-Cartesian surfaces of codimension 1.

In general by *quasi-stationary evolution* it is meant that at any instant the system is in a stationary configuration; that is, if the input variable were constant in some time interval, then also the output would be constant in the same time interval. However here the evolution of w is not always uniquely determined by that of u, namely $\tilde{F}_{\rho,\Psi}$ is multi-valued, cf. (3.29). So here the evolution is quasi-stationary in the following broader sense:

$$\begin{cases} \forall [t_1, t_2] \subset [0,T], \text{ if } u(x,t) = u(x,t_1) & \forall t \in [t_1, t_2], \\ \text{then } \exists w \in \tilde{F}_{\rho,\Psi}(u, w^0) : w(x,t) = w(x,t_1) & \forall t \in [t_1, t_2]. \end{cases} \tag{5.1}$$

Note that in general another $\hat{w} \in \tilde{F}_{\rho,\Psi}(u, w^0)$ might be non-constant in $[t_1, t_2]$. This can already occur in the space-independent case, namely for the operator $\tilde{f}_\rho$, cf. section 2; for instance, take u constantly equal to ρ_1 (ρ_2, respectively) and $w^0 = 1$ ($w^0 = -1$ respectively).

For a moment let us consider the corresponding stationary problem, namely (1.2) with $\Psi = I_X + V$. Let w be a solution and assume that the boundary S of $\Omega^- := \{x \in \Omega : w(x) = -1\}$ in Ω is of class C^2, so that its mean curvature H (assumed positive where Ω^- is convex) is defined at any point; let also

$u \in C^0(\Omega)$, for the sake of simplicity. Then, as it is well known [2,3], the Euler equation corresponding to (1.2) is

$$-(N-1)H = u \qquad \text{on } \mathsf{S}, \tag{5.2}$$

N being the Euclidean dimension of Ω.

Let us now consider the time-discretized problem (3.13). Still assuming regularity conditions, let us denote by H_m^n the local mean curvature of the boundary S_m^n of $\Omega_{m\cdot n}^- := \{x \in \Omega : w_m^n(x) = -1\}$; henceforth we shall imply the index m. By (3.13) one has

$$-(N-1)H^n = u^n + w^{n-1} \qquad \text{on } \mathsf{S}^n; \tag{5.3}$$

hence

$$|(N-1)H^n + u^n| \leq 1 \qquad \text{on } \mathsf{S}^n; \tag{5.4}$$

moreover where the domain Ω_n^- is locally advancing, $w^{n-1} = 1$, whence

$$-(N-1)H^n = u^n + 1; \tag{5.5}$$

similarly where Ω_n^- is locally receeding, $w^{n-1} = -1$, and so

$$-(N-1)H^n = u^n - 1. \tag{5.6}$$

Let us now consider the limit problem ($P3$); still assuming regularity conditions on S and u, let us denote by v_{S} the local normal velocity of S, taken positive for solidification and negative for melting. Here the limit moving boundary condition corresponding to (5.5) and (5.6) is

$$-(N-1)H - u \in S(v_{\mathsf{S}}) \qquad \text{on } \mathsf{S}, \tag{5.7}$$

(we recall that S denotes the sign graph and that several constants have been normalized). Note that if in (5.3) w^{n-1} is multiplied by a constant $C > 0$, then (5.7) is replaced by

$$-(N-1)H - u \in CS(v_{\mathsf{S}}) \qquad \text{on } \mathsf{S}; \tag{5.8}$$

thus the stationary condition (5.2) can be retrieved in the limit as $C \to 0$.

Condition (5.7) contains no time relaxation term and is rate-independent. Also note that here v_{S} is not required to be continuous in time; hence (5.7) is consistent with phenomena of *nucleation* and *annihilation*, namely with the formation and vanishing of (connected components of) phases. Indeed these effects imply a discontinuous set evolution, in the sense that $w \notin C^0([0,T];\ L^1(\Omega))$; otherwise diverging curvatures would appear on S, as it occurs for instance for a ball of vanishing radius.

5.2 *Pattern interpolation.*

We briefly examine a problem which arises for instance in the automatic production of animated movies. Given any couple (A^0, A^1) of subsets of a bounded domain $\Omega \subset \mathbf{R}^n$, we look for a *natural* interpolate family of subsets of Ω; namely we search for a *non-too-irregular* application $g : [0,T] \to \mathcal{P}(\Omega)$ such that $g(0) = A^0$, $g(T) = A^1$.

Note that there exists nothing like interpolation between sets. Obviously interpolation of the characteristic functions χ_{A^0} and χ_{A^1} does not solve the problem, since the interpolate functions are not characteristic functions (if $A^0 \not\equiv A^1$). Such a procedure would correspond to the smearing out of the image A^0 and the focusing of A^1, a technique which is also used in movie production, by the way. We also remark that only in special cases the interpolating application can be a homotopy.

We recall that problems $(P2)$ of section 3 and $(P3)$ of section 4 represent the (possibly discontinuous) evolution of an initial configuration under the control of a time-dependent field u. Therefore we suggest to reduce the problem of pattern interpolation to a control problem. Here either $(P2)$ or $(P3)$, with $w^0 := \chi_{A^0}$, are the *state equation*; u is the *control variable*, and must be such that the final configuration $A(T) := \{x \in \Omega : w(x,T) = 1\}$ approximate the desired one A^1 in a optimal way. This is a shooting problem.

Here we assume that (4.12) holds and that Λ fulfils the assumptions of theorem 5 of section 4. The cost functional we propose consists of several contributions. A first term is a suitable distance between the actual final state $w(\cdot,T) := \chi_{A(T)}$ and the desired one $w^1 := \chi_{A^1}$:

$$I^1_{w^1}(w) := C_1 \|w(T) - w^1\|_{L^1(\Omega)} + C_2 |\Lambda(w(T)) - \Lambda(w^1)|, \tag{5.9}$$

C_1 and C_2 being positive constants. Here the first addendum is proportional to the N-dimensional measure of the symmetric difference between the final configuration $A(T)$ and the desired one A^1. The second addendum can be regarded as a penalization of the difference between the shapes of the sets $A(T)$ and A^1. Note that, by theorem 5 of section 4, if $u \in L^\infty(Q) \cap L^1(\Omega; W^{1,1}(0,T))$ then problem $(P2)$ has a solution w such that $w \in L^\infty(Q)$, $\frac{\partial w}{\partial t} \in M(Q)$ and $\Lambda(w) \in BV([0,T])$; hence the traces $w(T)$ and $\Lambda(w(T)) := [\Lambda(w)]_{t=T}$ are meaningful.

Two further contributions to the cost functional are

$$I^{(2)}_{w^0}(w) := C_3 \left(\left\| \frac{\partial w}{\partial t} \right\|_{M(Q)} - \|w(T) - w^0\|_{L^1(\Omega)} \right), \tag{5.10}$$

$$I^{(3)}_{w^0}(w) := C_4 \left(\left\| \frac{d}{dt} \Lambda(w) \right\|_{M(0,T)} - |\Lambda(w(T)) - \Lambda(w^0)| \right), \tag{5.11}$$

C_3 and C_4 being positive constants. Note that the functionals $I^{(2)}_{w^0}$, $I^{(3)}_{w^0}$ are non-negative, convex and lower semi-continuous in $L^1(Q)$. These terms are aimed to force w to move from $w(0) = w^0$ to $w(T)$ in the most direct way, without unnecessary *deviations*. Note that the term $I^{(3)}_{w^0}$ penalizes shape perturbations.

Let us define the Banach space $U := L^\infty(Q) \cap L^1(\Omega; W^{1,1}(0,T))$. The last contribution to the cost functional is

$$I^{(4)}(u) := G(u), \tag{5.12}$$

with $G : U \to [0, +\infty]$ convex, lower semi-continuous and coercive with respect to the topology of $L^1(Q)$. For instance, one can take G equal to the norm of U. So the total cost functional is

$$\begin{aligned} &I(u,w) := I^{(1)}_{w^1}(w) + I^{(2)}_{w^0}(w) + I^{(3)}_{w^0}(w) + G(u) \\ &\forall u \in U,\ \forall w \in L^\infty(Q) \text{ such that } \tfrac{\partial w}{\partial t} \in M(Q),\ \Lambda(w) \in BV(0,T). \end{aligned} \tag{5.13}$$

Note that $I(u,w)$ is convex, lower semi-continuous and coercive in $L^1(Q)$ with respect to each variable. Under the assumptions of theorem 5 of section 4, the state equation, namely problem $(P3)$, defines a multi-valued operator $\tilde{F}_{\rho,\Psi}$; hence also the dependence of the cost from the control u is multi-valued:

$$\hat{I}(u) := \{I(u,w) : w \in \tilde{F}_{\rho,\Psi}(u, w^0)\}. \tag{5.14}$$

At this point it looks natural to consider a relaxed problem, which we shall not present here.

REFERENCES

[1] K.A. Brakke , *The motion of a surface by its mean curvature*, Math. Notes, Princeton University Press, Princeton (1978).

[2] E. De Giorgi, *Su una teoria generale della misura (r-1) dimensionale in uno spazio r dimensioni*, Ann. Mat. Pura Appl. (4), **36** (1954), 191-213.

[3] E. De Giorgi - F. Colombini - L.C. Piccinini, *Frontiere orientate di misura minima e questioni collegate*, Scuola Norm. Sup., Pisa (1972).

[4] G. Dziuk, *Finite element method for the Beltrami operator on arbitrary surfaces*, in Partial differential equations and calculus of variations (S. Hildebrandt, R. Leis, eds.), Lecture Notes in Math., Springer, Berlin (1988).

[5] H. Federer, *Geometric measure theory*, Springer-Verlag, Berlin (1969).

[6] W.H. Fleming - R. Rishel, *An integral formula for total gradient variation*, Arch. Math., **11** (1960), 218-222.

[7] E. Giusti, *Minimal surfaces and functions of bounded variation*, Birkhäuser, Boston (1984).

[8] G. Huisken, *Flow by mean curvature of convex surfaces into spheres*, J. Differential Geom., **20** (1984), 237-266.

[9] G. Huisken, *Contracting convex hypersurfaces in Riemannian manifolds by their mean curvature*, Invent. Math., **84** (1986), 463-480.

[10] G. HUISKEN, *Deforming hypersurfaces of the sphere by their mean curvature*, Math. Z., **195** (1987), 205-219.

[11] A. KRASNOSEL'SKIĬ - A.V. POKROVSKIĬ, *Systems with hysteresis* (Russian), Nauka, Moskow (1983). English Translation: Springer, Berlin (1989).

[12] A. VISINTIN, *Mathematical models of hysteresis*, in Topics in Non-smooth Mechanics (J.J. Moreau, G. Strang. P.D. Panagiotopoulos, Ed.s), Birkhäuser, Basel (1988).

[13] A. VISINTIN, *Surface tension effects in phase transitions*, in "Material Instabilities in Continuum Mechanics and Related Mathematical Problems" (J. Ball., ed.), Clarendon Press, Oxford (1988), 505-537.

[14] A. VISINTIN, *Non-convex functionals related to multi-phase system*, S.I.A.M. J. Math. Anal., September 1990 (in press).

[15] A. VISINTIN, *Surface tension effects in two-phase systems*, in Proceedings of the Colloquium on Free Boundary Problems held in Irsee in June 1987 (to appear).

[16] A. VISINTIN, *Generalized coarea formula and fractal sets*. Japan J. Appl. Math. (to appear).

[17] A. VISINTIN, *Generalized coarea formula*, In "*Recent Advances in Nonlinear Elliptic and Parabolic Problems*" (P. Benilan, M. Chipot, L.C. Evans and M. Pierre, eds.), Longman, Harlow (1989).

[18] A. VISINTIN, *Models of pattern formation*, C.R. Acad. Sci. Paris, **309** (1989), Série I, 429-434.

Dipartimento di Matematica
Università
38050 Povo (Trento)

Istituto di Analisi Numerica
del CNR
C.so Carlo Alberto, 5
27100 Pavia