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Limit Semigroups of Bernstein-Schnabl Operators Associated with Positive Projections (*)

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0. - Introduction

Let X be a convex compact subset of a locally convex Hausdorff space and let us denote by $\mathcal{C}(X,\mathbb{R})$ the Banach lattice of all continuous real-valued functions on X.

M.W. Grossman introduced in [11] a class of positive operators on $\mathcal{C}(X,\mathbb{R})$ which are called Bernstein-Schnabl operators on X and which are associated with an infinite lower triangular stochastic matrix and with a selection of representing measures.

These operators are a natural extension of similar ones introduced by R. Schnabl in [24] and by G. Feldbecker and W. Schempp in [9], in the context of particular convex compact subsets, namely the set of all probability measures on a compact Hausdorff space.

Of course all these operators extend the classical Bernstein polynomial operators on $\mathcal{C}([0,1],\mathbb{R})$.

In the particular case where X is a Bauer simplex, i.e. X is a Choquet simplex and the set of the extreme points of X is closed, the Bernstein-Schnabl operators on X have been extensively studied by T. Nishishiraho in [17], [18], [19], [20], [21] (see also R. Schnabl [25], [26]). In particular in these papers the Author investigates the limit behaviour of the sequences of the iterates of such operators, both in the case where the index of iteration is independent and where it is dependent on the order of the operators, extending similar results about the classical Bernstein operators obtained by P.C. Sikkema ([27]), R.P. Keliski and T.J. Rivlin ([13]), S. Karlin and Z. Ziegler ([12]), C.A. Micchelli ([14]), J. Nagel ([15]), M.R. Da Silva ([8]).

In this paper we study the Bernstein-Schnabl operators in a context which is more general than that of the Bauer simplexes.

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More precisely we simply suppose that the space of all continuous affine functions on X is contained in the range of a (non-trivial) positive projection T on $\mathcal{C}(X,\mathbb{R})$. This is the case, for example, not only for Bauer simplexes (cf. Ex. 2.1, 1), but also for the products of a finite number of Bauer simplexes (cf. Ex. 2.1, 2), for the convex compact subsets of $\mathbb{R}^p$ (cf. Ex. 2.1, 3) and others.

Under the hypothesis aforementioned, we investigate the asymptotic properties of the sequence of these operators and of their iterates.

In particular, we establish the existence of a (uniquely determined) positive contraction semigroup $(T(t))_{t\geq 0}$ of which we can indicate an explicit representation in terms of the Bernstein-Schnabl operators (cf. Th. 2.6).

This semigroup is mean-ergodic and strongly converges to the projection T as t goes to ∞. Moreover its infinitesimal generator is explicitly determined on a dense subspace of its domain.

Finally, in some concrete examples concerning convex compact subsets X of $\mathbb{R}^p$ we show that the generator is a degenerate elliptic second order differential operator on X. Consequently we obtain the solutions of the associated abstract Cauchy problems in terms of the Bernstein-Schnabl operators on X.

1. - Notations and Preliminary Results

Let X be a compact Hausdorff space and let us denote by $\mathcal{C}(X,\mathbb{R})$ the Banach lattice of all real continuous functions on X, endowed with the sup-norm topology and the natural order. Let $M^+(X)$ (resp. $M^1(X)$) be the set of all positive (resp. probability) Radon measures on X.

Let us consider a linear positive operator $T : \mathcal{C}(X,\mathbb{R}) \to \mathcal{C}(X,\mathbb{R})$ and a subsets S of $\mathcal{C}(X,\mathbb{R})$.

We say that S is a T-Korovkin set if for every net $(L_i)_{i\in I}^{\leq}$ of linear positive operators on $\mathcal{C}(X,\mathbb{R})$ satisfying the condition

$$\lim_{i\in I^{\leq}} L_i(h) = T(h), \qquad \text{for every } h \in S,$$

one also has

$$\lim_{i\in I^{\leq}} L_i(f) = T(f), \qquad \text{for every } f \in \mathcal{C}(X,\mathbb{R}).$$

If T is the identity operator on $\mathcal{C}(X,\mathbb{R})$ and S is a T-Korovkin set, we shall simply say that S is a Korovkin set.

In order to characterize the Korovkin sets it is useful to introduce the Choquet boundary $\partial_S X$ of X with respect to S which is defined by

$$\text{(1.1)} \quad \partial_S X = \{x \in X \mid \text{if } \mu \in M^+(X) \text{ and } \mu(h) = h(x) \text{ for every } h \in S \text{ then } \mu(f) = f(x) \text{ for every } f \in \mathcal{C}(X,\mathbb{R})\}.$$

Then we have (H. Bauer ([5]), H. Berens - G.G. Lorentz ([6]))

(1.2) S is a Korovkin set if and only if $\partial_S X = X$.

Another result which will be useful in the sequel is the following theorem.

THEOREM A. (*H. Bauer* ([5]), *W. Schempp* ([23]), *M.W. Grossman* ([11])). *Let S be a subset of $\mathcal{C}(X,\mathbb{R})$ which separates the points of X. Then $\{1\}\cup S\cup S^2$ is a Korovkin set, where $S^2 = \{f^2 \mid f \in S\}$ and 1 denotes the constant function one.*

We also recall some results concerning linear positive projections on $\mathcal{C}(X,\mathbb{R})$.

Let us consider a linear positive projection $T : \mathcal{C}(X,\mathbb{R}) \to \mathcal{C}(X,\mathbb{R})$ (i.e. T is a linear positive operator such that $T^2 = T$). Suppose that $T(1) = 1$ and the range $H = T(\mathcal{C}(X,\mathbb{R}))$ separates the points of X.

Under these hypotheses we know that ([1], Prop. 1.1)

(1.3) $$\partial_H X = \{x \in X \mid T(f)(x) = f(x) \text{ for every } f \in \mathcal{C}(X,\mathbb{R})\}.$$

In [3], Th. 1.3, we have proved the following result.

THEOREM B. *Let $T : \mathcal{C}(X,\mathbb{R}) \to \mathcal{C}(X,\mathbb{R})$ be a linear positive projection such that $T(1) = 1$ and the range $H = T(\mathcal{C}(X,\mathbb{R}))$ separates the points of X. Then for every function $\phi \in \mathcal{C}(X,\mathbb{R})$ such that $\phi \le T(\phi)$ and $\partial_H X = \{x \in X \mid T(\phi)(x) = \phi(x)\}$, the set $H \cup \{\phi\}$ is a T-Korovkin set.*

We also need the following lemma.

LEMMA 1.1. *Let X be a compact Hausdorff space and let $(h_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{C}(X,\mathbb{R})$ which separates the points of X, and such that the series $\sum_{n=0}^{\infty} h_n^2$ converges uniformly on X. Then $\{1\}\cup\{h_n \mid n \in \mathbb{N}\}\cup\left\{\sum_{n=0}^{\infty} h_n^2\right\}$ is a Korovkin set.*

PROOF. Let $x \in X$ and $\mu \in M^+(X)$ be such that $\mu(1) = 1$, $\mu(h_n) = h_n(x)$ for every $n \in \mathbb{N}$ and $\mu\left(\sum_{n=0}^{\infty} h_n^2\right) = \sum_{n=0}^{\infty} h_n^2(x)$. Then $\sum_{n=0}^{\infty} \mu\left((h_n - h_n(x))^2\right) = \sum_{n=0}^{\infty} \left(\mu(h_n^2) - h_n^2(x)\right) = 0$ and consequently $\mu\left((h_n - h_n(x))^2\right) = 0$ for every $n \in \mathbb{N}$.

From this it follows that the support $S(\mu)$ of μ is contained in $\bigcap_{n\in\mathbb{N}} \{y \in X \mid h_n(y) = h_n(x)\} = \{x\}$ since $(h_n)_{n\in\mathbb{N}}$ separates the points of X.

So, there exists $\lambda \ge 0$ such that $\mu(f) = \lambda f(x)$ for every $f \in \mathcal{C}(X,\mathbb{R})$ and finally, since $\mu(1) = 1$, we have $\lambda = 1$ and hence $\mu(f) = f(x)$ for every $f \in \mathcal{C}(X,\mathbb{R})$.

From (1.1) and (1.2) we have the desired result.

Before stating the next result it is necessary to point out that, if X is a metrizable compact space and H is a linear subspace of $\mathcal{C}(X,\mathbb{R})$ which separates the points of X, then it is always possible to construct a sequence $(h_n)_{n\in\mathbb{N}}$ in H which separates the points of X such that the series $\sum_{n=0}^{\infty} h_n^2$ is uniformly convergent on X. It sufficies to consider a countable dense subset $(\ell_n)_{n\in\mathbb{N}}$ of H (H being separable) and to put $h_n = \frac{\ell_n}{\|\ell_n\|\sqrt{2^n}}$ for every $n \in \mathbb{N}$.

PROPOSITION 1.2. *Let X be a metrizable compact Hausdorff space and $T : \mathcal{C}(X,\mathbb{R}) \to \mathcal{C}(X,\mathbb{R})$ a linear positive projection. Suppose that $T(1) = 1$ and the range $H = T(\mathcal{C}(X,\mathbb{R}))$ separates the points of X. Let $(h_n)_{n\in\mathbb{N}}$ be a sequence in H which separates the points of X and such that the series $\sum_{n=0}^{\infty} h_n^2$ converges uniformly to a function $\phi \in \mathcal{C}(X,\mathbb{R})$.*

Then $\phi \leq T(\phi)$ and $\partial_H X = \{x \in X \mid T(\phi)(x) = \phi(x)\}$.

Consequently $H \cup \{\phi\}$ is a T-Korovkin set. In particular $H \cup H^2$ is a T-Korovkin set.

PROOF. For every $x \in X$ let us consider the positive Radon measure μ_x on X defined by putting $\mu_x(f) = T(f)(x)$ for every $f \in \mathcal{C}(X,\mathbb{R})$.

For every $h \in H$ we have $\mu_x(h) = h(x)$ and hence, for every $n \in \mathbb{N}$,

$$|h_n(x)| = |\mu_x(h_n)| = \left|\int h_n \mathrm{d}\mu_x\right| \leq \int |h_n| \, \mathrm{d}\mu_x$$
$$\leq \left(\int h_n^2 \mathrm{d}\mu_x\right)^{1/2} \left(\int 1\mathrm{d}\mu_x\right) = \left(\mu_x \; (h_n^2)\right)^{1/2}$$

and so $h_n^2(x) \leq T(h_n^2)(x)$. Thus $\phi \leq T(\phi)$.

Now, if $x \in \partial_H X$, then $T(\phi)(x) = \phi(x)$ by (1.3). Conversely let us suppose that $T(\phi)(x) = \phi(x)$ for some $x \in X$. Then $\mu_x(\phi) = \phi(x)$.

Moreover $\mu_x(h_n) = h_n(x)$ for every $n \in \mathbb{N}$ and so, by virtue of Lemma 1.1 and (1.2), we conclude that $T(f)(x) = f(x)$ for every $f \in \mathcal{C}(X,\mathbb{R})$ which in turn implies that $x \in \partial_H X$ by (1.3).

The last assertion follows from Th. B.

2. - Bernstein-Schnabl Operators and their Iterates

Let X be a metrizable convex compact subset of some locally convex Hausdorff space and $T : \mathcal{C}(X,\mathbb{R}) \to \mathcal{C}(X,\mathbb{R})$ a linear positive projection. Let $H = T(\mathcal{C}(X,\mathbb{R}))$ be the range of T and let us denote by $A(X)$ the space of all continuous affine functions on X. We suppose that

(2.1) $$A(X) \subset H$$

(and hence H separates the points of X and $T(1) = 1$) and for every $\overline{x} \in X$, $\lambda \in [0,1]$ and $h \in H$

(2.2) the function $x \in X \mapsto h\left((1-\lambda)\overline{x} + \lambda x\right)$ belongs to H.

Let $P = (p_{nj})_{n\geq 1,\ j\geq 1}$ be an infinite lower triangular stochastic matrix, i.e. an infinite matrix of positive numbers satisfying $p_{nj} = 0$ whenever $j > n$ and $\sum_{j=1}^{\infty} p_{nj} = \sum_{j=1}^{n} p_{nj} = 1$ for every $n \geq 1$.

For every $n \geq 1$ let us consider the map $\pi_n : X^n \to X$ defined by putting for every $(x_1, \dots, x_n) \in X^n$

$$\pi_n(x_1, \dots, x_n) = \sum_{j=1}^{n} p_{nj}\ x_j.$$

For every $x \in X$ we shall denote by $\mu_x \in M^1(X)$ the probability Radon measure on X defined by putting

$$\mu_x(f) = T(f)(x), \qquad \text{for every } f \in \mathcal{C}(X, \mathbb{R}). \tag{2.3}$$

For every $n \in \mathbb{N}$, $n \geq 1$, let us consider the linear positive operator $B_n : \mathcal{C}(X, \mathbb{R}) \to \mathcal{C}(X, \mathbb{R})$ defined by putting for every $f \in \mathcal{C}(X, \mathbb{R})$ and $x \in X$

$$B_n(f)(x) = \int_{X^n} f \circ \pi_n \ \mathrm{d}\left(\bigotimes_{i=1}^{n} \mu_{x.i}\right) \tag{2.4}$$

where $\mu_{x.i} = \mu_x$ for every $i = 1, \dots, n$.

The linear operator B_n will be called the $n-th$ Bernstein-Schnabl operator with respect to the matrix P and the projection T, according to the definition suggested by M.W. Grossman in [11], p. 45 (cf. [9], p. 66, too).

Here we indicate some examples.

EXAMPLES 2.1.

1. - (Bauer simplexes). Let X be a metrizable Bauer simplex (i.e. the set $\partial_e X$ of the extreme points of X is closed and for every $x \in X$ there exists a unique probability Radon measure μ_x on X, concentrated on $\partial_e X$, the barycentre of which is x). Let us consider the linear positive projection $T : \mathcal{C}(X, \mathbb{R}) \to \mathcal{C}(X, \mathbb{R})$ defined by putting for all $f \in \mathcal{C}(X, \mathbb{R})$ and $x \in X$, $T(f)(x) = \mu_x(f)$. In this case $H = A(X)$ and (2.1) and (2.2) are satisfied.

Moreover, the Bernstein-Schnabl operators constructed according to (2.4) are exactly the Bernstein-Schnabl operators on Bauer simplexes. In particular, if X_p denotes the standard simplex of $\mathbb{R}^p$ $(p \geq 1)$ and if P denotes the arithmetic mean Toeplitz matrix, i.e. $p_{ni} = \frac{1}{n}$ if $n \geq 1$ and $i = 1, \dots, n$ and $p_{ni} = 0$ if $i > n$, then we obtain the classical Bernstein operators on X_p.

In this setting the approximation properties of the Bernstein-Schnabl operators and of their iterates have been extensively studied by T. Nishishiraho in [17], [18], [19], [20], [21], (see also R. Schnabl ([24], [25], [26]), G. Felbecker-W. Schempp ([9]), M.W. Grossman ([11]) and, for the classical Bernstein operators, P.C. Sikkema ([27]), R.P. Keliski-T.J. Rivlin ([13]), S. Karlin-Z. Ziegler ([12]), C.A. Micchelli ([14]), J. Nagel ([15]), M.R. Da Silva ([8])).

2. - (Convex compact subsets of $\mathbb{R}^p$). Let X be a convex compact subset of $\mathbb{R}^p$ $(p \geq 1)$ and let us consider the positive projection $T : \mathcal{C}(X, \mathbb{R}) \to \mathcal{C}(X, \mathbb{R})$ defined by associating to every $f \in \mathcal{C}(X, \mathbb{R})$ the unique solution $T(f)$ of the Dirichlet problem

$$\begin{cases} \Delta u = 0 \text{ on } \overset{\circ}{X}, \qquad u \in \mathcal{C}(X, \mathbb{R}) \cap \mathcal{C}^2(\overset{\circ}{X}, \mathbb{R}) \\ u_{|\partial X} = f_{|\partial X} \end{cases}$$

where, as usual, $\Delta = \sum\limits_{i=1}^{p} \frac{\partial^2}{\partial x_i^2}$ is the Laplacian in $\overset{\circ}{X}$.

In this case $H = \{u \in \mathcal{C}(X, \mathbb{R}) \cap \mathcal{C}^2(\overset{\circ}{X}, \mathbb{R}) \mid \Delta u = 0 \text{ on } \overset{\circ}{X}\}$; moreover (2.1) and (2.2) are fulfilled too.

For example, if $X = B(x_0, r)$ is a ball of $\mathbb{R}^p$ with centre x_0 and radius r, then by using the Poisson formula for the solutions of the Dirichlet problem for a ball, we have

$$(2.5) \quad B_n(f)(x) = \begin{cases} \left(\dfrac{r^2 - \|x_0 - x\|^2}{r\ \sigma_p}\right)^n \\ \displaystyle\int_{\partial X} \cdots \int_{\partial X} \frac{f(p_{n1}x_1 + \ldots + p_{nn}x_n)}{\|x_1 - x\|^p \ldots \|x_n - x\|^p} \, d\sigma(x_1) \ldots d\sigma(x_n) \\ \qquad\qquad\qquad\qquad \text{if } \|x - x_0\| < r \\ f(x) \qquad\qquad\qquad\quad \text{if } \|x - x_0\| = r, \end{cases}$$

where σ_p denotes the surface area of the sphere ∂X and σ is the surface measure on ∂X.

3. - (Products of Bauer simplexes). Let $(X_j)_{1 \leq j \leq p}$ be a finite family of metrizable Bauer simplexes and put $X = \prod\limits_{j=1}^{p} X_j$. Let us consider the positive projection $T : \mathcal{C}(X, \mathbb{R}) \to \mathcal{C}(X, \mathbb{R})$ defined by putting for every $f \in \mathcal{C}(X, \mathbb{R})$ and

$x = (x_1, \dots, x_p) \in X$

$$T(f)(x) = \int_X f \ \mathrm{d}\left(\bigotimes_{j=1}^{p} \mu_{x_j}\right),$$

where for every $j = 1, \dots, p$, μ_{x_j} is the only probability Radon measure concentrated on $\partial_e X_j$ with barycentre x_j.

In this case H is the space of all continuous functions on X which are affine with respect to each variable (cf. [2]).

Also in this case (2.1) and (2.2) are satisfied.

In particular, for $X = [0,1]^p$ and P the arithmetic mean Toeplitz matrix, we obtain the classical Bernstein operators on $[0,1]^p$.

Our next aim is to investigate the asymptotic behaviour of the sequence $(B_n)_{n\in\mathbb{N}}$ defined by (2.4), and the sequences of their iterates.

As usual we put

$$B_n^0 = I \text{ and } B_n^m = B_n \circ B_n^{m-1} \text{ for } m \geq 1, \ n \geq 1. \tag{2.6}$$

THEOREM 2.2. *Under the above assumptions* (2.1) *and* (2.2), *for every* $f \in C(X, \mathbb{R})$ *we have*

1) $\lim_{m\to\infty} B_n^m(f) = T(f)$ *uniformly on* X *for every* $n \geq 1$;

2) *if* $\lim_{n\to\infty} \sum_{i=1}^{n} p_{ni}^2 = 0$, *then* $\lim_{n\to\infty} B_n^m(f) = f$ *uniformly on* X *for every* $m \geq 1$.

PROOF. We first note that for every $h \in H$ and $x \in X$, since $T(h) = h$, we have $\mu_x(h) = h(x)$. So for every $n \geq 1$, by virtue of (2.2), we have

$$\begin{aligned} B_n(h)(x) &= \int \dots \int h(\pi_n(x_1, \dots, x_n)) \ \mathrm{d}\mu_x(x_1) \dots \mathrm{d}\mu_x(x_n) \\ &= h(\pi_n(x, \dots, x)) = h(x). \end{aligned}$$

Hence $B_n(h) = h$ for every $h \in H$ and $n \geq 1$, and so

$$\lim_{n\to\infty} B_n^m(h) = \lim_{m\to\infty} B_n^m(h) = h = T(h). \tag{1}$$

Consider now $h \in A(X) \subset H$; then, if for every $i = 1, \dots, n$ we denote by $pr_i : X^n \to X$ the $i-th$ projection, we have $h \circ \pi_n = \sum_{i=1}^{n} p_{ni} h \circ pr_i$ and hence

$$\begin{aligned} h^2 \circ \pi_n &= \left(\sum_{i=1}^{n} p_{ni} \ h \circ pr_i\right)^2 = \sum_{i=1}^{n} p_{ni}^2 \ h^2 \circ pr_i \\ &+ 2 \sum_{1 \leq i < j \leq n} p_{ni} p_{nj} (h \circ pr_i)(h \circ pr_j). \end{aligned}$$

From this it follows that for every $x \in X$

$$B_n(h^2)(x) = \int_{X^n} h^2 \circ \pi_n \mathrm{d}\left(\bigotimes_{i=1}^{n} \mu_{x.i}\right)$$
$$= \sum_{i=1}^{n} p_{ni}^2 \int_X h^2 \mathrm{d}\mu_x + 2\left(\sum_{1 \le i < j \le n} p_{ni} p_{nj}\right) h^2(x)$$
$$= \sum_{i=1}^{n} p_{ni}^2 \; T(h^2)(x) + \left(1 - \sum_{i=1}^{n} p_{ni}^2\right) h^2(x).$$

In conclusion

$$B_n(h^2) = \left(\sum_{i=1}^{n} p_{ni}^2\right) T(h^2) + \left(1 - \sum_{i=1}^{n} p_{ni}^2\right) h^2. \tag{2}$$

Consequently

$$B_n^2(h^2) = \sum_{i=1}^{n} p_{ni}^2 \; T(h^2) + \left(1 - \sum_{i=1}^{n} p_{ni}^2\right) B_n(h^2) = \left(\sum_{i=1}^{n} p_{ni}^2\right) T(h^2)$$
$$+ \left(1 - \sum_{i=1}^{n} p_{ni}^2\right)\left(\sum_{i=1}^{n} p_{ni}^2\right) T(h^2) + \left(1 - \sum_{i=1}^{n} p_{ni}^2\right)^2 h^2.$$

In general, for every $m \geq 1$

$$B_n^m(h^2) = \left(\sum_{k=0}^{m}\left(1 - \sum_{i=1}^{n} p_{ni}^2\right)^k\right)\left(\sum_{i=1}^{n} p_{ni}^2\right) T(h^2) + \left(1 - \sum_{i=1}^{n} p_{ni}^2\right)^m h^2 \tag{3}$$
$$= \left(1 - \left(1 - \sum_{i=1}^{n} p_{ni}^2\right)^{m+1}\right) T(h^2) + \left(1 - \sum_{i=1}^{n} p_{ni}^2\right)^m h^2.$$

Therefore for every $h \in A(X)$

$$\lim_{m \to \infty} B_n^m(h^2) = \left(\sum_{i=1}^{n} p_{ni}^2\right)^{-1} \sum_{i=1}^{n} p_{ni}^2 \; T(h^2) = T(h^2). \tag{4}$$

Consider now a sequence $(h_n)_{n \in \mathbb{N}}$ in $A(X)$ which separates the points of X and such that the series $\sum_{n=0}^{\infty} h_n^2$ uniformly converges to a function $\phi \in \mathcal{C}(X, \mathbb{R})$. Since $\|B_n\| = 1$ for every $n \in \mathbb{N}$, we also have $\lim_{n \to \infty} B_n^m(\phi) = T(\phi)$.

On the other hand, if $\lim_{n\to\infty} \sum_{i=1}^{n} p_{ni}^2 = 0$, from (3) it follows that

$$\lim_{n\to\infty} B_n^m(h^2) = h^2. \tag{5}$$

So the statement 1) follows from (1), (4) and Prop. 1.2, while the statement 2) follows from (1), (5) and Th. A. applied to $S = A(X)$.

REMARK 2.3. The second statement of Th. 2.2, for $m = 1$, has been obtained by M.W. Grossman ([11], Th. 2) in a more general context. Moreover in the setting of Ex. 2.1,1), Th. 2.2 has been obtained by T. Nishishiraho ([18], Th. 2 and Rem. 3).

We are now interested in studying the limit behaviour of the $k(n)-th$ iterates $B_n^{k(n)}$ of B_n as $n \to \infty$, $(k(n))_{n\in\mathbb{N}}$ being a sequence of positive integers. This seems to be interesting because, among other things, it is closely related to some aspects of the theory of stochastic processes an to the Trotter theorem concerning the convergence of contraction semigroups of operators ([12], pp. 312-313; [28]).

THEOREM 2.4. *Under the above assumptions* (2.1) *and* (2.2), *let* $(k(n))_{n\in\mathbb{N}}$ *be a sequence of positive integers. Moreover let us suppose that*

$$\lim_{n\to\infty} \sum_{i=1}^{n} p_{ni}^2 = 0.$$

For every $f \in C(X,\mathbb{R})$ *we have*:

1) *if* $\lim_{n\to\infty} k(n) \sum_{i=1}^{n} p_{ni}^2 = 0$, *then* $\lim_{n\to\infty} B_n^{k(n)}(f) = f$ *uniformly on* X;

2) *if* $\lim_{n\to\infty} k(n) \sum_{i=1}^{n} p_{ni}^2 = +\infty$, *then* $\lim_{n\to\infty} B_n^{k(n)}(f) = T(f)$ *uniformly on* X.

PROOF. By virtue of the above formula (3), for every $h \in A(X)$ and for every $n \in \mathbb{N}$, we have

$$B_n^{k(n)}(h^2) = \left[1 - \left(1 - \sum_{i=1}^{n} p_{ni}^2\right)^{k(n)} \left(1 - \sum_{i=1}^{n} p_{ni}^2\right)\right] T(h^2) + \left(1 - \sum_{i=1}^{n} p_{ni}^2\right)^{k(n)} h^2. \tag{1}$$

Having in mind that

$$\left(1 - \sum_{i=1}^{n} p_{ni}^2\right)^{k(n)} = \exp\left[-k(n)\left(\sum_{i=1}^{n} p_{ni}^2\right) \frac{\log\left(1 - \sum_{i=1}^{n} p_{ni}^2\right)}{-\sum_{i=1}^{n} p_{ni}^2}\right], \tag{2}$$

we can conclude as in the proof of Th. 2.2.

REMARK 2.5. In the setting of Ex. 2.1,1), Th. 2.4 has been obtained by T. Nishishiraho ([18]), Th. 2 and Rem. 3, [17], Rem. 2 to Th. 1; see also [22], Cor. 2).

Under additional assumptions it is possible to investigate what happens if $\lim_{n\to\infty} k(n) \sum_{i=1}^{n} p_{ni}^2 = t \in]0,+\infty[$.

For the sake of simplicity we restrict our attention to the case where P is the arithmetic mean Toeplitz matrix, i.e. $p_{ni} = \frac{1}{n}$ if $n \geq 1$ and $i = 1, \ldots, n$, and $p_{ni} = 0$ if $i > n$ (however the next result also holds for a general stochastic matrix P whose coefficients satisfy the conditions

$$\lim_{n\to\infty} \sum_{i=1}^{n} p_{ni}^2 = 0$$

and

$$\lim_{n\to\infty} \left(\sum_{i=1}^{n} p_{ni}^2\right)^{-1} \left(\sum_{i_1,\ldots,i_k=1}^{n} p_{ni_1}^{r_1} \cdots p_{ni_s}^{r_s}\, p_{ni_{s+1}} \cdots p_{ni_k}\right) = \delta_{k.m-1}$$

(where $\delta_{k.m-1}$ denotes the Kronecker symbol) for every $m \in \mathbb{N}$, with $m \geq 3$, $s \in \mathbb{N}$, $1 \leq s \leq \frac{m}{2}$, $k \in \mathbb{N}$, $s \leq k \leq m-s$ and $r_1, \ldots, r_s \in \mathbb{N}$ such that $r_1 + \ldots + r_s = m - k + s$).

We put for every $m \geq 1$

(2.7) $\qquad A_m =$ the linear subspace generated by

$$\left\{\prod_{i=1}^{m} h_i \mid h_i \in A(X),\ i = 1, \ldots, m\right\};$$

$(A_m)_{m\geq 1}$ is an increasing sequence of linear subspaces of $C(X,\mathbb{R})$.

Moreover the subspace

$$A_\infty = \bigcup_{m\geq 1} A_m \tag{2.8}$$

is a subalgebra of $C(X,\mathbb{R})$ which separates X and so is dense in $C(X,\mathbb{R})$.

We can now state our main result.

THEOREM 2.6. *Under the above assumptions* (2.1) *and* (2.2), *consider the sequence* $(B_n)_{n\in\mathbb{N}}$ *of Bernstein-Schnabl operators associated with* T *and the arithmetic mean Toeplitz matrix and suppose that*

$$T(A_2) \subset A(X) \tag{i}$$

or, alternatively,

(i)′ *$A(X)$ is finite dimensional and $T(A_m) \subset A_m$ for every $m \geq 1$.*

Then there exists a strongly continuous positive contraction semigroup $(T(t))_{t\geq 0}$ on $C(X,\mathbb{R})$ such that for every $t \geq 0$ and for every sequence $(k(n))_{n\in\mathbb{N}}$ of positive integers such that $\lim_{n\to\infty} \frac{k(n)}{n} = t$ one has

$$\lim_{n\to\infty} B_n^{k(n)} = T(t), \qquad \textit{strongly on } C(X,\mathbb{R}).$$

Moreover

$$\lim_{t\to+\infty} T(t) = T, \qquad \textit{strongly on } C(X,\mathbb{R})$$

and the generator of the semigroup $(T(t))_{t\geq 0}$ is the closure of the linear operator $Z : D(Z) \to C(X,\mathbb{R})$ defined by

$$Z(f) = \lim_{n\to\infty} n(B_n(f) - f)$$

for every $f \in D(Z) = \{g \in C(X,\mathbb{R}) \mid \lim_{n\to\infty} n(B_n(g) - g)$ exists in $C(X,\mathbb{R})\}$.

Finally, $A_\infty \subset D(Z)$ and for every $m \in \mathbb{N}$, $m \geq 1$ and $h_1,\dots,h_m \in A(X)$

$$(2.9) \qquad Z\left(\prod_{i=1}^m h_i\right) = \begin{cases} 0, & \textit{if } m = 1 \\ T(h_1h_2) - h_1h_2, & \textit{if } m = 2 \\ \sum_{1\leq i<j\leq m} (T(h_ih_j) - h_ih_j) \prod_{\substack{r=1\\ r\neq i,j}}^m h_r, & \textit{if } m \geq 3. \end{cases}$$

PROOF. Let us consider the linear operator $Z : D(Z) \to C(X,\mathbb{R})$ as defined above. If $f \in A_1(X) = A(X)$, then $B_n(f) = f$ for every $n \geq 1$ and so

$$(1) \qquad \lim_{n\to\infty} n(B_n(f) - f) = 0.$$

If $f = h_1h_2$ with $h_1, h_2 \in A(X)$, then for every $n \in \mathbb{N}$, $n \geq 1$,

$$f \circ \pi_n = \frac{1}{n^2} \left(\sum_{i=1}^n h_1 \circ pr_i\right) \left(\sum_{j=1}^n h_2 \circ pr_j\right)$$
$$= \frac{2}{n^2} \sum_{1\leq i<j\leq n}^{n} (h_1 \circ pr_i) \cdot (h_2 \circ pr_j) + \frac{1}{n^2} \sum_{i=1}^n (h_1 \cdot h_2) \circ pr_i.$$

Hence

$$B_n(f) = \frac{2}{n^2}\binom{n}{2} f + \frac{T(f)}{n} = \frac{n-1}{n} f + \frac{T(f)}{n}$$

and so

(2) $$\lim_{n\to\infty} n\,(B_n(f)-f)=T(f)-f.$$

If $f=\prod_{i=1}^{m} h_i$ with $h_1,\ldots,h_m\in A(X)$, $m\geq 3$, then for every $n\in\mathbb{N}$, $n\geq m$,

$$f\circ\pi_n=\frac{1}{n^m}\prod_{j=1}^{n}\left(\sum_{i=1}^{n}h_j\circ pr_i\right)$$

$$=\frac{1}{n^m}\sum_{s_1+\ldots+s_n=m}\sum\left(\prod_{1\leq u\leq s_1}h_{j_u^1}\right)\circ pr_1\ldots\left(\prod_{1\leq u\leq s_n}h_{j_u^n}\right)\circ pr_n,$$

where the last sum is extended to all subsets of integers $j_1^1,\ldots,j_{s_1}^1,\ldots,j_1^n,\ldots,j_{s_n}^n$ between 1 and m such that

$$\{j_1^1,\ldots,j_{s_1}^1\}\cap\ldots\cap\{j_1^n,\ldots,j_{s_n}^n\}=\emptyset,$$

with the convention that if some s_k is equal to zero then

$$\{j_1^k,\ldots,j_{s_k}^k\}=\emptyset \text{ and } \prod_{1\leq u\leq s_k}h_{j_u^k}=1.$$

So

(3) $$B_n(f)=\frac{1}{n^m}\sum_{s_1+\ldots+s_n=m}\sum T\left(\prod_{1\leq u\leq s_1}h_{j_u^1}\right)\ldots T\left(\prod_{1\leq u\leq s_n}h_{j_u^n}\right).$$

Applying Leibnitz formula to $n^m=(1+\ldots+1)^m$ one obtains

(4) $$1=\frac{1}{n^m}\sum_{s_1+\ldots+s_n=m}\binom{m}{s_1}\binom{m-s_1}{s_2}\binom{m-(s_1+s_2)}{s_3}\ldots\binom{m-(s_1+s_2+\ldots+s_{n-2})}{s_{n-1}}$$

and so

$$f=\frac{1}{n^m}\sum_{s_1+\ldots+s_n=m}\sum\left(\prod_{1\leq u\leq s_1}h_{j_u^1}\right)\ldots\left(\prod_{1\leq u\leq s_n}h_{j_u^n}\right),$$

where the integers j_u^v vary as above. Therefore

(5) $$B_n(f)-f=\frac{1}{n^m}\sum_{s_1+\ldots+s_n=m}\sum\left[T\left(\prod_{1\leq u\leq s_1}h_{j_u^1}\right)\ldots T\left(\prod_{1\leq u\leq s_n}h_{j_u^n}\right)\right.$$

$$- \left(\prod_{1 \le u \le s_1} h_{j_u^1} \right) \cdots \left(\prod_{1 \le u \le s_n} h_{j_u^n} \right) \Bigg] = \frac{1}{n^m} \sum_{s_1 + \ldots + s_n = m} g(s_1, \ldots, s_n)$$

where, of course,

$$\begin{aligned} &g(s_1, \ldots, s_n) \\ &= \sum \left[T\left(\prod_{1 \le u \le s_1} h_{j_u^1} \right) \ldots T\left(\prod_{1 \le u \le s_n} h_{j_u^n} \right) \right. \\ &\left. - \left(\prod_{1 \le u \le s_1} h_{j_u^1} \right) \cdots \left(\prod_{1 \le u \le s_n} h_{j_u^n} \right) \right] \end{aligned}$$

for all $s_1, \ldots, s_n \in \{0, 1, \ldots, m\}$ such that $s_1 + \ldots + s_n = m$.

For our purpose it is more convenient to write the last term of (5) in a different manner. We also remark that if $s_1, \ldots, s_n \in \{0, 1\}$ and $s_1 + \ldots + s_n = m$, then $g(s_1, \ldots, s_n) = 0$ because $T(h) = h$ for all $h \in A(X)$.

So, after putting $m_0 = \max \{k \in \mathbb{N} | 2k \le m\}$, using combinatorial arguments we have

$$\begin{aligned} &B_n(f) - f \\ &= \frac{1}{n^m} \sum_{s=1}^{m_0} \sum_{k=s}^{m-s} \sum_{r_1 + \ldots + r_s = m-k+s} n(n-1) \ldots (n-k+1)\ g^*(r_1, \ldots, r_s) \\ &= \sum_{s=1}^{m_0} \sum_{k=s}^{m-s} n^{k-m} \left(1 - \frac{1}{n}\right) \cdots \left(1 - \frac{k-1}{n}\right) \sum_{r_1 + \ldots + r_s = m-k+s} g^*(r_1, \ldots, r_s), \end{aligned}$$

where

$$\begin{aligned} &g^*(r_1, \ldots, r_s) \\ &= \sum \left[T\left(\prod_{1 \le u \le r_1} h_{j_u^1} \right) \ldots T\left(\prod_{1 \le u \le r_s} h_{j_u^n} \right) \right. \\ &\left. - \left(\prod_{1 \le u \le r_1} h_{j_u^1} \right) \cdots \left(\prod_{1 \le u \le r_s} h_{j_u^n} \right) \right] \prod_{j \ne j_u^v} h_j \end{aligned}$$

for every $r_1, \ldots, r_s \ge 2$ such that $r_1 + r_2 + \ldots + r_s = m - k + s$.

Since in the last sum $\lim\limits_{n \to \infty} n^{k-m+1} = 1$ only when $s = 1$ and $k = m - 1$ and the same limit is equal to zero otherwise, we infer that

$$\text{(6)} \qquad \lim_{n \to \infty} n\ (B_n(f) - f) = g^*(2) = \sum_{1 \le i < j \le m} [T(h_i h_j) - h_i h_j] \prod_{r \ne i.j} h_r.$$

Hence we have proved that $A_\infty \subset D(Z)$ and so $D(Z)$ is dense in $\mathcal{C}(X, \mathbb{R})$.

Suppose now that condition (i) is satisfied. We shall prove that for every $\lambda > 0$ the range $R(\lambda I - Z)$ is dense in $C(X,\mathbb{R})$, where I denotes the identity operator on $C(X,\mathbb{R})$.

In fact, fix $\lambda > 0$ and consider $\mu \in C(X,\mathbb{R})'$ such that $\mu(g) = 0$ for every $g \in R(\lambda I - Z)$, i.e. $\mu(f) = \frac{1}{\lambda}\mu(Z(f))$ for every $f \in D(Z)$. So for every $f \in A_1$, by virtue of (1), we have $\mu(f) = \frac{1}{\lambda}\ \mu(Z(f)) = 0$. Moreover, according to (2), for every $f \in A_2$ we have

$$\mu(f) = \frac{1}{\lambda}\ \mu(Z(f)) = \frac{1}{\lambda}\ \mu(T(f)) - \frac{1}{\lambda}\ \mu(f) = \frac{1}{\lambda}\ \mu(f)$$

and so again $\mu(f) = 0$.

Suppose now that $\mu = 0$ on A_m with $m \geq 2$ and let $f = \prod_{i=1}^{m+1} h_i$ with $h_i \in A(X)$ for every $i = 1,\ldots,m+1$. Then

$$\mu(f) = \frac{1}{\lambda}\ \mu(Z(f)) = \frac{1}{\lambda}\ \mu\left[\sum_{1\leq i<j\leq m+1} T(h_ih_j) \prod_{r\neq i.j} h_r - \binom{m+1}{2} f\right]$$
$$= -\frac{1}{\lambda}\ \frac{m(m-1)}{2}\ \mu(f)$$

since $T(h_ih_j) \prod_{r\neq i.j} h_r \in A_m$ for every $i,j = 1,\ldots,m+1$, by virtue of (i).

Consequently $\mu(f) = 0$. This implies that $\mu = 0$ on A_{m+1}, hence by induction on m, we have that $\mu = 0$ on A_∞ and so $\mu = 0$.

Thus we have proved that $R(\lambda I - Z)$ is dense in $C(X,\mathbb{R})$ for every $\lambda > 0$. Using a result of H.F. Trotter ([28], Th. 5.3) we infer that the closure of Z is the infinitesimal generator of a contraction semigroup $(T(t))_{t\geq 0}$ and

$$T(t) = \lim_{n\to\infty} B_n^{[nt]}, \qquad \text{strongly on } C(X,\mathbb{R}),$$

for all $t \geq 0$, where $[nt]$ denotes the integer part of nt.

In particular every $T(t)$ is positive. Consider now a sequence $(k(n))_{n\in\mathbb{N}}$ of positive integers such that $\lim_{n\to\infty} \frac{k(n)}{n} = t \geq 0$. Then for every $f \in A_\infty$,

$$\lim_{n\to\infty} k(n)(B_n(f) - f) = \lim_{n\to\infty} \frac{k(n)}{n}\ n(B_n(f) - f) = t\ Z(f).$$

Again according to Trotter's theorem, the closure of tZ is the infinitesimal generator of a semigroup $(S(u))_{u\geq 0}$ of contractions and for every $u \geq 0$

$$S(u) = \lim_{n\to\infty} B^{[k(n)u]}, \qquad \text{strongly on } C(X,\mathbb{R}).$$

Since the closure of tZ is also generated by $(T(tu))_{u\geq 0}$, we conclude that $S(u) = T(tu)$ for all $u \geq 0$ and $t \geq 0$ and so

$$T(t) = S(1) = \lim_{n\to\infty} B_n^{k(n)}, \qquad \text{strongly on } C(X,\mathbb{R}).$$

If, alternatively, condition (i)′ is satisfied, then for every $m \in \mathbb{N}$, A_m is finite dimensional and, by virtue of (3), it is invariant under B_n for every $n \in \mathbb{N}$. So, the existence of the semigroup $(T(t))_{t\geq 0}$ which satisfies the properties indicated in Th. 2.6, directly follows from a result of R. Schnabl ([26], Satz 4), (see also a result of T. Nishishiraho, [20], Th. 1).

Finally, let us consider $h \in H$; then for every $t \geq 0$

$$T(t)(h) = \lim_{n\to\infty} B_n^{[nt]}(h) = h = T(h).$$

After putting $k(n) = [nt]$ for every $n \in \mathbb{N}$ and taking into account formulas (1) and (2) of the proof of Th. 2.4, we have for every $f \in A(X)$

$$T(t)(h^2) = \lim_{n\to\infty} B_n^{k(n)}(h^2) = (1 - \exp(-t))\ T(h^2) + \exp(-t)\ h^2$$

because $\lim_{n\to\infty} \frac{k(n)}{n} = t$.

Therefore $\lim_{t\to+\infty} T(t)(h^2) = T(h^2)$. By arguing as in the final part of the proof of Th. 2.2 and by using Prop. 1.2, it follows that $\lim_{t\to+\infty} T(t)(f) = T(f)$ for every $f \in C(X,\mathbb{R})$.

REMARK 2.7.

1. - In the context of metrizable Bauer simplex (cf. Ex. 2.1,1) clearly condition (i) of Th. 2.6 is satisfied. However in this context Th. 2.6 has been obtained by T. Nishishiraho in [20], pp. 79-80, by a different method (see also R. Schnabl [25], [26]).

For the classical Bernstein operators on $[0,1]$, Th. 2.6 is substantially known (cf. S. Karlin - Z. Ziegler ([12]) and C.A. Micchelli ([14])).

In these articles a detailed analysis of the properties of the semigroup $(T(t))_{t\geq 0}$ can be found.

2. - Other results on the convergence of iterates of positive operators to semigroups can be found in [22].

3. - Applications

In this section we shall briefly indicate some applications of the results obtained previously.

3.1. *Bernstein operators on the standard simplex of* $\mathbb{R}^p$

Let us consider the standard simplex X_p of $\mathbb{R}^p$ $(p \geq 1)$ and the classical Bernstein operators on X_p, i.e.

$$B_n(f)(x_1, \dots, x_p) = \sum_{0 \leq h_1+\dots+h_p \leq n} \frac{n!}{h_1! \dots h_p!(n-h_1-\dots-h_p)!} f\left(\frac{h_1}{n}, \dots, \frac{h_p}{n}\right) x_1^{h_1} \dots x_p^{h_p} \left(1 - \sum_{i=1}^{p} x_i\right)^{n-\sum_{i=1}^{p} h_i}$$

for every $f \in C(X_p, \mathbb{R})$, $(x_1, \dots, x_p) \in X_p$, $n \geq 1$.

This sequence is generated by the linear positive projection $T : C(X_p, \mathbb{R}) \to C(X_p, \mathbb{R})$ defined by putting for every $f \in C(X_p, \mathbb{R})$ and $(x_1, \dots, x_p) \in X_p$

$$T(f)(x_1, \dots, x_p) = B_1(f)(x_1, \dots, x_p) = \sum_{0 \leq h_1+\dots+h_p \leq 1} \alpha_f(h_1, \dots, h_p) x_1^{h_1} \dots x_p^{h_p} \left(1 - \sum_{i=1}^{p} x_i\right)^{1-\sum_{i=1}^{p} h_i}, \tag{3.1}$$

were $\alpha_f(h_1, \dots, h_p) = f(\delta_{h_1 1}, \dots, \delta_{h_p 1})$, $\delta_{h_i 1}$ being the Kronecker symbol.

If for every $i = 1, \dots, p$ we denote by $pr_i : X_p \to \mathbb{R}$ the $i-th$ projection, then for every $i, j = 1, \dots, p$ we have

$$T(pr_i\ pr_j) = \begin{cases} 0 & \text{if } i \neq j \\ pr_i & \text{if } i = j. \end{cases} \tag{3.2}$$

In this context condition (i) of Th. 2.6 is satisfied and so let us consider the semigroup $(T(t))_{t \geq 0}$ and the operator Z appearing in Th. 2.6. Moreover let A be the closure of Z with domain $D(A)$.

By using formulas (3.2) and (2.9) it is easy to show that for every

$h_1, \ldots, h_p \geq 1$ and for every $(x_1, \ldots, x_p) \in X_p$

$$Z\left(\prod_{i=1}^{p} pr_i^{h_i}\right)(x_1, \ldots, x_p) = \sum_{\substack{1 \leq i \leq p \\ h_i \geq 2}} \frac{h_i(h_i-1)}{2} x_i (1-x_i) x_i^{h_i-2} \prod_{j \neq i} x_j^{h_j}$$

$$- \sum_{1 \leq i < j \leq p} h_i h_j x_i x_j x_i^{h_i-1} x_j^{h_j-1} \prod_{\lambda \neq i,j} x_\lambda^{h_\lambda}$$

$$= \left(\sum_{i=1}^{p} \frac{x_i(1-x_i)}{2} \frac{\partial^2}{\partial x_i^2} - \sum_{1 \leq i < i \leq p} x_i x_j \frac{\partial^2}{\partial x_i \partial x_j}\right) \left(\prod_{i=1}^{p} pr_i^{h_i}\right) (x_1, \ldots, x_p);$$

hence the operator Z coincides with the restriction to A_∞ (which is, in this case, the subalgebra generated by $pr_1, \ldots, pr_p$) of the differential operator

$$U(f)(x_1, \ldots, x_p) = \sum_{i=1}^{p} \frac{x_i(1-x_i)}{2} \frac{\partial^2 f}{\partial x_i^2} (x_1, \ldots, x_p) \tag{3.3}$$

$$- \sum_{1 \leq i < i \leq p} x_i x_j \frac{\partial^2 f}{\partial x_i \partial x_j} (x_1, \ldots, x_p)$$

which is an elliptic second-order differential operator which degenerates on the boundary of X_p.

Finally, using the theory of one parameter semigroups of operators ([10], II, Th. 1.2) we conclude that for every $u_0 \in D(A)$ the Cauchy problem

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = Au(t,x) \\ \\ u(0,x) = u_0(x) \end{cases} \qquad x \in X_p,$$

has a unique solution $u : [0, +\infty[\times X_p \to \mathbb{R}$ given by

$$u(t,x) = (T(t)u_0)(x) = \lim_{n \to \infty} (B_n^{[nt]}(u_0))(x)$$

for every $t \geq 0$ and $x \in X_p$. This solution is positive whenever u_0 is positive. For other consequences related to the positivity of the semigroup generated by A and which are of some interest for the above Cauchy problem, see [16].

3.2. *Bernstein operators on the hypercube of* $\mathbb{R}^p$

As another example let us consider the hypercube $X = [0,1]^p$ of $\mathbb{R}^p$ $(p \geq 1)$ and the sequence of classical Bernstein operators on X, i.e.

$$B_n(f)(x_1,\dots,x_p) = \sum_{h_1,\dots,h_p=0}^{n} \binom{n}{h_1}\binom{n}{h_2}\cdots\binom{n}{h_p} f\left(\frac{h_1}{n},\dots\frac{h_p}{n}\right) x_1^{h_1}(1-x_1)^{n-h_1}\dots x_p^{h_p}(1-x_p)^{n-h_p}$$

for every $f \in C(X,\mathbb{R})$, $(x_1,\dots,x_p) \in X$, $n \geq 1$.

In this case the positive projection $T : C(X,\mathbb{R}) \to C(X,\mathbb{R})$ is defined by putting for every $f \in C(X,\mathbb{R})$ and $(x_1,\dots,x_p) \in X$

$$T(f)(x_1,\dots,x_p) = B_1(f)(x_1,\dots,x_p) = \sum_{h_1,\dots,h_p=0}^{1} \beta_f(h_1,\dots,h_p)\; x_1^{h_1}(1-x_1)^{1-h_1}\dots x_p^{h_p}(1-x_p)^{1-h_p}, \tag{3.4}$$

where $\beta_f(h_1,\dots,h_p) = f(\delta_{h_1 1},\dots,\delta_{h_p p})$.

In this case we note that for every $i,j = 1,\dots p$

$$T(pr_i pr_j) = \begin{cases} pr_i pr_j & \text{if } i \neq j \\ pr_i & \text{if } i = j, \end{cases} \tag{3.5}$$

because the operator T leaves invariant the functions which are multiaffine on X (cf. Ex. 2.1,3).

Moreover for every $n_1,\dots,n_p \in \mathbb{N}$ we have

$$T\left(pr_1^{n_1}\cdot pr_2^{n_2}\cdot\ldots\cdot pr_p^{n_p}\right) = T(pr_1\cdot pr_2\cdot\ldots\cdot pr_p) = pr_1\cdot\ldots\cdot pr_p$$

since $\beta_{pr_1^{n_1}\dots pr_p^{n_p}} = \beta_{pr_1\dots pr_p}$.

Hence condition (i)′ (but not condition (i)) of Th. 2.6 is satisfied. So let us consider the semigroup $(T(t))_{t\geq 0}$ and the operator Z constructed in Th. 2.6.

Again by using formulas (2.9) and (3.5), it results that the operator Z coincides with the restriction to A_∞ (the subspace of all polynomials on X) of the differential operator

$$V(f)(x_1,\dots x_p) = \sum_{i=1}^{p} \frac{x_i(1-x_i)}{2}\,\frac{\partial^2 f}{\partial x_i^2}\,(x_1,\dots,x_p) \tag{3.6}$$

which also is a degenerate elliptic second-order differential operator.

Also in this case, denoted by B the closure of Z, we obtain the solution of the Cauchy problem

$$\begin{cases} \dfrac{\partial u}{\partial t}\ (t,x) = Bu(t,x) \\ u(0,x) = u_0(x) \qquad x \in X,\ u_0 \in D(B) \end{cases}$$

by

$$u(t,x) = \lim_{n\to\infty}\ (B_n^{[nt]}(u_0))(x)$$

for every $t \geq 0$ and $x \in X$.

3.3. *Bernstein-Schnabl operators on balls of* $\mathbb{R}^p$

The last example is concerned with a ball $X = B(x_0, r)$ of $\mathbb{R}^p$ with centre x_0 and radius r and with the sequence of Bernstein-Schnabl operators defined by (2.5) with respect to the arithmetic-mean Toeplitz matrix

$$B_n(f)(x) = \begin{cases} \left(\dfrac{r^2 - \|x_0 - x\|^2}{r\ \sigma_p}\right)^n \\ \displaystyle\int\limits_{\partial X} \cdots \int\limits_{\partial X} \frac{f\left(\frac{1}{n}(x_1 + \ldots + x_n)\right)}{\|x_1 - x\|^p \ldots \|x_n - x\|^p}\ \mathrm{d}\sigma(x_1)\ldots\mathrm{d}\sigma(x_n) \\ \qquad\qquad\qquad\qquad\qquad \text{if } \|x - x_0\| < r \\ f(x) \qquad\qquad\qquad\qquad\quad \text{if } \|x - x_0\| = r. \end{cases}$$

In this case the positive projection $T : C(X, \mathbb{R}) \to C(X, \mathbb{R})$ is defined by putting for every $f \in C(X, \mathbb{R})$ and $x \in X$

$$(3.7)\quad T(f)(x) = \begin{cases} \dfrac{r^2 - \|x_0 - x\|^2}{r\ \sigma_p} \displaystyle\int \frac{f(z)}{\|z - x\|^p}\ \mathrm{d}\sigma(z) & \text{if } \|x - x_0\| < r \\ f(x) & \text{if } \|x - x_0\| = r. \end{cases}$$

Moreover for every $i, j = 1, \ldots, p$ we have

$$(3.8)\qquad T(pr_i pr_j) = \begin{cases} pr_i pr_j & \text{if } i \neq j \\ \dfrac{1}{p}\left[r^2 - \displaystyle\sum_{\lambda \neq i}(pr_\lambda - pr_\lambda(x_0))^2 \right. \\ \left. + (p-1)(pr_i - pr_i(x_0))^2\right] + \\ +2pr_i(x_0)pr_i - pr_i^2(x_0) & \text{if } i = j. \end{cases}$$

Moreover from a result of M. Brelot and G. Choquet ([7], Th. 6; see also D.H. Armitage, [4], Th. 2 and Th. 4) ([1]), it follows that condition (i)′ of Th. 2.6. is satisfied.

If Z denotes the operator indicated in Th. 2.6, then, with the help of formulas (2.9) and (3.8), we deduce that the operator Z agrees on A_∞ with the degenerate elliptic second order differential operator

$$W(f)(x) = \frac{r^2 - \|x - x_0\|^2}{2p} \; \Delta f(x). \tag{3.9}$$

In this case, the function

$$u(t,x) = \lim_{n\to\infty} \; (B_n^{[nt]}(u_0))(x) \qquad\qquad t \geq 0, \;\; x \in X$$

is the unique solution of the Cauchy problem

$$\begin{cases} \dfrac{\partial u}{\partial t} \; (t,x) = C \;\; u(t,x) \\ u(0,x) = u_0(x) \qquad\qquad x \in X, \;\; u_0 \in D(C) \end{cases}$$

C being the closure of Z.

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