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On the Existence of Convex Hypersurfaces with Prescribed Mean Curvature

KAISING TSO

1. - Results

Let X be a smooth closed embedded hypersurface in $\mathbb{R}^{n+1}$ with mean curvature function H with respect to its inner normal. We are concerned with the following question.

Given a function F in $\mathbb{R}^{n+1}$, under what conditions does the equation

$$H(p) = F(p), \qquad p \in X \tag{1.1}$$

have a solution for a closed embedded hypersurface X?

The following result has been obtained.

THEOREM A. *Let $A = \{x \in \mathbb{R}^{n+1} : 0 < R_1 < |x| < R_2\}$. Suppose F is a positive function satisfying*

(a) $$\frac{\partial}{\partial \rho} \rho F(\rho x) \leq 0, \rho < 0, x \in S^n, \text{ and}$$

(b) $$F(x) \geq \frac{1}{R_1}, |x| = R_1;$$

$$F(x) \leq \frac{1}{R_2}, |x| = R_2.$$

There exists a hypersurface X which is starshaped with respect to the origin, lies in $\overline{A}$ and solves (1.1).

Furthermore, any two solutions are endpoints of homothetic dilations, all of which are solutions.

See [BK], [TW] or [CNS] for a proof. Here the monotonicity condition (a) is used not only in characterizing uniqueness but also in the proof of existence.

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It is known that (1.1) has a variational structure [Y, Problem 59]. Namely,

$$I(X) = \frac{1}{n}\int_X 1 - \int_{\tilde{X}} F \tag{1.2}$$

($\tilde{X}$ is the subset bounded by X) has (1.1) as its Euler-Lagrange equation. In this paper we shall establish results on the existence of *convex* hypersurfaces solving (1.1) via a study of the critical points of (1.2).

For simplicity we shall work within the smooth category. All functions and hypersurfaces are assumed to be smooth.

THEOREM B. *Let* $A = \{x \in \mathbb{R}^{n+1} : 0 < R_1 < |x| < R_2\}$. *Suppose* F *is a positive function satisfying*

(a) F *is concave in* A; *and*

(b)
$$F(x) \geq \frac{1}{R_1}, |x| = R_1;$$
$$F(x) \leq \frac{1}{R_2}, |x| = R_2.$$

There exists a convex hypersurface X *which encloses the origin, lies in* $\tilde{A}$ *and solves* (1.1). *In fact,* X *minimizes* I *among all convex hypersurfaces enclosing the origin and lying in* $\overline{A}$.

Theorem B is formulated so as to compare with Theorem A. The following result is much more satisfying from the variational point of view.

THEOREM C. *Let* F *be a concave function in* $\mathbb{R}^{n+1}$ *which becomes negative outside* $B_R = \{x : |x| < R\}$ *for some* $R > 0$. *Then*

(a) *Any absolute minimum of* I *among all convex hypersurfaces in* $\mathbb{R}^{n+1}$ *is a solution of* (1.1);

(b) I *has an absolute minimum if and only if there exists a convex hypersurface* Y *with* $I(Y) \leq 0$.

For concave functions F, the boundedness of I from below among convex hypersurfaces is equivalent to F being negative outside a bounded set. Since a small perturbation of a convex hypersurface may no longer be convex, (a) of Theorem C is a non-trivial assertion.

Let Γ be a subgroup of the orthogonal group $\mathcal{O}(n+1)$. F is called Γ-*invariant* if $F(xg) = F(x)$ for all $x \in \mathbb{R}^{n+1}$ and all $g \in \Gamma$. A subset C is called Γ-*symmetric* if $Cg = C$ for all $g \in \Gamma$.

THEOREM D. *Let* F *be given as in Theorem* C *and let* Γ *be a subgroup of* $\mathcal{O}(n+1)$ *such that* $\{xg : g \in \Gamma\}$ *spans* $\mathbb{R}^{n+1}$ *for some nonzero vector* x. *Suppose* *further that F is Γ-invariant and there is a Γ-symmetric convex hypersurface Y with $I(Y) \leq 0$. There exists a Γ-symmetric convex hypersurface Z solving* (1.1) *with* $I(Z) > 0$.

Therefore, for the F in Theorem D there are at least two solutions. In case of radial F's (i.e., $\Gamma = \mathcal{O}(n+1)$), there are exactly two solutions. See the discussion at the end of Section 4.

Both proofs of Theorems B and C rely on a study of the negative gradient flow associated with I:

$$\begin{aligned} &\frac{\partial X}{\partial t} = -(H - F)\nu \\ &X(\cdot, 0) \text{ given} \end{aligned} \tag{1.3}$$

(ν is the unit outer normal at $X(\cdot, t)$). In subsequent sections we shall show that (1.3) preserves convexity, admits a unique solution for all time for suitable initial data and contains a subsequence $\{X(\cdot, t_j)\}$ converging to a solution of (1.1). To prove Theorem D we shall use a mountain pass lemma. However, because the functional I is not continuously differentiable in the appropriate space, we shall work directly on the flow (1.3).

Sufficient conditions for the existence of convex hypersurfaces with prescribed mean curvature F have been given by Treibergs [T] when F is of homogeneous degree -1. Roughly speaking, his conditions are a priori inequalities between F and its derivatives (up to second order) which prevent the sought-after hypersurface derivated too much from the round sphere. This is close in spirit to the sufficient conditions of Pogorelov [P, Chapter 4] for the still unsolved intermediate Christoffel- Minkowski problem. Our result is not of this type. We use a parabolic equation to deform the convex hypersurfaces and rely mainly on the concavity of F.

That this flow preserves convexity is inspired by the computations in Hamilton [H] and Huisken [HU].

In subsequent sections, we shall establish Theorems C and D in a slightly generalized form. Namely, we shall replace the area integrand $\int_X 1$ in (1.2) by $\int_X \mathcal{F}$ where $\mathcal{F}$ is an elliptic parametric integrand depending only on the normals.

In the companion paper [TS] we treat the same problem for Gauss-Kronecker curvature, instead of mean curvature, where the concavity of F is not required.

2. - Preserving convexity

In this section we first prove a first variational formula for a parametric integral (Proposition 1). Then we show that the associated negative gradient flow preserves convexity (Proposition 2).

Let X be a closed embedded hypersurface oriented with respect to its unit inner normal $-\nu$. In a local coordinate $x = (x_1, \cdots, x_n)$ we shall use the following notations: for $i, j - 1, \cdots, n$,

g_{ij} (metric tensor)

$$= \langle \frac{\partial X}{\partial x_i}, \frac{\partial X}{\partial x_j} \rangle,$$

$\mathcal{D}$ (volume element)

$$= (\det\, g_{ij})^{1/2},$$

Γ^k_{ij} (Christoffel symbols)

$$= \frac{1}{2} g^{k\ell} \left(\frac{\partial g_{j\ell}}{\partial x_i} + \frac{\partial g_{i\ell}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_\ell} \right),$$

b_{ij} (second fundamental tensor)

$$= \langle \frac{\partial \nu}{\partial x_i}, \frac{\partial X}{\partial x_j} \rangle = -\langle \nu, \frac{\partial^2 X}{\partial x_i \partial x_j} \rangle,$$

H (mean curvature)

$$= \frac{1}{n} g^{ij} b_{ij},$$

R^ℓ_{kij} (Riemann-Christoffel curvature tensor)

$$= \frac{\partial \Gamma^\ell_{kj}}{\partial x_i} - \frac{\partial \Gamma^\ell_{ki}}{\partial x_j} + \Gamma^m_{kj}\Gamma^\ell_{im} - \Gamma^m_{ki}\Gamma^\ell_{jm},$$

$R_{\ell kij}$ (Riemann curvature tensor)

$$= R^m_{kij} g_{m\ell}.$$

Here $\langle \cdot, \cdot \rangle$ is the Euclidean metric in $\mathbb{R}^{n+1}$. As usual, we lower or upper indices due to contractions with the metric tensor or its inverse. Besides, the summation convention is always in effect.

We shall also use ∇_i to denote the covariant differentiation with respect to $\frac{\partial}{\partial x_i} (i = 1, \cdots, n)$. Thus, for instance,

$$\nabla_i \xi^j = \frac{\partial \xi^j}{\partial x_i} + \Gamma^j_{ik} \xi^k$$

for a vector field ξ on X. Recall the following fundamental formulas of a hypersurface in $\mathbb{R}^{n+1}$:

(2.1) $$\frac{\partial \nu}{\partial x_i} = b_{ij} \frac{\partial X}{\partial x_k} g^{kj} \text{ (Weingarten equation)},$$

(2.2) $$\frac{\partial^2 X}{\partial x_i \partial x_j} = \Gamma^k_{ij} \frac{\partial X}{\partial x_k} - b_{ij} \nu \text{ (Gauss formula-Weingarten equation)},$$

$$R_{ijk\ell} = b_{ik}b_{j\ell} - b_{i\ell}b_{jk} \text{ (Gauss equation)}, \tag{2.3}$$

$$\nabla_i b_{jk} = \nabla_k b_{ij} = \nabla_j b_{ki} \text{ (Codazzi equation)}. \tag{2.4}$$

Let $\mathcal{F}(y), y \in \mathbb{R}^{n+1}\backslash\{0\}$, be a function of homogeneous degree 1. It associates with a functional defined on the class of closed hypersurfaces in $\mathbb{R}^{n+1}$ given by

$$J = \int_X \mathcal{F}(-\nu^1, \cdots, -\nu^{n+1}).$$

This functional is called a parametric integral and $\mathcal{F}$ is its parametric integrand. For a general definition of a parametric integral we refer to [F].

DEFINITION. *Let $\mathcal{F}$ be a parametric integrand and let X be a hypersurface in $\mathbb{R}^{n+1}$. Denote a_{ij} the symmetric $\binom{2}{0}$-tensor given locally by*

$$a_{ij}(x) = \frac{\partial^2 \mathcal{F}(-\nu(x))}{\partial y_k \partial y_\ell} \frac{\partial X^k}{\partial x_i} \frac{\partial X^\ell}{\partial x_j}. \tag{2.5}$$

The $\mathcal{F}$-mean curvature of X is defined by

$$H_{\mathcal{F}} = \frac{1}{n} a^{ij} b_{ij}.$$

When $\mathcal{F}(y) = |y|^2, \mathcal{F}(-\nu)$ is equal to 1. The associated J is the area functional and $H_{\mathcal{F}}$ is simply H, the mean curvature of X.

PROPOSITION 1. *The following first variational formula holds. For a normal variation vector filed $\xi = \phi\nu$ on X,*

$$\delta J(\xi) = n \int_X H_{\mathcal{F}} \phi. \tag{2.6}$$

PROOF. Let $X(s)$ be a smooth family of closed hypersurfaces satisfying $X(0) = X$ and $\frac{\partial X}{\partial s}(0) = \xi$. We compute

$$\begin{aligned} \delta J(\xi) &= \frac{\mathrm{d}}{\mathrm{d}s}|_{s=0} J(X(s)) \\ &= -\int_X \frac{\partial \mathcal{F}}{\partial y_j} \frac{\partial \nu^j}{\partial s} + n \int_X \mathcal{F} H \phi. \end{aligned}$$

Notice we have used the first variation formula for area. Using $\langle \frac{\partial \nu}{\partial s}, \nu \rangle = 0$,

$$\begin{aligned} \frac{\partial \nu}{\partial s} &= \langle \frac{\partial \nu}{\partial s}, \frac{\partial X}{\partial x_k} \rangle g^{k\ell} \frac{\partial X}{\partial x_\ell} \\ &= -\langle \nu, \frac{\partial \xi}{\partial x_k} \rangle g^{k\ell} \frac{\partial X}{\partial x_\ell}. \end{aligned}$$

Therefore

$$\begin{aligned} -\frac{\partial \mathcal{F}}{\partial y_j} \frac{\partial \nu^j}{\partial s} &= \frac{\partial \mathcal{F}}{\partial y_j} \nu^m \frac{\partial \xi^m}{\partial x_k} g^{k\ell} \frac{\partial X^j}{\partial x_\ell} \\ &= \frac{\partial}{\partial x_k} \left(\phi \frac{\partial \mathcal{F}}{\partial y_j} g^{k\ell} \frac{\partial X^j}{\partial x_\ell} \right) + \phi \frac{\partial^2 \mathcal{F}}{\partial y_j \partial y_n} \frac{\partial \nu^n}{\partial x_k} g^{k\ell} \frac{\partial X^j}{\partial x_\ell} \\ &\quad - \frac{\partial \mathcal{F}}{\partial y_j} \frac{\partial \nu^m}{\partial x_k} g^{k\ell} \frac{\partial X^j}{\partial x_\ell} \xi^m - \phi \frac{\partial \mathcal{F}}{\partial y_j} \frac{\partial}{\partial x_k} \left(g^{k\ell} \frac{\partial X^j}{\partial x_\ell} \right). \end{aligned}$$

The second term is equal to $nH_{\mathcal{F}}\phi$ by (2.1). The third term vanishes because ξ is in the normal direction. Using (2.2) and the Euler's identity $-\nu^j \frac{\partial \mathcal{F}}{\partial y} = \mathcal{F}$ (notice that $\mathcal{F}$ is evaluated at $-\nu$) the fourth term is equal to

$$\frac{\partial \mathcal{F}}{\partial y_j} \frac{\partial g^{k\ell}}{\partial x_k} \frac{\partial X^j}{\partial x_j} \phi + \frac{\partial \mathcal{F}}{\partial y_j} g^{km} \Gamma^\ell_{km} \frac{\partial X^j}{\partial x_\ell} \phi - n\mathcal{F} H\phi.$$

Furthermore, by a direct computation we find that for $\ell = 1, \cdots, n$

$$\frac{\partial g^{kl}}{\partial x_k} + g^{km}\Gamma^\ell_{km} = g^{k\ell}\Gamma^m_{km}.$$

Henceforth, the fourth term is equal to

$$\frac{\partial \mathcal{F}}{\partial y_j} g^{k\ell} \Gamma^m_{km} \frac{\partial X^j}{\partial x_\ell} \phi - n\mathcal{F} H\phi.$$

Combining these, we have

$$-\frac{\partial \mathcal{F}}{\partial y_j} \frac{\partial \nu^j}{\partial s} = \text{ Div } \left(\phi \frac{\partial \mathcal{F}}{\partial y_j} g^{k\ell} \frac{\partial X^j}{\partial x_\ell} \right) + nH_{\mathcal{F}}\phi - n\mathcal{F} H\phi.$$

Here Div denotes the divergence of a vector field on X. (Recall for a vector field a, Div $a = \frac{\partial a^i}{\partial x_i} + a^j \Gamma^i_{ji}$). As X is closed, by divergence theorem we have

$$\begin{aligned} \delta J(\xi) &= \int_X \text{ Div } \left(\phi \frac{\partial \mathcal{F}}{\partial y_j} g^{k\ell} \frac{\partial X^j}{\partial x_\ell} \right) + \int_X nH_{\mathcal{F}}\phi \\ &= \int_X nH_{\mathcal{F}}\phi. \end{aligned}$$

The proof of Proposition 1 is completed. *q.e.d.*

For a parametric integrand $\mathcal{F}$ and a function F defined in $\mathbb{R}^{n+1}$, consider the functional $I = I_{\mathcal{F}}$ given by:

$$I(X) = \frac{1}{n} J_{\mathcal{F}}(X) - \int_{\tilde{X}} F$$

for a closed (connected) hypersurface X, where $\tilde{X}$ is the bounded component of $\mathbb{R}^{n+1} \backslash X$. From Proposition 1

$$\delta I(\xi) = \int_X (H_{\mathcal{F}} - F)\phi.$$

Therefore, its negative gradient flow is given by

$$\frac{\partial X}{\partial t} = -(H_{\mathcal{F}} - F)\nu. \tag{2.8}$$

We want to study under what conditions (2.8) preserves convexity. Following [HU] we look at the evolution of the second fundamental tensor inherited from (2.8):

$$\begin{aligned}\frac{\partial b_{ij}}{\partial t} &= -\frac{\partial}{\partial t}\langle \frac{\partial^2 X}{\partial x_i \partial x_j}, \nu\rangle \\ &= \frac{\partial^2}{\partial x_i \partial x_j}(H_{\mathcal{F}} - F) + (H_{\mathcal{F}} - F)\langle \frac{\partial^2 \nu}{\partial x_i \partial x_j}, \nu\rangle - \langle \frac{\partial^2 X}{\partial X_i \partial X_j}, \frac{\partial \nu}{\partial t}\rangle.\end{aligned}$$

By (2.1) and (2.2)

$$\frac{\partial \nu}{\partial t} = g^{ij}\frac{\partial X}{\partial x_j}\frac{\partial}{\partial x_i}(H_{\mathcal{F}} - F) \text{ and}$$

$$\langle \frac{\partial^2 \nu}{\partial x_i \partial x_j}, \nu\rangle = -b_{ik}g^{k\ell}b_{\ell j}.$$

Consequently,

$$\frac{\partial b_{ij}}{\partial t} = \nabla_i \nabla_j (H_{\mathcal{F}} - F) - (H_{\mathcal{F}} - F) b_{ik} g^{k\ell} b_{\ell j}. \tag{2.9}$$

To compute $\nabla_i \nabla_j H_{\mathcal{F}}$ we look at

$$\begin{aligned} n\nabla_i \nabla_j H_{\mathcal{F}} =& (\nabla_i \nabla_j a^{k\ell}) b_{k\ell} + \nabla_i a^{k\ell} \nabla_j b_{\ell k} \\ &+ \nabla_j a^{k\ell} \nabla_i b_{\ell k} + a^{k\ell} \nabla_i \nabla_j b_{k\ell}. \end{aligned} \tag{2.10}$$

First, by using (2.1) and (2.2) and the Euler's identities

$$\frac{\partial^2 \mathcal{F}}{\partial y_i \partial y_j}\nu_i = 0, \quad j = 1, \cdots, n,$$

we derive

$$\nabla_j a_{k\ell} = -\frac{\partial^3 \mathcal{F}}{\partial y_p \partial y_q \partial y_r} b_{sj} \frac{\partial X^p}{\partial x_t} \frac{\partial X^q}{\partial x_k} \frac{\partial X^r}{\partial x_\ell} g^{ts}. \tag{2.11}$$

By Codazzi equation (2.4),

$$\begin{aligned} \nabla_i \nabla_j a^{k\ell} = &- g^{km} g^{\ell n} \frac{\partial^3 \mathcal{F}}{\partial y_p \partial y_q \partial y_r} \frac{\partial X^p}{\partial x_t} g^{ts} \frac{\partial X^q}{\partial x_m} \frac{\partial X^r}{\partial x_n} \nabla_s b_{ij} \\ &- g^{km} g^{\ell n} \nabla_i \left(\frac{\partial^3 \mathcal{F}}{\partial y_p \partial y_q \partial y_r} \frac{\partial X^q}{\partial x_t} g^{ts} \frac{\partial X^q}{\partial x_m} \frac{\partial X^r}{\partial x_n} \right) b_{sj}. \end{aligned} \tag{2.12}$$

On the other hand, by (2.3), (2.4) and the Ricci identities, we have

$$\begin{aligned} \nabla_i \nabla_j b_{\ell k} &= \nabla_\ell \nabla_k b_{ij} - b_{mk} R^m_{ji\ell} - b_{jm} R^m_{ki\ell} \\ &= \nabla_\ell \nabla_k b_{ij} - g^{mn} \left(b_{mk} b_{ni} b_{j\ell} - b_{mk} b_{n\ell} b_{ij} \right. \\ &\quad \left. - b_{jm} b_{ni} b_{k\ell} + b_{jm} b_{n\ell} b_{ki} \right). \end{aligned} \tag{2.13}$$

Finally, we have

$$\nabla_i \nabla_j F = \frac{\partial^2 F}{\partial X_k \partial X_\ell} \frac{\partial X^k}{\partial x_i} \frac{\partial X^\ell}{\partial x_j} - b_{ij} \frac{\partial F}{\partial X_k} \nu^k. \tag{2.14}$$

By putting (2.10)-(2.14) into (2.9) we see that b_{ij} satisfies an evolutive equation of the form

$$\begin{aligned} \frac{\partial b_{ij}}{\partial t} &= \frac{1}{n} a^{k\ell} \nabla_k \nabla_\ell b_{ij} + u^k \nabla_k b_{ij} + N_{ij} \\ &\quad - \frac{\partial^2 F}{\partial X_k \partial X_\ell} \frac{\partial X^k}{\partial x_i} \frac{\partial X^\ell}{\partial x_j}, \end{aligned} \tag{2.15}$$

where u is the vector field given locally by

$$u^k = -\frac{1}{n} g^{hm} g^{\ell m} \frac{\partial^3 \mathcal{F}}{\partial y_p \partial y_q \partial y_r} \frac{\partial X^p}{\partial x_t} g^{tk} \frac{\partial X^q}{\partial x_m} \frac{\partial X^r}{\partial x_n} b_{h\ell}$$

and N is a $\binom{2}{0}$-tensor given locally by

$$N_{ij} = -g^{km}g^{\ell n}\nabla_i\left(\frac{\partial^3 \mathcal{F}}{\partial y_p \partial y_q \partial y_r}\frac{\partial X^p}{\partial x_t}g^{ts}\frac{\partial X^q}{\partial x_m}\frac{\partial X^r}{\partial x_n}\right)b_{sj}b_{k\ell}$$

$$-g^{km}g^{\ell n}\frac{\partial^3 \mathcal{F}}{\partial y_p \partial Y_q \partial y_r}\frac{\partial X^p}{\partial x_t}g^{ts}\frac{\partial X^q}{\partial x_m}\frac{\partial X^r}{\partial x_n}(b_{si}\nabla_j b_{\ell k} + b_{sj}\nabla_i b_{\ell k})$$

$$-\frac{1}{n}a^{k\ell}g^{mn}(b_{mk}b_{ni}b_{j\ell} - b_{mk}b_{n\ell}b_{ij} - b_{jm}b_{ni}b_{k\ell} + b_{jm}n_{n\ell}b_{ki})$$

$$+ b_{ij}\frac{\partial F}{\partial X_k}\nu^k - (H_{\mathcal{F}} - F)b_{ik}g^{k\ell}b_{\ell j}.$$

Clearly N fulfills the following condition: $N_{ij}\xi^i\xi^j = 0$ whenever ξ is a tangent vector satisfying $b_{ij}\xi^j = 0, i = 1, \cdots, n$.

A parametric integrand $\mathcal{F}$ is called *elliptic* (*semi-elliptic*) if there exists a positive (non-negative) number λ such that

$$\sum_{i,j}^{n+1}\frac{\partial^2 \mathcal{F}(-\nu)}{\partial y_i \partial y_j}\eta^i\eta^j \geq \lambda|\eta'|^2, \quad \nu \in S^n, \tag{2.16}$$

for all $\eta = (\eta^1, \cdots, \eta^{n+1})$ in $\mathbb{R}^{n+1}$. Here $\eta' = \eta - \langle\eta, \nu\rangle\nu$. When $\nu(x)$ is the unit outer normal for a hypersurface $X(x), \eta'$ is the projection of η onto the tangent space of X at x. Notice that (2.16) implies $a_{ij} \geq \lambda g_{ij}$.

Applying Hamilton's maximum principle [H, Theorem 4.1] with some straightforward modifications, we arrive at

PROPOSITION 2. *Suppose in* (2.8) $\mathcal{F}$ *is semi-elliptic and* F *is concave. Let* $b_{ij}(t)$ *be the second fundamental tensor of* $X(\cdot, t)$. *Then, if* $b_{ij}(0), b_{ij}(t) \geq 0$ *for all* $t > 0$. *In other words, if* $X(\cdot, 0)$ *is convex,* $X(\cdot, t)$ *remains convex for all time* t.

3. - Absolute minima

In this section we prove Theorem C′ (a generalization of Theorem C in Section 1) and Theorem B.

We shall follow the notations in Section 2. Let $\mathcal{F}$ be a parametric integrand which satisfies

$$\inf\{\mathcal{F}(\nu) : \nu \in S^n\} \geq a > 0 \text{ and} \tag{3.1}$$

$$\sum_{i,j}^{n+1}\frac{\partial^2 \mathcal{F}(-\nu)}{\partial y_i \partial y_j}\eta^i\eta^j \geq \lambda|\eta'|^2, \quad \nu \in S^n, \tag{3.2}$$

for some a and $\lambda > 0$. We shall solve the evolutive equation

$$\frac{\partial X(x,t)}{\partial t} = -\left[H_{\mathcal{F}}(x,t) - F(X(x,t))\right]\nu(x,t), \qquad X(\cdot,0) \text{ given} \tag{3.3}$$

for $t > 0$. Here x is a local parameter of $X(\cdot,t)$ during some time interval, $H_{\mathcal{F}}(x,t)$ and $\nu(x,t)$ are the respective $\mathcal{F}$-mean curvature and unit outer normal of $X(\cdot,t)$ at the point $X(x,t)$.

Along $X(\cdot,t), I$ satisfies

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}I(X(\cdot,t)) &= -\int_{X(\cdot,t)} |X_t(\cdot,t)|^2 \\ &= -\int_{X(\cdot,t)} (H_{\mathcal{F}} - F)^2 \leq 0\end{aligned} \tag{3.4}$$

and

$$I(X\cdot,0) - I(X(\cdot,T)) = \int_0^T \int_{X(\cdot,t)} (H_{\mathcal{F}} - F)^2. \tag{3.5}$$

THEOREM C$'$. *Let $\mathcal{F}$ be a positive elliptic parametric integrand and let F be a concave function in $\mathbb{R}^{n+1}$ which is negative outside B_R for some $R > 0$. Then*

(a) *Any absolute minimum of I among convex hypersurfaces is a solution of*

$$H_{\mathcal{F}}(X) = F(X); \tag{3.6}$$

(b) *I has an absolute minimum if and only if there exists a convex hypersurface Y with $I(Y) \leq 0$.*

The main body of the proof of this theorem is to establish the global existence of (3.3) and to obtain a priori estimates of $X(\cdot,t)$ which depend on initial data through I.

To begin with we establish local existence. This can be accomplished by writing (3.3) into a single equation for the radial function of $X(\cdot,t)$.

Let p be a point in $\mathbb{R}^{n+1}$ enclosed in the interior of $X(\cdot,t)$ for $t \in Q = (t_1,t_2)$. We use p as the origin and introduce radial function $\rho(\hat{x},t) > 0, \hat{x} \in S^n$, where $\hat{x} = \hat{x}(x,t)$ is defined implicitly by $X(x,t) = \rho(\hat{x},t)\hat{x}$. Extend $\rho(\cdot,t)$ as a function of homogeneous degree zero to $\mathbb{R}^{n+1}\backslash\{0\}$. From

(3.3) we have

$$\sum_{j}^{n+1} \frac{\partial \rho}{\partial \hat{x}_j} \frac{\partial \hat{x}^j}{\partial t} \hat{x} + \frac{\partial \rho}{\partial t} \hat{x} + \rho \frac{\partial \hat{x}}{\partial t} = -(H_{\mathcal{F}} - F)\nu. \tag{3.7}$$

Taking inner product with $\hat{x}$,

$$\frac{\partial \rho}{\partial t} = -(H_{\mathcal{F}} - F)\langle \nu, \hat{x} \rangle - \sum_{j}^{n+1} \frac{\partial \rho}{\partial \hat{x}_j} \frac{\partial \hat{x}^j}{\partial t}. \tag{3.8}$$

Let $z_i (i = 1, \cdots, n)$ be an orthonormal basis of the tangent space of S^n at $\hat{x}$. From (3.7)

$$\rho \langle \frac{\partial \hat{x}}{\partial t}, z_i \rangle = -(H_{\mathcal{F}} - F)\langle \nu, z_i \rangle.$$

Using the formula

$$\nu = \frac{1}{\sqrt{\rho^2 + |\nabla \rho|^2}}(-\nabla \rho + \rho \hat{x}), \ \ \nabla \rho = \left(\frac{\partial \rho}{\partial x_1}, \cdots, \frac{\partial \rho}{\partial x_{n+1}} \right), \tag{3.9}$$

we have

$$\frac{\partial \hat{x}}{\partial t} = \frac{(H_{\mathcal{F}} - F)}{\rho} \frac{\nabla \rho}{\sqrt{\rho^2 + |\nabla \rho|^2}}.$$

Putting this into (3.8), we obtain

$$\frac{\partial \rho}{\partial t} = -\frac{\sqrt{\rho^2 + |\nabla \rho|^2}}{\rho}(H_{\mathcal{F}} - F). \tag{3.10}$$

Because for each $t \in Q$, $x \to \hat{x}(x,t)$ is one-to-one and smooth, we can use $\hat{x}$ as a global parameter of $X(\cdot, t)$. Thus in (3.10) ρ and $\nabla \rho$ are evaluated at $(\hat{x}, t)$; $H_{\mathcal{F}}$ and F at $X(\hat{x}, t) = \rho(\hat{x}, t)\hat{x}$.

Let e_{ij} be the standard metric on S^n with respect to a local coordinate (U, ϕ). Then (U, ψ_t), $\psi_t(x) = \rho(\phi(x), t)\phi(x)$, is a local coordinate for $X(\cdot, t), t \in Q$. Using the computations in Section 2 we deduce

$$g_{ij} = e_{ij} + \nabla_i \rho \nabla_j \rho,$$

$$g^{ij} = e^{ij} - 1 + |\nabla \rho|^2 e^{i\ell} \nabla_\ell \rho e^{mj} \nabla_m \rho, \tag{3.11}$$

$$\left[|\nabla \rho|^2 = e^{ij} \nabla_i \rho \nabla_j \rho = \sum_{j}^{n+1} \left(\frac{\partial \rho}{\partial x_j} \right)^2 \right] \text{ and}$$

$$b_{ij} = \frac{1}{\sqrt{\rho^2 + |\nabla \rho|^2}} \ (-\rho \nabla_i \nabla_j \rho + 2\nabla_i \rho \nabla_j \rho + \rho^2 e_{ij}). \tag{3.12}$$

Putting (3.11) and (3.12) into (3.10) and (2.5) we see that (3.10) is a quasilinear parabolic equation for ρ on S^n. By applying implicit function theorem in a standard way we obtain

LEMMA 1. *Suppose $\mathcal{F}$ satisfies* (3.2). *There exists $T > 0$ such that* (3.10) *has a unique solution in $[0, T]$ for any initial data $\rho(\cdot, 0) > 0$. Here T depends only on n, λ^{-1}, a positive lower bound on $\inf_x \rho(x, 0)$, $\|\rho(\cdot, 0)\|_{C^{2,\alpha}(S^n)}$, $\|\mathcal{F}\|_{C^{3,\alpha}(S^n)}$ and $\|F\|_{C^{1,\alpha}(\overline{B_R(0)})}$ for some $\alpha \in (0, 1)$.*

To obtain a priori estimates we follow [CNS] by representing $X(\cdot, t)$ locally as a family of graphs.

Let X be a convex hypersurface bounded between spheres $S_{r_1}(0)$ and $S_{r_2}(0), 0 < r_1 < r_2$, where $S_r(p) = \{x \in \mathbb{R}^{n+1} : |x - p| = r\}$. For $\hat{x} \in S^n$ denote $P(\hat{x})$ the hyperplane passing through the origin with normal $\hat{x}$. There exists a neighbourhood G of the origin in $P(\hat{x})$ over which X can be expressed locally as $\{(x, u(x)) : x \in G\}$ with $\rho(\hat{x})\hat{x} = (0, u(0))$ for a function u. By the inequality

$$\langle \nu(\hat{y}), \hat{y} \rangle \geq \frac{r_1}{r_2}, \quad \hat{y} \in S^n, \tag{3.13}$$

($\nu(\hat{y})$ is the unit outer normal at $\rho(\hat{y})\hat{y}$) $\rho(\hat{y}) \geq r_1$ G can always be chosen as a ball $B_r(0)$ in $P(\hat{x})$ with r depending solely on r_1 and r_2.

Let $X(\cdot, t)$ be a solution of (3.3) which is convex for each t in $Q = (t_1, t_2)$. Suppose for some p in $\mathbb{R}^{n+1}$ all $X(\cdot, t)$ are bounded between $S_{r_1}(p)$ and $S_{r_2}(p), r_1 < r_2$. Choose p as the origin. Over any hyperplane $P(\hat{x})$ each $X(\cdot, t)$ can be expressed as a function near the origin at $P(\hat{x})$. As a typical case we take $\hat{x}$ to be the north pole. Using (D_r, ϕ), where $D_r = \{x \in \mathbb{R}^{n+1} : x_{n+1} = 0, |x| < r\}$ and $\phi(x) = \left(x, \sqrt{1 - |x|^2}\right)$ as a local coordinate, we convert (3.10) (with (3.11) and (3.12)) into a quasilinear parabolic equation of the form

$$\frac{\partial u}{\partial t} = a^{ij}(x, u, \nabla u) u_{ij} + b(x, u\nabla u), \tag{3.14}$$

$(x, t) \in D_r \times Q, i, j = 1, \cdots, n$, (subscripts denote partial differentiations) by the relation $u^2 = \rho^2 - |x|^2$. One verifies that there exist constants μ and M such that

$$0 < \mu|\xi|^2 \leq a^{ij}\xi_i\xi_j \leq \frac{1}{\mu}|\xi|^2, \xi \neq 0 \text{ in } \mathbb{R}^n$$

and

$$|b(x, u\nabla u)| \leq M,$$

in $(x, t) \in D_r \times Q$. Here μ and M depend only on n, r_1^{-1}, r_2, λ, and an upper bound on $\sup\{|\nabla u(x, t)| : (x, t) \in D_r \times Q\}$. The dependence on the gradient can be removed due to the convexity of $X(\cdot,t)$. For, by putting (3.9) and (3.13) together,

$$|\nabla\rho(x,t)| \leq \frac{r_2}{r_1^2}, \quad (x,t) \in D_r \times Q,$$

which clearly implies an estimate on $|\nabla u|$ in terms of r_1^{-1} and r_2.

At this point we appeal to Theorem 6.1.1. in [LSU, p. 517] to infer:

For u satisfying (3.14), there are constants $\gamma \in (0,1)$ and $C > 0$ such that

$$\sup\left\{\frac{|u_i(p) - u_i(q)|}{\mathrm{d}(p,q)^\gamma} : p,q \in \overline{D}_{r'} \times [t_3,t_2], i = 1,\cdots,n\right\} \leq C \tag{3.15}$$

($\mathrm{d}(p,q)$ is the parabolic distance between p and q) for all $r', 0 < r' < r$. Here γ and C depend only on $n, \lambda^{-1}, r_1^{-1}, r_2.\|\mathcal{F}\|_{C^3(S^n)}, \|F\|_{C(\overline{B_{r_2}(p)})}, t_2-t_1, (t_3-t_1)^{-1}$ and $(r-r')^{-1}$.

Starting with (3.15) we can use Schauder-type estimates for linear parabolic equations to obtain higher order estimation. Since a fixed number of neighbourhoods of the form $\phi(D_r)$ covers we conclude:

LEMMA 2. *Let $X(\cdot,t)$ be a solution of* (3.3) *which is convex and is bounded between $S_{r_1}(p)$ and $S_{r_2}(p)$, $r_1 < r_2$, for some p in $\mathbb{R}^{n+1}$ for $t \in [t_0, t_0+2\ell]$, $\ell > 0$. For each (k,α), $k \geq 0, 0 < \alpha < 1$, there exists a constant C such that*

$$\|X(\cdot,t)\|_{C^{k+2,\alpha}(S^n)} \leq C, \ \ t \in [t_0+\ell, t_0+2\ell],$$

where C depends only on $n, k, \alpha, \lambda^{-1}, r_1^{-1}, r_2, \ell, \ell^{-1}$, $\|\mathcal{F}\|_{C^3(S^n)\cap C^{k,\alpha}(S^n)}$ and $\|F\|_{C^{k\alpha}(\overline{B_{r_2}(p)})}$.

Thus, it boils down to estimating r_1^{-1}, r_2 and ℓ^{-1}. In the following we shall denote $r_{in}(C)$ and $D(C)$ respectively the inradius (i.e., the radius of a largest ball that can be fit inside) and the diameter of a subset C in $\mathbb{R}^{n+1}$. Also we write dist $(C,0)$ for the distance from the origin to C.

LEMMA 3. *Let $\mathcal{F}$ and F be given as in Theorem C$'$. Suppose $X(\cdot,t)$ is a solution of* (3.3) *in $S^n \times [0,\infty)$ where $X(\cdot,0)$ is convex and $I(X(\cdot,0)) < 0$. There exist three positive numbers δ_1, δ_2, and d such that*

$$r_{in}(X(\cdot,t)) > \delta_1, \tag{3.16}$$

$$D(X(\cdot,t)) < \delta_2, \tag{3.17}$$

$$\text{dist}\ (X(\cdot,t),0) < d, \tag{3.18}$$

for $t \geq 0$.

PROOF. By Proposition 2.2, $X(\cdot, t)$ is convex for all $t > 0$. From (3.5)

$$0 > I(X(\cdot, 0)) \geq I(X(\cdot, t)) \geq - \int\limits_{\tilde{X}(\cdot, t) \cap B_R} F.$$

Clearly $r_{in}(X(\cdot, t))$ cannot be too small, otherwise the last term would tend to zero. Therefore, (3.16) is valid for some $\delta_1 > 0$. On the other hand, if $D(X(\cdot, t))$ becomes unbounded, because of convexity and (3.16), the volume of $X(\cdot, t)$, $|X(\cdot, t)|$, becomes unbounded too. However, from

$$0 > I(X(\cdot, t)) \geq a|X(\cdot, t)| - \int\limits_{B_R} F$$

(a is given in (3.1)), we obtain a contradiction. (3.17) holds.

Finally, (3.18) follows by combining (3.17) and the assumption that F is negative outside B_R. *q.e.d.*

LEMMA 4. *Let C be an open convex set in $\mathbb{R}^{n+1}$ with centroid c. The ball with radius $\frac{2}{n+2} r_{in}(C)$ centred at c is contained in C.*

PROOF. Let $h_1(x)$ and $h_2(x)$ be the respective distance from c to the support hyperplane of C in the direction x and $-x$. By [P, p. 90]

$$\frac{h_1(x)}{n+1} \leq h_2(x) \leq (n+1) h_1(x).$$

Therefore, assuming $h_1(x) \leq h_2(x)$,

$$\begin{aligned} r_{in}(C) &\leq \inf_{x \in S^n} \frac{1}{2} [h_1(x) + h_2(x)] \\ &\leq \inf_{x \in S^n} \frac{n+2}{2} h_1(x) \leq \frac{n+2}{2} \operatorname{dist}(c, \partial C) \quad q.e.d. \end{aligned}$$

LEMMA 5. *Let $\mathcal{F}$ and F be given as in Theorem C′. Suppose $X(\cdot, t)$ is a solution of (3.3) which is convex for all $t \geq 0$. let $c(t)$ be the centroid of $X(\cdot, t)$. There exists $\ell > 0$ such that $X(\cdot, t)$ is bounded between $S_{\delta_3}(c(t_0))$ and $S_{2\delta_2}(c(t_0))$, ($\delta_3 = \frac{1}{n+2}\delta_1, \delta_1$ and δ_2 are given in Lemma 3), for t in $(t_0, t_0 + \ell)$ for any $t_0 \geq 0$. ℓ depends only on $n.\delta_1^{-1}, \delta_2, d$, and $\kappa = I(X(\cdot, 0)) - \inf\{I(X) : X$ is a convex hypersurface in $\mathbb{R}^{n+1}\}$.*

PROOF. In the following proof C_1, C_2, and C are constants which depend only on n, δ_1^{-1}, δ_2, and d. Using polar coordinate,

$$c(t) = \left(\int\limits_{\tilde{X}(\cdot, t)} x \right) \left(\int\limits_{\tilde{X}(\cdot, t)} 1 \right)^{-1} = \frac{n+1}{n+2} \left(\int\limits_{S^n} \rho^{n+2} \hat{x} \right) \left(\int\limits_{S^n} \rho^{n+1} \right)^{-1}.$$

Therefore,

$$\frac{\mathrm{d}c}{\mathrm{d}t} = (n+1)\left(\int_{S^n} \rho^{n+1}\rho_t\hat{x} - c(t)\int_{S^n} \rho^n \rho_t\right)\left(\int_{S^n} \rho^{n+1}\right)^{-1}.$$

Using (3.16)-(3.18) and (3.10), we have

$$\left|\frac{\mathrm{d}c}{\mathrm{d}t}\right| \le C_1 \int_{S^n} |\rho_t| \le C_2 \int_X |H_{\mathcal{F}} - F|.$$

By Cauchy inequality, for $t \ge t_0$,

$$\begin{aligned} |c(t) - c(t_0)| &\le C_2 \int_{t_0}^{t}\int_X |H_{\mathcal{F}} - F| \\ &\le C\left[\varepsilon \int_{t_0}^{t}\int_X |H_{\mathcal{F}} - F|^2 + \frac{1}{4\varepsilon}(t-t_0)\right] \\ &\le C\left[\varepsilon\kappa + \frac{1}{4\varepsilon}(t-t_0)\right]. \end{aligned}$$

The last line follows from (3.5). As a result, if we first choose ε satisfying $C\varepsilon\kappa = \frac{1}{2}\delta_3$ anf then determine ℓ from $\frac{C}{4\varepsilon}\ell = \frac{1}{2}\delta_3$, we have

$$|c(t) - c(t_0)| \le \delta_3.$$

Now Lemma 5 follows from Lemma 4. *q.e.d.*

PROOF OF THEOREM C$'$.

(i) Let X be a convex hypersurface which minimizes I among all convex hypersurfaces. Solve (3.3) by using X as the initial value. By Lemma 1 and Proposition 2.2, (3.3) has a solution $X(\cdot, t)$, which is convex for each small $t > 0$. However, X is already an absolute minimum. It follows from (3.5) that $X(\cdot, t) = X(\cdot)$ and it solves (3.6).

(ii) Let $\{S_k\}$ be a sequence of concentric spheres with radius decreasing to zero. Clearly $\lim\limits_{k\to\infty} I(S_k) = 0$.

Therefore, no absolute minima exist if I is positive for any convex hypersurfaces.

Conversely, for a convex $X(\cdot, 0)$ with $I(X(\cdot, 0)) < 0$, we apply Lemma 1 to show that (3.3) has a unique solution for some $t > 0$. From Proposition 2.2 each $X(\cdot, t)$ is convex. Hence we may use $c(0)$ as the origin, introduce the radial function of $X(\cdot, t)$, and use Lemmas 5 and 2, coupled with a standard argument, to conclude that a unique $X(\cdot,t)$ exists in an interval $[0,\alpha)$ with $\alpha > \ell$. Replace $c(0)$ by $c(\ell)$ as the new origin and repeat the same argument. (3.3) is uniquely solvable in $[0,\beta)$ with $\beta > 2\ell$. Keeping this way we eventually establish the solvability of (3.3) for all t.

Since I is bounded below, the left hand side of (3.5) (taking $T = \infty$) is finite. We can find a sequence $\{t_j\}$,

$$\lim_{t_j \to \infty} \int_{X(\cdot, t_j)} (H_{\mathcal{F}} - F)^2 = 0.$$

Applying Lemmas 5 and 2 to each interval $[t_j - \ell, t_j]$ and using (3.18), we conclude that $\{X(\cdot,t_j)\}$ is a uniformly bounded sequence in $C^{k,\alpha}(S^n)$ for each $k \geq 0, 0 < \alpha < 1$. Consequently it contains a subsequence which converges to a smooth solution of (3.6).

Finally, to show that an absolute minimum exists, we solve (3.3) with initial values a minimizing sequence of I. As just it has been shown, in this way we obtain a new minimizing sequence all of which are solutions of (3.6). Since they are uniformly bounded in $C^{k,\alpha}(S^n)$ for each $k \geq 0$ and $0 < \alpha < 1$, we can extract a convergent subsequence which tends to an absolute minimum of I. The proof of Theorem C$'$ is completed.

Now we turn to Theorem B. So we let $\mathcal{F} = 1$ (on S^n) in I and consider the flow (1.3). We shall prove Theorem B under a slightly restrictive condition (b)$'$:

$$\text{(b)}' \qquad \begin{aligned} F(x) &> \frac{1}{R_1}, |x| = R_1; \\ F(x) &< \frac{1}{R_2}, |x| = R_2. \end{aligned}$$

The general case be deduced by an approximation argument.

LEMMA 6. *Let $X(\cdot,t)$ be a solution of* (1.3) *with F satisfying* (b)$'$. *If $R_1 < \rho(x,0) < R_2, R_1 < \rho(x,t) < R_2$ for all $t > 0$. Here $\rho(x,t)$ is the radial function of $X(\cdot,t)$ with respect to the origin.*

PROOF. Suppose, for some $t, X(\cdot,t)$ touches $S_{R_1}(0)$ (or $S_{R_2}(0)$ - the proof is similar). Let (p,t^*) be a point of first contact. Without loss of generality we take $p = \rho(\hat{x},t^*)\hat{x}, \hat{x}$ being the north pole. Then (3.10) holds in $(x,t) \in D_r \times Q$, where $r > 0$ and Q is an open interval containing t^*. (Notice in (3.10) $H_{\mathcal{F}}$ should be replaced by H). Similarly, the radial function of $S_{R_1}(0), \rho^0$, satisfies

$$\text{(3.10)}' \qquad \frac{\partial \rho^0}{\partial t} = -\frac{\sqrt{\rho^{0^2} + |\nabla \rho^0|^2}}{\rho^0}\left(H^0 - \frac{1}{R_1}\right).$$

If we subtract (3.10) from (3.10)$'$ and use the fact that at (p,t^*), $\rho - \rho^0 = 0, \nabla_i(\rho - \rho^0) = 0, \frac{\partial}{\partial t}(\rho - \rho^0) \leq 0$ and $\nabla_i\nabla_j(\rho - \rho_0) \geq 0, i \cdot j = 1, \cdots, n$, we obtain

$$0 \leq F - \frac{1}{R_1} \text{ at } (p, t^*).$$

Contraction holds. Lemma 6 is established. *q.e.d.*

PROOF OF THEOREM B.

By Lemma 6 one can introduce the radial function with respect to the origin for all time. Then the global solvability of (3.3) follows from the global solvability of (3.10). Following similar lines as in the proof of Theorem C$'$ we can finish the proof of Theorem B.

4. - Mini-maxima

In this section we prove Theorem D$'$, a sharpened form of Theorem D.

THEOREM D$'$. *Let Γ be a subgroup of $\mathcal{O}(n+1)$ such that $\{xg : g \in \Gamma\}$ spans $\mathbb{R}^{n+1}$ for some vector x. Let $\mathcal{F}$ be a Γ-invariant parametric integrand satisfying* (3.1) *and* (3.2) *and F be a Γ-invariant concave function in $\mathbb{R}^{n+1}$ which is negative outside B_R for some $R > 0$.*

There exists positive constants $\mu = \mu(\mathcal{F}, \Gamma)$ and $r_0 = r_0(\mathcal{F}, \Gamma, F)$ such that

$$I(X) \geq \mu D^n(X)$$

for $D(X) \leq r_0$. Therefore, if there exists a Γ-symmetric convex hypersurface Y with $I(Y) < \mu r_0^n, D(Y) > r_0$ there exists a solution Z of (3.6) *with $I(Z) \geq \mu r_0^n$.*

PROOF. First we observe that for any Γ-symmetric convex X there exists $\beta = \beta(\Gamma) > 0$ such that

$$r_{in}(\tilde{X}) \geq \beta D(\tilde{X}). \tag{4.1}$$

Let $b = \max\{F(x) : x \in \mathbb{R}^{n+1}\}$ and ω_{n+1} be the volume of the unit ball in $\mathbb{R}^{n+1}$. By (4.1)

$$\begin{aligned} I(X) &\geq a\tau_n r_{in}^n(\tilde{X}) - \omega_{n+1} b D^{n+1}(\tilde{X}) \\ &\geq a\tau_n \beta^n D^n(\tilde{X}) - \omega_{n+1} b D^{n+1}(\tilde{X}). \end{aligned}$$

If we choose $\mu = \frac{1}{2} a\tau_n \beta^n$, for a sufficient small $r_0 > 0$.

$$I(X) \geq \mu D^n(\tilde{X}) \text{ for } D(\tilde{X}) \leq r_0.$$

Let $\mathcal{C}$ be the class of all continuous curves γ from $[0, 1]$ to the Frèchet space of all Γ-symmetric convex hypersurfaces such that $\gamma(0) = S_\rho, 2\rho < r_0, S_\rho$

satisfying $I(S_\rho) < \mu r_0^n$, and $\gamma(1) = Y$. Set $c = \inf_{\gamma \in C} \max_s I(\gamma(s))$. Clearly $c \geq \mu r_0^n$. We claim there exists a solution Z of (3.6) with $I(Z) = c$.

For $\epsilon > 0, c - \epsilon > 0$, we pick $\gamma \in \mathcal{C}$ such that

$$\max_s I(\gamma(s)) \leq c + \epsilon.$$

Solve (3.3) with initial curve $\gamma(s)$ to obtain a family of solutions $\gamma(t,s)$. By (4.1) and Lemma 2, $\gamma(t,s)$ ceases to exist only when both the inradius and diameter of $\gamma(t,s)$ tend to zero. But $I(\gamma(t,s))$ tends to zero too. Let $t^*(s) = \inf\{t : I(\gamma(t,s)) \leq c - \epsilon\}, 0 \leq t^*(s) \leq \infty$. The map $s \to t^*(s)$ cannot be continuous, otherwise $\gamma^*(s) = \gamma(t^*(s), s)$ belongs to $\mathcal{C}$ and yet $I(\gamma^*(s)) \leq c - \epsilon$, a contradiction with the definition of c. Let s_0 be a point of discontinuity of t^*. There are two possibilities: (a) $t^*(s_0) = \infty$ or (b) $t^*(s_0) < \infty$ and there exists $\{s_j\}, s_j \to s_0$, and $t^*(s_j) \to t_1 > t^*(s_0)$, ($t_1$ could be infinity). In case (a), write $X(t) = \gamma(t, s_0)$. $X(t)$ exists for all time. Consequently, from

$$2\epsilon \geq \int_0^\infty \int_{X(t)} (H_{\mathcal{F}} - F)^2,$$

we can extract a sequence $\{X^j\}, X^j = X(\cdot, t_j)$, which by Lemma 2 converges to a solution of (3.6). In case (b), we claim $\gamma(t^*(s_0), s_0)$ is in fact a solution of (3.6). For, if $\frac{\mathrm{d}}{\mathrm{d}t} I(\gamma(t, s_0))$ does not vanish at $t = t^*(s_0)$, for t_2 satisfying $t_1 > t_2 > t^*(s_0)$, we have $I(\gamma(t_2, s_0)) < c - \epsilon$. By continuity, $I(\gamma(t_2, s_j)) < c - \epsilon$ for large j. But this implies $t^*(s_j) \leq t_2$. Contradiction holds.

Thus, we have proved in both cases, for each ϵ satisfying $c - \epsilon > 0$, there exists $X(\epsilon)$ solving (3.6) with $|I(X(\epsilon)) - c| \leq \epsilon$. Letting $\epsilon \downarrow 0$, by (4.1) and Lemma 2, a subsequence of $X(\epsilon)$ converges smoothly to a solution Z of (3.6) with $I(Z) = c$. *q.e.d.*

An Example

Let $F(x) = f(r), r = |x|$, be a radial concave function which intersects the curve $r \to r^{-1}$ at two points r_1 and $r_2, r_1 < r_2$. We claim that $S_{r_1}(0)$ is the local maximum and $S_{r_2}(0)$ is the local minimum among all spheres $S_r(0), 0 < r < \infty$. For

$$I(S_r) = \frac{1}{n}\tau_n r^n - \tau_n \int_0^r f(r) r^n \mathrm{d}r,$$

at $r = r_i, i = 1, 2$,

$$I'(S_r) = \tau_n r^{n-1} - \tau_n f(r) r^n = 0$$

and

$$\begin{aligned} I''(S_r) &= (n-1)\tau_n r^{n-2} - \tau_n f'(r) r^n - \tau_n n f(r) r^{n-1} \\ &= -\tau_n r^{n-2} - \tau_n f'(r) r^n. \end{aligned}$$

By concavity, $I''(S_{r_2}) > 0$ and $I''(S_{r_1}) < 0$. Notice S_{r_2} is not necessary an absolute minimum - if r_2 is very close to $r_1, I(S_{r_2})$ becomes positive.

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