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HITOSHI ISHII

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A Boundary Value Problem of the Dirichlet Type for Hamilton - Jacobi Equations

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0. - Introduction

In this paper we consider the boundary value problem for Hamilton - Jacobi equations:

$$\text{(HJ)} \qquad H(x, u, \mathrm{D}u) = 0 \qquad \text{in } \Omega$$

and

$$\text{(BC)} \qquad u = h \qquad \text{or} \qquad H(x, u, \mathrm{D}u) = 0 \qquad \text{on } \partial\Omega.$$

Here Ω is an open subset of $\mathbb{R}^N, H : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ and $h : \partial\Omega \to \mathbb{R}$ are given functions, $u : \overline{\Omega} \to \mathbb{R}$ is the unknown and $\mathrm{D}u$ denotes the gradient of u.

The meaning of (solution of) problem (HJ) - (BC) will be made precise in Section 1 by modifying the notion of viscosity solution, introduced originally by M. G. Crandall - P.L. Lions [7], so that the boundary condition is appropriately taken into account. There are already several attempts of similar modifications of the notion of viscosity solution. For this we refer to R. Jensen [15], P.L. Lions [18], M.G. Crandall - R. Newcomb [8], H.M. Soner [20], P.E. Souganidis [21], I. Capuzzo Dolcetta - P.L. Lions [4] and G. Barles - B. Perthame [1]. Our introducing the boundary condition (BC) is strongly motivated by the analogy to the Neumann condition for Hamilton - Jacobi equations in P.L. Lions [18].

Problem (HJ) - (BC) is important in deterministic optimal control theory. The value functions of exit time problems satisfy (HJ) - (BC), where the first order PDE in (HJ) is the so-called Bellman equation, in the viscosity sense, and they are characterized to be viscosity solutions of (HJ) - (BC), under appropriate hypotheses. We prove these in Sections 3 and 5 where, denoting $H(x, u, p) - u$ again by $H(x, u, p)$, we treat the problem:

$$\text{(HJ)}' \qquad u + H(x, u, \mathrm{D}u) = 0 \qquad \text{in } \Omega$$

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and

$$\text{(BC)}' \qquad u = h \quad \text{or} \quad u + H(x, u, \mathrm{D}u) = 0 \qquad \text{on } \partial\Omega.$$

The boundary problem for (HJ), with the Dirichlet condition $u = h$, is not solvable in general. A compatibility condition on h is required to be satisfied in order that the Dirichlet problem for (HJ) is solvable (see P.L. Lions [16]), and the condition on h is usually hard to check. See also the recent work [9] by H. Engler. On the other hand, problem (HJ) - (BC) is solvable, in a weak sense, under quite general hypotheses (see Proposition 1.3 in Section 1). Moreover the uniqueness and existence of a continuous viscosity solution of (HJ) - (BC) is established under suitable assumptions in Sections 2 and 4. Therefore problem (HJ) - (BC) seems, at least for the author, a natural replacement of the Dirichlet problem for (HJ). Problem (HJ) - (BC) also arises naturally in connection with elliptic singular perturbation problems with the Dirichlet boundary condition or with the vanishing viscosity method. In this direction see Proposition 1.2 and refer to H. Ishii - S. Koike [14] for an application of our results to elliptic singular perturbation problems. Finally we remark that G. Barles - B. Perthame [2] and P.L. Lions [19] have recently treated (HJ) - (BC) independently.

1. - Viscosity solutions

In this section we define the viscosity solution of the boundary problem for a second order PDE:

$$(1.1) \qquad F(x, u, \mathrm{D}u, \mathrm{D}^2 u) = 0 \qquad \text{in } \Omega$$

and

$$(1.2) \qquad B(x, u, \mathrm{D}u) = 0 \quad \text{or} \quad F(x, u, \mathrm{D}u, \mathrm{D}^2 u) = 0 \quad \text{on} \quad \Sigma,$$

and then present some of its properties. Here Ω is an open subset of $\mathbb{R}^N, \Sigma$ is a subset of the boundary $\partial\Omega, F : \Omega \cup \Sigma \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{M}^N \to \mathbb{R}$, where $\mathbb{M}^N$ denotes the space of real $N \times N$ matrices, and $B : \Sigma \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ are given functions, $u : \Omega \cup \Sigma \to \mathbb{R}$ is the unknown, and $\mathrm{D}^2 u$ denotes the Hessian matrix of u. When F is independent of the last variables, i.e. $F(x, r, p, \xi) = F(x, r, p)$, and $B = F$, problem (1.1) - (1.2) will be simply written as

$$F(x, u, \mathrm{D}u) = 0 \qquad \text{on } \Omega \cup \Sigma.$$

Our main interest here is to study problem (HJ) - (BC). However, it seems natural to see the connection between viscosity solutions of (HJ) - (BC) and the vanishing viscosity method in light of the stability property of viscosity solutions of (1.1) - (1.2).

Let S, T and U be subsets of $\mathbb{R}^N$ satisfying $S \subset U$ and $S \subset T \subset \overline{S}$. For a function $f : U \to \mathbb{R} \cup \{-\infty, \infty\}$, we define

$$(f|S,T)^* : T \to \mathbb{R} \cup \{-\infty, \infty\}$$

by

$$(f|S,T)^*(x) = \lim_{r \downarrow 0} \ \sup \ \{f(y) : y \in S, |y-x| \leq r\}.$$

This function $(f|S,T)^*$ is upper semicontinuous (u.s.c. in short) on T, and we call it the *u.s.c. envelope of f restricted to S on T*. We note that $(f|S,T)^*(x) \geq f(x)$ for $x \in S$ and that, if f is u.s.c. on S, then $(f|S,T)^*(x) = f(x)$ for $x \in S$. We remark that, if $(f|S,T)^*(x) < \infty$ for $x \in T$, then

$$(f|S,T)^*(x) = \ \inf \ \{g(x) : g \in C(T), g \geq f \text{ on } S\}.$$

We define the *lower semicontinuous* (l.s.c. in short) *envelope*

$$(f|S,T)_* : T \to \mathbb{R} \cup \{-\infty, \infty\}$$

of f restricted to S on T by $(f|S,T)_* = -(-f|S,T)^*$. When $S = T = U$, we write f^* and f_*, respectively, for $(f|S,T)^*$ and $(f|S,T)_*$.

We call a function $u : \Omega \cup \Sigma \to \mathbb{R} \cup \{-\infty\}$ a *viscosity subsolution* of problem (1.1) - (1.2) if $u^*(x) < \infty$, for $x \in \Omega \cup \Sigma$, and whenever $\varphi \in C^2(\Omega \cup \Sigma)$ and $u^* - \varphi$ attains its local maximum at a point $y \in \Omega \cup \Sigma$, then

$$F_*(y, u^*(y), \mathrm{D}\varphi(y), \mathrm{D}^2\varphi(y)) \leq 0 \quad \text{if } y \in \Omega,$$

and

$$B_*(y, u^*(y), \mathrm{D}\varphi(y)) \leq 0 \quad \text{or} \quad F_*(y, u^*(y), \mathrm{D}\varphi(y), \mathrm{D}^2\varphi(y)) \leq 0 \quad \text{if} \quad y \in \Sigma.$$

Similarly, we call a function $u : \Omega \cup \Sigma \to \mathbb{R} \cup \{\infty\}$ a *viscosity supersolution* of (1.1) - (1.2) if $u_*(x) > -\infty$, for $x \in \Omega \cup \Sigma$, and whenever $\varphi \in C^2(\Omega \cup \Sigma)$ and $u_* - \varphi$ attains its local minimun at a point $y \in \Omega \cup \Sigma$, then

$$F^*(y, u_*(y), \mathrm{D}\varphi(y), \mathrm{D}^2\varphi(y)) \geq 0 \quad \text{if} \quad y \in \Omega,$$

and

$$B^*(y, u_*(y), \mathrm{D}\varphi(y)) \geq 0 \text{ or } F^*(y, u_*(y), \mathrm{D}\varphi(y), \mathrm{D}^2\varphi(y)) \geq 0 \text{ if } y \in \Sigma.$$

A *viscosity solution* of (1.1) - (1.2) is defined to be a function on $\Omega \cup \Sigma$ which is both a viscosity sub- and supersolution of (1.1) - (1.2). Note that this definition makes sense even for functions F, B which take values in $\mathbb{R} \cup \{-\infty, \infty\}$, and that if $F(x, r, p, \xi)$ is independent of ξ, then the space of "test functions", $C^2(\Omega \cup \Sigma)$, can be replaced by the space $C^1(\Omega \cup \Sigma)$. We say also that u satisfies $F(x, u, \mathrm{D}u, \mathrm{D}^2u) \le 0$ in Ω and $B(x, u, \mathrm{D}u) \le 0$ or $F(x, u, \mathrm{D}u, \mathrm{D}^2u) \le 0$ on Σ (resp., $F(x, u, \mathrm{D}u, \mathrm{D}^2u) \ge 0$ in Ω and $B(x, u, \mathrm{D}u) \ge 0$ or $F(x, u, \mathrm{D}u, \mathrm{D}^2u) \ge 0$ on Σ), in the viscosity sense, if u is a viscosity subsolution (resp., supersolution) of (1.1) - (1.2).

Of course, this definition is a modification of the original one introduced by M.G. Crandall - P.L. Lions [7] and by P.L. Lions [17] to the boundary value problem. The boundary condition (1.2) is motivated by the work [18] by P.L. Lions concerning the Neumann problem for Hamilton - Jacobi equations. In our notation, the Neumann condition for (HJ), in [18], can be written as

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{or} \quad H(x, u, \mathrm{D}u) = 0 \quad \text{on } \partial\Omega,$$

where ν denotes the outward unit normal of $\partial\Omega$ (assuming it exists). We should also remark that G. Barles - B. Perthame [2] and P.L. Lions [19] have independently introduced the same notion of viscosity solutions for (HJ) - (BC) as ours. We refer to H.M. Soner [20], I. Capuzzo Dolcetta - P.L. Lions [4] for another type of the boundary condition for Hamilton - Jacobi equations and to M.G. Crandall - H. Ishii - P.L. Lions [6], H. Ishii [10,12,13] and G. Barles - B. Perthame [1] for the work related to discontinuous viscosity solutions and Hamiltonians.

PROPOSITION 1.1. *Let Σ be an open subset of $\partial\Omega$. Let $\mathcal{S}$ be a family of viscosity subsolutions of* (1.1) - (1.2). *Set $u(x) = \sup\,\{v(x) : v \in \mathcal{S}\}$, for $x \in \Omega \cup \Sigma$. Assume u is locally bounded from above on $\Omega \cup \Sigma$. Then u is a viscosity subsolution of* (1.1) - (1.2).

This is a generalization of M.G. Crandall - L.C. Evans - P.L. Lions [5, Prop. 1.4] and H. Ishii [13, Prop. 2.4]. The proof of this proposition is similar to that of [13, Prop. 2.4], and we omit giving it here. An analogous assertion holds for supersolutions. To see this, observe that u is a viscosity subsolution of (1.1) - (1.2) if and only if $v = -u$ is a viscosity supersolution of (1.1) - (1.2), with F and B replaced, respectively, by the functions

$$(x, r, p, \xi) \to -F(x, -r, -p, -\xi) \quad \text{and} \quad (x, r, p) \to -B(x, -r, -p).$$

PROPOSITION 1.2. *(Stability of viscosity solutions). Let Σ be an open subset of $\partial\Omega$. For $0 < \varepsilon < 1$, let*

$$u_\varepsilon : \Omega \cup \Sigma \to \mathbb{R} \cup \{-\infty\},$$

$$F_\varepsilon : \Omega \cup \Sigma \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{M}^N \to \mathbb{R} \cup \{\infty\},$$

$$B_\varepsilon : \Sigma \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R} \cup \{\infty\}$$

be given functions. Assume that, for each $0 < \varepsilon < 1$, u_ε *is a viscosity subsolution of* (1.1) - (1.2), *with* F_ε *and* B_ε *in place of* F *and* B, *respectively. Set*

$$u(x) = \lim_{\delta \downarrow 0} \ \sup \ \{u_\varepsilon(y) : 0 < \varepsilon < \delta, y \in \Omega \cup \Sigma, |y - x| < \delta\},$$

$$\begin{aligned} F(x, r, p, \xi) = \lim_{\delta \downarrow 0} \ \inf \ \{&F_\varepsilon(y, s, q, \eta) : 0 < \varepsilon < \delta, \\ &(y, s, q, \eta) \in \Omega \cup \Sigma \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{M}^N, \\ &|y - x| < \delta, |s - r| < \delta, |q - p| < \delta, |\eta - \xi| < \delta\}, \end{aligned}$$

for $(x, r, p, \xi) \in \Omega \cup \Sigma \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{M}^N$, *and*

$$\begin{aligned} B(x, r, p,) = \lim_{\delta \downarrow 0} \ \inf \ \{&B_\varepsilon(y, s, q) : 0 < \varepsilon < \delta, (y, s, q) \in \Sigma \times \mathbb{R} \times \mathbb{R}^N, \\ &|y - x| < \delta, |s - r| < \delta, |q - p| < \delta\}, \end{aligned}$$

for $(x, r, p) \in \Sigma \times \mathbb{R} \times \mathbb{R}^N$. *Assume* u *is locally bounded from above on* $\Omega \cup \Sigma$. *Then* u *is a viscosity subsolution of* (1.1) - (1.2) *with these* F *and* B.

This generalizes a result of G. Barles - B. Perthame [1, Theorem A.2]. Of course, we have a proposition similar to the above for supersolutions. We can prove this proposition as in the proof of [1, Theorem A.2], and we do not give the details here.

The following observation is useful in applications of Proposition 1.2 to elliptic perturbation problems or the vanishing viscosity method.

Assume that $F : \Omega \cup \Sigma \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{M}^N \to \mathbb{R}$ and $B : \Sigma \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ are continuous and that $u \in C^1(\Omega \cup \Sigma) \cap C^2(\Omega)$ satisfies $F(x, u, \mathrm{D}u, \mathrm{D}^2 u) \le 0$ in Ω and $B(x, u, \mathrm{D}u) \le 0$ on Σ, in the classical sense. Assume, in addition, that $-F$ is elliptic, i.e.

$$F(x, r, p, \xi + \mathrm{D}^2\varphi(x)) \ge F(x, r, p, \xi),$$

for $(x, r, p, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{M}^N$ and $\varphi \in C^2(\Omega)$ such that φ attains its maximum at x, and that B satisfies the condition: $B(x, r, p) + \mathrm{D}\varphi(x)) \ge B(x, r, p)$, for $(x, r, p) \in \Sigma \times \mathbb{R} \times \mathbb{R}^N$ and $\varphi \in C^1(\Omega \cup \Sigma)$ such that φ attains its maximum at x. Then u is a viscosity subsolution of (1.1) - (1.2). From Proposition 1.2 we see, for instance, the following. Let $H : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ and $h : \partial\Omega \to \mathbb{R}$ be continuous. For each $0 < \varepsilon < 1$, let $u_\varepsilon \in C(\overline{\Omega}) \cap C^2(\Omega)$ satisfy - $\varepsilon \Delta u_\varepsilon + H(x, u_\varepsilon, \mathrm{D}u_\varepsilon) \le 0$ in Ω, where Δ denotes the N dimensional Laplacian, and $u_\varepsilon \le h$ on $\partial\Omega$, in the classical sense. Set $u(x) = \lim_{r \downarrow 0} \sup \{u_\varepsilon(y) : 0 < \varepsilon < r, |y - x| < r\}$, and assume $u(x) < \infty$, for $x \in \overline{\Omega}$. Then u is a viscosity subsolution of (HJ) - (BC).

PROPOSITION 1.3. *(Existence of viscosity solutions). Let* Σ *be an open subset of* $\partial\Omega$. *Let* f *and* g *be, respectively, viscosity sub- and supersolutions of*

(1.1) - (1.2). *Assume* $f \le g$ *on* $\Omega \cup \Sigma$ *and that* f *and* g *are locally bounded on* $\Omega \cup \Sigma$. *Define* $u : \Omega \cup \Sigma \to \Sigma$ *by*

$$u(x) = \sup \{v(x) : v \text{ a viscosity subsolution of } (1.1) - (1.2),\ v \le g \text{ on } \Omega \cup \Sigma\}.$$

Then u *is a viscosity solution of* (1.1) - (1.2).

This extends Theorem 3.1 of H. Ishii [13]. We leave it to the reader to prove this proposition as the proof is a simple modification of that of [13, Theorem 3.1].

2. - Comparison of viscosity solutions

Hereafter we study problem (HJ) - (BC). We need the following assumptions on H and Ω.

(H1) For each $(x,p) \in \overline{\Omega} \times \mathbb{R}^N$, the function $u \to H(x,u,p)$ is nondecreasing on $\mathbb{R}$.

(H2) There is a continuous nondecreasing function $m_1 : [0,\infty) \to [0,\infty)$, satisfying $m_1(0) = 0$, such that

$$|H(x,u,p) - H(y,u,p)| \le m_1(|x-y|(|p|+1)),$$

for $x, y \in \overline{\Omega}, u \in \mathbb{R}$ and $p \in \mathbb{R}^N$.

(H3) There is a continuous nondecreasing function $m_2 : [0,\infty) \to [0,\infty)$, satisfying $m_2(0) = 0$, such that

$$|H(x,u,p) - H(x,u,q)| \le m_2(|p-q|),$$

for $x \in \overline{\Omega}, u \in \mathbb{R}$ and $p, q \in \mathbb{R}^N$.

(H4) There is a continuous function $\eta : \overline{\Omega} \to \mathbb{R}^N$ and a constant $b > 0$ such that

$$B(x + t\eta(x), bt) \subset \Omega, \quad \text{for} \quad x \in \overline{\Omega} \quad \text{and} \quad 0 < t \le b,$$

(where $B(x,\delta)$ is the usual sphere centred at x with radius δ).

THEOREM 2.1. *Assume* Ω *is bounded and that* (H1) - (H4) *hold. Let* u *and* v *satisfy, respectively,*

$$(2.1) \qquad \begin{cases} H(x,u,\mathrm{D}u) \le -a & \text{in } \Omega \\ u \le h \text{ or } H(x,u,+\mathrm{D}u) \le -a & \text{on } \partial\Omega, \end{cases}$$

where $a > 0$ *is a constant, and*

$$(2.2) \qquad \begin{cases} H(x,v,\mathrm{D}v) \ge 0 & \text{in } \Omega \\ v \ge h \text{ or } H(x,v,\mathrm{D}v) \ge 0 & \text{on } \partial\Omega, \end{cases}$$

in the viscosity sense. Then:

(i) *If u and v are continuous at points of $\partial\Omega$ and h is continuous on $\partial\Omega$, then $u \le v$ on $\overline{\Omega}$.*

(ii) *If u (resp., v) is continuous at points of $\partial\Omega$, h is l.s.c. (resp., u.s.c.) on $\partial\Omega$ and $u \le h$ (resp., $v \ge h$) holds on $\partial\Omega$, then $u \le v$ on $\overline{\Omega}$.*

REMARK 2.1. As the proof below shows, the above theorem is still valid if the inequality in (H3) is satisfied only in a subset of $\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N$ of the form $U \times \mathbb{R} \times \mathbb{R}^N$, where U is a neighbourhood of $\partial\Omega$ in $\overline{\Omega}$.

REMARK 2.2. Of course, it is possible to formulate a comparison theorem like Theorem 2.1 for unbounded domains.

The proof below is a modification of the arguments in the proof of H.M. Soner [20, Theorem 2.2] to the current problem.

PROOF. Replacing u and v by u^* and v_*, we may assume that u is u.s.c. and v l.s.c. on $\overline{\Omega}$. First, we prove assertion (i). To this end, we suppose $\max_{\overline{\Omega}}(u - v) > 0$ and will obtain a contradiction in each of the following three possible cases. We will use the notation:

$$w(x, y) = u(x) - v(y) \quad \text{and} \quad \Delta(\varepsilon) = \{(x, y) \in \Omega \times \Omega : |x - y| < \varepsilon\}.$$

CASE 1. $\max_{\partial\Omega}(u - v) < \max_{\overline{\Omega}}(u - v)$. Set

$$\tilde{H}(x, y, r, p, q) = H_*(x, r + v(y), p) - H^*(y, u(x) - r, -q),$$

for $x, y \in \overline{\Omega}, r \in \mathbb{R}$ and $p, q \in \mathbb{R}^N$. Then $\tilde{H}$ is l.s.c. by (H1) and w satisfies

$$\tilde{H}(x, y, w, D_x w, D_y w) \le -a, \text{ in } \Omega \times \Omega, \text{ in the viscosity sense.} \tag{2.3}$$

Since $H_*(x, r, p) \ge \lim_{s\uparrow r} H(x, s, p)$ and $H^*(x, r, p) \le \lim_{s\downarrow r} H(x, s, p)$, for $(x, r, p) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N$ by (H1) - (H3), we have

$$H(x, s + v(y), p) - H(y, u(x) - s, -q) \le \tilde{H}(x, y, r, p, q), \tag{2.4}$$

for $x, y \in \overline{\Omega}$, $r, s \in \mathbb{R}$ satisfying $r > s$ and $p, q \in \mathbb{R}^N$.

Let $0 < \varepsilon, \delta < \frac{1}{2}$, and set $A = \max_{\partial\Omega}(u - v)$ and

$$\varphi(x, y) = A + \delta + \varepsilon^{-\frac{1}{2}} \left[\varepsilon^{\frac{1}{\varepsilon}} + |x - y|^2\right]^{\varepsilon}, \quad \text{for} \quad x, y \in \mathbb{R}^N.$$

Then we have

$$H(x, r, \mathrm{D}_x\varphi(x, y)) - H(y, r, -\mathrm{D}_y\varphi(x, y)) \ge -m_1(3\sqrt{\varepsilon}), \tag{2.5}$$

for $(x, y) \in \Delta(\varepsilon)$ and $r \in \mathbb{R}$, by (H2). Also, we have

$$\varphi(x,y) \geq A + \delta, \text{ for } (x,y) \in \overline{\Delta(\varepsilon)},$$
$$\varphi(x,x) = A + \delta + \sqrt{\varepsilon}, \text{ for } x \in \overline{\Omega}, \text{ and}$$
$$\varphi(x,y) \geq \varepsilon^{-\frac{1}{2}+2\varepsilon}, \text{ for } (x,y) \in \overline{\Delta(\varepsilon)}, \text{ with } |x-y| = \varepsilon.$$

Now we fix δ so small that $\max\limits_{\overline{\Omega}}(u-v) > A + 2\delta$ and then ε so small that $\sqrt{\varepsilon} \leq \delta$, $m_1(3\sqrt{\varepsilon}) < a$, $\varepsilon^{-\frac{1}{2}+2\varepsilon} \geq \max\limits_{\overline{\Omega}\times\overline{\Omega}} w$ and $w(x,y) \leq A + \delta$, if $(x,y) \in \partial\Delta(\varepsilon)$ and $|x-y| < \varepsilon$ (this is possible since w is u.s.c. and $w(x,x) \leq A$, for $x \in \partial\Omega$). Observe that $w - \varphi$ attains a positive maximum at some point $(\overline{x}, \overline{y}) \in \Delta(\varepsilon)$ since w is u.s.c. on $\overline{\Delta(\varepsilon)}$ and $\max\limits_{\partial\Delta(\varepsilon)}(w-\varphi) \leq 0$. Thus (2.3) and (2.5), together with (2.4), yield

$$\begin{aligned} -a &\geq \tilde{H}(\overline{x}, \overline{y}, w(\overline{x}, \overline{y}), D_x\varphi(\overline{x}, \overline{y}), D_y\varphi(\overline{x}, \overline{y})) \\ &\geq H(\overline{x}, u(\overline{x}), D_x\varphi(\overline{x}, \overline{y})) - H(\overline{y}, u(\overline{x}), -D_y\varphi(\overline{x}, \overline{y})) \\ &\geq -m_1(3\sqrt{\varepsilon}) > -a; \text{ a contradiction.} \end{aligned}$$

CASE 2. $\max\limits_{\overline{\Omega}}(u-v) = (u-v)(z)$ and $v(z) < h(z)$, for some $z \in \partial\Omega$. Let $b > 0$ and $\eta \in C(\overline{\Omega}, \mathbb{R}^N)$ be as in (H4). Define $\Phi_\varepsilon : \overline{\Omega} \times \overline{\Omega} \to \mathbb{R} \cup \{-\infty\}$ by

$$\Phi_\varepsilon(x,y) = w(x,y) - \left|\frac{x-y}{\varepsilon} - \eta(z)\right|^2 - |y-z|^2,$$

where ε is a small positive number. Let $(\overline{x}, \overline{y})$ be a point of $\overline{\Omega} \times \overline{\Omega}$ such that $\Phi_\varepsilon(\overline{x}, \overline{y}) = \max\limits_{\overline{\Omega}\times\overline{\Omega}} \Phi_\varepsilon$. Let us assume $\varepsilon \leq b$. Then $z + \varepsilon\eta(z) \in \Omega$ by (H4), and hence $\Phi_\varepsilon(\overline{x}, \overline{y}) \geq \Phi_\varepsilon(z + \varepsilon\eta(z), z)$. This yields

$$\left|\frac{\overline{x}-\overline{y}}{\varepsilon} - \eta(z)\right|^2 + |\overline{y}-z|^2 \leq w(\overline{x}, \overline{y}) - w(z,z) + m_3(C_1\varepsilon), \tag{2.6}$$

where m_3 is the modulus of continuity of u and $C_1 = \max\limits_{\overline{\Omega}} |\eta|$. Therefore

$$\left|\frac{\overline{x}-\overline{y}}{\varepsilon} - \eta(z)\right|^2 \leq \max_{\overline{\Omega}\times\overline{\Omega}} w + m_3(C_1\varepsilon),$$

and so

$$|\overline{x} - \overline{y}| \leq C_2\varepsilon, \tag{2.7}$$

for some constant $C_2 > 0$. Using again (2.6), we get

$$\left|\frac{\overline{x}-\overline{y}}{\varepsilon} - \eta(z)\right|^2 + |\overline{y}-z|^2 \le \max_{\Delta(C_2\varepsilon)} w - w(z,z) + m_3(C_1\varepsilon).$$

From this, we see

$$\left|\frac{\overline{x}-\overline{y}}{\varepsilon} - \eta(z)\right| \le m_4(\varepsilon), \tag{2.8}$$

and

$$|\overline{y}-z| \le m_4(\varepsilon). \tag{2.9}$$

for some real-valued continuous function m_4 on $[0,\infty)$ satisfying $m_4(0) = 0$. Since

$$|\overline{x} - (\overline{y} + \varepsilon\eta(\overline{y}))| \le \varepsilon(|\eta(\overline{y}) - \eta(z)| + m_4(\varepsilon))$$

by (2.8) and, choosing $0 < b_1 \le b$ small enough, $|\eta(\overline{y}) - \eta(z)| + m_4(\varepsilon) \le b$, for $0 < \varepsilon \le b_1$, in view of (2.9), we see from (H4) that $\overline{x} \in \Omega$, for $0 < \varepsilon \le b_1$. Since u satisfies (2.1), in the viscosity sense, we thus find

$$H_*\left(\overline{x}, u(\overline{x}), \frac{2}{\varepsilon}\left[\frac{\overline{x}-\overline{y}}{\varepsilon} - \eta(z)\right]\right) \le -a, \text{ for } 0 < \varepsilon \le b_1. \tag{2.10}$$

Using (2.6), (2.7) and (2.9), we find

$$\begin{aligned} v(\overline{y}) &\le v(z) + u(\overline{x}) - u(z) + m_3(C_1\varepsilon) \\ &\le v(z) + m_3(m_4(\varepsilon) + C_2\varepsilon) + m_3(C_1\varepsilon). \end{aligned}$$

Hence, selecting $0 < b_2 \le b$ small enough, we see that $v(\overline{y}) < h(\overline{y})$ if $0 < \varepsilon \le b_2$ and $\overline{y} \in \partial\Omega$. Since v solves (2.2), in the viscosity sense, we now see

$$H^*\left(\overline{y}, v(\overline{y}), \frac{2}{\varepsilon}\left[\frac{\overline{x}-\overline{v}}{\varepsilon} - \eta(z)\right] - 2(\overline{y}-z)\right) \ge 0, \text{ for } 0 < \varepsilon \le b_2. \tag{2.11}$$

Choose $0 < b_3 \le \min\{b_1, b_2\}$ so small that $m_3(C_1\varepsilon) < w(z,z)$, for $0 < \varepsilon \le b_3$. We henceforth assume $0 < \varepsilon \le b_3$. Now (2.6) guarantees $u(\overline{x}) > v(\overline{y})$, for $0 < \varepsilon \le b_3$. Setting $\delta = \dfrac{u(\overline{x}) - u(\overline{y})}{2}$ and combining (2.10), (2.11) and (2.4), we get

$$\begin{aligned} &H\left(\overline{x}, u(\overline{x}) - \delta, \frac{2}{\varepsilon}\left[\frac{\overline{x}-\overline{y}}{\varepsilon} - \eta(z)\right]\right) \\ &-H\left(\overline{y}, u(\overline{x}) - \delta, \frac{2}{\varepsilon}\left[\frac{\overline{x}-\overline{y}}{\varepsilon} - \eta(z)\right] - 2(\overline{y}-z)\right) \le -a. \end{aligned}$$

Thus, using (H3), (H2), (2.7), (2.8) and (2.9), we find

$$-a \geq -m_2(2|\overline{y} - z|) - m_1 \left(2\frac{|\overline{x} - \overline{y}|}{\varepsilon}\left|\frac{\overline{x} - \overline{y}}{\varepsilon} - \eta(z)\right|\right)$$
$$\geq -m_2(2m_4(\varepsilon)) - m_1(2C_2 m_4(\varepsilon));$$

this yields a contradiction by selecting ε small enough.

CASE 3. $\max_{\overline{\Omega}}(u - v) = (u - v)(z)$ and $u(z) > h(z)$, for some $z \in \partial\Omega$. Set $\overline{h} = -h$ on $\partial\Omega$, $\overline{u} = -v$ and $\overline{v} = -u$ on $\overline{\Omega}$, and

$$\overline{H}(x, r, p) = -H(x, -r, -p) - a, \qquad \text{for} \qquad (x, r, p) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N.$$

As remarked before, $\overline{u}$ and $\overline{v}$, solve, respectively, (2.1) and (2.2) with $\overline{H}$ and $\overline{h}$ in place of H and h, in the viscosity sense. We easily see that (H1) - (H3) are satisfied for $H = \overline{H}$ and that $\max_{\overline{\Omega}}(\overline{u} - \overline{v}) = (\overline{u} - \overline{v})(z)$ and $\overline{v}(z) < \overline{h}(z)$. Thus the present case is reduced to Case 2, and we have a contradiction.

Now we turn to the proof of assertion (ii). If we assume $\max_{\overline{\Omega}}(u - v) > 0$ and argue as in the proof of assertion (i), then the assumption $u \leq h$ on $\partial\Omega$ (resp., $v \geq h$ on $\partial\Omega$) eliminates, from the arguments, the possibility that Case 3 (resp., Case 2) occurs. Meanwhile we only need the following continuity of u, v and h: the continuity of u (resp., v) at points of $\partial\Omega$ and the lower (resp., upper) semicontinuity of h in the arguments of Cases 1 and 2 (resp., Cases 1 and 3). □

A more precise result is needed in Section 4.

THEOREM 2.2. *Let a be a positive number and $h : \partial\Omega \to \mathbb{R}$ be l.s.c.. Assume Ω is bounded and that* (H1) - (H3) *hold. Let u and v, respectively, be u.s.c. and l.s.c. on $\overline{\Omega}$, and satisfy* (2.1) *and* (2.2), *in the viscosity sense. Let Γ be an open subset of $\partial\Omega$, and assume:* (i) *$u \leq v$, on $\partial\Omega\backslash\Gamma$;* (ii) *$u \leq h$, on Γ; and* (iii) *for each $z \in \Gamma$, there is a positive number b, a bounded sequence $\{\eta_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ and a sequence $\{t_n\}_{n\in\mathbb{N}} \subset (0, \infty)$ converging to 0, such that*

$$(2.12) \qquad B(x + t_n\eta_n, bt_n) \subset \Omega, \qquad \text{for } x \in B(z, b) \cap \overline{\Omega}, \quad n \in \mathbb{N},$$

and $\lim_{n\to\infty} u(z + t_n\eta_n) = u(z)$. Then $u \leq v$ on $\overline{\Omega}$.

The proof of this theorem is similar to that of Theorem 3.1. We just present its outline here and leave the details to the reader.

OUTLINE OF PROOF. Suppose $\max_{\overline{\Omega}}(u - v) > 0$. Let $z \in \overline{\Omega}$ be a point where the maximum of $u - v$ is attained. By assumption, $z \in \Omega \cup \Gamma$. If $\max_{\partial\Omega}(u - v) < \max_{\overline{\Omega}}(u - v)$, then the same argument, as in Case 1 of the proof of Theorem 2.1, yields a contradiction. If $z \in \Gamma$, then we go as in Case 2 of the proof of Theorem 2.1, except that we now use

$$\Phi_n(x, y) = u(x) - v(y) - \left|\frac{x-y}{t_n} - \eta_n\right|^2 - |y - z|^2, \qquad \text{with } n \in \mathbb{N},$$

instead of Φ_ε, and will get a contradiction. Thus $\max_{\overline{\Omega}}(u - v) \leq 0$. □

COROLLARY 2.1. *Let Ω be bounded and conditions* (H1) - (H4) *be satisfied. Let u and $v : \overline{\Omega} \to \mathbb{R}$ be, respectively, viscosity subsolution and supersolution of* (HJ)′ - (BC)′. *Then the same assertions, as* (i) *and* (ii) *of Theorem* 2.1, *hold.*

REMARK 2.3. We have an assertion, similar to this collorary, which corresponds to Theorem 2.2. This remark also applies to Corollary 2.2. below.

PROOF. Let $\varepsilon > 0$ and define $u_\varepsilon : \overline{\Omega} \to \mathbb{R}$ by $u_\varepsilon(x) = u(x) - \varepsilon$. Then u_ε solves

$$\begin{cases} u + H(x, u, \mathrm{D}u) \leq -\varepsilon & \text{in } \partial\Omega, \\ u \leq h \quad \text{or} \quad u + H(x, u, \mathrm{D}u) \leq -\varepsilon & \text{on } \partial\Omega, \end{cases}$$

in the viscosity sense. From Theorem 2.1 we find that $u_\varepsilon \leq v$ on $\overline{\Omega}$ in all cases. Sending $\varepsilon \downarrow 0$, we conclude the proof. □

COROLLARY 2.2. *Let Ω be bounded and* (H1) - (H4) *be satisfied. Let u and $v : \overline{\Omega} \to \mathbb{R}$ be, respectively, viscosity sub- and supersolutions of* (HJ) - (BC). *Assume moreover that $p \to H(x, r, p)$ is convex on $\mathbb{R}^N$, for each $(x, r) \in \overline{\Omega} \times \mathbb{R}$, and that there is a function $\psi \in C^1(\overline{\Omega})$ and a constant $a > 0$ for which*

$$H(x, r, \mathrm{D}\psi(x)) \leq -a, \quad \textit{for} \quad (x, r) \in \overline{\Omega} \times \mathbb{R}.$$

Then the same assertions, as (i) *and* (ii) *in Theorem* 2.1, *hold.*

PROOF. Note that h is bounded below on $\partial\Omega$, in all cases. Therefore we may assume $\psi(x) \leq h(x)$, for $x \in \partial\Omega$, by adding a negative constant to h, if necessary. Then ψ is a viscosity subsolution of (HJ) - (BC) and hence $u \vee \psi$, the maximum of u and ψ taken pointwise, is a viscosity subsolution of (HJ) - (BC) by Proposition 1.1. Now let $0 < \theta < 1$ and set $w_\theta(x) = \theta u \vee \psi(x) + (1 - \theta)\psi(x)$, for $x \in \overline{\Omega}$. By the convexity of H, we see that w_θ satisfies

$$(2.13) \qquad \begin{cases} H(x, w_\theta, \mathrm{D}w_\theta) \leq -(1 - \theta)a, & \text{in } \Omega, \\ w_\theta \leq h \quad \text{or} \quad H(x, w_\theta, \mathrm{D}w_\theta) \leq -(1 - \theta)a, & \text{on } \partial\Omega, \end{cases}$$

in the viscosity sense. To check this, let $\varphi \in C^1(\overline{\Omega})$ and suppose $w_\theta^* - \varphi$ attains its maximum at $y \in \overline{\Omega}$. Suppose further

$$w_\theta^*(y) > h^*(y)$$

(this implies $(u \vee \psi)^*(y) > h^*(y)$), if $y \in \partial\Omega$; otherwise we are done. Then

$$(u \vee \psi)^* - \frac{1}{\theta}\varphi - \left(1 - \frac{1}{\theta}\right)\psi$$

attains its maximum at y, and hence

$$H_*\left(y, (u \vee \psi)^*(y), \frac{1}{\theta}\mathrm{D}\varphi(y) + \left(1 - \frac{1}{\theta}\right)\mathrm{D}\psi(y)\right) \leq 0.$$

Thus

$$H_*(y, w_\theta^*(y), \mathrm{D}\varphi(y)) \leq \theta H_*\left(y, (u \vee \psi)^*(y), \frac{1}{\theta}\mathrm{D}\varphi(y) + \left(1 - \frac{1}{\theta}\right)\mathrm{D}\psi(y)\right)$$
$$+(1-\theta)H_*(y, (u \vee \psi)^*(y), \mathrm{D}\psi(y)) \leq -(1-\theta)a,$$

by (2.12); this proves (2.13). Now Theorem 2.1 guarantees $w_\theta \leq v$, on $\overline{\Omega}$. Sending $\theta \uparrow 1$, we see that $v \geq u \vee \psi \geq u$ on $\overline{\Omega}$. □

REMARK 2.4. *If u (resp., v) is Lipschitz continuous on $\overline{\Omega}$, in assertion* (ii) *of Theorem* 2.1, *then we have the same conclusion with the weaker assumption* (2.14) *below in place of* (H2) *and* (H3).

(2.14) *For each $R > 0$, there is a continuous function $m_R : [0, \infty) \to [0, \infty)$, satisfying $m_R(0) = 0$, such that*

$$|H(x, r, p) - H(y, r, q)| \leq m_R(|x - y| + |p - q|),$$

for $x, y \in \overline{\Omega}, r \in \mathbb{R}$ and $p, q \in B(0, R)$.

Analogous remarks are valid for Corollaries 2.1 and 2.2.

PROOF OF REMARK 2.4. Assume all the hypotheses of Theorem 2.1, except (H2) - (H3). Assume further (2.14) and, for definiteness, that u is Lipschitz continuous on $\overline{\Omega}$, with Lipschitz constant $C > 0$. Set $R = C + a$, and define $\theta : \mathbb{R} \to \mathbb{R}$ by

$$\theta(r) = \begin{cases} 1 & \text{for } r \leq R \\ R + 1 - r & \text{for } R \leq r \leq R + 1 \\ 0 & \text{for } r \geq R + 1. \end{cases}$$

Set

$$G(x, r, p) = H(x, r, \theta(|p|)p) \vee (|p| - R),$$

for $(x, r, p) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N$. Then G satisfies (H2) and (H3), with appropriate functions m_1 and m_2. Also, u and v satisfy, respectively, (2.1) and (2.2), with G in place of H, in the viscosity sense. Theorem 2.1 now guarantees our assertions. □

3. - Value functions of exit time problems

In this section we will show that value functions of exit time problems, in deterministic optimal control theory, satisfy the associated Hamilton - Jacobi (Bellman) equations.

Let Ω be, as usual, an open subset of $\mathbb{R}^N$, A a compact topological space, f a real-valued function on $\mathbb{R}^N \times A$, g a function on $\mathbb{R}^N \times A$ into $\mathbb{R}^N$ and h a real-valued function on $\partial\Omega$. We assume:

(A1) Functions f, g and h are continuous. Moreover, there is a constant $L > 0$ such that

$$|g(x,a) - g(y,a)| \leq L|x - y|, \quad \text{for} \quad x, y \in \mathbb{R}^N \quad \text{and} \quad a \in A.$$

Let $\mathcal{A}$ denote the set of controls, i.e.

$$\mathcal{A} = \{\alpha : [0, \infty) \to A \quad \text{measurable}\}.$$

For $x \in \mathbb{R}^N$ and $\alpha \in \mathcal{A}$, we consider the initial value problem

$$\text{(3.1)} \qquad \begin{cases} \dot{X}(t) &= g(X(t), \alpha(t)), \quad \text{for a.e.} \quad t > 0, \\ X(0) &= x. \end{cases}$$

The unique solution of (3.1) will be denoted by $X(t) = X(t; x, \alpha)$. The exit times τ from Ω and $\overline{\tau}$ from $\overline{\Omega}$ are defined, respectively, by

$$\tau = \tau(x, \alpha) = \inf\, \{t \geq 0 : X(t) \in \Omega^c\}$$

and

$$\overline{\tau} = \overline{\tau}(x, \alpha) = \inf\, \{t \geq 0 : X(t) \in \overline{\Omega}^c\}.$$

Associated with these are cost functionals:

$$J(x, \alpha) = \int_0^\tau e^{-t} f(X(t), \alpha(t)) dt + e^{-\tau} h(X(t))$$

and

$$\overline{J}(x, \alpha) = \int_0^{\overline{\tau}} e^{-t} f(X(t), \alpha(t)) dt + e^{-\overline{\tau}} h(X(\overline{\tau})).$$

With these cost functionals at hand, the main purpose of optimal control is stated as follows: Find controls or sequences of controls which minimize $J(x, \alpha)$ and $\overline{J}(x, \alpha)$ over $\mathcal{A}$. Our interest here, however, is restricted to characterizing value functions:

$$\text{(3.2)} \qquad V(x) = \inf_{\alpha \in \mathcal{A}} J(x, \alpha) \quad \text{and} \quad \overline{V}(x) = \inf_{\alpha \in \mathcal{A}} \overline{J}(x, \alpha), \quad \text{on} \quad \overline{\Omega},$$

as viscosity solutions of the boundary value problem

$$\text{(3.3)} \qquad \begin{cases} u + H(x, \mathrm{D}u) = 0 & \text{in } \Omega, \\ u = h \quad \text{or} \quad u + H(x, \mathrm{D}u) = 0 & \text{on } \partial\Omega, \end{cases}$$

where

$$\text{(3.4)} \qquad H(x,p) = \max_{a \in A} \{-g(x,a) \cdot p - f(x,a)\}.$$

The first order PDE in (3.3) is called the *Bellman equation*. Note that $H : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}$ satisfies (H1) - (H3), under assumption (A1).

LEMMA 3.1. *(Dynamic programming principle). Assume* (A1). *For any* $t > 0$ *and* $x \in \overline{\Omega}$, *one has*

$$\text{(3.5)} \qquad V(x) = \inf_{\alpha \in \mathcal{A}} \left\{ \int_0^{t \wedge \tau} e^{-s} f(X(s), \alpha(s)) \mathrm{d}s + 1_{\{t < \tau\}} e^{-t} V(X(t)) + 1_{\{t \geq \tau\}} e^{-\tau} h(X(\tau)) \right\}$$

and

$$\text{(3.6)} \qquad \overline{V}(x) = \inf_{\alpha \in \mathcal{A}} \left\{ \int_0^{t \wedge \overline{\tau}} e^{-s} f(X(s), \alpha(s)) \mathrm{d}s + 1_{\{t < \overline{\tau}\}} e^{-t} \overline{V}(X(t)) + 1_{\{t \geq \overline{\tau}\}} e^{-\overline{\tau}} h(X(\overline{\tau})) \right\},$$

where, for any subset B *of* $\mathcal{A}$, 1_B *denotes the characteristic function of* B.

We refer to P.L. Lions [16] for a proof of this lemma.

THEOREM 3.1. *Assume* (A1). *The functions* V *and* $\overline{V}$, *defined by* (3.2), *are both viscosity solutions of the boundary value problem* (3.3), *with* H *defined by* (3.4).

PROOF. We only prove that V is a viscosity solution of (3.3). The proof for $\overline{V}$ is similar, and we leave it to the reader. To see that V is a supersolution of (3.3), let $\varphi \in C^1(\overline{\Omega})$, $y \in \overline{\Omega}$ and $V_* - \varphi$ attains its minimum at y. We may assume $\min_{\overline{\Omega}}(V_* - \varphi) = 0$ by adding a constant to φ. We may also assume either $y \in \Omega$, or $y \in \partial\Omega$ and $V_*(y) < h(y)$, because otherwise we have nothing to prove.

We suppose

$$\varphi(y) + H(y, \mathrm{D}\varphi(y)) < 0,$$

and will get a contradiction. By continuity, there is a positive constant $\varepsilon > 0$ such that

$$\varphi(x) + H(x, D\varphi(x)) < -\varepsilon, \quad \text{for} \quad x \in B(y,\varepsilon) \cap \overline{\Omega}.$$

Moreover we assume

$$B(y,\varepsilon) \subset \Omega, \quad \text{if} \quad y \in \Omega,$$

and

$$\varphi(x) < h(x) - \varepsilon, \text{ for } x \in B(y,\varepsilon) \cap \partial\Omega, \text{ if } y \in \partial\Omega.$$

Select $M > 0$ so that $|g(x,a)| \leq M$, for $(x,a) \in \overline{\Omega} \times A$. Note that, if $\alpha \in \mathcal{A}$ and $x \in B(y, \frac{\varepsilon}{2})$, then $X(s) = X(s; x, \alpha) \in B(y, \varepsilon)$, for $0 \leq s \leq \frac{\varepsilon}{2M}$. Set $t = \frac{\varepsilon}{2M}$, and choose a $\delta > 0$ so that $\varepsilon(1 - e^{-t}) > 2\delta$. By Lemma 3.1, for any $x \in B(y, \frac{\varepsilon}{2}) \cap \overline{\Omega}$, there is an $\alpha = \alpha_x \in \mathcal{A}$ such that

$$V(x) > \begin{cases} -\delta + \int_0^t e^{-s} f(X(s), \alpha(s)) ds + e^{-t} V(X(t)), & \text{if } t < \tau, \\ -\delta + \int_0^\tau e^{-s} f(X(s), \alpha(s)) ds + e^{-\tau} h(X(\tau)), & \text{if } t \geq \tau, \end{cases}$$

Therefore, if $t < \tau$, then we have

$$V(x) - \varphi(x) > -\delta + \int_0^t e^{-s} f(X(s), \alpha(s)) ds + e^{-t} \varphi(X(t)) - \varphi(x)$$

$$= -\delta + \int_0^t e^{-s} \{f(X(s), \alpha(s)) - \varphi(X(s)) + g(X(s), \alpha(s)) \cdot D\varphi(X(s))\} ds$$

$$\geq -\delta - \int_0^t e^{-s} \{\varphi(X(s)) + H[X(s), D\varphi(X(s))]\} ds \geq -\delta + \varepsilon \int_0^t e^{-s} ds > \delta.$$

Also, if $t \geq \tau$, then we have

$$\begin{aligned} V(x) - \varphi(x) &> -\delta + \int_0^\tau e^{-s} f(X(s), \alpha(s)) ds + e^{-\tau} \varphi(X(\tau)) - \varphi(x) \\ &\quad + e^{-\tau} [h(X(\tau)) - \varphi(X(\tau))] \\ &\geq -\delta + \varepsilon \int_0^\tau e^{-s} ds + \varepsilon e^{-\tau} = \varepsilon - \delta > \delta. \end{aligned}$$

In both cases, we thus obtain

$$V_*(y) > \varphi(y) + \delta; \text{ a contradiction.}$$

Similarly we can check that V is a subsolution of (3.3), and we omit giving the proof here. $\square$

4. - Continuity of viscosity solutions

We continue to study problem (HJ)′ - (BC)′ for Bellman equations. Let Ω, A, f, g and h be as in §3, and H be the function defined by (3.4). We present a sufficient condition under which (HJ)′ - (BC)′ has a continuous viscosity solution. In order to state our assumptions, we need the notation: For a subset S of $\mathbb{R}^N$, we write $\overline{\text{co}}S$ for the closed convex hull of S. Throughout this paper, the term "cone" will stand for a cone with vertex at the origin. For a subset S of $\mathbb{R}^N$ and $\varepsilon > 0$, we denote by $\text{cn}_\varepsilon S$ the ε - convex conic neighbourhood of S, i.e. the set

$$\overline{\text{co}}\left\{t\left(\frac{p}{|p|}+q\right) : t \geq 0, p \in S\backslash\{0\}, q \in B(0,\varepsilon)\right\}.$$

We note that $\text{cn}_\varepsilon S$ is a closed convex cone of $\mathbb{R}^N$. We set

$$G(x) = \overline{\text{co}}\{g(x,a) : a \in A\}, \quad \text{for} \quad x \in \mathbb{R}^N.$$

The following are pieces of the sufficient condition mentioned above. Let $z \in \partial\Omega$ and $\Lambda \subset \partial\Omega$.

(A2) There is an open convex cone K and a constant $\varepsilon > 0$ for which $G(z) \subset K$ and $(z+K) \cap B(z,\varepsilon) \cap \Omega = \emptyset$.

(A3) There is an open convex cone K and a constant $\varepsilon > 0$ such that $G(z) \cap K \neq \emptyset$ and $(x+K) \cap B(z,\varepsilon) \subset \Omega$, for $x \in B(z,\varepsilon) \cap \overline{\Omega}$.

(A4) There is an open convex cone K and a constant $\varepsilon > 0$ for which $G(z) \subset K$, $(z+K) \cap B(z,r) \cap \Omega \neq \emptyset$, for $r > 0$, and $(x+K) \cap B(z,\varepsilon) \cap \Lambda = \emptyset$, for $x \in B(z,\varepsilon) \cap \overline{\Omega}$.

(A5) There is an open convex cone K and a constant $\varepsilon > 0$ such that $G(z) \cap K \neq \emptyset$ and $(z+K) \cap B(z,\varepsilon) \cap \Omega = \emptyset$.

It is easy to check that if z is an interior point (relative to $\partial\Omega$) of Λ, then condition (A4) is equivalent to

(A4)′ There is an open convex cone K and a constant $\varepsilon > 0$ such that $G(z) \subset K$ and $(x+K) \cap B(z,\varepsilon) \subset \Omega$, for $x \in B(z,\varepsilon) \cap \overline{\Omega}$.

THEOREM 4.1. *Assume* (A1). *Let* Γ *and* Σ *be, respectively, an open and a closed subsets of* $\partial\Omega$ *such that* $\partial\Omega = \Gamma \cup \Sigma$ *and* $\Gamma \cap \Sigma = \emptyset$. *Let* Γ_0 *be an* *open subset of* Γ. *Assume that* (A2), (A3), (A4), *with* $\Lambda = \Gamma_0$, *and* (A5) *hold, respectively, for* $z \in \Sigma, z \in \Gamma, z \in \overline{\Gamma}_0$ *and* $z \in \Gamma \backslash \Gamma_0$. *If* u *is a viscosity solution of* (HJ)′ - (BC)′, *with* H *defined by* (3.4), *then* $u \in C(\Omega)$, *and* $u|\Omega$ *has a unique continuous extension to* $\overline{\Omega}$ *which is also a viscosity solution of* (HJ)′ - (BC)′.

It may happen that some of Γ, Σ and Γ_0 are empty in this theorem. Proposition 1.3 or Theorem 3.1 guarantees the existence of a viscosity solution of (HJ)′ - (BC)′. Thus we see, from this theorem, that under the above hypotheses (HJ)′ - (BC)′ has a unique viscosity solution in $C(\overline{\Omega})$.

The assumption of Theorem 4.1 is motivated by the following observation. Let $T > 0$, and set $\tilde{\Omega} = (0, T) \times \Omega$. Let $\Sigma = \{0\} \times \overline{\Omega}, \Gamma = (0, T] \times \partial\Omega \cup \{T\} \times \Omega$ and $\Gamma_0 = \{T\} \times \Omega$. Clearly Σ and Γ are, respectively, a closed and an open subsets of $\partial\tilde{\Omega}$ and Γ_0 is an open subset of Γ. Let f and g be as above and satisfy (A1). Let $\lambda \in \mathbb{R}$, and define $\tilde{f} : \mathbb{R}^{N+1} \times A \to \mathbb{R}$ and $\tilde{g} : \mathbb{R}^{N+1} \times A \to \mathbb{R}^{N+1}$ by $\tilde{f}(t, x, a) = e^{(\lambda-1)t} f(x, a)$ and $\tilde{g}(t, x, a) = (-1, g(x, a))$, for $(t, x, a) \in \mathbb{R}^{N+1} \times A$. The associated Hamiltonian $\tilde{H} : \mathbb{R}^{N+1} \times \mathbb{R}^{N+1} \to \mathbb{R}$ is given by

$$\tilde{H}(t, x, s, p) = s + \max_{a \in A} \left\{ -g(x, a) \cdot p - \tilde{f}(t, x, a) \right\},$$

for $t, s \in \mathbb{R}$ and $x, p \in \mathbb{R}^N$, and the corresponding Bellman equation is:

$$u + \tilde{H}\left(t, x, \frac{\partial u}{\partial t}, \mathrm{D}u\right) = 0.$$

On the other hand, if (A3) and (A5) hold for $z \in \partial\Omega$, then (A2), (A3), (A4), with $\Lambda = \Gamma_0$, and (A5) hold, respectively, for $z \in \Sigma, z \in \Gamma, z \in \overline{\Gamma}_0$ and $z \in \Gamma \backslash \Gamma_0$. This observation and Theorem 4.1 imply that if (A3) and (A5) hold for $z \in \partial\Omega$ and h is a given continuous function on $\partial\tilde{\Omega}$, then the problem

$$\begin{cases} u + \tilde{H}\left(t, x, \frac{\partial u}{\partial t}, \mathrm{D}u\right) = 0, & \text{in } \tilde{\Omega}, \\ u = \tilde{h} \quad \text{or} \quad u + \tilde{H}\left(t, x, \frac{\partial u}{\partial t}, \mathrm{D}u\right) = 0, & \text{on } \partial\tilde{\Omega}, \end{cases}$$

has a unique viscosity solution in $C(\tilde{\Omega})$, where $\tilde{h}(t, x) = e^{(\lambda-1)t} h(t, x)$. Note that this problem for u is equivalent to the problem

$$\begin{cases} \lambda v + \frac{\partial v}{\partial t} + H(x, \mathrm{D}v) = 0, & \text{in } \tilde{\Omega}, \\ v = h \quad \text{or} \quad \lambda v + \frac{\partial v}{\partial t} + H(x, \mathrm{D}v) = 0, & \text{on } \partial\tilde{\Omega}, \end{cases}$$

for $v = e^{(1-\lambda)t} u(t, x)$, where $H(x, p) = \max_{a \in A}\{-g(x, a) \cdot p - f(x, a)\}$. Also note that the following theorem and Lemmas 4.1 and 4.2 assert that the solution of this problem does not depend on the values of h on $\overline{\Gamma}_0$ and equals to h on Σ. Thus our result covers the initial-boundary value problem for Hamilton - Jacobi equations.

THEOREM 4.2. *Assume the hypotheses of Theorem* 4.1. *Let* u *and* v *be, respectively, viscosity sub- and supersolutions of* (HJ)$'$ - (BC)$'$, *with* H *defined by* (3.4). *Then*

$$(u|\Omega \cup \Sigma, \overline{\Omega})^* \le (v|\overline{\Omega}\backslash\Gamma_0, \overline{\Omega})_*, \quad \text{on } \overline{\Omega}. \tag{4.1}$$

Moreover, $(u|\Omega \cup \Sigma, \overline{\Omega})^*$ *and* $(v|\overline{\Omega}\backslash\Gamma_0, \overline{\Omega})_*$ *are, respectively, viscosity sub- and supersolutions of* (HJ)$'$ - (BC)$'$.

For the time being, we assume Theorem 4.2 is correct and prove Theorem 4.1.

PROOF. Let u be a viscosity solution of (HJ)$'$ - (BC)$'$. By virtue of Theorem 4.2 we have

$$(u|\Omega, \overline{\Omega})^* \le (u|\Omega \cup \Sigma, \overline{\Omega})^* \le (u|\overline{\Omega}\backslash\Gamma_0, \overline{\Omega})_* \le (u|\Omega, \overline{\Omega})_*, \quad \text{on } \overline{\Omega},$$

while $(u|\Omega, \overline{\Omega})_* \le (u|\Omega, \overline{\Omega})^*$, on $\overline{\Omega}$, by definition. Therefore

$$(u|\Omega, \overline{\Omega})^* = (u|\Omega \cup \Sigma, \overline{\Omega})^* = (u|\overline{\Omega}\backslash\Gamma_0, \overline{\Omega})_* = (u|\Omega, \overline{\Omega})_*, \quad \text{on } \overline{\Omega}.$$

Thus $(u|\Omega, \overline{\Omega})^*$ is continuous on $\overline{\Omega}$ and is a viscosity solution of (HJ)$'$ - (BC)$'$, by Theorem 4.2. The uniqueness follows immediately from Theorem 2.1. □

In what follows we always assume the hypotheses of Theorem 4.2 and will prove the theorem.

LEMMA 4.1. *One has*

$$u^*(z) \le h(z), \quad \text{for } z \in \partial\Omega\backslash\Gamma_0. \tag{4.2}$$

PROOF. We begin by observing that

$$H(x,p) = \max\{-\xi \cdot p - \eta : (\xi, \eta) \in \mathcal{H}(x)\}, \tag{4.3}$$

where $\mathcal{H}(x) = \overline{\text{co}}\{(g(x,a), f(x,a)) : a \in A\}$.

Fix $z \in \partial\Omega\backslash\Gamma_0$. By translation, we may assume $z = 0$. Since

$$\partial\Omega\backslash\Gamma_0 = \Sigma \cup (\Gamma\backslash\Gamma_0),$$

by assumptions (A2) and (A5) there is an open convex cone K and an $\varepsilon \in (0,1)$ such that

$$G(0) \cap K \ne \emptyset \quad \text{and} \quad K \cap B(0,\varepsilon) \cap \Omega = \emptyset.$$

Choose finite sequences $\{t_1\}_{i=1}^n \subset (0,1]$ and $\{a_i\}_{i=1}^n \subset A$ so that $\sum_{i=1}^n t_i = 1$ and $\sum_{i=1}^n t_i g(0,a_i) \in K$. Set $\xi(x) = \sum_{i=1}^n t_i g(x,a_i)$ and $\eta(x) = \sum_{i=1}^n t_i f(x,a_i)$, for

$x \in \mathbb{R}^N$. Note that $(\xi(x), \eta(x)) \in \mathcal{H}(x)$, for $x \in \mathbb{R}^N$. By the continuity of g, we see that $\xi(x) \in K$, for x in a neighbourhood of 0. Set $C_\varepsilon = \mathrm{cn}_\varepsilon\{\xi(x) : x \in B(0,\varepsilon)\}$, $E_\varepsilon = \mathrm{cn}_\varepsilon C_\varepsilon$ and $K_\varepsilon = \mathrm{cn}_\varepsilon E_\varepsilon$. Then $C_\varepsilon, E_\varepsilon$ and K_ε are nonempty closed convex cones. Replacing $\varepsilon > 0$ by a smaller one, we may assume that

$$K_\varepsilon \cap B(0,\varepsilon) \cap \Omega = \emptyset, \tag{4.4}$$

and

$$|\xi(x)| \geq \varepsilon, \qquad \text{for } x \in B(0,\varepsilon) \cap \overline{\Omega}. \tag{4.5}$$

Define $d : \mathbb{R}^N \to [0,\infty)$ by $d(x) = \operatorname{dist}(x, E_\varepsilon)$, for $x \in \mathbb{R}^N$. It is well-known that $d \in C^1(E_\varepsilon^c)$ and

$$\mathrm{D}d(x) = \frac{x - P(x)}{|x - P(x)|}, \qquad \text{for } x \in E_\varepsilon^c,$$

where $P(x)$ denotes the nearest point of E_ε from x. (One way to see these, for instance, consists in observing that

$$d(x)^2 = \min\{|x-y|^2 : y \in E_\varepsilon\},$$

and then applying general results (e.g., H. Brézis [3, Prop. 2.11]) for convex functions to d^2). Note also that d is positively homogeneous of degree 1.

We claim that

$$d(x) \geq \frac{\varepsilon}{2}|x|, \qquad \text{for } x \in B(0,\varepsilon) \cap \overline{\Omega}. \tag{4.6}$$

In view of (4.4), by continuity, it is enough to prove (4.6) for $x \in K_\varepsilon^c$. To this end, recall that $0 < \varepsilon < 1$, and set $\beta = \frac{1}{1+\varepsilon}$. If $|P(x)| \leq \beta|x|$, then

$$d(x) = |x - P(x)| \geq |x| - |P(x)| \geq (1-\beta)|x| = \frac{\varepsilon}{1+\varepsilon}|x|.$$

Assume $x \in K_\varepsilon^c$ and that $|P(x)| \geq \beta|x|$. Then

$$d\left(\frac{x}{|P(x)|}\right) = \left|\frac{x}{|P(x)|} - \frac{P(x)}{|P(x)|}\right|.$$

As $P(x) \in E_\varepsilon$ and $\frac{x}{|P(x)|} \in K_\varepsilon^c$, we find that $d\left(\frac{x}{|P(x)|}\right) \geq \varepsilon$. The homogeneity of d yields that

$$d(x) = |P(x)| d\left(\frac{x}{|P(x)|}\right) \geq \varepsilon|P(x)| \geq \varepsilon\beta|x| = \frac{\varepsilon}{1+\varepsilon}|x|.$$

Thus we have proved (4.6).

Let $x \in E_\varepsilon^c$ and $p \in C_\varepsilon$ satisfy $|p| = 1$. Let $q \in B(0,1)$. Since $p + \varepsilon q \in E_\varepsilon$ and $P(x) \cdot (P(x) - x) = 0$, we have

$$0 \le (p + \varepsilon q - P(x)) \cdot (P(x) - x) = (p + \varepsilon q) \cdot (P(x) - x).$$

Taking $q = \mathrm{D}d(x)$, with $x \in E_\varepsilon^c$, in this inequality, we find that

$$-p \cdot \mathrm{D}d(x) \ge \varepsilon |p|, \quad \text{for} \quad x \in E_\varepsilon^c \quad \text{and} \quad p \in C_\varepsilon. \tag{4.7}$$

Now fix $M > 0$, so that

$$M \ge \max\left\{u^*(x) - h(0) : x \in B(0,\varepsilon) \cap \overline{\Omega}\right\}.$$

Let $\alpha > 0$, and select a constant $C > 0$ so that

$$h(x) \le h(0) + \alpha + C|x|, \quad \text{for} \quad x \in B(0,\varepsilon) \cap \partial\Omega.$$

Let $p \in E_\varepsilon^\circ$, and set $w(x) = h(0) + \alpha + Bd(x - p)$, for $x \in \mathbb{R}^N$, with $B > 0$ to be fixed later on. Since $p + E_\varepsilon \subset E_\varepsilon^\circ$, we have

$$x - p \in E_\varepsilon^c, \quad \text{for} \quad x \in (E_\varepsilon^\circ)^c,$$

and hence, by (4.4),

$$x - p \in E_\varepsilon^c, \quad \text{for} \quad x \in B(0,\varepsilon) \cap \overline{\Omega}.$$

By the same reason, we have

$$d(x - p) = \text{ dist } (x, p + E_\varepsilon) \ge \text{ dist } (x, E_\varepsilon) = d(x), \quad \text{for} \quad x \in \mathbb{R}^N.$$

Therefore, using (4.6), we see that if $B \ge 2\max\left\{\frac{C}{\varepsilon}, \frac{M}{\varepsilon^2}\right\}$, then $w \ge h$ on $B(0,\varepsilon) \cap \partial\Omega$ and $w \ge u^*$ on $\partial B(0,\varepsilon) \cap \overline{\Omega}$. Using (4.7) and (4.5), we calculate that

$$\begin{aligned} w(x) + H(x, \mathrm{D}w(x)) &\ge h(0) - B\xi(x) \cdot \mathrm{D}w(x) - \eta(x) \\ &\ge h(0) - \eta(x) + B\varepsilon^2, \quad \text{for} \quad x \in B(0,\varepsilon) \cap \overline{\Omega}. \end{aligned}$$

Fix

$$B = 2\max\left\{\frac{C}{\varepsilon}, \frac{M}{\varepsilon^2}, \max\left[\frac{\eta(x) - h(0)}{\varepsilon^2} : x \in B(0,\varepsilon)\right]\right\},$$

and set $\Omega_0 = \Omega \cap B(0,\varepsilon)^\circ$ and $h_0(x) = w(x)$, for $x \in \partial\Omega_0$. Then w satisfies

$$\begin{cases} w(x) + H(x, \mathrm{D}w(x)) > 0 & \text{in} \quad \overline{\Omega}_0, \\ w(x) \ge h_0(x) & \text{on} \quad \partial\Omega_0, \end{cases}$$

in the classical sense, while u^* is a viscosity subsolution of (HJ)′ - (BC)′, with Ω_0 and h_0 in place of Ω and h. A simple argument which leads to a contradiction now shows that $u^* \le w$ on $\overline{\Omega}_0$ and, in particular, $u^*(0) \le h(0) + \alpha + Bd(-p)$. Sending $p \to 0$, while keeping $p \in E_\varepsilon^\circ$, and then $\alpha \downarrow 0$, we conclude (4.2). □

LEMMA 4.2. *One has*

$$v_*(z) \ge h(z), \quad \textit{for} \quad z \in \Sigma. \tag{4.8}$$

Our proof is similar to the above one, and we only give its outline here.

OUTLINE OF PROOF. We may assume $z = 0$. Fix $\varepsilon > 0$. We set

$$C_\varepsilon = \mathrm{cn}_\varepsilon \left(\bigcup_{x \in B(0,\varepsilon)} G(x) \right), \quad E_\varepsilon = \mathrm{cn}_\varepsilon C_\varepsilon \quad \text{and} \quad K_\varepsilon = \mathrm{cn}_\varepsilon E_\varepsilon.$$

By assumption (A2), we can choose an $\varepsilon > 0$ so that

$$K_\varepsilon \cap B(0,\varepsilon) \cap \Omega = \emptyset$$

and

$$|p| \ge \varepsilon, \quad \text{for} \quad p \in \bigcup_{x \in B(0,\varepsilon)} G(x).$$

Let $p \in E_\varepsilon^\circ, \alpha > 0$ and $B > 0$, and define $w : \mathbb{R}^N \to \mathbb{R}$ by

$$w(x) = h(0) - \alpha - B \text{ dist } (x - p, E_\varepsilon).$$

There is a positive constant B_α, independent of p, such that if $B \ge B_\alpha$, then $w \le h$ on $B(0,\varepsilon) \cap \partial\Omega$, $w \le v_*$ on $\partial B(0,\varepsilon) \cap \overline{\Omega}$ and $w(x) + H(x, Dw(x)) < 0$ in $\overline{\Omega} \cap B(0,\varepsilon)$, in the classical sense. Thus we conclude, as in the above proof, that $w \le v_*$ on $B(0,\varepsilon) \cap \overline{\Omega}$, and hence $h(0) \le v_*(0)$. □

In what follows, we write

$$\hat{u} = (u | \Omega \cup \Sigma, \overline{\Omega})^* \quad \text{and} \quad \hat{v} = (v | \overline{\Omega} \backslash \overline{\Gamma}_0, \overline{\Omega})_*.$$

We note that $\hat{u} \le u^*$ and $\hat{v} \ge v_*$ on $\partial\Omega$ and that $\hat{u} = u^*$ and $\hat{v} = v_*$ in Ω.

LEMMA 4.3. *Assume $u^*(x) \ge -C$ for $x \in \overline{\Omega}$ and some constant $C > 0$. Then, for each $z \in \Gamma$, there is an open convex cone K and a constant $\varepsilon > 0$ such that*

$$\hat{u}(z) = \lim_{r \downarrow 0} \sup \{u(x) : x \in B(z,r) \cap (z + K)\} \tag{4.9}$$

and

$$(x + \mathrm{cn}_\varepsilon K) \cap B(z,\varepsilon) \subset \overline{\Omega}, \quad \textit{for} \quad x \in B(z,\varepsilon) \cap \overline{\Omega}. \tag{4.10}$$

PROOF. Let $z \in \Gamma$. We may assume $z = 0$. By assumption (A3), there is an open convex cone K and a constant $\varepsilon \in (0,1)$ such that $G(0) \cap K \neq \emptyset$ and

$$(x + K) \cap B(0,\varepsilon) \subset \Omega, \quad \text{for} \quad x \in B(0,\varepsilon) \cap \overline{\Omega}.$$

We select finite sequences

$$\{t_i\}_{i=1}^m \subset (0,1] \text{ and } \{a_i\}_{i=1}^m \subset A \text{ so that}$$
$$\sum_{i=1}^m t_i = 1 \text{ and } \sum_{i=1}^m t_i g(0, a_i) \in K.$$

Set

$$\xi(x) = \sum_{i=1}^m t_i g(x, a_i) \text{ and } \eta(x) = \sum_{i=1}^m t_i f(x, a_i), \text{ for } x \in \mathbb{R}^N.$$

Also, set $n = \frac{\xi(0)}{|\xi(0)|}$, and define

$$C_\varepsilon = \mathrm{cn}_\varepsilon\{\xi(x) : x \in B(0,\varepsilon)\},$$
$$E_\varepsilon = \mathrm{cn}_\varepsilon C_\varepsilon, F_\varepsilon = \mathrm{cn}_\varepsilon E_\varepsilon \text{ and } K_\varepsilon = \mathrm{cn}_\varepsilon F_\varepsilon.$$

Replacing $\varepsilon > 0$ by a new one, we may assume that

$$(4.11) \qquad (x + K_\varepsilon) \cap B(0,\varepsilon) \subset \overline{\Omega}, \quad \text{for} \quad x \in \overline{\Omega} \cap B(0,\varepsilon),$$

and

$$(4.12) \qquad n \cdot \xi(x) \geq \varepsilon, \quad \text{for} \quad x \in B(0,\varepsilon) \cap \overline{\Omega}.$$

In view of (4.11), it is enough to show that (4.9) holds for $K = F_\varepsilon^\circ$. To this end, we will prove that, for any $r \in (0,\varepsilon]$,

$$(4.13) \qquad u(x) \leq M_r + 2Br, \quad \text{for} \quad x \in B(0,b) \cap \Omega,$$

where $b = \frac{\varepsilon r}{4}$,

$$M_r = \sup\ \{u^*(x) : x \in F_\varepsilon^\circ \cap B(0,r)\} \text{ and}$$
$$B = \max\ \left\{\frac{C + |\eta(x)|}{\varepsilon} : x \in \overline{\Omega}\right\}.$$

Note that (4.9), with $K = F_\varepsilon^\circ$, is a direct consequence of (4.13). Also, notice that $M_r \geq -C$.

Let $r \in (0,\varepsilon]$, and let b, M_r and B be as above. Observe that

$$(4.14) \qquad (y + C_\varepsilon) \cap \partial B(0,r) \subset E_\varepsilon, \quad \text{for} \quad y \in B(0,2b).$$

Indeed, if $y \in B(0, 2b), p \in C_\varepsilon$ and $|y+p| = r$, then $|p| \geq \frac{r}{2}$ and hence, $\frac{|y|}{|p|} \leq \varepsilon$, from which we conclude that $y + p = |p| \left(\frac{y}{|p|} + \frac{p}{|p|} \right) \in E_\varepsilon$.

Fix $\varsigma \in B(0, b) \cap \Omega$ and choose a $p \in B(0, b) \cap E_\varepsilon^\circ$ so that $\varsigma - p \in \Omega$. We write $y = \varsigma - p, C_\varepsilon(y) = y + C_\varepsilon$ and $\Omega_0 = B(0, r)^\circ \cap C_\varepsilon(y)^\circ$. Note that $\overline{\Omega}_0 \subset \overline{\Omega}$ by (4.11). For $L > 0$, we define a C^1 function $w : \overline{\Omega}_0 \to \mathbb{R}$ by

$$w(x) = M_r + B(r - n \cdot x) + L \text{ dist } (x, \xi + C_\varepsilon)^2.$$

Clearly, $w(x) \geq M_r$, for $x \in \overline{\Omega}_0$. Hence, by (4.14), $w \geq u^*$ on $\partial B(0, r) \cap \overline{\Omega}_0$. Since $\varsigma + C_\varepsilon = p + C_\varepsilon(y) \subset C_\varepsilon^\circ$, we see that dist $(C_\varepsilon^c, \varsigma + C_\varepsilon) > 0$. Therefore, choosing L large enough, we may assume that $u^* \leq w$, on $\partial C_\varepsilon(y) \cap \overline{\Omega}_0$. Let us write $d(x) =$ dist $(x, C_\varepsilon)^2$. Then, dist $(x, \varsigma + C_\varepsilon)^2 = d(x - \varsigma)$. Also, $\mathrm{D}d(x) = 2(x - P(x))$, where $P(x)$ denotes the nearest point of C_ε from x, and hence, $\mathrm{D}d(x) \cdot q \leq 0$ for $q \in C_\varepsilon$. In particular, we see that

$$\xi(x) \cdot \mathrm{D} \text{ dist } (x, \varsigma + C_\varepsilon)^2 \leq 0, \qquad \text{for} \qquad x \in \mathbb{R}^N.$$

Hence, using (4.12), we find that

$$w(x) + H(x, \mathrm{D}w(x)) \geq w(x) + B\xi(x) \cdot n - |\eta(x)| \geq 0, \qquad \text{for} \qquad x \in \Omega_0.$$

Thus, by the standard comparison result (see, e.g. [6]) or by Theorem 2.2 with $\Omega = \Omega_0$ and $\Gamma = \emptyset$, we see that $u^* \leq w$ on $\overline{\Omega_0}$ and, in particular, $u^*(\varsigma) \leq M_r + B(r - n \cdot x) \leq M_r + 2Br$, proving (4.13). □

LEMMA 4.4. *Assume that $v_*(x) \leq C$ for $x \in \overline{\Omega}$ and some constant $C > 0$. Then, for any $z \in \Gamma_0$ there is an open convex cone K and a constant $\varepsilon > 0$ such that $G(z) \subset K, (z + K) \cap B(z, \varepsilon) \subset \Omega$ and*

$$\hat{v}(z) = \lim_{r \downarrow 0} \inf \{v(x) : x \in B(z, r) \cap (z + K)\}. \tag{4.15}$$

The proof of this lemma is similar to that of Lemma 4.3, and so we present only the outline.

OUTLINE OF PROOF. Let $0 < \varepsilon < 1$ and $z \in \Gamma_0$, and assume $z = 0$ as usual. Set $C_\varepsilon = \mathrm{cn}_\varepsilon \left(\bigcup_{x \in B(0,\varepsilon)} G(x) \right)$, $E_\varepsilon = \mathrm{cn}_\varepsilon C_\varepsilon, F_\varepsilon = \mathrm{cn}_\varepsilon E_\varepsilon$ and $K_\varepsilon = \mathrm{cn}_\varepsilon F_\varepsilon$. Choosing ε small enough, from (A4)$'$ we may assume that

$$(x + K_\varepsilon) \cap B(0, \varepsilon) \subset \overline{\Omega}, \qquad \text{for} \qquad x \in \overline{\Omega} \cap B(0, \varepsilon),$$

and

$$|p| \geq \varepsilon, \qquad \text{for} \qquad p \in \bigcup_{x \in B(0,\varepsilon)} G(x).$$

Since K_ε is convex and $0 \in K_\varepsilon \neq \mathbb{R}^N$, there is a unit vector $n \in \mathbb{R}^N$ such that $n \cdot p \leq 0$ for $p \in K_\varepsilon$. This implies that $n \cdot p \leq -\varepsilon|p|$ for $p \in F_\varepsilon$. Indeed, if $p \in F_\varepsilon$ satisfies $|p| = 1$, then $p + \varepsilon n \in K_\varepsilon$ and so $n \cdot p + \varepsilon \leq 0$. Thus

$$n \cdot p \leq -\varepsilon^2, \quad \text{for} \quad p \in \bigcup_{x \in B(0,\varepsilon)} G(x).$$

Fix $r \in (0, \varepsilon]$, and set $b = \frac{\varepsilon r}{4}$,

$$M_r = \inf\{v_*(x) : x \in F_\varepsilon^\circ \cap B(0,r)\},$$

$$B = \max\left\{\frac{|g(x,a)| + C}{\varepsilon^2} : x \in \overline{\Omega}, a \in A\right\}.$$

Fix $\varsigma \in \Omega \cap B(0,b)$. Choose a $p \in B(0,b) \cap E_\varepsilon^\circ$ so that $\varsigma - p \in \Omega$. We denote $y = \varsigma - p$ and $\Omega_0 = B(0,r)^\circ \cap (y + C_\varepsilon)^\circ$. For $L > 0$, we define $w \in C^1(\overline{\Omega}_0)$ by

$$w(x) = M_r - B(r + n \cdot x) - L \text{ dist } (x, \varsigma + E_\varepsilon)^2.$$

As in Lemma 4.3, assuming L large enough, we see that $w \leq v_*$ on $\partial\Omega_0$ and

$$w(x) + H(x, \mathrm{D}w(x)) \leq 0, \quad \text{in} \quad \Omega_0.$$

Thus, by comparison, we have $w \leq v_*$ on Ω_0. From this we conclude (4.15) for $K = F_\varepsilon^\circ$. □

LEMMA 4.5. *Assume $v_*(x) \leq C$ for $x \in \overline{\Omega}$ and some constant $C > 0$. Let $z \in \overline{\Gamma}_0 \backslash \Gamma_0$, and assume $\hat{v}(z) < h(z)$. Then there is an open convex cone K and a constant $\varepsilon > 0$ such that $G(z) \subset K$, $(z + K) \cap B(z,\varepsilon) \cap \overline{\Gamma}_0 = \emptyset$ and*

$$\hat{v}(z) = \lim_{r \downarrow 0} \inf \{v(x) : x \in (z + K) \cap B(z,r) \cap \overline{\Omega}\}. \tag{4.16}$$

The proof is similar to the above one and we give here its outline.

OUTLINE OF PROOF. By Lemma 4.2, we see that $z \in \Gamma$, since $\hat{v}(z) < h(z)$. As usual, we assume $z = 0$. For $\varepsilon > 0$, define $C_\varepsilon, E_\varepsilon, F_\varepsilon$ and K_ε as in the proof of Lemma 4.4. In view of (A4), we can choose an $\varepsilon \in (0,1)$ so that

$$(x + K_\varepsilon^\circ) \cap B(x,\varepsilon) \cap \overline{\Gamma}_0 = \emptyset, \quad \text{for} \quad x \in B(0,\varepsilon) \cap \overline{\Omega},$$

and

$$|p| \geq \varepsilon, \quad \text{for} \quad p \in \bigcup_{x \in B(0,\varepsilon)} G(x).$$

We choose a unit vector $n \in \mathbb{R}^N$, as in Lemma 4.4. Fix $r \in (0, \varepsilon]$. Let $b = \frac{\varepsilon r}{4}$,

$$M_r = \min\{\inf\,[v_*(x) : x \in F_\varepsilon^\circ \cap B(0,r) \cap \overline{\Omega}],$$
$$\inf\,[h(x) : x \in F_\varepsilon^\circ \cap B(0,r) \cap \partial\Omega]\},$$

and $B = \max\left\{\dfrac{|g(x,a)| + C}{\varepsilon^2} : x \in \overline{\Omega}, a \in A\right\}$. Fix $\varsigma \in \Omega \cap B(0,b)$ and choose a $p \in B(0,b) \cap E_\varepsilon^\circ$ so that $\varsigma - p \in \Omega$. Set $y = \varsigma - p$ and $\Omega_0 = (\varsigma + E_\varepsilon)^\circ \cap B(0,b)^\circ$. For $L > 0$, we define $w \in C^1(\overline{\Omega}_0)$ by

$$w(x) = M_r - B(r + n \cdot x) - L \text{ dist } (x, y + E_\varepsilon)^2.$$

As before, choosing L large enough, we find that $w \le v^*$ on $\partial\Omega_0 \cap \overline{\Omega}$ and

$$w(x) + H(x, \mathrm{D}w(x)) \le 0, \quad \text{in} \quad \Omega_0 \cap \Omega.$$

Moreover we have $w \le h$ on $\Omega_0 \cap \partial\Omega$. Using Theorem 2.2, we find that $w \le v_*$ on $\Omega_0 \cap \Omega$. From this, we conclude that (4.16) holds for $K = F_\varepsilon^\circ$. □

LEMMA 4.6. *Assume that $u^*(x) \ge -C$ and $v_*(x) \le C$ for $x \in \overline{\Omega}$ and some constant $C > 0$. Then $\hat{u}$ and $\hat{v}$ are, respectively, viscosity sub- and supersolutions of*

$$u + H(x, \mathrm{D}u) = 0, \quad \textit{on} \quad \Omega \cap \Gamma_0. \tag{4.17}$$

PROOF. We only prove that $\hat{u}$ is a subsolution of (4.17). The proof of the assertion for $\hat{v}$ is similar. By assumption, $\hat{u} + H(x, \mathrm{D}\hat{u}) \le 0$, in Ω, in the viscosity sense. Hence it suffices to show that, for each $z \in \Gamma_0$, there is an open convex cone K and a constant $\varepsilon > 0$ such that $(z + K) \cap B(z, \varepsilon) \subset \overline{\Omega}$ and

$$\hat{u} + H(x, \mathrm{D}\hat{u}) \le 0, \quad \text{in} \quad [(z + K) \cap B(z, \varepsilon)^\circ] \cup \{z\}, \tag{4.18}$$

in the viscosity sense. To see this, fix $z \in \Gamma_0$. Let K and ε be, respectively, an open convex cone and a positive number, as in Lemma 4.3. The proof of Lemmas 4.3 and 4.4 shows that we may assume that $G(x) \subset K$ for $x \in B(z, \varepsilon) \cap \overline{\Omega}$. Set $\Omega_0 = (z + K) \cap B(z, \varepsilon)^\circ$. Let $\varphi \in C^1(\Omega_0 \cup \{z\})$ and assume $\hat{u} - \varphi$ attains its maximum at some point of $\Omega_0 \cup \{z\}$ (notice that $\hat{u}$ is u.s.c. on $\Omega_0 \cup \{z\}$). We may assume that z is the maximum point; otherwise we are done. We may also assume $\varphi \in C^1(\overline{\Omega}_0)$. By (4.9), there is a sequence $\{z_n\} \subset z + K$ such that $\hat{u}(z_n) \to \hat{u}(z)$ and $z_n \to z$, as $n \to \infty$. Since dist $(z + K^c, z_n + K) > 0$, we can choose a sequence $\{L_n\} \subset (0, \infty)$ so that

$$\hat{u}(x) - \varphi(x) - L_n \text{ dist } (x, z_n + K)^2 \le \hat{u}(z) - \varphi(z) - 1,$$

for $x \in \partial(z + K) \cap \overline{\Omega}_0$. For each n, we define $\Phi_n \in C^1(\overline{\Omega}_0)$ by

$$\Phi_n(x) = \hat{u}(x) - \varphi(x) - L_n \text{ dist } (x, z_n + K)^2 - |x - z|^2.$$

Let $x_n \in \overline{\Omega}_0$ be a maximum point of Φ_n. Since

$$\begin{aligned}\hat{u}(x_n) - \varphi(x_n) - |x_n - z|^2 &\ge \Phi_n(x_n) \ge \Phi_n(z_n) \\ &= \hat{u}(z_n) - \varphi(z_n) - |z_n - z|^2,\end{aligned}$$

we see that $\hat{u}(x_n) - \varphi(x_n) \to \hat{u}(z) - \varphi(z)$ and $x_n \to z$, as $n \to \infty$. Therefore, in view of our choice of L_n, we see that $x_n \in \Omega_0$, for large n. Note that

$$p \cdot \mathrm{D} \text{ dist } (x, z_n + K)^2 \leq 0, \quad \text{for} \quad p \in K \quad \text{and} \quad x \in \mathbb{R}^N.$$

Thus, if n is large enough, we have

$$\begin{aligned} 0 &\geq \hat{u}(x_n) + H(x_n, \mathrm{D}\varphi(x_n) + L_n \mathrm{D} \text{ dist } (x_n, z_n + K)^2 + 2(x_n - z)) \\ &\geq \hat{u}(x_n) + H(x_n, \mathrm{D}\varphi(x_n) + 2(x_n - z)). \end{aligned}$$

Sending $n \to \infty$, we conclude that $\hat{u}(z) + H(z, \mathrm{D}\varphi(z)) \leq 0$. □

REMARK 4.1. For the above proof we borrowed some ideas from M.G. Crandall - R. Newcomb [8] and P.E. Souganidis [21].

LEMMA 4.7. *Assume $v_*(x) \leq C$ for $x \in \overline{\Omega}$ and some constant $C > 0$. Let $z \in \overline{\Gamma}_0 \backslash \Gamma_0$. Then $\hat{v}$ satisfies*

$$\text{(4.19)} \qquad \begin{cases} \hat{v} + H(x, \mathrm{D}\hat{v}) \geq 0, & \textit{in } \Omega, \\ \hat{v} \geq h \quad \textit{or} \quad \hat{v} + H(x, \mathrm{D}\hat{v}) \leq 0, & \textit{at } z, \end{cases}$$

in the viscosity sense.

By using Lemma 4.5, the proof parallels that of Lemma 4.6, and we leave the details to the reader.

PROOF OF THEOREM 4.2. If we set

$$C = \max \{|H(x,0)| : x \in \overline{\Omega}\} + \max \{|h(x)| : x \in \partial\Omega\}$$

and $\eta(x) = -C$ (resp., $\varsigma(x) = C$) for $x \in \overline{\Omega}$, then η (resp., ς) is a viscosity subsolution (resp., supersolution) of (HJ)′ - (BC)′ and so is $u \vee \eta$ (resp., $v \wedge \varsigma$). Therefore, we may assume u and v are bounded on $\overline{\Omega}$. We see, from Lemmas 4.1 and 4.2, that $u^* \leq v_*$ on Σ and that $u^* \leq h$ on $\Gamma \backslash \Gamma_0$. From Lemma 4.3, we find that condition (iii) of Theorem 2.2 is satisfied for $\hat{u}$ and $z \in \Gamma$. Since u^* is u.s.c. on $\overline{\Omega}$ and $u^* \leq h$ on $\partial\Omega \backslash \Gamma_0$, there is a continuous function $\hat{h}$ on $\partial\Omega$ which satisfies $u^* \leq \hat{h}$ on $\partial\Omega$ and $\hat{h} = h$ on $\partial\Omega \backslash \Gamma_0$. By virtue of Lemmas 4.6 and 4.7, we find that $\hat{u}$ and $\hat{v}$ are, respectively, viscosity sub- and supersolutions of (HJ)′ - (BC)′, with $\hat{h}$ in place of h. Applying Theorem 2.2, we conclude that $\hat{u} \leq \hat{v}$ on $\overline{\Omega}$. □

5. - Identification of value functions

By virtue of Theorems 3.1, 2.2 and 4.1, the value functions V and $\overline{V}$, defined by (3.2), are characterized as follows.

THEOREM 5.1. *Assume the hypotheses of Theorem* 4.1. *Then:*

(i) *The function V is the unique viscosity solution of* (HJ)′ - (BC)′, *with H defined by* (3.4), *satisfying $V = h$ on $\partial\Omega$.*

(ii) *The function $\overline{V}$ is continuous on $\overline{\Omega}$ and it is the unique viscosity solution of* (HJ)′ - (BC)′, *with H defined by* (3.4), *in $C(\overline{\Omega})$.*

REMARK 5.1. G. Barles and B. Perthame [2] have obtained results of similar nature.

PROOF OF ASSERTION (i). By Theorem 3.1 and the definition of V, we see that V is a viscosity solution of (HJ)′ - (BC)′ satisfying $V = h$ on $\partial\Omega$. Meanwhile such a solution is unique because of Theorems 4.1 and 2.2. □

To prove assertion (ii), we need a lemma.

LEMMA 5.1 *Assume* (A1). *Let $\{\lambda_i\}_{i=1}^m$ be a sequence of positive numbers satisfying $\sum_{i=1}^m \lambda_i = 1$ and $\{a_i\}_{i=1}^m$ a sequence of elements of A. Let $x \in \mathbb{R}^N$, and set*

$$Y(t) = x + t\sum_{i=1}^m \lambda_i g(x, a_i), \quad \textit{for} \quad t \geq 0.$$

Then, for any $\varepsilon > 0$, there is a $\delta > 0$ and a control $\alpha \in \mathcal{A}$ such that

$$|X(t) - Y(t)| \leq \varepsilon t, \quad \textit{for} \quad 0 \leq t \leq \delta, \tag{5.1}$$

where $X(t) = X(t; x, \alpha)$.

PROOF. For notational simplicity we assume $m = 2$. The proof in the general case is similar, which we omit giving here. We write $\lambda = \lambda_1$. Fix $k \in \mathbb{N}$ and define $\alpha_k \in \mathcal{A}$ by

$$\alpha_k(t) = \begin{cases} a_1 & \text{if } \left(1 + \frac{j}{k}\right) 2^{-n} < t \leq \left(1 + \frac{j+\lambda}{k}\right) 2^{-n}, \text{ for some } n \in \mathbb{N} \\ & \text{and } j = 0, 1, \cdots, k-1, \\ a_2 & \text{otherwise.} \end{cases}$$

Then we find that

$$\int_{(1+\frac{j}{k})2^{-n}}^{(1+\frac{j+1}{k})2^{-n}} g(x, \alpha_k(s))ds = \frac{1}{k2^n}\{\lambda g(x, a_1) + (1-\lambda)g(x, a_2)\}, \tag{5.2}$$

for $j = 0, 1, \cdots, k-1$ and $n \in \mathbb{N}$. Choose $M \geq 1$, so that

$$M \geq \max\ \{|g(y, a)| : y \in B(x, 1), a \in A\}.$$

Let $X_k(t) = X(t; x, \alpha_k)$. Setting $t_0 = \frac{1}{M}, \left(\frac{1}{M} \in (0,1]\right)$, we see easily that $X_k(t) \in B(x, 1)$, for $0 \leq t \leq t_0$, and that

$$|X_k(t) - x| \leq \int_0^t |g(X_k(s), \alpha_k(s))| \mathrm{d}s \leq Mt, \qquad \text{for} \quad 0 \leq t \leq t_0.$$

Therefore, we have

$$\left| \int_0^t g(X_k(s), \alpha_k(s)) \mathrm{d}s - \int_0^t g(x, \alpha_k(s)) \mathrm{d}s \right| \leq \frac{ML}{2} t^2, \qquad \text{for} \quad 0 \leq t \leq t_0.$$

Now let $0 < t \leq t_0$ and select $n \in \mathbb{N}$ and $j = 0, 1, \cdots, k-1$, so that $\left(1 + \frac{j}{k}\right) 2^{-n} < t \leq \left(1 + \frac{j+1}{k}\right) 2^{-n}$. We compute that

$$\begin{aligned}
|X_k(t) - Y(t)| &\leq \left| \int_0^t g(x, \alpha_k(s)) \mathrm{d}s - t[\lambda g(x, a_1) + (1-\lambda) g(x, a_2)] \right| + \frac{ML}{2} t^2 \\
&\leq \left| \int_0^{(1+\frac{j}{k})2^{-n}} g(x, \alpha_k(s)) \mathrm{d}s - \left(1 + \frac{j}{k}\right) \frac{1}{2^n} [\lambda g(x, a_1) + (1-\lambda) g(x, a_2)] \right| \\
&\qquad + \frac{ML}{2} t^2 + \frac{2M}{k} \frac{1}{2^n} \leq M \left(\frac{L}{2} t + \frac{2M}{k}\right) t,
\end{aligned}$$

by (5.2).

Fix $\varepsilon > 0$ and choose $k \in \mathbb{N}$ and $0 < \delta \leq t_0$ so that $M\left(\frac{L\delta}{2} + \frac{2M}{k}\right) < \varepsilon$. Then the last inequality proves (5.1) for $X = X_k$. □

PROOF OF ASSERTION (ii). Let us set $u = \overline{V}$. Recall that

$$(u|\Omega \cup \Sigma, \overline{\Omega})^* = (u|\overline{\Omega} \backslash \overline{\Gamma}_0, \overline{\Omega})_*$$

is the unique viscosity solution of (HJ)$'$ - (BC)$'$ in $C(\overline{\Omega})$ (see Theorem 3.1 and the proof of Theorem 4.1). It is enough to show that $u \leq (u|\Omega \cup \Sigma, \overline{\Omega})^*$ on $\overline{\Gamma}$ and $u \geq (u|\overline{\Omega} \backslash \overline{\Gamma}_0, \overline{\Omega})_*$ on $\overline{\Gamma}_0$.

First, we prove $u \leq (u|\Omega \cup \Sigma, \overline{\Omega})^*$ on $\overline{\Gamma}$. If $z \in \overline{\Gamma} \backslash \Gamma (\subset \Sigma)$, then

$$u^*(z) \leq h(z) = (u|\Omega \cup \Sigma, \overline{\Omega})^*(z),$$

by Lemmas 4.1 and 4.2, and so $u(z) \leq (u|\Omega \cup \Sigma, \overline{\Omega})^*(z)$. Now we let $z \in \Gamma$. By (A3), there is an open convex cone K and a constant $\varepsilon > 0$ such that $G(z) \cap K \neq \emptyset$ and

$$(z + K) \cap B(z, \varepsilon) \subset \overline{\Omega}. \tag{5.3}$$

Let $p \in G(z) \cap K$, and choose a constant $\delta > 0$ such that $B(p, 2\delta) \subset K$. Choose a sequence $\{\lambda_i\}_{i=1}^m \subset (0,1]$ satisfying $\sum_{i=1}^m \lambda_i = 1$ and a sequence $\{a_i\}_{i=1}^m \subset A$ such that $\left|p - \sum_{i=1}^m \lambda_i g(z, a_i)\right| < \delta$. By virtue of Lemma 5.1, there is a control $\alpha \in \mathcal{A}$ and a constant $t_0 > 0$ such that $X(t) = X(t; z, \alpha)$ satisfies

$$\left|X(t) - z - t\sum_{i=1}^m \lambda_i g(z, a_i)\right| < \delta t, \qquad \text{for} \qquad 0 < t \le t_0.$$

Hence $|X(t) - z - tp| < 2\delta t$, for $0 < t \le t_0$, and so $X(t) \in z + K$, for $0 < t \le t_0$. From this and (5.3), we see that $X(t) \in \Omega$, for all $0 < t \le t_1$ and some $0 < t_1 \le t_0$. We now apply Lemma 3.1 to find

$$u(z) \le \int_0^t e^{-s} f(X(s), \alpha(s))\mathrm{d}s + e^{-t}u(X(t)), \qquad \text{for} \qquad 0 \le t \le t_1.$$

Therefore

$$u(z) \le \limsup_{t\downarrow 0} u(X(t)) \le (u|\Omega \cup \Sigma, \overline{\Omega})^*(z).$$

Now let us prove that $u \ge (u|\overline{\Omega}\backslash\Gamma_0, \overline{\Omega})_*$ on $\overline{\Gamma}_0$. Let $z \in \overline{\Gamma}_0\backslash\Gamma_0$. We have $(u|\overline{\Omega}\backslash\Gamma_0, \overline{\Omega})_* \le h(z)$, by Lemma 4.1. Therefore we may assume $u(z) < h(z)$. Let K and ε be an open convex cone and a positive number from (A4). We may assume that $G(z) + B(0, \varepsilon) \subset K$. Set $M = \max\ \{|g(x, a)| : x \in \overline{\Omega}, a \in A\}$. Let $\alpha \in \mathcal{A}$ and $X(t) = X(t; z, \alpha)$. It is easy to see that

$$|X(t) - z| \le Mt,$$

and so

$$\left|X(t) - \left[z + \int_0^t g(z, \alpha(s))\mathrm{d}s\right]\right| \le \frac{LMt^2}{2},$$

as far as $X(s) \in \overline{\Omega}$, for $0 \le s \le t$. Also we see

$$\frac{1}{t}\int_0^t g(z, \alpha(s))\mathrm{d}s \in G(z), \qquad \text{for} \qquad t > 0.$$

Thus, setting $t_0 = \min\ \{\frac{\varepsilon}{M+1}, \frac{2\varepsilon}{LM+1}\}$, we see that, if $0 < t \le t_0$ and $X(s) \in \overline{\Omega}$, for $0 \le s \le t$, then $X(t) \in (z + K) \cap B(z, \varepsilon)$. By Lemma 3.1, for each $0 < t \le t_0$, there is an $\alpha = \alpha_t \in \mathcal{A}$ such that

$$(5.4)\quad u(z) > \begin{cases} -t + \int_0^t e^{-s} f(X(s), \alpha(s))ds + e^{-t}u(X(t)), & \text{if } \overline{\tau} > t, \\ -t + \int_0^{\overline{\tau}} e^{-s} f(X(s), \alpha(s))ds + e^{-\overline{\tau}}h(X(\overline{\tau})), & \text{otherwise,} \end{cases}$$

where $X(s) = X(s; z, \alpha_t)$ and $\overline{\tau} = \overline{\tau}(z, \alpha_t)$ is the first exit time from $\overline{\Omega}$. Since $u(z) < h(z)$, we see, from (5.4), that $\liminf_{t\downarrow 0} \overline{\tau}(z, \alpha_t) > 0$. Therefore, sending $t \downarrow 0$ in (5.4) and taking into account that $(z + K) \cap B(z, \varepsilon) \cap \Gamma_0 = \emptyset$, we find that

$$(5.5)\qquad u(z) \ge \liminf_{t\downarrow 0} u(X(t)) \ge (u|\overline{\Omega}\backslash\overline{\Gamma}_0, \overline{\Omega})_*(z).$$

Finally, let $z \in \Gamma_0$. Let K and ε be an open convex cone and a positive number from (A4)'. Defining $t_0 > 0$ as above and repeating the same argument, we see that $X(t) \in \Omega$, for $0 < t \le t_0$, and that (5.4) holds, for each $0 < t \le t_0$ and some $\alpha = \alpha_t \in \mathcal{A}$. Therefore we have (5.5). The proof is now completed. □

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Department of Mathematics
Chuo University
Bunkyo-ku,
Tokio 112, Japan