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1. - Introduction

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded open set. We denote the Lebesgue measure of a subset E of $\mathbb{R}^n$ by $|E|$. Let Ω^* be the ball centred at the origin such that $|\Omega^*| = |\Omega|$. Let $u : \Omega \to \mathbb{R}$ be a given function and let $u^* : \Omega^* \to \mathbb{R}$ be its spherically symmetric decreasing rearrangement.

Amongst the useful results arising out of the symmetrization technique is one due to Talenti [10] which states that if, for given $f \geq 0$, $f \in L^2(\Omega)$, $u \in H_0^1(\Omega)$ solves

$$(1.1) \qquad \begin{cases} -\Delta u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{cases}$$

and if $v \in H_0^1(\Omega^*)$ solves

$$(1.2) \qquad \begin{cases} -\Delta v &= f^* \quad \text{in } \Omega^* \\ v &= 0 \quad \text{on } \partial\Omega^*, \end{cases}$$

then

$$(1.3) \qquad u^* \leq v \text{ a.e. in } \Omega^*.$$

Infact a more general result is true but we shall restrict our attention to the one mentioned above. At least two proofs of this result are available. The original proof by Talenti [10] uses an isoperimetric inequality involving the De Giorgi perimeter of Ω. Another proof which does not use this inequality is due to Lions [6]. The essential ingredient of this proof is a differential inequality between the distribution functions of u and v.

The main result we prove here is that if u and v have the same distribution function, then Ω is a ball and u is radial (i.e. spherically symmetric). After a preliminary version of this paper was prepared the author learnt that this result was also proved in Alvino *et al* [2]. However the proof we give here

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differs from theirs and we feel it is conceptually simpler. An application of this result is that, if the *first* eigenvalue of the Laplacian on Ω (with homogeneous Dirichlet boundary conditions) equals that on Ω^*, then Ω is a ball. In particular this proves that a ball in $\mathbb{R}^n$ is isospectral only with balls of the same radius (a result known if $n = 2$, but not proved, to the best of the author's knowledge, for higher dimensions).

Another problem we consider is the following. If $u \in H_0^1(\Omega)$ satisfies $-\Delta u = 1$ (in the sense of distributions) in Ω, we set

$$J(\Omega) = \max_{x \in \Omega} u(x) \tag{1.4}$$

and look for domains Ω of given measure such that J is maximized. An immediate consequence of Talenti's result is that the maximum is attained for a ball. However using the differential inequality proved by Lions [6] we will also show that it is attained *only* for the ball. (Again this is also contained in Alvino *et al* [2] but the proof depends on their proof of the original result. Our approach is different.) This result shows that of all the domains with given measure having a particular point x_0 in their interior, the ball centred at x_0 is the only one for which the mean exit time of the Wiener process starting at x_0 is maximized.

In Section 2 we briefly recall results on the spherically symmetric decreasing rearrangement which we shall use. In section 3 we present the main result and applications follow in Sections 4 and 5.

2. - Preliminaries

Let $\Omega \subset \mathbb{R}^n$ be an open set. Let $\phi : \Omega \to \mathbb{R}$ be a non-negative measurable function. The unidimensional decreasing rearrangement of ϕ is a function $\phi^\# : [0, |\Omega|] \to \mathbb{R}$ defined by

$$\begin{cases} \phi^\#(s) = \inf\{t : \ |\phi > t| < s\}, \ s > 0 \\ \phi^\#(0) = \text{ess.sup } u \ (\leq +\infty), \end{cases} \tag{2.1}$$

where $|\phi > t|$ is the measure of the set $\{\phi > t\} \overset{\text{def}}{=} \{x \in \Omega | \phi(x) > t\}$. If $\phi \in L^1(\Omega)$, as it will be the case in our discussions to follow, $\phi^\#$ is well-defined and finite. The spherically symmetric, decreasing rearrangement of ϕ is defined on Ω^*, the ball centred at the origin and such that $|\Omega^*| = |\Omega|$, and is denoted by ϕ^*. It is defined by

$$\phi^*(x) = \phi^*(|x|) = \phi^\#(\omega_n |x|^n) \tag{2.2}$$

where ω_n is the measure of the unit ball in $\mathbb{R}^n$.

The important properties of ϕ^* are listed below.

1. ϕ^* is a radial (i.e. spherically symmetric) and radially decreasing function such that

$$|\phi^* > t| = |\phi > t| \text{ for all } t. \tag{2.3}$$

2. If $\phi \in L^p(\Omega)$ for any $1 \leq p \leq \infty$, then $\phi^* \in L^p(\Omega^*)$ and

$$\|\phi\|_{L^p(\Omega)} = \|\phi^*\|_{L^p(\Omega^*)}. \tag{2.4}$$

3. If $\phi, \psi \in L^2(\Omega)$, then

$$\int_\Omega \phi\psi \leq \int_0^{|\Omega|} \phi^\# \psi^\# = \int_{\Omega^*} \phi^* \psi^*. \tag{2.5}$$

In particular for any $E \subset \Omega$,

$$\int_E \phi \leq \int_0^{|E|} \phi^\# \tag{2.6}$$

with equality if, and only if, $(\phi|_E)^* = \phi^*|_{E^*}$.

4. If $\phi \in H_0^1(\Omega)$ and $\phi \geq 0$, then $\phi^* \in H_0^1(\Omega^*)$ and

$$\int_{\Omega^*} |\nabla\phi^*|^2 \leq \int_\Omega |\nabla\phi|^2. \tag{2.7}$$

These results are standard ones. The reader is referred to, for instance, Hardy *et al* [4] or to Polya and Szego [8]. See also Mossino [7] for a more readable version.

Another notion we will need is the perimeter of a set $E \subset \mathbb{R}^n$ in the sense of De Giorgi. If $E \subset \Omega$, Ω an open subset of $\mathbb{R}^n$, then the De Giorgi perimeter of E with respect to Ω, denoted by $P_\Omega(E)$, is given by

$$P_\Omega(E) = \sup_{\substack{\phi \neq 0 \\ \phi \in (\mathcal{D}(\Omega))^n}} \frac{|\int_E \operatorname{div} \phi|}{\|\phi\|_{(L^\infty(\Omega))^n}}. \tag{2.8}$$

An important isoperimetric inequality involving this perimeter states that (cf. De Giorgi [3]) if E is bounded then

$$P_{\mathbb{R}^n}(E) \geq n\omega_n^{1/n} |E|^{1-1/n}, \tag{2.9}$$

with equality if, and only if, E is a ball. If E were a smooth bounded open set, the $P_{\mathbb{R}^n}(E)$ is none other than the $(n-1)$-dimensional measure of ∂E. Thus when $n = 2$, (2.9) is the classical isoperimetric inequality linking the perimeter L and the area A of a smooth bounded domain:

$$L^2 \geq 4\pi A, \tag{2.10}$$

with equality only for a disc.

If $\Omega \subset \mathbb{R}^n$ is a bounded open set and $u \in W^{1,1}(\Omega)$ with $u > 0$, it can be shown that for $t > 0$, (cf. Talenti [10])

$$\int_{\{u>t\}} |\nabla u| = \int_t^\infty P_\Omega(\{u > \tau\}) \mathrm{d}\tau, \tag{2.11}$$

whence we deduce that, for $t > 0$,

$$-\frac{\mathrm{d}}{\mathrm{d}t} \int_{\{u>t\}} |\nabla u| = P_\Omega(\{u > t\}). \tag{2.12}$$

If, in addition, $u \in H_0^1(\Omega)$, we also have that

$$P_\Omega(\{u > t\}) = P_{\mathbb{R}^n}(\{u > t\}) \tag{2.13}$$

for $t > 0$.

3. - The Main Result

Let Ω be a bounded open and smooth set in $\mathbb{R}^n$ and let $f \in L^2(\Omega)$, $f \geq 0$. Let $u \in H_0^1(\Omega)$ and $v \in H_0^1(\Omega^*)$ be the respective solutions of

$$-\Delta u = f \text{ in } \Omega, \quad -\Delta v = f^* \text{ in } \Omega^*. \tag{3.1}$$

We set

$$\begin{aligned} \mu(t) &= |u > t| = |u^* > t| \\ \nu(t) &= |v > t|. \end{aligned} \tag{3.2}$$

Notice that $u > 0$ in Ω and $v > 0$ in Ω^*. We can give an explicit representation for v. In fact as v is radial, we can write,

$$-v'' - \frac{n-1}{r} v' = f^*, \quad v'(0) = v(R) = 0, \tag{3.3}$$

where $r = |x|$ and the prime denotes differentiation with respect to r, and $|\Omega| = |\Omega^*| = \omega_n R^n$. Setting $F(\xi) = \int_0^{\xi} f^{\#}(\eta)\mathrm{d}\eta$, we get (cf. Talenti [10])

$$v(r) = \frac{1}{n^2\omega_n^{2/n}} \int_{\omega_n r^n}^{|\Omega|} \xi^{(2/n)-2} F(\xi)\mathrm{d}\xi. \tag{3.4}$$

It follows from (3.4) that for any t such that $0 \le t \le v_{\max} = \max_{0\le r\le R} v(r)$, there exists a unique $r_0(t)$ such that

$$t = \frac{1}{n^2\omega_n^{2/n}} \int_{\omega_n (r_0(t))^n}^{|\Omega|} \xi^{(2/n)-2} F(\xi)\ \mathrm{d}\xi. \tag{3.5}$$

We can now prove our main result.

THEOREM 3.1. *If u is smooth enough and $\mu(t) = \nu(t)$ for all $t \ge 0$, then Ω is a ball and u is radial.*

PROOF. Retracing the proof of Talenti [10] we get that

$$\begin{aligned} -\mu'(t) \int_{\{u>t\}} f &\le -\mu'(t) \int_0^{\mu(t)} f^{\#} \quad \text{(cf. (2.6))} \\ &= -\mu'(t) F(\mu(t)) \\ &= -\nu'(t) F(\nu(t)) \end{aligned} \tag{3.6}$$

since $\mu(t) = \nu(t)$.

Since $f \ge 0$, the expression (3.4) obtained for $v(r)$ shows that it is a decreasing function. Hence

$$\nu(t) = \omega_n (r_0(t))^n \tag{3.7}$$

where $r_0(t)$ was defined in (3.5). If we differentiate the relation (3.5) on both sides with respect to t, we get, in view of (3.7)

$$1 = \frac{-1}{n^2\omega_n^{2/n}}\ \omega_n^{(2/n)-2} (r_0(t))^{2-2n} F(\nu(t))(\nu'(t)).$$

Thus (3.6) yields

$$-\mu'(t) \int_{\{u>t\}} f \le (n\omega_n^{1/n}(\nu(t))^{1-1/n})^2 = (n\omega_n^{1/n}(\mu(t))^{1-1/n})^2.$$

On the other hand, by the isoperimetric inequality (2.9), we have

$$P_{\mathbb{R}^n}(\{u > t\}) \geq n\omega_n^{1/n}(\mu(t))^{1-1/n}.$$

Thus it follows from the above that

$$(3.8) \qquad (n\omega_n^{1/n}(\mu(t))^{1-1/n})^2 \leq (P_{\mathbb{R}^n}(\{u > t\}))^2 \leq (n\omega_n^{1/n}(\mu(t))^{1-1/n})^2.$$

which gives

$$P_{\mathbb{R}^n}(\{u > t\}) = n\omega_n^{1/n}(\mu(t))^{1-1/n}.$$

which is possible only if the set $\{u > t\}$ is a ball. Thus the level sets $\{u = t\}$ are all spheres and so $\partial\Omega$, which is precisely the set $\{u = 0\}$, is also a sphere and so Ω is a ball.

We complete the proof by showing that u is radial (infact then $u = u^*$ and $f = f^*$). This will be done once we show that all the level sets $\{u = t\}$, which are now spheres, are concentric.

We set $\Omega_t = \{u > t\}$ and let x_t be its centre. We assume that u is at least continuous. Then as both u^* and v are radially decreasing and equimeasurable, we have $u^* = v$. Also since $f \geq 0$ we have from (3.5) that $r_0(t) \to 0$ as $t \to v_{\max} = u^*_{\max} = u_{\max}$. Thus $\{\Omega_t\}$ forms a decreasing family of balls with $|\Omega_t| \to 0$ and so $\bigcap_{t\geq 0} \Omega_t$ is a single point where alone $u_{\max}$ is attained. We show that $u_{\max}$ is attained at x_t, $t \geq 0$, so that all the balls will be concentric.

Observe, first of all, that it suffices to do this for $t = 0$. Indeed, since we have $u^* = v$, we see that

$$|v|^2_{1,\Omega} \leq |u|^2_{1.\Omega} = \int_\Omega fu \leq \int_{\Omega^*} f^*u^* = \int_{\Omega^*} f^*v = |v|^2_{1.\Omega}.$$

But then

$$\int_\Omega fu = \int_0^{v_{\max}}\int_{\Omega_t} f = \int_0^{v_{\max}}\int_{\Omega_t^*} f^* = \int_{\Omega^*} f^*v$$

and as $\int_{\Omega_t} f \leq \int_{\Omega_t^*} f^*$ always, we deduce that

$$(3.9) \qquad \int_{\Omega_t} f = \int_{\Omega_t^*} f^*.$$

But this is true if, and only if, $(f|_{\Omega_t})^* = f^*|_{\Omega_t}$. We can now consider for any $t > 0$ the problem

$$\begin{cases} -\Delta(u-t) &= f \quad \text{in } \Omega_t \\ \quad u - t &= 0 \quad \text{on } \partial\Omega_t, \end{cases}$$

which is such that

$$\begin{cases} -\Delta(u^* - t) &= f^* \quad \text{in } \Omega_t^* \\ \quad u^* - t &= 0 \quad \text{on } \partial\Omega_t^* \end{cases}$$

and by a similar argument, $(u - t)_{\max}$ will be attained at x_t; i.e. $u_{\max}$ will be attained at x_t.

So we finally prove that $u_{\max} = u(0)$. Let $w \in H_0^1(\Omega)$ be the solution of

$$-\Delta w = 1 \text{ in } \Omega$$

and let $\phi \in H_0^1(\Omega)$ be the first eigenfunction of the Laplace operator with Dirichlet boundary conditions, i.e. $-\Delta\phi = \lambda_1\phi$, and recall that $\phi > 0$ in $\Omega(= \Omega^*)$.

We remark that all concentric balls centred at 0 in $\Omega(= \Omega^*)$ are level sets of w and ϕ and vice-versa.

Now,

$$\int_\Omega fw = \int_\Omega \nabla u \cdot \nabla w = \int_\Omega u = \int_\Omega u^* = \int_\Omega v = \int_\Omega \nabla v \cdot \nabla w = \int_\Omega f^* w$$

and so

$$\int_0^{w_{\max}} \int_{\{w>t\}} f = \int_0^{w_{\max}} \int_{\{w>t\}} f^*$$

and as before we deduce that

$$\int_{\{w>t\}} f = \int_{\{w>t\}} f^*. \tag{3.10}$$

Then by our preceding remark,

$$\int_{\{\phi>t\}} f = \int_{\{\phi>t\}} f^*, \text{ for all } 0 \le t \le \phi_{\max}. \tag{3.11}$$

Hence

$$\int_\Omega f\phi = \int_\Omega f^*\phi.$$

Now

$$\begin{aligned} \lambda_1 \int_\Omega u\phi &= \int_\Omega \nabla u \cdot \nabla\phi = \int_\Omega f\phi \\ &= \int_\Omega f^*\phi = \int_\Omega \nabla v \cdot \nabla\phi = \lambda_1 \int_\Omega v\phi = \lambda_1 \int_\Omega u^*\phi \end{aligned}$$

and it follows that

$$\int_{\{\phi>t\}} u = \int_{\{\phi>t\}} u^*, \text{ for all } t. \tag{3.12}$$

If $u(0) < u^*(0) = u_{\max}$, by choosing t close enough to $\phi_{\max}$ we will get a contradiction of (3.12). Hence $u(0) = u_{\max}$ and the theorem is proved. ∎

REMARK 3.1. In the case $f \equiv 1$, the solution v of (1.2) is given by $v(r) = (R^2 - r^2)/2n$ and so, for $0 < t \leq R^2/2n$, we have

$$\nu(t) = \omega_n(R^2 - 2nt)^{n/2}.$$

Also $\int_{\{u>t\}} f = \mu(t)$ and so the right-hand side of (3.6) is nothing but

$$-\mu'(t)\mu(t) = -\frac{1}{2}\frac{d}{dt}((\mu(t))^2) = -\frac{1}{2}\frac{d}{dt}(\nu(t))^2$$

which can be computed explicity as

$$n^2\omega_n^2(R^2 - 2nt)^{n-1} = (n\omega_n^{1/n}(\mu(t))^{1-1/n})^2$$

and the result is proved as in Theorem 3.1. In this case since $f = f^*$ already we automatically have $u = u^*$ without further proof.

4. - Hearing the shape of a sphere

In a celebrated paper, Kac [5] posed the following problem. Let Ω be a bounded open set in $\mathbb{R}^n$. Consider the eigenvalue problem

$$\begin{cases} -\Delta u &= \lambda u \quad \text{in } \Omega \\ \quad u &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

It is known that there exists a sequence of eigenvalues

$$0 < \lambda_1(\Omega) < \lambda_2(\Omega) \leq \lambda_3(\Omega) \leq \cdots$$

with $\lambda_k(\Omega) \to \infty$ as $k \to \infty$ and a corresponding sequence of eigenfunctions forming an orthonormal basis for $L^2(\Omega)$. The first eigenvalue $\lambda_1(\Omega)$ is simple with an eigenfunction of constant sign. The problem is to recover information on Ω given the spectrum $\{\lambda_k(\Omega)\}$. In particular if two domains Ω_1 and Ω_2 have the same spectrum, are they isometric?

It has since been shown that for $n \geq 4$, isospectral domains (i.e. domains with the same spectrum) need not be isometric. The problem remains open in dimensions $n = 2$ and $n = 3$. However it is known that when $n = 2$ isospectral domains have the same area and the same perimeter. Thus by the isoperimetric inequality (2.10) it follows that a disc is isospectral only with itself. For details on this and other related problems, the reader is referred to a recent survey article by Protter [9].

We will now show that in all dimensions n, a sphere is completely characterized by its measure and its *first* eigenvalue of the Laplace operator (for homogeneous Dirichlet boundary conditions).

THEOREM 4.1. *Let Ω be a smooth bounded domain in $\mathbb{R}^n$ such that $\lambda_1(\Omega) = \lambda_1(\Omega^*) = \lambda_1$. Then Ω is also a ball.*

PROOF. Let $w_1 > 0,\ w_1 \in H_0^1(\Omega)$ be such that

$$-\Delta w_1 = \lambda_1 w_1 \text{ in } \Omega, \quad \int_\Omega w_1^2 = 1.$$

Let $w > 0,\ w \in H_0^1(\Omega^*)$ be such that

$$-\Delta w = \lambda_1 w_1^* \text{ in } \Omega^*.$$

Then by Talenti's result, $w_1^* \leq w$. Thus if we denote

$$|v|_{1.\Omega^*}^2 = \int_{\Omega^*} |\nabla v|^2$$

we get

$$|w|_{1.\Omega^*}^2 = \lambda_1 \int_{\Omega^*} w_1^* w \leq \lambda_1 \int_{\Omega^*} w^2.$$

But by the Rayleigh-quotient characterization of the first eigenvalue,

$$|w|_{1.\Omega^*}^2 \geq \lambda_1 \int_{\Omega^*} w^2.$$

Thus

$$|w|_{1.\Omega^*}^2 = \lambda_1 \int_{\Omega^*} w^2$$

and since λ_1 is the first eigenvalue, i.e. the infimum of the Rayleigh quotient over all of $H_0^1(\Omega^*)\backslash\{0\}$, we get that w is an eigenfunction, i.e. $-\Delta w = \lambda_1 w$. Hence $w = w_1^*$ and the result follows from Theorem 3.1. ∎

REMARK 4.1. To deduce Theorem 4.1, it is indeed sufficient to have proved the easier and more special case of Theorem 3.1 where $f \equiv 1$. Indeed $\lambda_1(\Omega) = \lambda_1(\Omega^*) = \lambda_1$, then as above, we saw that $w = w_1^*$. Now if $-\Delta u = 1$

in Ω and $-\Delta v = 1$ in Ω^*, $u \in H_0^1(\Omega)$, $v \in H_0^1(\Omega^*)$, we have by Talenti's result that $u^* \leq v$. Hence by (2.4) and (2.5) we get

$$\int_{\Omega^*} w_1^* = \int_{\Omega} w_1 = \int_{\Omega} \nabla u \nabla w_1 = \lambda_1 \int_{\Omega} u w_1 \leq \lambda_1 \int_{\Omega^*} u^* w_1^*$$
$$\leq \lambda_1 \int_{\Omega^*} v w_1^* = \int_{\Omega^*} \nabla v \nabla w_1^* = \int_{\Omega^*} w_1^*.$$

Thus equality holds everywhere and so

$$\int_{\Omega^*} (v - u^*) w_1^* = 0,$$

which gives that $v = u^*$ a.e. (since $w_1^* > 0$ in Ω^*) and thus $\nu(t) = |u^* > t| = |u > t| = \mu(t)$ for $t > 0$, and the result follows.

REMARK 4.2. The essential ingredient in the above proof is that the extreme case of the isoperimetric inequality (2.9) pertains to the ball alone. This is also the fact used in the proof of the case when $n = 2$. There one uses the extension of the Weyl formula of the form (cf. Kac [5])

$$\sum_{k=1}^{\infty} e^{-\lambda_k(\Omega)t} \sim \frac{|\Omega|}{4\pi t} - \frac{L}{4} \frac{1}{\sqrt{2\pi t}}.$$

The conclusion arises out of the fact that $\{\lambda_k(\Omega)\}$ is the same as the sequence for Ω^*. But we prove it using only $\lambda_1(\Omega)$.

5. - On Maximal Mean Exit Times

We return to the problem

$$\text{(5.1)} \qquad \begin{cases} -\Delta u = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

By the strong maximum principle, $u > 0$ in Ω. Let

$$u(x_0) = \max_{x \in \overline{\Omega}} u(x).$$

This quantity has an interpretation in probability theory, which we briefly outline.

Let $\{X(t)\}$ be a Wiener process with $X(0) = \overline{x} \in \Omega$. Let τ be the first exit time, i.e. $X(\tau) \in \partial\Omega$ and $X(t) \in \Omega$ for $t < \tau$. If $-\frac{1}{2}\Delta w = f$ in Ω with

$w = 0$ on $\partial\Omega$, then it follows from Dynkin's formula that

$$E(w(X(0))) - E(w(X(\tau))) = E\Big(\int_0^\tau f(X(s))\mathrm{d}s\Big).$$

Since $w(X(\tau)) = 0$, we get that

$$2u(\overline{x}) = E(\tau) \tag{5.2}$$

when $f \equiv 1$, where u is the solution of (5.1). Thus x_0 is that point in Ω for which the mean exit time of a Wiener process starting from it is maximal. We wish to determine domains Ω of given measure containing a given point x_0 such that the mean exit time is maximized. In other words we need to maximize the functional

$$J(\Omega) = \max_{x\in\overline{\Omega}} \; u(x) \tag{5.3}$$

over domains Ω of constant measure, u being the solution of (5.1).

Using probabilistic arguments, Aizenman and Simon [1] have shown (in fact for a more general case) that the maximum is attained for a ball. This can be proved as an immediate consequence of Talenti's result. Indeed

$$\max_{x\in\Omega} \; u(x) = \max_{y\in\Omega^*} \; u^*(y) = u^*(0) \leq v^*(0) = \frac{R^2}{2n}, \quad |\Omega| = \omega_n R^n,$$

since $v(r) = (R^2 - r^2)/2n$ is the solution of $-\Delta v = 1$ in Ω^*, $v \in H_0^1(\Omega)$. Thus $J(\Omega) \leq R^2/2n$ and it is attained for the ball Ω^*. We improve on this.

THEOREM 5.1. *Let* $u \in H_0^1(\Omega)$ *be the solution of* (5.1). *Let* $|\Omega| = \omega_n R^n$. *If* $\max_{x\in\Omega} \; u(x) = \frac{R^2}{2n}$, *then* Ω *is a ball.*

PROOF. As before we use $\mu(t) = |u > t|$ and $\nu(t) = |v > t|$ where $v(r) = (R^2 - r^2)/2n$. Since both u^* and v are radially decreasing and $u^* \leq v$, we have that

$$\mu(t) = |u^* > t| \leq \nu(t).$$

Now recall the proof of Theorem 3.1, where we saw that (cf. (3.6) with $f \equiv 1$)

$$-\mu'(t)\mu(t) \geq (P_{\mathbb{R}^n}(\{u > t\}))^2 \geq n^2\omega_n^{2/n}(\mu(t))^{2-\frac{2}{n}}.$$

Thus

$$\frac{2}{n} \; \mu'(t)\mu(t)^{(2/n)-1} \leq -2n\omega_n^{2/n}$$

i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}((\mu(t))^{2/n}) \leq -2n\omega_n^{2/n} = \frac{\mathrm{d}}{\mathrm{d}t}(\nu(t))^{2/n}),$$

since $\nu(t)^{2/n} = \omega_n^{2/n}(R^2 - 2nt)$. Thus if we set $\varsigma(t) = (\nu(t))^{2/n} - (\mu(t))^{2/n}$, we get that

$$\varsigma(t) \geq 0,\ \varsigma'(t) \geq 0,\ \varsigma(0) = \varsigma(R^2/2n) = 0 \tag{5.4}$$

since $\mu(R^2/2n) = \nu(R^2/2n) = 0$. It follows from (5.4) that $\varsigma(t) \equiv 0$ which gives $\mu(t) = \nu(t)$ for $t > 0$ and the result follows from Theorem 3.1. ■

REMARK 5.1. This result is contained in Alvino *et al* [1] but the proof is different.

We can generalize the above result further. If L is a second-order elliptic operator in divergence form then again, by Talenti's result, $u^* \leq v$ in Ω^* where $u \in H_0^1(\Omega)$ solves $Lu = f$ in Ω $(f \in L^2(\Omega),\ f \geq 0)$ and $v \in H_0^1(\Omega^*)$ solves $-\Delta v = f^*$ in Ω^*. We thus have

$$\max_{x\in\Omega} u(x) = u^*(0) \leq v(0) = M,$$

where M is defined by

$$M = \frac{1}{n^2\omega_n^{2/n}} \int_0^{|\Omega|} \xi^{(2/n)-2} F(\xi)\mathrm{d}\xi. \tag{5.5}$$

If $\mu(t) = |u > t|$ and $\nu(t) = |v > t|$, we have $\mu(t) \leq \nu(t)$ as before and also (cf. Lions [6])

$$\frac{\mathrm{d}}{\mathrm{d}t}\ (\tilde{H}(\mu(t))) \leq \frac{\mathrm{d}}{\mathrm{d}t}\ (\tilde{H}(\nu(t))),$$

where $\tilde{H}$ is such that $\frac{\mathrm{d}\tilde{H}}{\mathrm{d}t} = H$ and H is given by

$$H(\lambda) = F(\lambda)\ (n\omega_n^{1/n}\ \lambda^{1-1/n})^{-2},$$

F as in Section 3. Now $\tilde{H}$ is a strictly increasing function of λ and so if we set

$$\varsigma(t) = \tilde{H}(\nu(t)) - \tilde{H}(\mu(t)),$$

we again have (5.4) verified (with $R^2/2n$ replaced by M), provided that $\max_{x\in\Omega} u(x) = M$. Thus u is radial and Ω a ball.

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