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On the Asymptotic Behavior of Solutions of Linear Parabolic Equations in L^1 Space

DONG GUN PARK - HIROKI TANABE (*)

The object of this paper is to investigate the asymptotic behavior of the solution of the initial-boundary value problem for the linear parabolic equation

$$\partial u/\partial t + A(x,t,D)u = f(x,t), \text{ in } \Omega \times [0,\infty), \tag{0.1}$$

$$B_j(x,t,D)u = 0, \qquad j = 1,\dots,m/2, \text{ on } \partial\Omega \times [0,\infty), \tag{0.2}$$

$$u(x,0) = u_0(x), \text{ on } \Omega, \tag{0.3}$$

in $L^1(\Omega)$ as $t \to \infty$.

This type of problem for an abstract parabolic evolution equation

$$du(t)/dt + A(t)u(t) = f(t) \tag{0.4}$$

was first treated in [9], and the convergence of the solution $u(t)$ to a stationary state was shown under the assumption that the domain $D(A(t))$ of $A(t)$ is independent of t. Pazy [8] established the asymptotic expansion of the solution of (0.4) assuming a certain asymptotic behavior of $A(t)$ and $f(t)$, and as its application he obtained the asymptotic expansion of the solution of the parabolic problem (0.1)-(0.3) in $L^p(\Omega)$, $1 < p < \infty$, in case when the boundary conditions (0.2) are independent of t.

Recently, Guidetti [4] extended the above results to the case when $D(A(t))$ and the boundary conditions (0.2) depend on time. We show that analogous results for the solution of (0.1)-(0.3) hold in $L^1(\Omega)$ using the method of [7], [11] of estimating the Green function of the problem considered.

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1. - Notations

Let Ω be a not necessarily bounded domain in $\mathbb{R}^n$ locally regular of class C^{2m} and uniformly regular of class C^m in the sense of Browder [2]. The boundary of Ω is denoted by $\partial\Omega$. We put

$$D = (\partial/\partial x_1, \ldots, \partial/\partial x_n).$$

Let

$$A(x,t,D) = \sum_{|\alpha|\le m} a_\alpha(x,t) D^\alpha$$

be a linear differential operator of even order m with coefficients defined in $\overline{\Omega}$ for each fixed $t \in [0,\infty)$, and let

$$B_j(x,t,D) = \sum_{|\beta|\le m_j} b_{j,\beta}(x,t) D^\beta, \qquad j = 1, \ldots, \frac{m}{2},$$

be a set of linear differential operators of respective orders $m_j < m$ with coefficients defined on $\partial\Omega$ for each fixed $t \in [0,\infty)$.

The principal parts of $A(x,t,D)$ and $B_j(x,t,D)$ are denoted by $A^{\#}(x,t,D)$ and $B_j^{\#}(x,t,D)$ respectively.

Let k be a nonnegative integer. For $1 \le p \le \infty$, $W^{k,p}(\Omega)$ stands for the Banach space consisting of all measurable functions defined in Ω whose distribution derivatives of order up to k belong to $L^p(\Omega)$.

The norm of $W^{k,p}(\Omega)$ is defined by

$$\|u\|_{k,p} = \begin{cases} \left(\sum_{|\alpha|\le k} \int_\Omega |D^\alpha u|^p dx\right)^{1/p} & \text{if } 1 \le p < \infty, \\ \max_{|\alpha|\le k} \operatorname{ess\,sup}_\Omega |D^\alpha u| & \text{if } p = \infty. \end{cases}$$

We simply write $\| \ \|_p$ instead of $\| \ \|_{0,p}$ to denote L^p norm for $1 < p \le \infty$.

We use the notation $\| \ \|$ to denote both the norm of $L^1(\Omega)$ and that of bounded linear operators from $L^1(\Omega)$ to itself.

We denote by $B^k(\overline{\Omega})$ the set of all functions which are bounded and uniformly continuous in $\overline{\Omega}$ together with their derivatives of order up to k. $B^k(\overline{\Omega})$ is a Banach space with norm

$$|u|_k = \max_{|\alpha|\le k} \sup_{x\in\Omega} |D^\alpha u(x)|.$$

For $0 < h < 1$, $B^{k+h}(\overline{\Omega})$ is the set of all functions in $B^k(\overline{\Omega})$ whose kth order derivatives are uniformly Hölder continuous of order h. The norm of $B^{k+h}(\overline{\Omega})$ is defined by

$$|u|_{k+h} = |u|_k + \max_{|\alpha|=k} \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x-y|^h}.$$

$B^k(\partial\Omega)$ denotes the Banach space consisting of all functions having bounded and uniformly continuous derivatives of order up to k on $\partial\Omega$.

$B^k(\partial\Omega)$ is a Banach space with norm

$$|u|_{k,\partial\Omega} = \max_{|\alpha|\leq k} \sup_{x\in\partial\Omega} |D^\alpha u(x)|.$$

We denote the set of all bounded linear operators from $L^p(\Omega)$ to $L^p(\Omega)$, $W^{m,p}(\Omega)$ by $B(L^p, L^p)$, $B(L^p, W^{m,p})$ respectively.

For a Banach space X we denote by $B^k(I : X)$ the set of all functions with values in X which are bounded and continuous in the interval I together with their derivatives of order up to k.

2. - Convergence as $t \to \infty$

We assume the following:

(I.1) $A(x,t,D)$ is uniformly strongly elliptic, i.e. there exists an angle $\theta_0 \in (0, \frac{\pi}{2})$ such that for all real vectors $\xi \neq 0$ and all $(x,t) \in \overline{\Omega} \times [0,\infty)$

$$|\arg(-1)^{m/2} A^{\#}(x,t,\xi)| < \theta_0.$$

(I.2) $\{B_j(x,t,D)\}_{j=1}^{m/2}$ is a normal set of boundary operators, i.e. $\partial\Omega$ is noncharacteristic for each $B_j(x,t,D)$ and $m_j \neq m_k$ for $j \neq k$.

(I.3) For any $(x,t) \in \partial\Omega \times [0,\infty)$ let ν be the normal to $\partial\Omega$ at x and $\xi \neq 0$ be parallel to $\partial\Omega$ at x. The polynomials in τ

$$B_j^{\#}(x,t,\xi + \tau\nu), \qquad j = 1, \ldots, m/2$$

are linearly independent modulo the polynomial in τ, $\prod_{j=1}^{m/2} (\tau - \tau_k^+(\xi,\lambda; x,t))$ for any complex number λ with $\theta_0 \leq \arg \lambda \leq 2\pi - \theta_0$ where $\tau_k^+(\xi,\lambda;x,t)$ are the roots with positive imaginary part of the polynomial in τ, $(-1)^{m/2}A^{\#}(x,t,\xi + \tau\nu) - \lambda$.

(I.4) For each $t \in [0,\infty)$ the formal adjoint of $A(x,t,D)$

$$A'(x,t,D) = \sum_{|\alpha|\leq m} a'_\alpha(x,t) D^\alpha$$

and the adjoint system of boundary operators

$$B'_j(x,t,D) = \sum_{|\beta|\leq m'_j} b'_{j,\beta}(x,t) D^\beta, \qquad j = 1, \ldots, \frac{m}{2},$$

can be constructed.

(I.5) For $|\alpha| = m$, $a_\alpha \in B^0(\overline{\Omega} \times [0,\infty))$. For $|\alpha| \le m$, a_α, $a'_\alpha \in B^1([0,\infty); L^\infty(\Omega))$, and

$$\lim_{t\to\infty} \|\dot{a}_\alpha(\cdot,t)\|_\infty = 0, \qquad \lim_{t\to\infty} \|\dot{a}'_\alpha(\cdot,t)\|_\infty = 0,$$

where $\dot{a}_\alpha = \partial a_\alpha/\partial t$, $\dot{a}'_\alpha = \partial a'_\alpha/\partial t$.

(I.6) For $|\beta| \le m_j$, $j = 1,\dots,\frac{m}{2}$, $b_{j,\beta} \in B^1([0,\infty); B^{m-m_j}(\partial\Omega))$, and

$$\lim_{t\to\infty} |\dot{b}_{j,\beta}(\cdot,t)|_{m-m_j,\partial\Omega} = 0.$$

Similarly, for $|\beta| \le m'_j$, $j = 1,\dots,\frac{m}{2}$, $b'_{j,\beta} \in B^1([0,\infty); B^{m-m'_j}(\partial\Omega))$, and

$$\lim_{t\to\infty} |\dot{b}'_{j,\beta}(\cdot,t)|_{m-m'_j,\partial\Omega} = 0.$$

(II) For each $p \in (1,\infty)$ there exists a constant C_p such that for $t \in [0,\infty)$, $\theta_0 \le \arg\lambda \le 2\pi - \theta_0$, $u, v \in W^{m,p}(\Omega)$

$$(2.1)\quad \sum_{j=0}^{m} |\lambda|^{(m-j)/m}\|u\|_{j,p} \le C_p\{\|(A(\cdot,t,D) - \lambda)u\|_p + \sum_{j=1}^{m/2} |\lambda|^{(m-m_j)/m}\|g_j\|_p + \sum_{j=1}^{m/2} \|g_j\|_{m-m_j,p}\},$$

$$(2.2)\quad \sum_{j=0}^{m} |\lambda|^{(m-j)/m}\|v\|_{j,p} \le C_p\{\|(A'(\cdot,t,D) - \lambda)v\|_p + \sum_{j=1}^{m/2} |\lambda|^{(m-m'_j)/m}\|h_j\|_p + \sum_{j=1}^{m/2} \|h_j\|_{m-m'_j,p}\},$$

where g_j and h_j are arbitrary functions in $W^{m-m_j,p}(\Omega)$ and $W^{m-m'_j,p}(\Omega)$ such that $B_j(x,t,D)u = g_j$ and $B'_j(x,t,D)v = h_j$ on $\partial\Omega$ respectively.

REMARK. It is known that under the hypothesis (I.1)-(I.6) the inequalities (2.1), (2.2) hold if we add some positive constant to $A(x,t,D)$ if necessary.

For $1 < p < \infty$ let $A_p(t)$ be the operator defined by

$$D(A_p(t)) = \{u \in W^{m,p}(\Omega) : B_j(x,t,D)u|_{\partial\Omega} = 0,\ j = 1,\dots,\frac{m}{2}\},$$

for $u \in D(A_p(t))$, $(A_p(t)u)(x) = A(x,t,D)u(x)$ in the distribution sense.

Similarly, the operator $A'_p(t)$ is defined by replacing $A(x,t,D)$ and $\{B_j(x,t,D)\}_{j=1}^{m/2}$ by $A'(x,t,D)$ and $\{B'_j(x,t,D)\}_{j=1}^{m/2}$.

From the assumptions above it follows that $-A_p(t)$, $-A'_p(t)$ generate analytic semigroups in $L^p(\Omega)$, and the resolvent sets $\rho(A_p(t))$, $\rho(A'_p(t))$ contain the closed sector

$$\sum = \{\lambda : \theta_0 \leq \arg \lambda \leq 2\pi - \theta_0\} \cup \{0\}.$$

The operator $A(t)$ is defined as follows:

The domain $D(A(t))$ is the totality of functions u satisfying the following three conditions:

(i) $u \in W^{m-1,q}(\Omega)$ for any q with $1 \leq q < n/(n-1)$,

(ii) $A(x,t,D)u \in L^1(\Omega)$ in the sense of distributions,

(iii) for any p with $0 < (n/m)(1 - 1/p) < 1$ and any $v \in D(A'_{p'}(t))$, $p' = p/(p-1)$,

$$(A(x,t,D)u, v) = (u, A'(x,t,D)v).$$

For $u \in D(A(t))$ $(A(t)u)(x) = A(x,t,D)u(x)$ in the distribution sense.

It is known that $-A(t)$ generates an analytic semigroup $\exp(-\tau A(t))$ in $L^1(\Omega)$ ([10], [11]). It can be shown without difficulty that for some positive constant c_0 the inequalities (2.1) and (2.2) hold if we replace $A(x,t,D)$ by $A(x,t,D) - c_0$ and C_p by some other constant. Hence, there exists a constant C_0 such that for $\tau > 0$, $0 \leq t < \infty$

$$\|\exp(-\tau A(t))\| \leq C_0 \exp(-c_0\tau), \tag{2.3}$$

$$A(t)\exp(-\tau A(t))\| \leq C_0\tau^{-1}\exp(-c_0\tau). \tag{2.4}$$

Let $U(t,s)$ be the evolution operator of the evolution equation in $L^1(\Omega)$:

$$du(t)/dt + A(t)u(t) = f(t). \tag{2.5}$$

The existence of such an operator was shown in [6] and it is constructed as follows:

$$U(t,s) = \exp(-(t-s)A(t)) + W(t,s), \tag{2.6}$$

$$W(t,s) = \int_s^t \exp(-(t-\tau)A(t))R(\tau,s)d\tau, \tag{2.7}$$

$$R(t,s) - \int_s^t R_1(t,\tau)R(\tau,s)d\tau = R_1(t,s), \tag{2.8}$$

$$R_1(t,s) = -(\partial/\partial t + \partial/\partial s)\exp(-(t-s)A(t)). \tag{2.9}$$

Our first main result is the following:

THEOREM 2.1. *Suppose that the hypotheses* (I.1)-(I.6), (II) *are satisfied. Let* $f(t)$ *be a uniformly Hölder continuous functions with values in* $L^1(\Omega)$ *defined in* $[0,\infty)$:

$$\|f(t) - f(s)\| \leq C_1(t-s)^h, \qquad 0 \leq s < t < \infty,$$

where C_1 *and* h *are constants with* $C_1 > 0$, $0 < h \leq 1$. *Moreover, assume that the strong limit* $f_0 = \lim_{t\to\infty} f(t)$ *exists. Then, for any solution* $u(t)$ *of the evolution equation* (2.5), *we have*

$$\lim_{t\to\infty} A(t)u(t) = f_0$$

in the strong topology of $L^1(\Omega)$.

Following the argument of [4] we can prove Theorem 2.1 with the aid of (2.3), (2.4) and the following lemma.

LEMMA 2.1. *For each fixed* $s \geq 0$

$$\lim_{t\to\infty} \|A(t)W(t,s)\| = 0 \tag{2.10}$$

For any $\varepsilon > 0$ *there exists a constant* $s_0 \geq 0$ *such that*

$$\int_s^t \|A(t)W(t,\sigma)\| d\sigma < \varepsilon \qquad \text{for } s_0 \leq s < t < \infty. \tag{2.11}$$

We plan to prove Lemma 2.1 as follows. First we note that

$$A(t)W(t,s) = A(t)\int_s^t \exp(-(t-\tau)A(t))R_1(\tau,s)d\tau$$
$$+ \int_s^t A(t)\int_\sigma^t \exp(-(t-\tau)A(t))R_1(\tau,\sigma)d\tau \; R(\sigma,s)d\sigma.$$

If we have a desired estimate of $A(t)^\rho R_1(\tau,s)$ for some $0 < \rho < 1$, then we can write the first term of the right side of (2.12) as

$$\int_s^t A(t)^{1-\rho}\exp(-(t-\tau)A(t))A(t)^\rho R_1(\tau,s)d\tau.$$

Let $W^{\theta,1}(\Omega) = (L^1(\Omega), W^{1,1}(\Omega))_{\theta,1}$ be the real interpolation space of $L^1(\Omega)$ and $W^{1,1}(\Omega)$ with norm denoted by $\| \; \|_{\theta,1}$.

Then, in view of Grisvard [3] we have for $0 < \theta < 1$

$$W^{\theta,1}(\Omega) = (L^1(\Omega), W_0^{m,1}(\Omega))_{\theta/m,1}. \tag{2.13}$$

It is easy to show that for $0 < \rho < \theta/m$

$$(L^1(\Omega), D(A(t)))_{\theta/m,1} \subset D(A(t)^\rho). \tag{2.14}$$

Clearly,

$$W_0^{m,1}(\Omega) \subset D(A(t)). \tag{2.15}$$

Combining (2.13), (2.14), (2.15) we get

$$W^{\theta,1}(\Omega) \subset D(A(t)^\rho). \tag{2.16}$$

Consequently, in order to establish an estimate of $\|A(t)^\rho R_1(\tau, s)\|$ it suffices to obtain that of $\|R_1(\tau, s)\|_{B(L^1, W^{\theta,1})}$ where $B(L^1, W^{\theta,1})$ is the set of all bounded linear operators from $L^1(\Omega)$ to $W^{\theta,1}(\Omega)$. Since

$$\|R_1(\tau, s)f\|_{\theta,1} \le C\|R_1(\tau, s)f\|_{1,1}^{\theta}\|R_1(\tau, s)f\|^{1-\theta}, \tag{2.17}$$

the problem is reduced to estimating $\|R_1(t, s)\|_{B(L^1, W^{1,1})}$ and $\|R(t, s)\|$ for $0 \le s < t < \infty$. In view of (2.1) the desired result follows from the estimates of $\partial^2 G(x, y, \tau; t)/\partial x_i \partial t$, $\partial G(x, y, \tau; t)/\partial t$ where $G(x, y, \tau; t)$ is the kernel of $\exp(-\tau A(t))$.

3. - Proof of Lemma 2.1.

In what follows we let the notation C_p stand for constants depending only on the hypothesis (I.1)-(I.6), (II) and $p \in (1, \infty)$.

Arguing as in [7], [11] we see that for each $p \in (1, \infty)$ there exists a positive constant δ_p such that for each $t \in [0, \infty)$, $\lambda \in \sum$, a complex vector η with $|\eta| \le \delta_p |\lambda|^{1/m}$ and $u, v \in W^{m,p}(\Omega)$

$$\sum_{j=0}^{m} |\lambda|^{(m-j)/m}\|u\|_{j,p} \le C_p\{\|(A(\cdot, t, D+\eta) - \lambda)u\|_p + \sum_{j=1}^{m/2} |\lambda|^{(m-m_j)/m}\|g_j\|_p + \sum_{j=1}^{m/2} \|g_j\|_{m-m_j,p}\}, \tag{3.1}$$

$$\sum_{j=0}^{m} |\lambda|^{(m-j)/m}\|v\|_{j,p} \le C_p\{\|(A'(\cdot, t, D+\eta) - \lambda)v\|_p + \sum_{j=1}^{m/2} |\lambda|^{(m-m'_j)/m}\|h_j\|_p + \sum_{j=1}^{m/2} \|h_j\|_{m-m'_j,p}\}, \tag{3.2}$$

where g_j and h_j are arbitrary functions satisfying $B_j(x,t,D+\eta)u|_{\partial\Omega} = g_j$ and $B'_j(x,t,D+\eta)v|_{\partial\Omega} = h_j$ for $j = 1,\dots,\frac{m}{2}$.

We define the operator $A^\eta_p(t)$ by

$$D(A^\eta_p(t)) = \{u \in W^{m,p}(\Omega) : B_j(x,t,D+\eta)u|_{\partial\Omega} = 0,\ j = 1,\dots,\frac{m}{2}\},$$

for $u \in D(A^\eta_p(t))$ $(A^\eta_p(t)u)(x) = A(x,t,D+\eta)u(x)$ in the sense of distributions.

Similarly replacing $A(x,t,D+\eta)$, $\{B_j(x,t,D+\eta)\}_{j=1}^{m/2}$ by $A'(x,t,D+\eta)$, $\{B'_j(x,t,D+\eta)\}_{j=1}^{m/2}$ the operator $A'^\eta_p(t)$ is defined.

It follows from (3.1), (3.2) that if $|\eta| \le \delta_p|\lambda|^{1/m}$, $\lambda \in \sum$, then

$$\left.\begin{array}{l} \|(A^\eta_p(t)-\lambda)^{-1}\|_{B(L^p,L^p)} \\ \|(A'^\eta_p(t)-\lambda)^{-1}\|_{B(L^p,L^p)} \end{array}\right\} \le \frac{C_p}{|\lambda|}, \tag{3.3}$$

$$\left.\begin{array}{l} \|(A^\eta_p(t)-\lambda)^{-1}\|_{B(L^p,W^{m,p})} \\ \|(A'^\eta_p(t)-\lambda)^{-1}\|_{B(L^p,W^{m,p})} \end{array}\right\} \le C_p. \tag{3.4}$$

Let $\omega(t)$ be a function defined in $[0,\infty)$ such that

$$\begin{aligned} &\lim_{t\to\infty}\omega(t) = 0 \\ &\|\dot a_\alpha(\cdot,t)\|_\infty \le \omega(t),\ \|\dot a'_\alpha(\cdot,t)\|_\infty \le \omega(t) \qquad \text{for } |\alpha| \le m, \\ &|\dot b_{j,\beta}(\cdot,t)|_{m-m_j,\partial\Omega} \le \omega(t) \qquad \text{for } |\beta| \le m_j,\ j = 1,\dots,\frac{m}{2}, \\ &|\dot b'_{j,\beta}(\cdot,t)|_{m-m'_j,\partial\Omega} \le \omega(t) \qquad \text{for } |\beta| \le m'_j,\ j = 1,\dots,\frac{m}{2}. \end{aligned}$$

Since the derivative $\dot w = \partial w/\partial t$ of the function $w(t) = (A^\eta_p(t)-\lambda)^{-1}f$ satisfies

$$\begin{aligned} &(A(x,t,D+\eta)-\lambda)\dot w(x,t) = -\dot A(x,t,D+\eta)w(x,t) \qquad x \in \Omega, \\ &B_j(x,t,D+\eta)\dot w(x,t) = -\dot B_j(x,t,D+\eta)w(x,t),\ j = 1,\dots,\frac{m}{2},\ x \in \partial\Omega, \end{aligned}$$

it follows from (3.1), (3.3) that

$$\|(\partial/\partial t)(A^\eta_p(t)-\lambda)^{-1}\|_{B(L^p,L^p)} \le \frac{C_p\omega(t)}{|\lambda|}, \tag{3.6}$$

$$\|(\partial/\partial t)(A^\eta_p(t)-\lambda)^{-1}\|_{B(L^p,W^{m,p})} \le C_p\omega(t) \tag{3.7}$$

for $|\eta| \le \delta_p|\lambda|^{1/m}$, $\lambda \in \sum$. Similarly for those values of η, λ

$$\|(\partial/\partial t)(A'^\eta_p(t)-\lambda)^{-1}\|_{B(L^p,L^p)} \le \frac{C_p\omega(t)}{|\lambda|}, \tag{3.8}$$

$$\|(\partial/\partial t)(A'^\eta_p(t)-\lambda)^{-1}\|_{B(L^p,W^{m,p})} \le C_p\omega(t). \tag{3.9}$$

We choose natural numbers ℓ, s and exponents $2 = q_1 < q_2 < \ldots < q_s < q_{s+1} = \infty$, $2 = r_1 < r_2 < \ldots < r_{\ell-s} < r_{\ell-s+1} = \infty$ as follows (Beals [1]):

(i) in case $n/2m < 1 - 1/m : \ell = 2$ and $s = 1$, hence $2 = q_1 < q_2 = \infty$, $2 = r_1 < r_2 = \infty$, $q_1^{-1} < (m-1)/n$.

(ii) in case $1 - 1/m \le n/2m < 1 : \ell = 3$ and $s = 2$, $2 = q_1 < q_2 < q_3 = \infty$, $q_1^{-1} - q_2^{-1} < m/n$, $q_2^{-1} < (m-1)/n$, $2 = r_1 < r_2 = \infty$.

(iii) in case $n/2m > 1 : s > n/2m + 1/m$, $\ell - s > n/2m$, $q_j^{-1} - q_{j+1}^{-1} < m/n$ for $j = 1, \ldots, s-1$, $q_{s-1}^{-1} > m/n$, $q_s^{-1} < (m-1)/n$, $m - n/q_s$ is not a nonnegative integer, $r_j^{-1} - r_{j+1}^{-1} < m/n$ for $j = 1, \ldots, \ell - s - 1$, $r_{\ell-s-1}^{-1} > m/n > r_{\ell-s}^{-1}$, $m - n/r_{\ell-s}$ is not a nonnegative integer.

(iv) in case $n/2m = 1 : \ell = 4$ and $s = 2$, $2 = q_1 < q_2 < q_3 = \infty$, $2 = r_1 < r_2 < r_3 = \infty$, $q_2^{-1} < 1/2 - 1/n = (m-1)/n$.

In what follows we consider only the case (iii).

According to Sobolev's imbedding theorem there exists a positive constant γ such that for $j = 1, \ldots, s-1$

$$W^{m,q_j}(\Omega) \subset L^{q_{j+1}}(\Omega), \quad \|u\|_{q_{j+1}} \le \gamma \|u\|_{m,q_j}^{a_j} \|u\|_{q_j}^{1-a_j}, \tag{3.10}$$

where $0 < a_j = (n/m)(q_j^{-1} - q_{j+1}^{-1}) < 1$,

$$\begin{aligned} &W^{m,q_s}(\Omega) \subset B^{m-n/q_s}(\overline{\Omega}) \subset W^{1,\infty}(\Omega), \\ &\|u\|_{1,\infty} \le \gamma \|u\|_{m,q_s}^{a_s+1/m} \|u\|_{q_s}^{1-a_s-1/m}, \end{aligned} \tag{3.11}$$

where $0 < a_s = (n/m)q_s^{-1} < 1 - 1/m$, and for $j = 1, \ldots, \ell - s$

$$W^{m,r_j}(\Omega) \subset L^{r_{j+1}}(\Omega), \quad \|u\|_{r_{j+1}} \le \gamma \|u\|_{m,r_j}^{a_{s+j}} \|u\|_{r_j}^{1-a_{s+j}}, \tag{3.12}$$

where $0 < a_{s+j} = (n/m)(r_j^{-1} - r_{j+1}^{-1}) < 1$.

Let $\delta = \min\{\delta_p : p = q_1, \ldots, q_s, r_1, \ldots, r_{\ell-s}\}$. By virtue of (3.3), (3.4), (3.6), (3.7), (3.10), (3.11) for $\lambda \in \sum$ and $|\eta| \le \delta |\lambda|^{1/m}$

$$\|(A_{q_j}^{\eta}(t) - \lambda)^{-1}\|_{B(L^{q_j}, L^{q_{j+1}})} \le C|\lambda|^{a_j - 1}, \quad j = 1, \ldots, s-1, \tag{3.13}$$

$$\|(A_{q_s}^{\eta}(t) - \lambda)^{-1}\|_{B(L^{q_s}, W^{1,\infty})} \le C|\lambda|^{a_s + 1/m - 1}, \tag{3.14}$$

$$\|(\partial/\partial t)(A_{q_j}^{\eta}(t) - \lambda)^{-1}\|_{B(L^{q_j}, L^{q_{j+1}})} \le C\omega(t)|\lambda|^{a_j - 1}, \quad j = 1, \ldots, s-1, \tag{3.15}$$

$$\|(\partial/\partial t)(A_{q_s}^{\eta}(t) - \lambda)^{-1}\|_{B(L^{q_s}, W^{1,\infty})} \le C\omega(t)|\lambda|^{a_s + 1/m - 1}. \tag{3.16}$$

Similarly, by virtue of (3.3), (3.4), (3.8), (3.9), (3.12) we obtain

$$\|(A'^{\eta}_{r_j}(t) - \lambda)^{-1}\|_{B(L^{r_j}, L^{r_{j+1}})} \le C|\lambda|^{a_{s+j} - 1}, \tag{3.17}$$

$$\|(\partial/\partial t)(A'^{\eta}_{r_j}(t) - \lambda)^{-1}\|_{B(L^{r_j}, L^{r_{j+1}})} \le C\omega(t)|\lambda|^{a_{s+j} - 1} \tag{3.18}$$

for $j = 1, \ldots, \ell - s$.

As is easily seen

$$\exp(-\ell\tau A_2(t)) = (1/2\pi i)^\ell \int_\Gamma \dots \int_\Gamma e^{-\lambda_1\tau - \dots - \lambda_\ell\tau} (A_2(t)-\lambda_1)^{-1} \dots (A_2(t)-\lambda_\ell)^{-1} d\lambda_1 \dots d\lambda_\ell, \tag{3.19}$$

where Γ is a smooth contour running in $\sum\backslash\{0\}$ from $\infty e^{-i\theta_0}$ to $\infty e^{i\theta_0}$.

For $\lambda_1, \dots, \lambda_\ell \in \sum$ and η with

$$|\eta| \leq \delta \min\{|\lambda_1|^{1/m}, \dots, |\lambda_\ell|^{1/m}\} \tag{3.20}$$

set

$$S(t) = (A_2^\eta(t) - \lambda_s)^{-1} \dots (A_2^\eta(t) - \lambda_1)^{-1},$$
$$T(t) = (A_2^\eta(t) - \lambda_{s+1})^{-1} \dots (A_2^\eta(t) - \lambda_\ell)^{-1}.$$

Let $K^\eta_{\lambda_1,\dots,\lambda_\ell}(x,y;t)$ be the kernel of

$$S(t)T(t) = (A_2^\eta(t) - \lambda_1)^{-1} \dots (A_2^\eta(t) - \lambda_\ell)^{-1}.$$

Then $(\partial^2/\partial x_j \partial t) K^\eta_{\lambda_1 \dots \lambda_\ell}(x,y;t)$ is the kernel of

$$D_j(d/dt)S(t) \cdot T(t) + D_j S(t) \cdot (d/dt)T(t).$$

By an elementary calculus

$$\begin{aligned} D_j(d/dt)S(t) = {} & D_j(\partial/\partial t)(A_2^\eta(t) - \lambda_s)^{-1} \\ & \cdot (A_2^\eta(t) - \lambda_{s-1})^{-1} \dots (A_s^\eta(t) - \lambda_1)^{-1} \\ & + \dots + D_j(A_2^\eta(t) - \lambda_s)^{-1} \\ & \dots (A_2^\eta(t) - \lambda_2)^{-1} (\partial/\partial t)(A_2^\eta(t) - \lambda_1)^{-1}. \end{aligned} \tag{3.21}$$

With the aid of (3.13), (3.14), (3.15), (3.16)

$$\begin{aligned} & \|D_j(\partial/\partial t)(A_2^\eta(t) - \lambda_s)^{-1} \cdot (A_2^\eta(t) - \lambda_{s-1})^{-1} \dots (A_2^\eta(t) - \lambda_1)^{-1}\|_{B(L^2,L^\infty)} \\ & \leq \|(\partial/\partial t)(A_{q_s}^\eta(t) - \lambda_s)^{-1}\|_{B(L^q, W^{1,\infty})} \\ & \cdot \prod_{j=1}^{s-1} \|(A_{q_j}^\eta(t) - \lambda_j)^{-1}\|_{B(L^{q_j}, L^{q_{j+1}})} \\ & \leq C\omega(t)|\lambda_s|^{1/m} \prod_{j=1}^{s} |\lambda_j|^{s_j - 1}. \end{aligned}$$

Estimating other terms of the right side of (3.21) analogously we obtain

$$\|D_j(d/dt)S(t)\|_{B(L^2,L^\infty)} \leq C\omega(t)|\lambda_s|^{1/m} \prod_{j=1}^{s} |\lambda_j|^{a_j - 1}. \tag{3.22}$$

Similarly we get

$$\|T^*(t)\|_{B(L^2,L^\infty)} \le C \prod_{j=s+1}^{\ell} |\lambda_j|^{a_j-1}, \tag{3.23}$$

$$\|D_j S(t)\|_{B(L^2,L^\infty)} \le C|\lambda_s|^{1/m} \prod_{j=1}^{s} |\lambda_j|^{a_j-1}, \tag{3.24}$$

$$\|(d/dt)T^*(t)\|_{B(L^2,L^\infty)} \le C\omega(t) \prod_{j=s+1}^{\ell} |\lambda_j|^{a_j-1}. \tag{3.25}$$

Hence

$$|(\partial^2/\partial x_j \partial t)K^{\eta}_{\lambda_1,\ldots,\lambda_\ell}(x,y;t)| \le C|\lambda_s|^{1/m} \prod_{j=1}^{\ell} |\lambda_j|^{a_j-1}. \tag{3.26}$$

It is easy to show

$$|(\partial/\partial t)K^{\eta}_{\lambda_1,\ldots,\lambda_\ell}(x,y;t)| \le C\omega(t) \prod_{j=1}^{\ell} |\lambda_j|^{a_j-1}. \tag{3.27}$$

Let $K_{\lambda_1,\ldots,\lambda_\ell}(x,y;t)$ be the kernel of $(A_2(t)-\lambda_1)^{-1}\ldots(A_2(t)-\lambda_\ell)^{-1}$. Then as was shown in [7], [11]

$$K_{\lambda_1,\ldots,\lambda_\ell}(x,y;t) = e^{(x-y)\eta} K^{\eta}_{\lambda_1,\ldots,\lambda_\ell}(x,y;t). \tag{3.28}$$

With the aid of (3.20), (3.26), (3.27), (3.28) we obtain

$$|(\partial^2/\partial x_j \partial t)K_{\lambda_1,\ldots,\lambda_\ell}(x,y;t)| \le C\omega(t) e^{(x-y)\eta} |\lambda_s|^{1/m} \prod_{j=1}^{\ell} |\lambda_j|^{a_j-1}.$$

Minimizing the right side of the above inequality with respect to η we get (Hörmander [5])

$$\begin{aligned} &|(\partial^2/\partial x_j \partial t)K_{\lambda_1,\ldots,\lambda_\ell}(x,y;t)| \\ &\le C\omega(t)|\lambda_s|^{1/m} \prod_{j=1}^{\ell} |\lambda_j|^{a_j-1} \exp\{-\delta \min(|\lambda_1|^{1/m},\ldots,|\lambda_\ell|^{1/m})|x-y|\} \\ &\le C\omega(t)|\lambda_s|^{1/m} \prod_{j=1}^{\ell} |\lambda_j|^{a_j-1} \sum_{k=1}^{\ell} \exp(-\delta|\lambda_k|^{1/m}|x-y|) \end{aligned} \tag{3.29}$$

In view of (3.19) we have

$$\begin{aligned} &G(x,y,\ell\tau;t) \\ &= (1/2\pi i)^\ell \int_\Gamma \cdots \int_\Gamma e^{-\lambda_1\tau-\cdots-\lambda_\ell\tau} K_{\lambda_1,\ldots,\lambda_\ell}(x,y;t)\,d\lambda_1\cdots d\lambda_\ell. \end{aligned} \tag{3.30}$$

For any fixed $x, y \in \Omega, \tau > 0$ let $\Gamma_{x,y,\tau}$ be the contour defined by

$$\Gamma_{x,y,\tau} = \{\lambda : |\arg \lambda| = \theta_0,\ |\lambda| \geq a\} \cup \{\lambda : \lambda = ae^{i\theta},\ \theta_0 \leq \theta \leq 2\pi - \theta_0\}$$

where $a = \varepsilon(|x-y|/\tau)^{m/(m-1)} = \varepsilon\rho/\tau$, $\rho = |x-y|^{m/(m-1)}/\tau^{1/(m-1)}$ and ε is a positive constant which will be fixed later. Differentiating both sides of (3.30) with respect to x_j and t, deforming the integral path Γ to $\Gamma_{x,y,\tau}$, and using (3.29) yield

$$\begin{aligned}&|(\partial^2/\partial x_j \partial t)G(x,y,\ell\tau;t)|\\ &\leq C\omega(t)\sum_{k=1}^{\ell}\int_{\Gamma_{x,y,\tau}}\cdots\int_{\Gamma_{x,y,\tau}} e^{-Re\lambda_1\tau-\ldots-Re\lambda_\ell\tau}|\lambda_s|^{1/m}\\ &\times\prod_{j=1}^{\ell}|\lambda_j|^{a_j-1}\exp(-\delta|\lambda_k|^{1/m}|x-y|)|d\lambda_1\ldots d\lambda_\ell|.\end{aligned}$$

Estimating the right side of the above inequality as in section 5 of [6] we conclude

$$(3.31)\qquad |(\partial^2/\partial x_j\partial t)G(x,y,\tau;t)| \leq \frac{C\omega(t)}{\tau^{(n+1)/m}}\exp(-c\,\frac{|x-y|^{m/(m-1)}}{\tau^{1/(m-1)}}).$$

Similarly

$$(3.32)\qquad |(\partial/\partial t)G(x,y,\tau;t)| \leq \frac{C\omega(t)}{\tau^{n/m}}\exp(-c\,\frac{|x-y|^{m/(m-1)}}{\tau^{1/(m-1)}}).$$

If we denote the kernel of $R_1(t,s)$ by $R_1(x,y,t,s)$, then in view of (2.9)

$$R_1(x,y,t,s) = -(\partial/\partial t)G(x,y,\tau;t)|_{\tau=t-s}.$$

By virtue of (3.31) and (3.32)

$$\begin{aligned}(3.33)\qquad &|(\frac{\partial}{\partial x_j})R_1(x,y,t,s)| = |\,(\frac{\partial^2}{\partial x_j\partial t})G(x,y,\tau;t)\Big|_{\tau=t-s}|\\ &\leq \frac{C\omega(t)}{(t-s)^{(n+1)/m}}\exp(-c\,\frac{|x-y|^{m/(m-1)}}{(t-s)^{1/(m-1)}}),\end{aligned}$$

$$(3.34)\qquad |R_1(x,y,t,s)| \leq \frac{C\omega(t)}{(t-s)^{n/m}}\exp(-c\,\frac{|x-y|^{m/(m-1)}}{(t-s)^{1/(m-1)}}).$$

With the aid of (3.33), (3.34) we conclude

$$\|(\frac{\partial}{\partial x_j})R_1(t,s)f\| \leq C\omega(t)(t-s)^{-1/m}\|f\|, \tag{3.35}$$

$$\|R_1(t,s)f\| \leq C\omega(t)\|f\| \tag{3.36}$$

for any $f \in L^1(\Omega)$.

We choose constants ρ and θ so that $0<\rho<\theta/m,\ 0<\theta<1$. Combining (2.16), (2.17), (3.35), (3.36) we obtain

$$\|A(t)^\rho R_1(\tau,s)\| \leq C\omega(\tau)\{1+(\tau-s)^{-1/m}\}^\theta. \tag{3.37}$$

By virtue of (2.8), (3.36) and Gronwall's inequality we get

$$\|R(t,s)\| \leq C\omega(t)\exp(C\int_s^t \omega(\tau)d\tau). \tag{3.38}$$

Using (3.37) and the inequality

$$\|A(t)^{1-\rho}\exp\ (-(t-\tau)A(t))\| \leq C(t-\tau)^{\rho-1}$$

we get

$$\begin{aligned} &\|A(t)\int_s^t \exp\ (-(t-\tau)A(t))R_1(\tau,s)d\tau\| \\ &= \|\int_s^t A(t)^{1-\rho}\exp\ (-(t-\tau)A(t))A(t)^\rho R_1(\tau,s)d\tau\| \\ &\leq C\int_s^t (t-\tau)^{\rho-1}\{1+(\tau-s)^{-\theta/m}\}\omega(\tau)d\tau \\ &\leq C\{(t-s)^\rho+(t-s)^{\rho-\theta/m}\}\sup_{s\leq\tau\leq t}\omega(\tau). \end{aligned} \tag{3.39}$$

Making use of (3.37) and (3.39) yields

$$\begin{aligned} &\|\int_s^t A(t)\int_\sigma^t \exp\ (-(t-\tau)A(t))R_1(\tau,\sigma)d\tau\ R(\sigma,s)d\sigma\| \\ &\leq C\int_s^t \{(t-\sigma)^\rho+(t-\sigma)^{\rho-\theta/m}\}\sup_{s\leq\tau\leq t}\omega(\tau)\omega(\sigma)\exp\ (C\int_s^\sigma \omega(\tau)d\tau)d\sigma \\ &\leq C\{(t-s)^{\rho+1}+(t-s)^{\rho-\theta/m+1}\}\sup_{s\leq\tau\leq t}\omega(\tau)^2\exp\ (C\int_s^t \omega(\tau)d\tau). \end{aligned} \tag{3.40}$$

Combining (2.12), (3.39), (3.40) we get

$$\|A(t)W(t,s)\| \le C\{(t-s)^{\rho} + (t-s)^{\rho-\theta/m}\} \sup_{s\le\tau\le t} \omega(\tau)$$

$$+ C\{(t-s)^{\rho+1} + (t-s)^{\rho-\theta/m+1}\} \sup_{s\le\tau\le t} \omega(\tau)^2 \exp\ (C\int_s^t \omega(\tau)d\tau)$$

With the aid of (2.7) and (3.38) we get

$$\|W(t,s)\| \le \{\exp\ (C\int_s^t \omega(\tau)d\tau) - 1\} \sup \|\exp\ (-\tau A(t))\| \tag{3.42}$$

$$\|U(t,s)\| \le \exp\ (C\int_s^t \omega(\tau)d\tau) \sup \|\exp\ (-\tau A(t))\|. \tag{3.43}$$

As was mentioned in section 1 all the hypothesis (I.1)-(I.6), (II) are satisfied by $A(x,t,D) - c_0$, $\{B_j(x,t,D)\}_{j=1}^{m/2}$ for some $c_0 > 0$ if we replace C_p by some other constant. If we denote by $U^0(t,s)$, $W^0(t,s)$, $R_1^0(t,s)$, $R^0(t,s)$ operators obtained by replacing $A(t)$ by $A^0(t) = A(t) - c_0$ in the definition of $U(t,s)$, $W(t,s)$, $R_1(t,s)$, $R(t,s)$, then

$$U(t,s) = U^0(t,s)e^{-c_0(t-s)}, \quad W(t,s) = W^0(t,s)e^{-c_0(t-s)},$$
$$R_1(t,s) = R_1^0(t,s)e^{-c_0(t-s)}, \quad R(t,s) = R^0(t,s)e^{-c_0(t-s)},$$

and (3.41), (3.42) hold with $A^0(t)$, $W^0(t,s)$ in place of $A(t)$, $W(s,t)$.
Hence

$$\begin{aligned} \|A(t)W(t,s)\| &= \|((A^0(t)+c_0)W^0(t,s)e^{-c_0(t-s)}\| \\ &\le \|A^0(t)W^0(t,s)\|e^{-c_0(t-s)} + c_0\|W^0(t,s)\|e^{-c_0(t-s)} \\ &\le C\{(t-s)^{\rho} + (t-s)^{\rho-\theta/m}\} \sup_{s\le\tau\le t} \omega(\tau)e^{-c_0(t-s)} \\ &+ C\{(t-s)^{\rho+1} + (t-s)^{\rho-\theta/m+1}\} \sup_{s\le\tau\le t} \omega(\tau)^2 \\ &\times \exp\ (C\int_s^t \omega(\tau)d\tau - c_0(t-s))) \\ &+ c_0C_0\{\exp\ (C\int_s^t \omega(\tau)d\tau) - 1\}e^{-c_0(t-s)}, \end{aligned} \tag{3.44}$$

which implies (2.10).

With the aid of (3.44) we have

$$\int_s^t \|A(t)W(t,\sigma)\|d\sigma \le C\int_s^t \{(t-\sigma)^\rho + (t-\sigma)^{\rho-\theta/m}\}e^{-c_0(t-s)}d\sigma \sup_{s\le\tau\le t}\omega(\tau)$$

$$+C\int_s^t\{(t-\sigma)^{\rho+1} + (t-\sigma)^{\rho-\theta/m+1}\}\exp\left(C\int_s^t\omega(\tau)d\tau - c_0(t-\sigma)\right)d\sigma \sup_{s\le\tau\le t}\omega(\tau)^2$$

$$+c_0C_0\int_s^t\{\exp\left(C\int_s^t\omega(\tau)d\tau\right)-1\}e^{-c_0(t-\sigma)}d\sigma.$$

Let ε be an arbitrary positive number. If s is so large that $\sup_{s\le\tau\le\infty}\omega(\tau)<\varepsilon$, then the right side of (3.45) does not exceed

$$C\int_0^\infty(\tau^\rho+\tau^{\rho-\theta/m})e^{-c_0\tau}d\tau\ \varepsilon + C\int_0^\infty(\tau^{\rho+1}+\tau^{\rho-\theta/m+1})e^{-(c_0-C\varepsilon)\tau}d\tau\ \varepsilon^2$$

$$+c_0C_0\int_0^\infty(e^{-(c_0-C\varepsilon)\tau}-e^{-c_0\tau})d\tau,$$

from which the second half of the assertion of Lemma 2.1 follows.

Thus the proof of Lemma 2.1 is complete.

4. - Asymptotic expansion at $t=\infty$

In this section we consider the asymptotic expansion at $t=\infty$.

In addition to (I.1)-(I.6), (II) we make the following assumptions.

(III.1) For $|\alpha|\le m$

$$a_\alpha(x,t)=\sum_{k=0}^{\nu}t^{-k}a_{\alpha,k}(x)+t^{-\nu}r_\alpha(x,t), \tag{4.1}$$

$$a'_\alpha(x,t)=\sum_{k=0}^{\nu}t^{-k}a'_{\alpha,k}(x)+t^{-\nu}r'_\alpha(x,t) \tag{4.2}$$

with $a_{\alpha,k}$, $a'_{\alpha,k}\in L^\infty(\Omega)$ for $k=0,\dots,\nu$ and r_α, $r'_\alpha\in B^1([0,\infty):L^\infty(\Omega))$. If $|\alpha|=m$, $a_{\alpha,k}\in B^0(\overline{\Omega})$ and $r_\alpha\in B^0(\overline{\Omega}\times[0,\infty))$, and hence so do $a'_{\alpha,k}$ and r'_α.

(III.2) For $|\alpha| \leq m$

$$\lim_{t\to\infty} \|r_\alpha(\cdot,t)\|_\infty = 0, \qquad \lim_{t\to\infty} \|\dot{r}_\alpha(\cdot,t)\|_\infty = 0, \tag{4.3}$$

$$\lim_{t\to\infty} \|r'_\alpha(\cdot,t)\|_\infty = 0, \qquad \lim_{t\to\infty} \|\dot{r}'_\alpha(\cdot,t)\|_\infty = 0, \tag{4.4}$$

where $\dot{r}_\alpha = \partial r_\alpha/\partial t$, $\dot{r}'_\alpha = \partial r'_\alpha/\partial t$.

(III.3) For $|\beta| \leq m_j$, $j = 1,\ldots,\frac{m}{2}$,

$$b_{j,\beta}(x,t) = \sum_{k=0}^{\nu} t^{-k} b_{j,\beta,k}(x) + \rho_{j,\beta}(x,t) \tag{4.5}$$

with $b_{j,\beta,k} \in B^{m-m_j}(\partial\Omega)$ for $k = 0,\ldots,\nu$ and $\rho_{j,\beta} \in B^1([0,\infty); B^{m-m_j}(\partial\Omega))$.
For $|\beta| \leq m'_j$, $j = 1,\ldots,\frac{m}{2}$,

$$b'_{j,\beta}(x,t) = \sum_{k=0}^{\nu} t^{-k} b'_{j,\beta,k}(x) + \rho'_{j,\beta}(x,t) \tag{4.6}$$

with $b'_{j,\beta,k} \in B^{m-m'_j}(\partial\Omega)$ for $k = 0,\ldots,\nu$ and $\rho'_{j,\beta} \in B^1([0,\infty); B^{m-m'_j}(\partial\Omega))$.

(III.4) For $|\beta| \leq m_j$, $j = 1,\ldots,\frac{m}{2}$,

$$\lim_{t\to\infty} |\rho_{j,\beta}(x,t)|_{m-m_j,\partial\Omega} = 0, \qquad \lim_{t\to\infty} |\dot{\rho}_{j,\beta}(x,t)|_{m-m_j,\partial\Omega} = 0, \tag{4.7}$$

and for $|\beta| \leq m'_j$, $j = 1,\ldots,\frac{m}{2}$,

$$\lim_{t\to\infty} |\rho'_{j,\beta}(\cdot,t)|_{m-m'_j,\partial\Omega} = 0, \qquad \lim_{t\to\infty} |\dot{\rho}'_{j,\beta}(\cdot,t)|_{m-m'_j,\partial\Omega} = 0, \tag{4.8}$$

where $\dot{\rho}_{j,\beta} = \partial\rho_{j,\beta}/\partial t$, $\dot{\rho}'_{j,\beta} = \partial\rho'_{j,\beta}/\partial t$.

THEOREM 4.1. *Suppose that the hypotheses* (I.1)-(I.6), (II), (III.1)-(III.4) *are satisfied. Let* $f(t)$ *be such that*

$$f(t) = \sum_{k=0}^{\nu} t^{-k} f_k + t^{-\nu} r(t)$$

with $f_k \in L^1(\Omega)$ *for* $k = 0,\ldots,\nu$ *and* $r \in B^0([0,\infty); L^1(\Omega))$, $\lim_{t\to\infty} \|r(t)\| = 0$.
Then for any mild solution of (1.5)

$$u(t) = \sum_{k=0}^{\nu} t^{-k} u_k + t^{-k} \rho(t)$$

with $u_k \in L^1(\Omega)$ *for* $k = 0, \ldots, \nu$ *and* $\rho \in B^0([0,\infty); L^1(\Omega))$, $\lim_{t\to\infty} \|\rho(t)\| = 0$.

According to the argument of theorem 1.4 of [4] it suffices to show that

$$A(t)^{-1} = \sum_{k=0}^{\nu} t^{-k} T_k + t^{-\nu} R(t) \tag{4.9}$$

with T_k, $R(t) \in B(L^1, L^1)$ for $k = 0, \ldots, \nu$ and $t \in [0, \infty)$, $\lim_{t\to\infty} R(t) = 0$, $\lim_{t\to\infty} (d/dt)R(t) = 0$ in the strong operator topology. Actually we shall prove this convergence in the uniform operator topology.

In what follows we assume that $b_{j,\beta,k}$, $\rho_{j,\beta}$, $b'_{j,\beta,k}$, $\rho'_{j,\beta}$ are extended to the whole of $\overline{\Omega}$ or $\overline{\Omega} \times [0,\infty)$ so that

$$\begin{aligned}
&|b_{j,\beta,k}|_{m-m_j} \le 2|b_{j,\beta,k}|_{m-m_j,\partial\Omega},\\
&|\rho_{j,\beta}(\cdot,t)|_{m-m_j} \le 2|\rho_{j,\beta}(\cdot,t)|_{m-m_j,\partial\Omega},\\
&|b'_{j,\beta,k}|_{m-m'_j} \le 2|b'_{j,\beta,k}|_{m-m'_j,\partial\Omega},\\
&|\rho'_{j,\beta}(\cdot,t)|_{m-m'_j} \le 2|\rho'_{j,\beta}(\cdot,t)|_{m-m'_j,\partial\Omega}.
\end{aligned}$$

We put

$$A(x,D) = \sum_{|\alpha|\le m} a_{\alpha,0}(x) D^\alpha, \qquad B_j(x,D) = \sum_{|\beta|\le m_j} b_{j,\beta,0}(x) D^\beta$$

$$A_k(x,D) = \sum_{|\alpha|\le m} a_{\alpha,k}(x) D^\alpha, \qquad B_{j,k}(x,D) = \sum_{|\beta|\le m_j} b_{j,\beta,k}(x) D^\beta$$

for $k = 1, \ldots, \nu$, and

$$\tilde{A}(x,t,D) = \sum_{|\alpha|\le m} r_\alpha(x,t) D^\alpha, \ \tilde{B}_j(x,t,D) = \sum_{|\beta|\le m_j} \rho_{j,\beta}(x,t) D^\beta.$$

Analogously, the operators $A'(x,D)$, $A'_k(x,D)$, $\tilde{A}'(x,t,D)$, $B'_j(x,D)$, $B'_{j,k}(x,D)$, $\tilde{B}'_j(x,t,D)$ are defined.

It is obvious that (3.1) holds also for $A(x, D+\eta)$, $\{B_j(x, D+\eta)\}_{j=1}^{m/2}$ in place of $A(x,t,D+\eta)$, $\{B_j(x,t,D+\eta)\}_{j=1}^{m/2}$: for $\lambda \in \sum$, $\eta \in \mathbf{C}^n$ with $|\eta| \le \delta_p |\lambda|^{1/m}$ and $u \in w^{m,p}(\Omega)$

$$\begin{aligned}
\sum_{j=0}^{m} |\lambda|^{(m-j)/m} \|u\|_{j,p} \le C_p\{\|(A(x,D+\eta) - \lambda)u\|_p \\
+ \sum_{j=1}^{m/2} |\lambda|^{(m-m_j)/m} \|g_j\|_p + \sum_{j=1}^{m/2} \|g_j\|_{m-m_j}\},
\end{aligned} \tag{4.10}$$

where g_j is an arbitrary function in $W^{m-m_j,p}(\Omega)$ satisfying $B_j(x, D+\eta)u|_{\partial\Omega} = g_j$ for each $j = 1, \ldots, m/2$. This is the same for the inequality (3.2).

Let A_p be the operator defined by

$$D(A_p) = \{u \in W^{m,p}(\Omega);\ B_j(x,D)u|_{\partial\Omega} = 0,\ j = 1,\dots,\frac{m}{2}\},$$

$(A_p u)(x) = A(x,D)u(x)$ for $u \in D(A_p)$ in the distribution sense, and A_p^η be the operator defined analogously with $A(x,D+\eta)$ and $\{B_j(x,D+\eta)\}_{j=1}^{m/2}$ in place of $A(x,D)$ and $\{B_j(x,D)\}_{j=1}^{m/2}$. Similarly, the operator A'_p, A'^η_p are defined.

For $\lambda \in \sum \setminus \{0\}$, $\eta \in \mathbf{C}^n$ with $|\eta| \le \delta_p |\lambda|^{1/m}$ and $f \in L^p(\Omega)$, $1 < p < \infty$, we put $v(t) = (A_p^\eta(t) - \lambda)^{-1} f$ and $v_0 = (A_p^\eta - \lambda)^{-1} f$. In view of (3.1) and (4.10)

$$\sum_{j=0}^{m} |\lambda|^{(m-j)/m} \|v(t)\|_{j,p} \le C_p \|f\|_p, \tag{4.11}$$

$$\sum_{j=0}^{m} |\lambda|^{(m-j)/m} \|v_0\|_{j,p} \le C_p \|f\|_p. \tag{4.12}$$

We define a finite sequence of functions v_i, $i = 1,\dots,\nu$, successively as the solutions of the following boundary value problems:

$$(A(x,D+\eta) - \lambda)v_i(x) = -\sum_{k=0}^{i=1} A_{i-k}(x,D+\eta)v_k(x), \qquad x \in \Omega$$

$$B_j(x,D+\eta)v_i(x) = -\sum_{k=0}^{i-1} B_{j,i-k}(x,D+\eta)v_k(x),\ j = 1,\dots,\frac{m}{2},\ x \in \partial\Omega.$$

Since the functions v_i, $i = 1,\dots,\nu$, are uniquely determined by f, we may denote them as

$$v_i = H_{i,\lambda,p}^\eta\, f, \qquad i = 1,\dots,\nu.$$

We put

$$H_{0,\lambda,p}^\eta = (A_p^\eta - \lambda)^{-1}, \qquad H_{0,\lambda,p} = (A_p - \lambda)^{-1},$$

$$R_{\lambda,p}^\eta(t) = t^\nu (A_p^\eta(t) - \lambda)^{-1} - \sum_{i=0}^{\nu} t^{\nu-i} H_{i,\lambda,p}^\eta,\ R_{\lambda,p}(t) = R_{\lambda,p}^0(t).$$

Clearly

$$(A_p^\eta(t) - \lambda)^{-1} = \sum_{i=0}^{\nu} t^{-i} H_{i,\lambda,p}^\eta + t^{-\nu} R_{\lambda,p}^\eta(t). \tag{4.13}$$

Applying (4.10) to v_i yields

$$\sum_{j=0}^{m}|\lambda|^{(m-j)/m}\|v_i\|_{j,p} \le C_p\{\|\sum_{k=0}^{i-1} A_{i-k}(x,D+\eta)v_k\|_p$$
$$+\sum_{j=1}^{m/2}|\lambda|^{(m-m_j)/m}\|\sum_{k=0}^{i-1} B_{j,i-k}(x,D+\eta)v_k\|_p$$
$$+\sum_{j=1}^{m/2}\|\sum_{k=0}^{i-1} B_{j,i-k}(x,D+\eta)v_k\|_{m-m_j,p}\}$$
$$\le C_p\sum_{=0}^{i-1}\sum_{j=0}^{m}|\lambda|^{(m-j)/m}\|v_k\|_{j,p}.$$

It follows from (4.12) and the above inequality that

$$\sum_{j=0}^{m}|\lambda|^{(m-j)/m}\|H^{\eta}_{i,\lambda,p}\ f\|_{j,p} \le C_p\|f\|_p, \qquad i=0,\ldots,\nu. \tag{4.14}$$

We put

$$\omega_1(t) = \max\{\|r_\alpha(\cdot,t)\|_\infty,\ \|\dot r_\alpha(\cdot,t)\|_\infty,\ |\rho_{j,\beta}(\cdot,t)|_{m-m_j},$$
$$|\dot\rho_{j,p}(\cdot,t)|_{m-m_j};\ |\alpha|\le m,\ |\beta|\le m_j,\ j=1,\ldots,\frac{m}{2}\}.$$
$$\omega_2(t) = \max\{\|r'_\alpha(\cdot,t)\|_\infty,\ \|\dot r'_\alpha(\cdot,t)\|_\infty,\ |\rho'_{j,\beta}(\cdot,t)|_{m-m'_j},$$
$$|\dot\rho'_{j,\beta}(\cdot,t)|_{m-m'_j};\ |\alpha|\le m,\ |\beta|\le m'_j,\ j=1,\ldots,\frac{m}{2}\}.$$

An elementary calculus yields

$$(A(x,t,D+\eta)-\lambda)(t^\nu v(x,t)-\sum_{i=0}^{\nu} t^{\nu-i}v_i(x))$$
$$= -\tilde A(x,t,D+\eta)v_0(x),\quad x\in\Omega$$
$$B_j(x,t,D+\eta)(t^\nu v(x,t)-\sum_{i=0}^{\nu} t^{\nu-i}v_i(x))$$
$$= -\tilde B_j(x,t,D+\eta)v_0(x),\qquad j=1,\ldots,\frac{m}{2},\qquad x\in\partial\Omega.$$

Hence, applying (4.10) to

$$R^{\eta}_{\lambda,p}(t)f = t^\nu v(t)-\sum_{i=0}^{\nu} t^{\nu-i}v_i \qquad \text{and} \qquad (\partial/\partial t)R^{\eta}_{\lambda,p}(t)f,$$

we easily get

$$\sum_{j=0}^{m} |\lambda|^{(m-j)/m} \|R^{\eta}_{\lambda,p}(t)f\|_{j,p} \le C_p \omega_1(t) \|f\|_p, \tag{4.15}$$

$$\sum_{j=0} |\lambda|^{(m-j)/m} \|(\partial/\partial t)R^{\eta}_{\lambda,p}(t)\|_{j,p} \le C_p \omega_1(t) \|f\|_p. \tag{4.16}$$

The inequality (4.15) implies

$$\lim_{t\to\infty} \|R^{\eta}_{\lambda,p}(t)f\|_{m,p} = 0, \quad . \lim_{t\to\infty} \|(\partial/\partial t)R^{\eta}_{\lambda,p}(t)f\|_{m,p} = 0. \tag{4.17}$$

Similarly, replacing $(A,\{B_j\})$ by its adjoint $(A',\{B'_j\})$ we define operators $H'^{\eta}_{i,\lambda,p}$, $i=0,\dots,\nu$, and $R'^{\eta}_{\lambda,p}(t)$ so that

$$(A'^{\eta}_{p}(t)-\lambda)^{-1} = \sum_{=0}^{\nu} t^{-\nu} H'^{\eta}_{i,\lambda,p} + t^{-\nu} R'^{\eta}_{\lambda,p}(t).$$

We obtain

$$\sum_{j=0}^{m} |\lambda|^{(m-j)/m} \|H'^{\eta}_{i,\lambda,p} f\|_{j,p} \le C_p \|f\|_p, \; i=0,\dots,\nu, \tag{4.18}$$

$$\sum_{j=0}^{m} |\lambda|^{(m-j)/m} \|R'^{\eta}_{\lambda,p}(t)f\|_{j,p} \le C_p \omega_2(t) \|f\|_p, \tag{4.19}$$

$$\sum_{j=0}^{m} |\lambda|^{(m-j)/m} \|(\partial/\partial t)R'^{\eta}_{\lambda,p}(t)f\|_{j,p} \le C_p \omega_2(t) \|f\|_p. \tag{4.20}$$

Since $((A^{\eta}_{p}(t)-\lambda)^{-1})^* = (A'^{-\overline{\eta}}_{p'}(t)-\overline{\lambda})^{-1}$ we see that

$$(H^{\eta}_{i,\lambda,p})^* = H'^{-\overline{\eta}}_{i,\overline{\lambda},p'}, \qquad (R^{\eta}_{\lambda,p}(t))^* = R'^{-\overline{\eta}}_{\overline{\lambda},p'}(t). \tag{4.21}$$

We first establish the asymptotic expansion of the kernels of the semigroup $\exp(-\tau A_2(t))$ at $t=\infty$. We choose natural numbers ℓ, s and exponents $2=q_1<q_2<\dots<q_s<q_{s+1}=\infty$, $2=r_1<r_2<\dots<r_{\ell-s}<r_{\ell-s+1}=\infty$ as in [7], [11] (Beals [1]) i.e.

(i) in case $m>n/2$. $\ell=2$ and $s=1$, hence $2=q_1<q_2=\infty$ and $2=r_1<r_2=\infty$;

(ii) in case $m<n/2$. $s>n/2m$, $\ell-s>n/2m$, $q_j^{-1}-q_{j+1}^{-1}<m/n$ for $j=1,\dots,s-1$, $q_{s-1}^{-1}>m/n>q_s^{-1}$, $m-n/q_s$ is not a nonnegative integer, $r_j^{-1}-r_{j+1}^{-1}<m/n$ for $j=1,\dots,\ell-s-1$, $r_{\ell-s-1}^{-1}>m/n>r_{\ell-s}^{-1}$, $m-n/r_{\ell-s}$ is not a nonnegative integer;

(iii) in case $m = n/2$. $\ell = 4$, $s = 2$, $2 = q_1 < q_2 < q_3 = \infty$, $2 = r_1 < r_2 < r_3 = \infty$.

In what follows we only consider the case (ii). According to Sobolev's imbedding theorem the inequalities (3.10) and (3.12) hold for $j = 1, \ldots, s$ and $j = 1, \ldots, \ell - s$ respectively.

Put $\delta = \min\{\delta_p;\ p = q_1, \ldots, q_s, r_1, \ldots, r_{\ell-s}\}$. Let $\lambda_1, \ldots, \lambda_\ell \in \sum \backslash \{0\}$ and η be a complex n-vector satisfying

$$|\eta| \leq \delta \ \min\{|\lambda_1|^{1/m}, \ldots, |\lambda_\ell|^{1/m}\}. \tag{4.22}$$

In view of (4.13)

$$\begin{aligned}(A_2^\eta(t) - \lambda_1)^{-1} \cdots (A_2^\eta(t) - \lambda_\ell)^{-1} &= (A_2^\eta(t) - \lambda_s)^{-1} \cdots (A_2^\eta(t) - \lambda_1)^{-1}\ (A_2^\eta(t) - \lambda_{s+1})^{-1} \cdots \\ &\qquad \cdots (A_2^\eta(t) - \lambda_\ell)^{-1} \\ &= \sum_{i_1, \ldots i_\ell = 0}^{\nu} t^{-i_1 - \cdots - i_\ell} H^\eta_{i_s, \lambda_s, 2} \cdots H^\eta_{i_1, \lambda_1, 2}\ H^\eta_{i_{s+1}, \lambda_{s+1}, 2} \cdots \\ &\qquad \cdots H^\eta_{i_\ell, \lambda_\ell, 2} + t^{-\nu} R^\eta_{\lambda_1, \ldots, \lambda_\ell, 2}(t),\end{aligned} \tag{4.23}$$

where $R^\eta_{\lambda_1, \ldots \lambda_\ell, 2}(t)$ is the sum of terms which contain at least one of $R^\eta_{\lambda_k, 2}(t)$'s as a factor. With the aid of (4.14) and (3.10) we see that

$$\|H^\eta_{i_k, \lambda_k, q_k} f\|_{q_{k+1}} \leq C|\lambda_k|^{a_k - 1}\|f\|_{q_k}, \ k = 1, \ldots, s. \tag{4.24}$$

Hence

$$\begin{aligned}&\|H^\eta_{i_s, \lambda_s, 2} \cdots H^\eta_{i_1, \lambda_1, 2}\|_{B(L^2, L^\infty)} \\ &\qquad \leq \prod_{k=1}^{s} \|H^\eta_{i_k, \lambda_k, q_k}\|_{B(L^{q_k}, L^{q_{k+1}})} \leq C \prod_{k=1}^{s} |\lambda_k|^{a_k - 1}.\end{aligned} \tag{4.25}$$

Analogously

$$\begin{aligned}&\|(H^\eta_{i_{s+1}, \lambda_{s+1}, 2} \cdots H^\eta_{i_\ell, \lambda_\ell, 2})^*\|_{B(L^2, L^\infty)} \\ &\qquad = \|H'^{-\overline{\eta}}_{i_\ell, \overline{\lambda}_\ell, 2} \cdots H'^{-\overline{\eta}}_{i_{s+1}, \overline{\lambda}_{s+1}, 2}\|_{B(L^2, L^\infty)} \\ &\qquad \leq C \prod_{k=s+1}^{\ell} |\lambda_k|^{a_k - 1}.\end{aligned} \tag{4.26}$$

Therefore, if we denote the kernel of

$$H^\eta_{i_s, \lambda_s, 2} \cdots H^\eta_{i_1, \lambda_1, 2}\ H^\eta_{i_{s+1}, \lambda_{s+1}, 2} \cdots H^\eta_{i_\ell, \lambda_\ell, 2}$$

by $M^\eta(x,y)$, then in view of (4.25), (4.26) and Lemma 2 of [1]

$$|M^\eta(x,y)| \le C \prod_{k=1}^{\ell} |\lambda_k|^{a_k - 1} \tag{4.27}$$

As was shown in [7], [11] if η is pure imaginary, then

$$(A_2 - \lambda)^{-1} = e^{\cdot\eta}(A_2^\eta - \lambda)^{-1} e^{-\cdot\eta} \tag{4.28}$$

$$(A_2(t) - \lambda)^{-1} = e^{\cdot\eta}(A_2^\eta(t) - \lambda)^{-1} e^{-\cdot\eta} \tag{4.29}$$

(if Ω is bounded, η need not be pure imaginary). Since

$$H_{1,\lambda,2}^\eta = \lim_{t\to\infty} \{(A_2^\eta(t) - \lambda)^{-1} - (A_2^\eta - \lambda)^{-1}\},$$

it follows from (4.28), (4.29) that $H_{1,\lambda,2} = e^{\cdot\eta} H_{1,\lambda,2}^\eta \, e^{-\cdot\eta}$.
Similarly we obtain

$$H_{i,\lambda,2} = e^{\cdot\eta} H_{i,\lambda,2}^\eta e^{-\cdot\eta}, \qquad i = 2,\dots,\nu, \qquad R_{\lambda,2}(t) = e^{\cdot\lambda} R_{\lambda,2}^\eta e^{-\cdot\eta}.$$

Hence, arguing as in [7], [11] if we denote the kernel of

$$H_{i_s,\lambda_s,2}\cdots H_{i_1,\lambda_1,2} \; H_{i_{s+1},\lambda_{s+1},2}\cdots H_{i_\ell,\lambda_\ell,2}$$

by $M(x,y)$, then we have

$$M(x,y) = e^{(x-y)\eta} M^\eta(x,y) \tag{4.30}$$

for any complex n-vector satisfying (4.22). Combining (4.27), (4.30) and arguing as in section 4 of [7] we get

$$|M(x,y)| \le C \prod_{k=1}^{\ell} |\lambda_k|^{a_k-1} \exp\{-\delta \min(|\lambda_1|^{1/m},\dots,|\lambda_\ell|^{1/m})|x-y|\}. \tag{4.31}$$

Analogously, if we denote the kernel of $R_{\lambda_1,\dots,\lambda_\ell}^\eta(t)$ by $\tilde{M}(x,y;t)$ we get

$$\begin{aligned} &|\tilde{M}(x,y;t)|, \; |(\partial/\partial t)\tilde{M}(x,y;t)| \\ &\qquad \le C\tilde{\omega}(t) \prod_{k=1}^{\ell} |\lambda_k|^{a_k-1} \exp\{-\delta \min(|\lambda_1|^{1/m},\dots,|\lambda_\ell|^{1/m})|x-y|\}, \end{aligned} \tag{4.32}$$

where $\tilde{\omega}(t) = \max\{\omega_1(t), \; \omega_2(t), \; t^{-1}\}$.
It follows from (4.23), (4.31), (4.32) that

$$\begin{aligned} (A_2(t) - \lambda_1)^{-1}\cdots(A_2(t) - \lambda_\ell)^{-1} &= (A_2 - \lambda_1)^{-1}\cdots(A_2 - \lambda_\ell)^{-1} \\ &\quad + \sum t^{-i} H_{i,\lambda_1,\dots,\lambda_\ell,2} + t^{-\nu} R_{\lambda_1,\dots,\lambda_\ell,2}(t), \end{aligned} \tag{4.33}$$

where $H_{i,\lambda_1,\dots,\lambda_\ell,2}$, $R_{\lambda_1,\dots,\lambda_\ell,2}(t)$ are operators with kernels $K_{i,\lambda_1,\dots,\lambda_\ell}(x,y)$, $\tilde{K}_{\lambda_1,\dots,\lambda_\ell}(x,y;t)$ satisfying

$$(4.34)\quad |K_{i,\lambda_1,\dots,\lambda_\ell}(x,y)| \le C\prod_{k=1}^{\ell}|\lambda_k|^{a_k-1}\exp\{-\delta\ \min(|\lambda_1|^{1/m},\dots,|\lambda_\ell|^{1/m})|x-y|\},$$

$$(4.35)\quad |\tilde{K}_{\lambda_1,\dots,\lambda_\ell}(x,y;t)|,\ |(\partial/\partial t)\tilde{K}_{\lambda_1,\dots,\lambda_\ell}(x,y;t)| \le C\tilde{\omega}(t)\prod_{k=1}^{\ell}|\lambda_k|^{a_k-1}\exp\{-\delta\ \min(|\lambda_1|^{1/m},\dots,|\lambda_\ell|^{1/m})|x-y|\}.$$

If we denote by $K_{\lambda_1,\dots,\lambda_\ell}(x,y;t)$, $K_{\lambda_1,\dots,\lambda_\ell}(x,y)$ the kernels of $(A_2(t)-\lambda_1)^{-1}\dots(A_2(t)-\lambda_\ell)^{-1}$, $(A_2-\lambda_1)^{-1}\dots(A_2-\lambda_\ell)^{-1}$, then in view of (4.33)

$$(4.36)\quad K_{\lambda_1,\dots,\lambda_\ell}(x,y;t) = K_{\lambda_1,\dots,\lambda_\ell}(x,y) + \sum_{i=1}^{\nu} t^{-i}K_{i,\lambda_1,\dots,\lambda_\ell}(x,y) + t^{-\nu}\tilde{K}_{\lambda_1,\dots,\lambda_\ell}(x,y;t).$$

We denote the kernels of the operators $\exp(-\tau A_2(t))$, $\exp(-\tau A_2)$,

$$(1/2\pi i)^\ell \int_\Gamma\dots\int_\Gamma \exp(-\lambda_1\tau-\dots-\lambda_\ell\tau)\ H_{i,\lambda_1,\dots,\lambda_\ell,2}\ d\lambda_1\dots d\lambda_\ell,$$

$$(1/2\pi i)^\ell \int_\Gamma\dots\int_\Gamma \exp(-\lambda_1\tau-\dots-\lambda_\ell\tau)\ R_{\lambda_1,\dots,\lambda_\ell,2}\ (t)d\lambda_1\dots d\lambda_\ell$$

by $G(x,y,\tau;t)$, $G(x,y,\tau)$, $G_i(x,y,\ell\tau)$, $\tilde{G}(x,y,\ell\tau;t)$. Then by virtue of the equality (3.19) and that with A_2 in place of $A_2(t)$ we have

$$(4.37)\quad G(x,y,\tau;t) = G(x,y,\tau) + \sum_{i=1}^{\nu} t^{-i}G_i(x,y\cdot\tau) + t^{-\nu}\tilde{G}(x,y,\tau;t).$$

Arguing as in [7], [11] we get

$$\left.\begin{array}{l}|G(x,y,\tau;t)|\\ |G(x,y,\tau)|\\ |G_i(x,y,\tau)|\end{array}\right\} \le \frac{C}{|\tau|^{n/m}}\exp\left(-c\,\frac{|x-y|^{m/(m-1)}}{|\tau|^{1/(m-1)}}\right)$$

$$\left.\begin{array}{l}|\tilde{G}(x,y,\tau;t)|\\ |(\partial/\partial t)\tilde{G}(x,y,\tau;t)|\end{array}\right\} \le \frac{C\tilde{\omega}(t)}{|\tau|^{n/m}}\exp\left(-c\,\frac{|x-y|^{m/(m-1)}}{|\tau|^{1/(m-1)}}\right)$$

for τ in the region $|\arg\tau| < \frac{\pi}{2}-\theta_0$.

Let $K_\lambda(x,y;t)$, $K_\lambda(x,y)$ be the kernels of $(A_2(t)-\lambda)^{-1}$, $(A_2-\lambda)^{-1}$, and put

$$K_{i,\lambda}(x,y)=\int_0^\infty e^{\lambda\tau}G_i(x,y,\tau)d\tau,\ \tilde{K}_\lambda(x,y;t)=\int_0^\infty e^{\lambda\tau}\tilde{G}(x,y,\tau;t)d\tau.$$

With the aid of the argument of [7], [11] we obtain

$$\begin{aligned} &|K_\lambda(x,y;t)|,\ |K_\lambda(x,y)|,\ |K_{i,\lambda}(x,y)| \\ &\le C\ \exp(-c|\lambda|^{1/m}|x-y|) \\ &\times\begin{cases} |x-y|^{m-n} & \text{if } m<n, \\ |\lambda|^{n/m-1} & \text{if } m>n, \\ (1+\log^+(|\lambda|^{-1/m}|x-y|^{-1})) & \text{if } m=n. \end{cases}\end{aligned} \tag{4.38}$$

$$\begin{aligned} &|\tilde{K}_\lambda(x,y;t)|,\ |(\partial/\partial t)\tilde{K}_\lambda(x,y;t)| \\ &\le C\tilde{\omega}(t)\exp(-c|\lambda|^{1/m}|x-y|) \\ &\times\begin{cases} |x-y|^{m-n} & \text{if } m<n, \\ |\lambda|^{n/m-1} & \text{if } m>n, \\ (1+\log^+(|\lambda|^{-1/m}|x-y|^{-1})) & \text{if } m=n. \end{cases}\end{aligned} \tag{4.39}$$

From (4.13) with $\eta=0$ and the equality

$$K_\lambda(x,y;t)=K_\lambda(x,y)+\sum_{i=1}^{\nu}t^{-i}K_{i,\lambda}(x,y)+t^{-\nu}\tilde{K}_\lambda(x,y;t),$$

it follows that $K_{i,\lambda}(x,y)$, and $K_\lambda(x,y;t)$ are the kernels of $H_{i,\lambda,p}$ and $R_{\lambda,p}(t)$ respectively for $1<p<\infty$. If we define the operators $H_{i,\lambda,1}$, $R_{\lambda,1}(t)$ as integral operators in $L^1(\Omega)$ with kernels $K_{i,\lambda}(x,y)$, $K_\lambda(x,y;t)$, then

$$(A_1(t)-\lambda)^{-1}=(A_1-\lambda)^{-1}+\sum_{i=1}^{\nu}t^{-i}H_{i,\lambda,1}+t^{-\nu}R_{\lambda,1}(t). \tag{4.40}$$

In view of (4.39) we have

$$\lim_{t\to\infty}\|R_{\lambda,1}(t)\|=0,\qquad \lim_{t\to\infty}\|(\partial/\partial t)R_{\lambda,1}(t)\|=0.$$

Since the above argument remains valid if we replace $A(x,t,D)$ by $A(x,t,D)-c_0$ for some $c_0>0$ (section 1), the expansion (4.40) also holds for $\lambda=0$. Thus the proof of (4.9) is complete.

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