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Pointwise Estimates for a Class of Strongly Degenerate Elliptic Operators: a Geometrical Approach

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1. Introduction

In this paper we extend to a class of strongly degenerate elliptic operators of the second order some classical pointwise estimates: the Harnack inequality and the Hölder continuity of the weak solutions (De Giorgi - Nash - Moser Theorem). This problem is the subject of many papers: see, e.g., the references in [F.K.S.] and [F.L.2]. See also the survey contained in the book by Stredulinsky [Str] and the recent results in [Ch.W.] and [Sc]. Most of these papers are concerned with classes of "not too degenerate" operators. Usually this means that the inverse of the lowest eigenvalue of the quadratic form of the operator is supposed to belong to some suitable L^p space, for a large p. On the other hand, in [F.K.S.] and in [F.L.2] stronger degenerations are allowed: a typical operator satisfying the hypotheses in [F.K.S.] is

$$Lu = \text{ div } (|x|^{\alpha}\nabla u) \text{ in } R^n, \text{ for } -n < \alpha < n,$$

while a typical example in [F.L.2] is

$$Lu = \text{ div } (\nabla_x u + |x|^{2\sigma}\nabla_y u) \text{ in } R^{n+m}_{(x,y)} \text{ for } \sigma > 0$$

We note explicitly that these two classes are completely different ones. Roughly speaking, the operators of [F.K.S.] are good operators when working with a degenerate measure as $w(x)dx$, where w is an A_2 weight in the sense of Muckenhaupt, while an appropriate metric not comparable with the euclidean one is the right tool for the operators in [F.L.2]. These two approaches can be unified in the setting of homogeneous metric spaces in a way that seems to be the most natural one. (We note that a theory of A_p weights in homogeneous spaces was developed by A. Calderon in [C]). Thus we can deal simultaneously

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with the two classes of operators and we are able to enlarge them in an essential way. A typical example of the new class is given by

$$Lu = \operatorname{div}(|(x,y)|^{\alpha}(\nabla_x u + |x|^{2\sigma}\nabla_y u) \text{ in } R^{n+m}_{(x,y)}.$$

The proofs of the main results are obtained adjusting the classical Moser technique to the geometry of the homogeneous space. We just observe here that we had also the different possibility of using the new approach by Di Benedetto and Trudinger ([D.T.]). An essential tool in both these proofs is a weighted Sobolev type embedding theorem. We obtain such a result using a representation formula for a function u closely fitting the geometry of the operator and substituting the usual rapresentation of u as a fractional integral of its gradient.

The material of the paper is divided as follows: in Section 2 we describe our hypotheses and we recall some known results on the metric associated to a degenerate elliptic operator and on A_p-weights in homogeneous spaces. In Section 3, we state the main results and give some examples. In Section 4 we prove the representation formula and the Sobolev-Poincaré Theorem. In Sections 5 and 6 we prove the results of Section 3 and in Section 7 we give the proof of a number of technical estimates used throughout the paper.

Section 2.

In this paper we consider the differential operator:

(2.a)
$$L \equiv \sum_{i,j=1}^{N} D_i \ (a_{ij}(x)D_j),$$

where $a_{ij} = a_{ji}$ for $i,j = 1,\cdots,N$, are real, measurable functions defined in a bounded open set $\Omega \subseteq R^N$. We assume the following hypotheses on the structure of L:

(2.b) there exist $\nu \geq 1$ and $N+1$ real non negative functions $w,\ \lambda_1,\ \cdots,\ \lambda_N$ defined on R^N such that:
$$\nu^{-1}w(x) \sum_{j=1}^{N} \lambda_j^2(x)\xi_j^2 \leq \sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \leq \nu w(x)\sum_{j=1}^{N}\lambda_j^2(x)\xi_j^2$$
$\forall\ x \in \Omega,\ \forall\ \xi \in R^N.$

We will say that $L \in \mathcal{L}(\Omega, N, \nu, \Lambda,\ p,\ c_{w,2})$ if the functions $\lambda = (\lambda_1,\cdots,\lambda_N)$ and w satisfy the following assumptions (2.c),$\cdots$,(2.f).

(2.c)
$$\lambda_1 \equiv 1,\ \lambda_j(x) = \lambda_j(x_1,\cdots,x_{j-1}) \qquad j = 2,\cdots,N$$
$$\forall\ x = (x_1,\cdots,x_N) \in R^N.$$

(2.d) put $\Pi = \{x \in R^N : \prod_{k=1}^{N} x_k = 0\}$, then
$\lambda_j \in C(R^N) \cap C^1(R^N/\Pi)$;
$0 < \lambda_j(x) \leq \Lambda, \quad \forall\ x \in R^N/\Pi, \quad j = 1, \cdots, N.$
$\lambda_j(x_1, \cdots, x_i, \cdots, x_{j-1}) = \lambda_j(x_1, \cdots, -x_i, \cdots, x_{j-1})$
for $j = 2, \cdots, N$ and $i = 1, \cdots, j-1$.

(2.e) there is a family P of $N(N-1)/2$ non negative numbers $\rho_{j,i}$ such that:
$0 \leq x_i(D_i\lambda_j)(x) \leq \rho_{j,i}\ \lambda_j(x)$ for $2 \leq j \leq N$,
$1 \leq i \leq j-1$ and $\forall\ x \in \mathbb{R}^N/\Pi$.

REMARK 2.1. The right hand side of condition (2.e) can be stated in an equivalent way as:

(2.e)$'$
$$\begin{gathered} \forall\ \theta \in\ ]0,1[\text{ and } \forall\ x \in R^N : \\ \theta^{\rho_{j,i}}\ \lambda_j(x) \leq \lambda_j(x_1, \cdots, \theta x_i, \cdots, x_{j-1}) \\ 2 \leq j \leq N,\ 1 \leq i \leq j-1. \end{gathered}$$

For a proof of this fact see [F.L.3] Prop. 4.2. For a further discussion about the meaning of (2.e) see also [F.L.3] Remark 2.9.

In order to formulate the last hypothesis on the function w we anticipate that it is possible to associate a natural distance d on R^N (See Def. 2.1) to the vector valued function $\lambda = (\lambda_1, \cdots, \lambda_N)$ in such a way that the triple $(R^N, d, \mathcal{L})$ consisting of R^N equipped with the distance d and the Lebesgue measure $\mathcal{L}$ is an homogeneous space in the sense of Coifman and Weiss (see e.g. [C.W.1] and [C.W.2]). Given the existence of this d we make the following hypothesis on w:

(2.f) There is $c_{w,2} \geq 1$ such that $\forall\ x \in R^N$ and $\forall\ r > 0$
$$\frac{1}{|S(x,r)|} \int_{S(x,r)} w\ dy \cdot \frac{1}{|S(x,r)|} \int_{S(x,r)} 1/w\ dy \leq c_{w,2}.$$

Here $S(x,r) = \{y \in R^N : d(x,y) < r\}$ are the d-balls in R^N while we indicate by $B(x,r)$ the euclidean balls. Moreover, if $E \subseteq R^N$ is a measurable set, $|E|$ is its Lebesgue measure.

Observe that (2.f) is a Muckenhaupt A_2-condition in the setting of the homogeneous space $(R^N, d, \mathcal{L})$. More properties of functions satisfying (2.f) will be mentioned in Lemmas (2.9) and (2.10).

Our next task is to give the definition of the natural distance d and to state some of its properties. Let's start introducing the notions of λ-subunit vector and λ-subunit curve (see also [F.P.] and [N.S.W.]):
a vector $v \in R^N$ is a λ-subunit vector at a point x if

$$< v, \xi >^2 \leq \sum_{j=1}^{N} \lambda_j^2(x)\xi_j^2, \; \forall \; \xi \in R^N.$$

Let $\gamma : [0,T] \rightarrow R^N$ be an absolutely continuous curve; then γ is a λ-subunit curve if $\dot{\gamma}(t)$ is a λ-subunit vector at $\gamma(t)$ for a.e. $t \in [0,T]$.

DEFINITION 2.2. For any $x, y \in R^N$ we define $d : R^N \times R^N \rightarrow R^+$ as:

$$\begin{aligned} d(x,y) = \operatorname{Inf} \{ & T \in R_+ | \text{ there exists a } \lambda\text{-subunit} \\ & \text{curve } \gamma : [0,T] \rightarrow R^N, \; \gamma(0) = x, \; \gamma(T) = y \}. \end{aligned}$$

REMARK 2.3.: d is a well defined distance. In fact our hypotheses on λ guarantee the existence of a λ-subunit curve joining x and y, for any couple of points x and y. This has been proved in [F.L.1] and [F.L.3]. Moreover there is a positive $b = b(\Lambda)$ such that $\forall \; x, y : |x - y| \leq b \; d(x,y)$.

For our purposes it is useful to introduce a (non symmetric) quasi-distance δ, more explicitely defined and sometimes easier than d to work with. Observe that δ has been defined and throughly studied in [F.L.1]; we just recall here the relevant results.

If $x \in R^N$ and $t \in R$ put $H_0(x,t) = x$ and

$$H_{k+1}(x,t) = H_k(x,t) + t\lambda_{k+1}(H_k(x,t))e_{k+1} \text{ for } k = 0, \cdots, N-1.$$

Here $\{e_k\}_k$ is the standard base in R^N. The function

$$s \rightarrow F_j(x,s) = s \; \lambda_j(H_{j-1}(x,s))$$

is strictly increasing on $]0, +\infty[$ for any

$$x \in R_j^N = \{ x \in R^N | x_k \geq 0 \; k = 1, \cdots, j-1 \}$$

and for $j = 1, \cdots, N$. Hence it is possible to define the inverse function of $F_j(x, \cdot)$ that is

$$\varphi_j(x, \cdot) = (F_j(x, \cdot))^{-1} \text{ for } j = 1, \cdots, N \text{ and for } x \in R_j^N.$$

Now we give the definition of the new quasi distance:

DEFINITION 2.4. For any $x, y \in R^N$ we define $\delta : R^N \times R^N \rightarrow R^+$ as

$$\delta(x,y) = \operatorname*{Max}_{j=1,\cdots N} \; \varphi_j(x*, \; |x_j - y_j|);$$

where $x* = (|x_1|, \cdots, |x_n|)$.

Associated with δ we will consider the rectangular neighbourhoods $Q(x,r) = \{y \in R^N | \delta(x,y) < r\}$. Some properties of δ are collected in the following lemmas and proposition:

LEMMA 2.5. ([F.L.1] *Theorems* 2.6 *and* 2.7). *There exists* $a \geq 1$ *(a depends only on P) such that for any* $x, y \in R^N$:

(2.5a) $$a^{-1} \leq d(x,y)/\delta(x,y) \leq a.$$

LEMMA 2.6. ([F.L.2], *Proposition* 4.3). *Put*

$$G_1 = 1 \text{ and } G_j = 1 + \sum_{\ell=1}^{j-1} G_\ell \, \rho_{j,\ell} \text{ for } j = 2, \cdots, N$$

Then

$$\forall\ x \in R^N,\ \forall\ s > 0,\ \forall\ \theta \in]0,1[$$

we get:

(2.6a) $$\theta^{G_j} \leq F_j(x*, \theta s)/F_j(x*, s) \leq \theta$$

(2.6b) $$\theta \leq \varphi_j(x*, \theta s)/\varphi_j(x*, s) \leq \theta^{1/G_j}.$$

PROPOSITION 2.7. *For any* $x \in R^N$ *and any* $r > 0$

(2.7a) $$S(x, r/a) \subseteq Q(x,r) \subseteq S(x, ar)$$

where a is the constant in Lemma 2.5.

Moreover there is $b > 1$ *such that if* $|x - y| \leq 1$:

(2.7b) $$b^{-1}|x-y| \leq \dot{d}(x,y) \leq b|x-y|^\eta$$

where $\eta = \min_j \{1/G_j\}$.

Finally, there is a constant $A > 1$*, such that the two following doubling properties hold*:

(2.7c) $$|S(x,\ 2r)| \leq A\ |S(x,r)|$$

(2.7d) $$|Q(x,\ 2r)| \leq A\ |Q(x,r)|$$

for any $x \in R^N$ *and for any* $r > 0$.

PROOF: Follows easily from (2.6a), (2.6b) and Lemma 2.5.

Observe that all the constants depend only on N, Λ and P. □

It follows from (2.7c) and (2.7d) that both $(R^N, d, \mathcal{L})$ and $(R^N, \delta, \mathcal{L})$ are homogeneous structures in the sense of Coifman and Weiss (see [C.W.2] pag. 587).

Now we have to recall some results related with condition (2.f).

Let's start with a definition:

DEFINITION 2.8: Let $1 < p < \infty$. We say that a non negative function w defined on R^N belongs to $A_p = A_p(R^N, d, \mathcal{L})$ or is an A_p weight if there is a constant $c_{w,p} \geq 1$, such that:

$$\left(\frac{1}{|S(x,r)|} \int_{S(x,r)} w \, dy\right) \cdot \left(\frac{1}{|S(x,r)|} \int_{S(x,r)} w^{-\frac{1}{p-1}} \, dy\right)^{p-1} \leq c_{w,p}.$$

It is obvious that $A_p \subset A_q$ for $p < q$; on the other hand a crucial point of the theory of A_p weights is the following "almost reverse" result

LEMMA 2.9. *(See* [C], *Theorem* 2): *If $w \in A_p$ then $w \in A_r$ for any $r > p_0$, $p_0 < p$ and p_0 depending only on $c_{w,p}$, p and A (A is the constant in* (2.7c)).

The following doubling property for the measure $w(x)dx$ is a consequence of Lemma 2.9:

LEMMA 2.10. *If $w \in A_p(R^N, d, \mathcal{L})$ then there is a constant $B = B(p, c_{w,p}, A)$ such that*:

$$w(S(x,2r)) \leq B \, w(S(x,r)) \tag{2.10a}$$

for any $x \in R^N$, for any $r > 0$. (We use the notation $w(E) = \int_E w \, dx$ for any measurable set $E \subseteq R^N$).

It follows from (2.10a) that the triple $(R^N, d, w dx)$ consisting of R^N equipped with the distance d and the weighted measure $w(x)dx$ is an homogeneous structure. Moreover since (2.7a) the same is true for the triple $(R^N, \delta, w dx)$.

We note explicitely that (2.10a) implies that there exist $\alpha > 0$ such that, $\forall \, r > 0$, $\forall \, \theta \in]0,1[$, $\forall \, x \in R^N$.

$$w(S(x,\theta r)) \geq \theta^\alpha \, w(S(x,r)). \tag{2.10b}$$

We want now to give the definitions of solution, subsolution and supersolution. Let's start with some fact about weighted Sobolev spaces.

Given a measurable set $E \subseteq R^N$ we denote by $L^p(E)$, $1 \leq p \leq \infty$ the usual Lebesgue spaces, while we denote by $L^p(E,w)$, $1 \leq p < \infty$, the Banach space of the measurable functions f, defined on E, for which:

$$\|f\|_{L^p(E,w)} = \left(\int_E |f|^p w dx\right)^{1/p} < +\infty.$$

Observe that since $\frac{1}{w} \in L^1_{\text{loc}}(R^N)$ then $L^2(E,w) \subset L^1(E)$ for any bounded measurable set E. We will denote also by ∇_λ the differential operator $\nabla_\lambda \equiv (\lambda_i D_i)_{i=1,\cdots N}$ and by div_λ the differential operator acting on vector valued functions $f = (f_1, \cdots, f_N)$ as

$$\text{div}_\lambda\ f = \sum_{i=1}^{N} \lambda_i D_i f_i.$$

Given an open set Ω again we use the notations $H^{1,p}(\Omega)$, $\overset{0}{\mathrm{H}}{}^{1,p}(\Omega)$, $1 \le p_1 \le \infty$, for the usual Sobolev spaces, while we indicate by $H^1_\lambda(\Omega,w)$ (respectively $\overset{0}{\mathrm{H}}{}^1_\lambda(\Omega,w)$) the closure of the space $\text{Lip}(\Omega)$ of the Lipschitz continuous functions on Ω (respectively $\text{Lip}(\Omega) \cap \mathcal{E}'(\Omega)$) with respect to the norm:

$$\|f\|_{H^1_\lambda(\Omega,w)} = \|f\|_{L^2(\Omega,w)} + \|\nabla_\lambda f\|_{L^2(\Omega,w)}.$$

Moreover the spaces $L^p_{\text{loc}}(E,w)$ and $H^1_{\lambda,\text{loc}}(\Omega,w)$ are defined in the usual way.

The following assertion is straightforward:

PROPOSITION 2.11: *The bilinear form $\mathcal{B}$ on $C^1(\Omega) \cap H^1_\lambda(\Omega,w)$ defined as:*

$$\mathcal{B}(u,v) = \sum_{i,j=1}^{N} \int_\Omega a_{ij}\ D_i u D_j v\ dx$$

can be continued on all of $H^1_\lambda(\Omega,w)$.

DEFINITION 2.12. Let $f = (f_1, \cdots, f_N)$ be a vector valued function such that $|f|/w \in L^2(\Omega,w)$ and let $g \in H^1_\lambda(\Omega,w)$. We say that u is a solution of the Dirichlet problem:

$$\text{(2.g)} \qquad \begin{aligned} Lu &= -\text{div}_\lambda f && \text{in } \Omega \\ u &= g && \text{in } \partial\Omega \end{aligned}$$

if $u - g \in \overset{0}{\mathrm{H}}{}^1_\lambda(\Omega,w)$ and

$$\mathcal{B}(u,\varphi) = \sum_{i=1}^{N} \int_\Omega \lambda_i f_i D_i \varphi \qquad \forall\ \varphi \in \overset{0}{\mathrm{H}}{}^1_\lambda(\Omega,w).$$

DEFINITION 2.13. We say that $v \in H^1_{\lambda,\text{loc}}(\Omega,w)$ is a local subsolution (local supersolution) for L if for any open set $\Omega' \subset\subset \Omega$ and $\forall \varphi \ge 0$ $\varphi \in \overset{0}{\mathrm{H}}{}^1_\lambda(\Omega',w) : \mathcal{B}(v,\varphi) \le 0$ (≥ 0).

DEFINITION 2.14. We say that $u \in H^1_{\lambda,\mathrm{loc}}(\Omega, w)$ is a local solution for L if it is both a local subsolution and a local supersolution.

The following theorem on existence and uniqueness of solutions follows from Lax-Milgram theorem and the results of section 4:

THEOREM 2.15. *Given $f = (f_1, \cdots, f_N)$ such that $|f|/w \in L^2(\Omega, w)$ and $g \in H^1_\lambda(\Omega, w)$, there exists a unique solution in $H^1_\lambda(\Omega, w)$ of the Dirichlet problem* (2.g).

Section 3.

In this section, we shall formulate the main theorems of our paper; moreover, some typical applications will be discussed. The basic point is an invariant Harnack inequality for the d-balls (i.e. a Harnack inequality where the constant does not depend on the radius of the ball). This result will be proved in Section 5.

THEOREM 3.1. *If the hypotheses* (2.b)-(2.f) *are satisfied, there exist $C, M > 0$ such that, if $u \in H^1_{\lambda,\mathrm{loc}}(\Omega, w)$ is a nonnegative weak solution of $Lu = 0$ in a bounded open subset Ω, then*

$$\sup_{S(x_0,\rho)} \mathrm{u} \le M \inf_{S(x_0,\rho)} \mathrm{u}$$

for any $x_0 \in R^N$, $\rho > 0$, $\rho < Cd(x_0, \partial\Omega)$.

The classical Harnack inequality follows straightforwardly from the above result. We get:

THEOREM 3.2. *Let the hypotheses of the above Theorem be verified. If Ω is connected, then for every compact subset K of Ω, there exist $M_K > 0$ (independent of u) such that*

$$\sup_K u \le M_K \inf_K \mathrm{u}\,.$$

Finally, the Holder continuity of the weak solutions follows from the invariant Harnack inequality by standard methods (see, e.g., [Sta]). We have:

THEOREM 3.3. *Let the hypotheses* (2.b)-(2.f) *be satisfied. If $u \in H^1_{\lambda,\mathrm{loc}}(\Omega, \omega)$ is a weak solution of $Lu = 0$ in an open subset Ω, then u is locally Holder continuous.*

Let us now describe some relevant examples where our hypotheses are verified.

EXAMPLE 3.4. Let us choose $\lambda_1 = \lambda_2 = \cdots = \lambda_N = 1$; in this case (2.c)-(2.e) are obviously satisfied and the metric d is the usual euclidean metric.

Thus, hypothesis (2.f) is an usual A_2-condition in R^N and the operators we consider are the same as in [F.K.S.].

EXAMPLE 3.5. Let us choose $\omega = 1$. In this case, if (2.c)-(2.e) are satisfied, (2.f) is always verified and we obtain, in particular, the results of [F.L.3].

EXAMPLE 3.6. Let us suppose $N = n + m,\ n, m \geq 1$.

Denote by (x, y) the generic point in R^N, by $|(x, y)|$ and $|x|$ the usual euclidean norms in R^N and in R^n respectively. If we choose $\lambda_1 = \cdots = \lambda_n = 1$ and $\lambda_{n+1} = \cdots = \lambda_N = |x|^\sigma$, $\sigma > 0$, the hypotheses (2.c)-(2.e) are satisfied. We will prove later that, in this case, $w(x, y) = |(x, y)|^\alpha$ is an A_2-weight with respect to the metric d defined by $\lambda_1, \cdots, \lambda_N$ if $-n < \alpha < n$. Thus, our results may be applied, e.g., to operators such as

$$L = \text{ div } (|(x, y)|^\alpha(\nabla_x + |x|^{2\sigma}\nabla_y));\ -n < \alpha < n,\ 0 < \sigma.$$

Let us prove that $|(x, y)|^\alpha$ is an A_2-weight if $-n < \alpha < n$.

Obviously, we need only to prove the assertion if $0 < \alpha < n$. Now, let r be a positive real number and let (ξ, η) be a fixed point such that $|(\xi, \eta)| \leq 2rb$, where b is the constant we introduced above such that $d((x', y'),\ (x'', y'')) \geq 1/b|(x' - x'',\ y' - y'')|$ for any $(x', y'),\ (x'', y'') \in R^N$; then $S((\xi, \eta),\ r) \subseteq B(0,\ 3br)$; hence

$$|S((\xi, \eta),\ r)|^{-1} \int_{S((\xi,\eta),r)} |(x, y)|^\alpha\ dx\ dy \leq (3br)^\alpha.$$

On the other hand, by Lemma 2.5, there exist two constants $c_1, c_2 > 0$ (depending only on P) such that $B(\xi, c_1 r) \times B(\eta, c_1\rho) \subseteq S((\xi, \eta),\ r) \subseteq B(\xi, c_2 r) \times B(\eta, c_2\rho))$, where $\rho = r(|x_1| + r, \cdots, |x_n| + r)^\sigma$. Then, we get (the constants depending only on P and α):

$$\begin{aligned} &|S((\xi, \eta),\ r)|^{-1} \int_{S((\xi,\eta),r)} |(x, y)|^{-\alpha} dx\ dy \\ &\leq c_3 r^{-n} |B(\eta, c_1\rho)|^{-1} \int_{B(\xi,c_2 r)\times B(\eta,c_2\rho)} (|x|^\alpha + |y|^\alpha)^{-1} dx\ dy \\ &\leq c_4 r^{-n} \int_{B(\xi,c_2 r)} |x|^{-\alpha} dx = c_5 r^{-\alpha}. \end{aligned}$$

Thus, we proved that

$$(|S((\xi, \eta), r)|^{-1} \int_{S((\xi,\eta),r)} |(x, y)|^\alpha dx\ dy)(|S((\xi, \eta), r)|^{-1} \int_{S((\xi,\eta),r)} |(x, y)|^{-\alpha}\ dx\ dy) < c_6$$

for any $(\xi, \eta) \in R^N$, $\forall\ r > 0$ such that $|(\xi, \eta)| \leq 2rb$.

Now, if $|(\xi,\eta)| > 2rb$ and $(x,y) \in B((\xi,\eta),\ r)$, we get:

$$|(x,y)| \leq |(x-\xi,\ y-\eta)| + |(\xi,\eta)| \leq bd((x,y),\ (\xi,\eta)) + |(\xi,\eta)| < br + |(\xi,\eta)|;$$

moreover

$$|(x,y)| \geq |(\xi,\eta)| - |(x-\xi,\ y-\eta)| \geq |(\xi,\eta)| - b\ d((x,y),(\xi,\eta)) \geq |(\xi,\eta)| - rb;$$

Hence

$$(|S((\xi,\eta),r)|^{-1} \int\limits_{S((\xi,\eta),r)} |(x,y)|^{\alpha} dx\ dy)(|S((\xi,\eta),r)|^{-1} \int\limits_{S((\xi,\eta),r)} |(x,y)|^{-\alpha} dx\ dy)$$
$$\leq (\frac{rb + |(\xi,\eta)|}{|(\xi,\eta)| - rb})^{\alpha} \leq 3^{\alpha}, \text{ due to our choice of } |(\xi,\eta)|.$$

We note also that our choice of α cannot be improved, as we can see, by choosing $\xi = \eta = 0$.

EXAMPLE 3.7. Let $\lambda_1 \cdots, \lambda_N$ be given functions satisfying (2.c)-(2.e). Then, if ξ is a fixed point in R^N, the function $d(\bullet,\xi)^{\alpha}$ is an A_2-weight with respect to the d-balls if $-N < \alpha < N$.

Due to the doubling property (2.7c), the proof of the assertion can be reduced in a standard way to the estimate of the means

$$\frac{1}{|S(\xi,r)|} \int\limits_{S(\xi,r)} d(x,\xi)^{\alpha} dx \text{ and } \frac{1}{|S(\xi,r)|} \int\limits_{S(\xi,r)} d(x,\xi)^{-\alpha} dx.$$

where $\alpha > 0$.

By Lemma 2.5 we have:

$$\frac{1}{|S(\xi,r)|} \int\limits_{S(\xi,r)} d(x,\xi)^{\alpha} dx \leq \frac{C_1}{|S(\xi,r)|} \sum_{j=1}^{N} \int\limits_{|x_k - \xi_k| \leq F_k(\xi *, c_2 r)} \varphi_j^{\alpha}(\xi *, |x_j - \xi_j|) dx \leq$$

(defining y_j such that $x_j = \xi_j + y_j F_j(\xi *, c_2 r)$)

$$\leq c_3 \sum_{j=1}^{N} \int\limits_{|y_k| \leq 1} \varphi_j^{\alpha}(\xi *, |y_j| F_j(\xi *, c_2 r))\ dy \leq$$

(by Lemma 2.6 and the very definition of φ_j)

$$\leq c_4 r^{\alpha} \sum_{j=1}^{N} \int\limits_{|y_k| \leq 1} |y_j|^{a/G_j} dy = c_5 r^{\alpha}.$$

Analogously,

$$\frac{1}{|S(\xi,r)|}\int\limits_{S(\xi,r)} d(x,\xi)^{-\alpha}dx \le c_4 r^{-\alpha} \int\limits_{|y_k|\le 1} |y|^{-\alpha}dy,$$

and the assertion is proved.

EXAMPLE 3.8. For sake of simplicity, let us choose $N = 2$ and let us consider the operator

$$L = D_1(|x_1|^{\alpha_1}|x_2|^{\alpha_2}D_1) + D_2(|x_1|^{\beta_1}|x_2|^{\beta_2}D_2),$$

where α_i, β_i are real numbers, $i = 1, 2$.

If

$$\alpha_1 \le \beta_1,\ \alpha_2 = \beta_2,\ -1 < \alpha_i < 1,\ i = 1, 2,$$

the hypotheses (2.c)-(2.f) are satisfied choosing

$$\lambda_1 = 1,\ \lambda_2 = |x_1|^{\beta_1-\alpha_1},\ \omega = |x_1|^{\alpha_1}|x_2|^{\alpha_2}.$$

In an analogous way we can deal with the case

$$\alpha_1 = \beta_1,\ \alpha_2 \ge \beta_2,\ -1 < \beta_i < 1,\ i = 1, 2.$$

We note that the results in [F.K.S.] contain the case

$$\alpha_1 = \beta_1,\ \alpha_2 = \beta_2,\ -1 < \alpha_i < 1,\ i = 1, 2$$

and the results in [F.L.3] the cases

$$\alpha_1 = \alpha_2 = \beta_2 = 0 \text{ or } \beta_1 = \beta_2 = \alpha_1 = 0.$$

Section 4.

In this Section, we will prove the basic results we use in order to adapt Moser's machinery to our geometry. The main result is a Sobolev-Poincaré estimate. It is stated in two equivalent versions in Theorems 4.1 and 4.5. We will show first how Theorem 4.1 follows easily from Theorem 4.5. To prove Theorem 4.5 we obtain first a weak version of Theorem 4.1, i.e. statement 4.a; from this one and an appropriate covering argument Theorem 4.5 follows

THEOREM 4.1. *Let β be a fixed positive real number and let w belong to $A_q(R^N, d, \mathcal{L})$ for a given $q > 1$. Then, there exists $c_\beta > 0$ (depending only* *on β, N, $c_{w,q}$ and p) such that, if $S = S(\xi, r)$ is a fixed d-ball and $u : S \to R$ is Lipschitz continuous and such that $|E| = |\{x \in S : u(x) = 0\}| \geq \beta|S|$, we get*

$$(w(S)^{-1} \int_S |u|^{kp} w)^{1/kp} \leq c_\beta r (w(S)^{-1} \int_S |\nabla_\lambda u|^p w)^{1/p}. \tag{4.1a}$$

for any $p \geq q$ and for a suitable $k > 1$ depending only on p, q and P.

The admissible range for k is:

$$\begin{aligned} &k \in [1,\ (1 - (\sum_j G_j)^{-1} p/q)^{-1}] && \text{if } \sum_j G_j < q/p \\ &k \in [1, +\infty) && \text{if } \sum_j G_j \geq q/p \end{aligned} \tag{4.1b}$$

where the constants G_j are the ones in Lemma 2.6.

PROOF. Let $u_s = |S|^{-1} \int_S u$. Then:

$$\begin{aligned} |u_s| = |E|^{-1} \int_E |u_s - u| &\leq \beta^{-1} |S|^{-1} \int_S |u_s - u| \\ &\leq \beta^{-1} c((w(S))^{-1} \int_S |u_s - u|^{kp} w)^{1/kp}. \end{aligned}$$

In the last inequality $w \in A_q$ is used. Now triangular inequality and Theorem 4.5 yield the thesis:

$$\begin{aligned} &(w(S)^{-1} \int_S |u|^{kp} w)^{1/kp} \leq (w(S)^{-1} \int_S |u - u_s|^{kp} w)^{1/kp} + |u_s| \\ &\leq c\ r (w(S)^{-1} \int_S |\nabla_\lambda u|^p w)^{1/p}. \end{aligned}$$

□

The main tool used in this section is the representation formula, closely fitting the geometry of the homogeneous space $(R^N, d, \mathcal{L})$, stated in Lemma 4.3. Preliminarly, we recall the following definition.

DEFINITION 4.2. Let v belong to $L^1_{\text{loc}}(R^N)$. If $0 < \alpha \leq 1$, define the fractional maximal funtion of v as:

$$(M_\alpha v)(x) = \sup_{r>0} |S(x,r)|^{-\alpha} \int_{S(x,r)} |v(y)| dy.$$

Now, we have:

LEMMA 4.3. *Assume that $S = S(\xi, r)$ is a fixed d-ball of radius r. For any $\beta \in]0,1[$ and any $Q \in]1 - (\sum_j G_j)^{-1},\ 1[$, there are constants $c_0 = c_0(\beta, Q)$ and $\vartheta = \vartheta(\beta, Q) > 1$ (depending also on p and N) such that if $u : \vartheta S = S(\xi, \vartheta r) \to R$ is Lipschitz continuous and $|E| = |\{x \in S : u(x) = 0\}| \geq \beta|S|$ then:*

$$|u(x)| \leq c_0 r\ |S|^{Q-1} M_Q(|\nabla_\lambda u| \chi_{\vartheta S})(x), \qquad \forall\ x \in S. \tag{4.3a}$$

Here we used the symbol $\chi_{\vartheta S}$ for the characteristic function of ϑS.

PROOF. In the sequel, we shall use some of the notations of Section 7. All the constants $c_1,\ c_2, \cdots$ depend on P; we will specify explicitly the (possible) dependence on β.

Let x be a point of S. Since $S(\xi, r) \subseteq S(x, 2r)$ and $S(x,r) \subseteq S(\xi, 2r)$, by the doubling property (2.7c), $c_1|S| \leq |S(x,r)| \leq c_2|S|$.

Now, we note that there exists $\sigma \in \{-1,1\}^N$ such that

$$|E \cap Q^\sigma(x, 2ar)| \geq \beta 2^{-N} |S(x,r)|,$$

since

$$\bigcup_\sigma Q^\sigma(x,\ 2ar) = Q(x,\ 2ar) \supseteq S(x,\ 2r) \supseteq S(\xi, r) \supseteq E.$$

Thus, by the doubling property,

$$|E \cap Q^\sigma(x,\ 2ar)| \geq c_3\beta |Q^\sigma(x,\ 2ar)|.$$

Let us now choose in Proposition 7.4 $\lambda = c_3\beta/2$; (thus α and ε are fixed, depending on β); then, we get:

$$\begin{aligned}
&|E \cap Q^\sigma(x,\ 2ar) \cap H(2ar,\ x,\ \Delta^\alpha_\varepsilon(\sigma))| \geq (\beta c_3/2)|Q^\sigma(x,\ 2ar)| \\
&(\text{in fact, } |Q^\sigma| \geq |(Q^\sigma \cap E) \cup (Q^\sigma \cap H(\cdots))| = \\
&= |Q^\sigma \cap E| + |Q^\sigma \cap H(\cdots)| - |E \cap Q^\sigma \cap H(\cdots)| \geq \\
&\geq |Q^\sigma|(\beta c_3 + 1 - \beta c_3/2) - |E \cap Q^\sigma \cap H(\cdots)|).
\end{aligned}$$

Now, we can suppose $x \notin E$ (otherwise (4.3a) is obvious) and we put

$$\sum = \{y \in \Delta^\alpha_\varepsilon(\sigma);\ H(2ar,\ x, y) \in E\}.$$

Let

$$y \to K(y) = K_1(y_1) \cdots K_N(y_N)$$

be a smooth function supported in $\Delta^{2\alpha}_{\varepsilon/2}$, $0 \leq K \leq 1$, $K = 1$ on $\Delta^\alpha_\varepsilon(\sigma)$.

Let's assume for the moment that $u \in C^1(\Omega)$. If $y \in \sum$, we have

$$u(x) = u(x) - u(H(2ar, x, y)), \text{ hence}$$
$$|u(x)| = |u(x) - u(H(2ar, x, y))| K(y).$$

Integrating on $\sum$, we get

$$|\textstyle\sum| \; |u(x)| = \int\limits_{\Sigma} |u(x) - u(H(2ar, x, y))| K(y) \, dy$$
$$\leq \int\limits_{\text{supp } K} |u(x) - u(H(2ar, x, y))| K(y) \, dy$$
$$\leq \int\limits_{0}^{2ar} dt \int\limits_{\text{supp } K} dy | < \nabla u(H(t, x, y)), \; \dot{\mathrm{H}}(t, x, y) > | K(y)$$
$$\leq \int\limits_{0}^{2ar} dt \int\limits_{\text{supp } K} dy |\nabla_\lambda u(H(t, x, y))| \; |y| K(y).$$

Now, we note that $y \to H(t, x, y)$ is a good change of variables in $\Delta^{2\alpha}_{\varepsilon/2}(\sigma)$, $\forall \; t > 0$; moreover,

$$|\det \frac{\partial H}{\partial y}(t, x, y)| = \prod_{j=1}^{N} \int\limits_{0}^{t} \lambda_j(H(s, x, y)) ds$$

and, by Proposition 7.5, the last product is equivalent to $|S(x,t)|$ with equivalence constants depending only on P, α and ε; hence the equivalence constants depend only on P and on β.

Thus, we get:

$$|u(x)| \leq c_4(\beta) \frac{1}{|\sum|} \int\limits_{0}^{2ar} dt \frac{1}{|S(x,t)|} \int\limits_{H(t,x,\Delta^{2\alpha}_{\varepsilon/2}(\sigma))} |\nabla_\lambda u(y)| dy$$

Now, there is $c_5 = c_5(\beta)$ such that $H(t, x, \Delta^{2\alpha}_{\varepsilon/2}(\sigma)) \subseteq S(x, c_5 t)$. In fact, if $\xi \in R^N$ and $y \in \Delta^{2\alpha}_{\varepsilon/2}(\sigma)$, we get:

$$< \dot{\mathrm{H}}(s, x, y), \; \xi >^2 = (\sum_j \lambda_j(H(s, x, y)) y_j \xi_j)^2$$
$$\leq (\sum_j \lambda_j^2(H(s, x, y)) \; \xi_j^2) |y|^2.$$

Thus, $s \to H(s/|y|^2, x, y)$ is a sub-unit curve starting from x and attaining $H(t, x, y)$ at the time $s = t|y|^2$; then, by the very definition of d, it follows that $d(x, H(t, x, y)) \le t|y|^2 \le 4\alpha^2 t$.

So that we obtain

$$|u(x)| \le c_4(\beta) \frac{1}{|\sum|} \int_0^{2ar} dt \frac{1}{|S(x,t)|} \int_{S(x,c_5 t)} |\nabla_\lambda u(y)|\, dy. \tag{4.3b}$$

A standard approximation argument shows that the above inequality holds also for Lipschitz continous functions.

Now define $\vartheta = 1 + 2ac_5$, then

$$S(x, c_5 t) \subseteq \vartheta S, \ \forall t \in \ [0, 2ar]$$

and $\forall x \in S$; hence in (4.3b), under the integral, we can substitute $|\nabla_\lambda (u \cdot \chi_{\vartheta S})|$ for $|\nabla_\lambda u|$.

Moreover, by the doubling property, $|S(x,t)|$ is comparable to $|S(x, c_5 t)|$ and we obtain:

$$|u(x)| \le c_6(\beta) \frac{1}{|\sum|} \int_0^{2ar} dt \ |S(x,t)|^{Q-1} (M_Q |\nabla_\lambda u \cdot \chi_{\vartheta S}|)(x). \tag{4.3c}$$

The last step is an estimate of $|\sum|$; we have:

$$|\Sigma| = \int_\Sigma dy \ge$$

(changing variable once more $H(2ar, x, y) = y'$)

$$\begin{aligned} &\ge c_7(\beta) \frac{|H(2ar, x, \sum)|}{|S(x,r)|} = c_7(\beta) \frac{|E \cap H(2ar, x, \Delta_\varepsilon^\alpha(\sigma))|}{|S(x,r)|} \\ &\ge c_7(\beta) \ c_3 \beta \frac{|Q^\sigma(x, \ 2ar)|}{2 \ |S(x,r)|} \ge c_8(\beta) > 0. \end{aligned}$$

Substituting this estimate in formula (4.3c) we obtain

$$\begin{aligned} |u(x)| &\le c_9(\beta) \int_0^{2ar} |S(x,t)|^{Q-1} dt (M_Q(|\nabla_\lambda u| \chi_{\vartheta S}))(x) \\ &= c_9(\beta) r \int_0^{2a} |S(x, r\tau)|^{Q-1} d\tau (M_Q(|\nabla_\lambda u| \chi_{\vartheta S}))(x) \\ &\le c_{10}(\beta, Q) r |S(x,r)|^{Q-1} (M_Q(|\nabla_\lambda u| \chi_{\vartheta S}))(x) \end{aligned}$$

The last inequality follows from:

$$|S(x,r\tau)| \geq c_{11}\tau^{\sum_j G_j}|S(x,r)|, \ \forall \tau \in [0,1],$$

and from our choice of Q.

Estimate (4.3a) now follows using once more that

$$|S(x,r)| \geq c_1|S|.$$

□

Next Lemma is almost identical to Lemma 1.1 of [F.K.S].

LEMMA 4.4. *Let w belong to $A_q(R^N,d,\mathcal{L})$ for a given $q>1$.*

Let f belong to $L^1_{\text{loc}}(R^N)$ and be compactly supported in the closure of a given ball $S=S(\xi,R)$. Then if $Q\in\,]0,1]$, for any $p\geq q$ we have:

$$\text{(4.4a)} \qquad \Big(\frac{1}{w(S)}\int_S |(M_Qf)(x)|^{kp}w(x)dx\Big)^{1/kp} \leq C|S|^{1-Q}\Big(\frac{1}{w(S)}\int_S |f(x)|^p w(x)dx\Big)^{1/p},$$

for $1\leq k\leq (1-(1-Q)p/q)^{-1}$ if $1-Q<q/p$ and for any $k\geq 1$ if $1-Q\geq q/p$. The constant C depends only on N and on $c_{w,p}$.

PROOF. Fix $\lambda\geq 0$ and let $E_\lambda=\{x\in S: M_\alpha f(x)>\lambda\}$. By definition of maximal function, for any $x\in E_\lambda$ there is a positive number $r(x)$ and a d-ball $S(x,r(x))$ such that:

$$|S(x,r(x))|^{-Q}\int_{S(x,r(x))}|f(y)|dy>\lambda.$$

Since f is compactly supported in S, without loss of generality we can suppose that $r(x)\leq 2R$ for any $x\in E_\lambda$. By a "Vitali type" covering lemma (see e.g. [C.W.1] pag. 69) we find a sequence $\{x_j\}_j$, $x_j\in E_\lambda$, such that for the associated d-balls $S_j=S(x_j,r(x_j))$:

$$\text{(4.4b)} \qquad S_i\cap S_j=0 \text{ for } i\neq j$$

$$\text{(4.4c)} \qquad \bigcup_j \theta\, S_j=\bigcup_j S(x_j,\ \theta r(x_j))\supseteq E_\lambda.$$

Here $\theta>1$ is a constant depending only on the homogeneous structure $(R^n,d,\mathcal{L})$ (more precisely, θ depends only on the constant A in (2.7c)). From (4.4c) and the doubling property (2.7c) it follows for any $k\geq 1$ and for any

$p > q$:

$$\begin{aligned}
w(E_\lambda)^{1/k} &\le c(\sum_j w(S_j))^{1/k} \le c\sum_j (w(S_j))^{1/k} \\
&\le c\sum_j w(S_j)^{1/k}(1/\lambda)^p |S_j|^{-pQ}(\int_{S_j} |f|)^p \\
&\le c(1/\lambda)^p \sum_j w(S_j)^{1/k}|S_j|^{-pQ}(\int_{S_j} (\frac{1}{w})^{1/p-1})^{p-1}(\int_{S_j} |f|^p w) \\
&\le c(1/\lambda)^p \sum_j w(S_j)^{(1-k)/k}|S_j|^{-pQ+p}\int_{S_j} |f|^p w.
\end{aligned}$$

Note that in the last inequality we used $w \in A_p$; now from Jensen inequality:

$$(\frac{1}{|S_j|}\int_{S_j} w)^{-1} \le (\frac{1}{|S_j|}\int_{S_j} (\frac{1}{w})^{1/q-1})^{q-1}$$

So that we obtain:

$$w(E_\lambda)^{1/k} \le c(1/\lambda)^p \sum_j |S_j|^{p(1-Q)-q(1-1/k)}(\int_{S_j} (\frac{1}{w_j})^{1/q-1})^{(p-1)(k-1)/k}$$
$$\cdot \int_{S_j} |f|^p w.$$

Now we can choose k such that $p(1-Q) - q(1-1/k) \ge 0$ i.e.

$$1 < k \le (1-(1-Q)p/q)^{-1}$$

if $Q > 1 - q/p$ and $k \ge 1$ otherwise. Now from (4.4b):

$$w(E_\lambda)^{1/k} \le c(1/\lambda)^p |S|^{p(1-Q)-(1-1/k)}(\int_S (\frac{1}{w})^{\frac{1}{q-1}})^{(q-1)(k-1)/k}$$
$$\cdot \int_S |f|^p w.$$

Finally from $w \in A_q$ we obtain:

$$w(E_\lambda) \le c(1/\lambda)^{pk}|S|^{pk(1-Q)}(w(S))^{1-k}(\int_S |f|^p w)^k; \tag{4.4d}$$

that is M_Q is continuous from $L^p(w(x)dx)$ in weak-$L^{pk}(w(x)dx)$. Now the conclusion follows from a standard use of Marcinkiewicz interpolation theorem (see [Ste] pag. 272) and from Lemma 2.9. □

From Lemma 4.3 and 4.4 it follows readily the following:

WEAK VERSION OF THEOREM 4.1: *Assume* $w \in A_q(R^N, d, \mathcal{L})$ *for some* $q > 1$ *and* $S = S(\xi, r)$ *is a fixed d-ball. For any* $\beta \in]0,1[$ *there are constants* $c_\beta = c_\beta(\beta, q)$ *and* $\vartheta > 1$ *(both depending on P and N) such that if* $u : \vartheta S \to R$ *is Lipschitz continuous and* $|E| = |\{x \in S : u(x) = 0\}| \geq \beta |S|$ *then*:

$$(4.a) \qquad (w(S)^{-1} \int_S |u|^{kp} w)^{1/kp} \leq c_\beta r (w(\vartheta S)^{-1} \int_{\vartheta S} \nabla_\lambda u|^p w)^{1/p}$$

for any $p \geq q$ *and* $k \geq 1$ *satisfying the limitations in* (4.1b).

The proof is obvious but for the use of Lemma 2.9 to get the full range of values of k.

Now we are in position to prove:

THEOREM 4.5. *Assume* $w \in A_q(R^N, d, \mathcal{L})$ *for some* $q > 1$ *and* $S = S(\xi, r)$ *is a fixed d-ball. Let* $u : S \to R$ *be Lipschitz continuous, then there are constants* $c_0 = c_0(c_{w,q}, P, N)$ *and* u_s *such that*:

$$(4.5a) \qquad (w(S)^{-1} \int_S |u - u_s|^{kp} w)^{1/kp} \leq c_0 r (w(S)^{-1} \int_S |\nabla_\lambda u|^p)^{1/p},$$

for any $p \geq q$ *and* $k \geq 1$ *satisfying the limitations in* (4.1b).

Moreover u_s *can be chosen to be either the Lebesgue average of* u, *i.e.* $u_s = |S|^{-1} \int_S u$ *or the weighted average, i.e.* $u_s = w(S)^{-1} \int_S u\, w$.

PROOF. First we obtain from (4.a) the following inequality (4.5b) for any d-ball B such that $\vartheta B \subset S$

$$(4.5b) \qquad \begin{aligned} &(w(B)^{-1} \int_B |u - \mu|^{kp} w)^{1/kp} \\ &\leq c r(B)\ (w(\vartheta B)^{-1} \int_{\vartheta B} |\nabla_\lambda u|^p w)^{1/p}. \end{aligned}$$

Here $r(B)$ indicates the radius of B and $\mu = \mu(u, B)$ is the median of u in B. To get (4.5b) just observe that, B and u given, there is a number $\mu = \mu(u, B)$, the median of u in B, such that if $B^+ = \{x \in B : u(x) \geq \mu\}$ and $B^- = \{x \in B : u(x) \leq \mu\}$ then:

$$(4.5c) \qquad |B^+| \geq |B|/2 \text{ and } |B^-| \geq |B|/2.$$

Hence both the functions $u^+ = \max\{u - \mu, 0\}$ and $u^- = \max\{\mu - u, 0\}$ satisfy the hypotheses of the weak version of Theorem 4.1 with $\beta = 1/2$ and we get:

$$(w(B)^{-1} \int_B |u^+|^{kp} w)^{1/kp} \leq cr(B)(w(B)^{-1} \int_B |\nabla_\lambda u^+|^p w)^{1/p}$$

and the same for u^-. Adding these two we get (4.5b).

Now, by a minor modification of a tecnique developed by Jerison in [J], we show that (4.5b) is actually equivalent to a Sobolev-Poincaré inequality "on the same ball", i.e. to (4.5a).

We just sketch here the main steps in Jerison's argument.

First observe that by a standard covering argument and Lemma 2.10, (4.5b) yields:

$$(4.5d) \qquad (\int_B |u - \mu|^{kp} w)^{1/kp} \leq c\ r(B) w(B)^{1/kp - 1/p} (\int_{2B} |\nabla_\lambda u|^p w)^{1/p}.$$

Here and in the following c indicates a constant, not the same at any appearance, depending on the doubling constant of Lemma 2.10. By Whitney decomposition there is a pairwise disjoint family F of d-balls B and a constant M such that

(i) $S = \bigcup_{B \in F} 2B$.

(ii) If $B \in F$ then $10^2 r(B) \leq d(B, \vartheta S) \leq 10^3 r(B)$; where $d(B, \partial S)$ is the distance in the metric d from B to ∂E.

(iii) for any $x \in S : \#\{B \in F : x \in 10B\} \leq M$; here $\#\{\ \}$ is the number of elements of the set $\{\ \}$.

For any $B \in F$ fix a subunit curve γ_B from the center of B to ξ, the center of S, of length $\leq r$, the radius of S. Denote

$$F(B) = \{A \in F : 2A \cap \gamma_B \neq 0\}$$

and for any $A \in F$ denote

$$A(F) = \{B \in F : A \in F(B)\}.$$

Then (see Coroll. 5.8 and Lemma 5.9 in [J]) there are constants c and ε depending only on the doubling constant in Lemma 2.10 such that for any $A, B \in F$.

$$(4.5e) \qquad \#\ F(B) \leq c \log (r/r(B)).$$

$$(4.5f) \qquad \sum w(B) \leq c(r(B)/r(A))^\varepsilon w(A)$$

where the sum is extended to all $B \in A(F)$ such that

$$r \leq r(B) \leq 2r.$$

Now choose $B_0 \in F$ such that $\xi \in 2B_0$. Let μ_0 be the median of u on B. From (4.5d) and arguing as in the proof of Theorem 2.1 in [J] we get for any $B \in F$:

$$\int_{2B} |u-\mu_0|^{pk} w \leq$$

$$\leq c\,(\#F(B))^{pk-1} w(B) \sum_{A\in F(B)} r(A)^{pk} w(A)^{-k} \Big(\int_{10A} |\nabla_\lambda u|^p w\Big)^k.$$

Adding over all $B \in F$ and raising to $1/k$:

$$\Big(\int_S |u-\mu_0|^{pk} w\Big)^{1/k} \leq \sum_{B\in F} \Big(\int_{2B} |u-\mu_0|^{pk} w\Big)^{1/k}$$

$$\leq c \sum_{B\in F} \sum_{A\in F(B)} (\#F(B))^{p-1/k} w(B)^{1/k} r(A)^p w(A)^{-1} \int_{10A} |\nabla_\lambda u|^p w$$

$$= c \sum_{A\in F} \sum_{B\in A(F)} (\#F(B))^{p-1/k} w(B)^{1/k} r(A)^p w(A)^{-1} \int_{10A} |\nabla_\lambda u|^p w.$$

Now by (4.5e) and (4.5f):

$$\sum_{B\in A(F)} (\#F(B))^{p-1/k} w(B)^{1/k} \leq c(\log\,(r/r(A)))^{p-1/k} w(A)^{1/k}$$

hence:

$$\Big(\int_S |u-\mu_0|^{pk} w\Big)^{1/k}$$

$$\leq c. \sum_{A\in F} r(A)^p w(A)^{1/k-1} (\log\,(r/r(A)))^{p-1/k} \int_{10A} |\nabla_\lambda u|^p w.$$

Since $w \in A_q$, from Lemma 2.9 $w \in A_t$ for some t, $1 < t < q$; hence:

$$|A|^t \Big(\int_A w^{-1/(t-1)}\Big)^{1-t} \leq w(A) \leq c_{w,t} |A|^t \Big(\int_A w^{-1/(t-1)}\Big)^{1-t}. \tag{4.5g}$$

From the left side of (4.5g):

$$r(A)^p w(A)^{1/k-1} (\log\,(r/r(A)))^{p-1/k}$$

$$\leq c\; r(A)^p |A|^{-t(1-1/k)} \Big(\int_A w^{-1/(t-1)}\Big)^{(t-1)\,(1-1/k)} (\log\,(r/r(A)))^{p-1/k}$$

$$\leq c\; r(A)^p |A|^{-t(1-1/k)} (\log\,(r/r(A)))^{p-1/k} \Big(\int_S w^{-1/(t-1)}\Big)^{(t-1)\,(1-1/k)}.$$

Note that, by Lemma 2.6 and Proposition 2.7:

$$c\ r(A)^{\sum_j G_j} \le |A| \le c\ r(A)^N$$

hence:

$$c\ r(A)^{p-t(1-1/k)\sum_j G_j} \le r(A)^p|A|^{-t(1-1/k)} \le c\ r(A)^{p-Nt(1-1/k)}$$

where both the exponents on the left and on the right are positive for k as in (4.1b). From the above inequality and $r(A) < r/10$:

$$r(A)^p|A|^{-t(1-1/k)}(\log\ (r/r(A)))^{p-1/k} \le c\ r^p|S|^{-t(1-1/k)}.$$

In all:

$$\Big(\int_S |u-\mu_0|^{pk}w\Big)^{1/k} \le$$

$$\le c\ r^p|S|^{-t(1-1/k)}\Big(\int_S w^{-1/(t-1)}\Big)^{(t-1)\ (1-1/k)} \sum_{A\in F} \int_{10A} |\nabla_\lambda u|^p w \le$$

by (4.5g):

$$\le c\ r^p w(S)^{1/k-1} \sum_{A\in F} \int_{10A} |\nabla_\lambda u|^p w$$

$$\le c\ r^p w(S)^{1/k-1} \int_S |\nabla_\lambda u|^p w.$$

Finally it is easy to show that μ_0 can be substituted by an average u_s of u:

$$\int_S |u-u_s|^{pk}w \le 2^{pk-1}\Big(\int_S |u-\mu_0|^{pk}w + \int_S |\mu_0-u_s|^{pk}w\Big).$$

If $u_s = w(S)^{-1}\int_S uw$ then:

$$\int_S |\mu_0-u_s|^{pk}w \le w(S)\ \Big(w(S)^{-1}\int_S |\mu_0-u|w\Big)^{pk} \le \int_S |\mu_0-u|^{pk}w.$$

If $u_s = |S|^{-1}\int_S u$ then:

$$\int_S |\mu_0-u_s|^{pk}w \le w(S)\ \Big(|S|^{-1}\int_S |\mu_0-u|\Big)^{pk} \le \int_S |\mu_0-u|^{pk}w$$

since $w \in A_q$ and $pk \ge q$. $\square$

THEOREM 4.6. *Assume* $w \in A_q(R^N, d, \mathcal{L})$ *for some* $q > 1$ *and* $S = S(\xi, r)$ *is a d-ball. Let* $u : S \to R$ *be Lipschitz continuous and supported in* S, *then there is a constant* $c_0 = c_0(c_{w,q}, p, N)$ *such that*:

$$(w(S)^{-1} \int_S |u|^{pk} w)^{1/pk} \le c_0 r (w(S)^{-1} \int_S |\nabla_\lambda u|^p w)^{1/p}, \tag{4.6a}$$

for $p \ge q$ *and* $k > 1$ *as in* (4.1b).

Moreover, if Ω *is a bounded open subset of* R^N, *and*

$$w \in A_2(R^N, d, \mathcal{L}), \text{ then } \overset{0}{\mathrm{H}}{}^1_\lambda(\Omega; w)$$

is compactly embedded in

$$L^p(\Omega, w) \text{ if } 2 \le p < p_0 = 2 \sum_j G_j / (\sum_j G_j - 1).$$

PROOF. In order to prove the first assertion, we need only to apply Theorem 4.1 in the ball $S(\xi, 2r)$, keeping in mind that, by the doubling property (2.7c), it follows immediatly that $|S(\xi, 2r) - S(\xi, r)|$ is equivalent to $|S(\xi, 2r)|$.

Now, we note that the continuous embedding of $\overset{0}{\mathrm{H}}{}^1_\lambda(\Omega, w)$ in $L^p(\Omega, w)$, $2 \le p < p_0$, follows from (4.6a), if we choose r sufficiently large.

Then, we need only to prove compacteness. By an interpolation argument, we can reduce ourselves to the case $p = 2$.

Let now $(u_n)_{n \in N}$ be a sequence in the unit ball of $\overset{0}{\mathrm{H}}{}^1_\lambda(\Omega, w)$; without loss of generality, we may suppose

$$u_n \in C_0^\infty(R^N), \text{ supp } u_n \subseteq \Omega \subseteq S(0,1) \text{ for any } n \in N.$$

By the reflexivity of $L^2(\Omega, w)$, we may suppose that u_n converges to u weakly in $L^2(\Omega, w)$.

Now, by the "Vitali type" theorem of [C.W.1], pag. 69, for every $r \in]0,1]$, there exist $x_1, \cdots, x_{m(r)} \in S(0,1)$ such that:

i) $S(x_k, r/5) \cap S(x_h, r/5) = 0$ if $k \neq h$;

ii) $S(0,1) \subseteq \bigcup_k S(x_k, r)$.

We can prove that:

iii) $m(r) \le c_0 r^{-\alpha}$ and

iv) $S(x_k, r) \cap S(x_i, r) \neq 0$ for at most M different indices $i \neq k$;

here α is the constant of (2.10b) and c_0 and M depend only on α. In fact,

$$S(x_k,\ r/5) \subseteq S(0,\ 6/5)\ \forall\ k,\ \forall\ r \in]0,1],$$

so that

$$\begin{aligned} w(S(0,\ 6/5)) &\geq \sum_k w(S(x_k, r/5)) \geq (r/11)^\alpha \sum_k w(S(x_k, 11/5)) \\ &\geq (r/11)^\alpha m(r) w(S(0,\ 6/5)), \end{aligned}$$

and iii) follows.

Arguing in the same way, if we put

$$I(k) = \{i \leq m(r) : S(x_k, r) \cap S(x_i, r) \neq 0\},$$

since $d(x_k, x_i) < 2r$ if $i \in I(k)$, we get:

$$\begin{aligned} w(S(x_k, 11r/5) &\geq \sum_{i \in I(k)} w(S(x_i, r/5)) \\ &\geq 21^{-\alpha} \sum_{i \in I(k)} w(S(x_i, 21r/5)) \geq 21^{-\alpha} w(S(x_k, 11r/5)) \bullet \text{ card } I(k). \end{aligned}$$

Then, iv) follows.

Now, we can prove that $(u_n)_{n \in N}$ is a Cauchy sequence in $L^2(\Omega, w)$. Let $\varepsilon > 0$ be fixed. By (4.5a), we have:

$$\begin{aligned} \int_\Omega |u_n - u_m|^2 w(x)dx &\leq \sum_{j=1}^{m(r)} \int_{S(x_j,r)} |u_n - u_m|^2 w(x)dx \\ &\leq \sum_{j=1}^{m(r)} \frac{1}{w(S(x_j, r))} \Big(\int_{S(x_j,r)} (u_n - u_m) w(x)dx \Big)^2 \\ &+ C\ r^2 \sum_{j=1}^{m(r)} \int_{S(x_j,r)} |\nabla(u_n - u_m)|^2 w(x)dx \\ &\leq \sum_{j=1}^{m(r)} \frac{1}{w(S(x_j, r))} \Big(\int_{S(x_j,r)} (u_n - u_m) w(x)dx \Big)^2 + 4\ C(M+1) r^2. \end{aligned}$$

Now, let us choose $r = \varepsilon(2C(M+1))^{-1/2}$, so that last term is estimated by $\varepsilon^2/2$. On the other hand, since $S(x_j, 2) \supseteq S(0,1)$, we get:

$$\sum_{j=1}^{m(r)} \frac{1}{w(S(x_j, r))} (\cdots)^2 \leq (\frac{2}{r})^\alpha w(S(0,1))^{-1} \sum_{j=1}^{m(r)} \Big(\int_{S(x_j,r)} (u_n - u_m) w(x)dx \Big)^2.$$

Since the sequence $(u_n)_{n \in N}$ is weakly convergent in $L^2(\Omega, w)$, there exists

$n(\varepsilon)$ such that, if $n, m > n(\varepsilon)$,

$$|\int\limits_{S(x_j,r)} (u_n - u_m)w(x)dx|^2 < \varepsilon^2 w(S(0,1))r^{2\alpha}/2^\alpha c_0, \ j = 1, \cdots, m(r).$$

Keeping in mind the estimate of $m(r)$, the assertion follows. □

REMARK 4.7. It is a consequence of the comparability of the two distances d and δ and of the doubling properties that Theorems 4.1, 4.5 and 4.6 yield similar statements with the δ-balls $Q(\xi, r)$ instead of the d-balls $S(\xi, r)$.

Let us see how this works for Theorem 4.6. Let $u : Q = Q(\xi, r) \to R$ be Lipschitz continous and supported in Q and extend it equal to zero to all of R^N.

From (2.7a) and Lemma 2.10 there exists a constant $c(a, B)$, $0 < c(a, B) < 1$, such that:

$$w(Q) \geq w(S(\xi, r/a)) \geq c(a, B)w(S(\xi, ar)). \tag{4.7a}$$

Hence thinking u as supported in $S(\xi, ar)$, from (4.7a) and Theorem 4.6 we have:

$$(\frac{1}{w(Q)}) \int\limits_Q |u|^{pk}w)^{1/pk} \leq c\ r(\frac{1}{w(Q)} \int\limits_Q |\nabla_\lambda u|^p w)^{1/p}. \tag{4.7b}$$

The analogous versions of Theorems 4.1 and 4.5 are less precise than the original ones, but sufficient to our purposes.

Let's see first Theorem 4.1:

Assume $u : R^N \to R$ be Lipschitz continuous and

$$|\{x \in Q(\xi, r) | u(x) = 0\}| \geq \beta |Q(\xi, r)|, \ 0 < \beta \leq 1.$$

Once more from (2.7a) and (4.7a) we have:

$$\begin{aligned} &|\{x \in S(\xi, ar) : u(x) = 0\}| \geq |\{x \in Q(\xi, r) : u(x) = 0\}| \\ &\geq \beta |Q(\xi, r)| \geq \beta\ c(a, B)|S(\xi, ar)| \end{aligned}$$

Hence from Theorem 4.1 in $S(\xi, ar)$ and from (4.7a):

$$\begin{aligned} &(\frac{1}{w(Q(\xi, r))} \int\limits_{Q(\xi,r)} |u|^{pk}w)^{1/pk} \\ &\leq (\frac{1}{c(a, B)})^{1/pk} (\frac{1}{w(S(\xi, ar))} \int\limits_{S(\xi,ar)} |u|^{pk}\ w)^{1/pk} \\ &\leq c\ r(\frac{1}{w(S(\xi, ar))} \int\limits_{S(\xi,ar)} |\nabla_\lambda u|^p\ w)^{1/p} \\ &\leq c' r(\frac{1}{w(Q(\xi, a^2 r))} \int\limits_{Q(\xi,a^2 r)} |\nabla_\lambda u|^p w)^{1/p}. \end{aligned} \tag{4.7c}$$

From (4.7c) as in the proof of Theorem 4.6 one gets:

$$\int\limits_{Q(\xi,r)} |u-u_Q|^p w \leq cr^p \int\limits_{Q(\xi,a^2r)} |\nabla_\lambda u|^p w. \tag{4.7d}$$

The last result we formulate explicitly is an embedding near the boundary. We will need the following definition.

DEFINITION 4.8. A bounded open subset $\Omega \subseteq R^N$ is said to belong to the class S if there exist $\alpha, r_0 > 0$ such that for each $x_0 \in \partial\Omega$ and for each $r \in \,]0, r_0[$, we have

$$|S(x_0,r)-\Omega| \geq \alpha |S(x_0,r)|.$$

THEOREM 4.9. *Assume $w \in A_q$, for $q > 1$, Ω belong to the class S with constant α and $x_0 \in \partial\Omega$. If $u : \Omega \cap S(x_0,r)$ is Lipschitz continuous and $u = 0$ on $\partial\Omega \cap S(x_0,r)$, then there is $c = c(\alpha, N, c_{wq}, p)$ such that*

$$(w(S)^{-1} \int\limits_{S\cap\Omega} |u|^{pk} w)^{1/pk} \leq c\; r(w(S)^{-1} \int\limits_{S\cap\Omega} |\nabla_\lambda u|^p w)^{1/p} \tag{4.8a}$$

for $p \geq q$ and $k > 1$ as in (4.1b).

PROOF. Just extend $u = 0$ on $S - \Omega$ and use Theorem 4.1. □

Section 5.

We devote this section and the following one to the study of global and local properties of the solutions defined in section 2. Our main result here is Harnack inequality from which interior Holder continuity of solutions follows. The study of the boundary behavior of solutions is the argument of the next section.

Since the techniques we use are well known we often merely state the main theorems and lemmas. After the statement is given, we give a reference to a corresponding nondegenerate theorem and explain the differences in the proof (if there are any). The references we use in this section are [K.S.] chapter 2 and [Sta]; see also [F.K.S.].

Let's begin with some technical lemmas:

LEMMA 5.1 (Chain rule) *Let $f : R \to R$, be either a C^1 function with bounded derivative, or a piecewise linear function whose derivative has discontinuities at $\{a_1, \cdots, a_n\}$. Let $u \in H^1_\lambda(\Omega, w)$.*

Then

$$fou \in H^1_\lambda(\Omega,w) \text{ and } \nabla_\lambda(fou)(x) = f'(u(x))\nabla_\lambda u(x),$$

with the convention that both sides are 0 when $x \in \bigcup_j \{y : u(y) = a_j\}$.

The proof of this lemma follows the lines of Lemma A.3 and Corollary A.5 of [K.S.].

COROLLARY 5.2. *Let* $u \in H^1_\lambda(\Omega, w)$, *then* $u^+ = \max(u, 0)$, $u^- = \max(-u, 0)$ *and* $|u| \in H^1_\lambda(\Omega, w)$. *Moreover* $\nabla_\lambda u^+ = \nabla_\lambda u$ *where* u *is positive, otherwise* $\nabla_\lambda u^+ = 0$. *An analogous statement holds for* u^- *and* $|u|$. *Moreover if* $u \in \overset{0}{\mathrm{H}}{}^1_\lambda(\Omega, w)$ *so does* $u^+, u^-, |u|$. *If* v *also belongs to* $H^1_\lambda(\Omega, w)$ *so does* $\mathrm{Sup}(u, v)$ *and* $\mathrm{Inf}(u, v)$.

Given $E \subset \overline{\Omega}$ and $u, v \in H^1_\lambda(\Omega, w)$, the notions of $u \leq v$ on E in $H^1_\lambda(\Omega, w)$ and of $\mathrm{Sup}\, u$ in $H^1_\lambda(\Omega, w)$ are defined as usual (see [K.S.] Definition 5.1). The following lemma summarizes the main properties of the concepts mentioned above.

LEMMA 5.3 *Let* $E \subset \overline{\Omega}$, $u \in H^1_\lambda(\Omega, w)$.

(i) *If* $u \geq 0$ *on* E *in* $H^1_\lambda(\Omega, w)$, *then* $u \geq 0$ *a.e. on* E.

(ii) *If* $u \geq 0$ *on* Ω *a.e. then* $u \geq 0$ *in* $H^1_\lambda(\Omega, w)$.

(iii) *If* $u \geq 0$ *on* Ω *a.e.,* $u \in \overset{0}{\mathrm{H}}{}^1_\lambda(\Omega, w)$, *then there exists a sequence* $\{\varphi_k\} \in \mathrm{Lip}(\overline{\Omega})$, $\mathrm{supp}\, \varphi_k \subset\subset \Omega$, *such that* $\varphi_k \geq 0$ *in* Ω *and* $\varphi_k \to u$ *in* $\overset{0}{\mathrm{H}}{}^1_\lambda(\Omega, w)$.

(iv) *If* E *is open in* Ω *and* $u \geq 0$ *on* E *a.e., then* $u \geq 0$ *on* K *in the* $H^1_\lambda(\Omega, w)$ *sense, for any compact* $K \subset E$.

PROOF. (i), (ii), (iii) have identical proofs to the corresponding statements in Proposition 5.2 of [K.S.]. For the proof of (iv) see [F.K.S.] Lemma 2.13. □

The last lemma we need is the following:

LEMMA 5.4. (Existence of cut-off functions. See [F.L.2] pag. 537). *Let* $\overline{x} \in R^n$ *and* $0 < r_1 < r_2$, *then there is a function* $\alpha \in C_0^\infty(R^N)$ *such that*

(i) Supp $\alpha \subset Q(\overline{x}, r_2)$; $\alpha \equiv 1$ *in* $Q(\overline{x}, r_1)$

(ii) $|\nabla_\lambda \alpha| \leq 2(r_2 - r_1)^{-1}$.

Now it follows a version of the classical Stampacchia's weak maximum principle and after a global boundedness theorem.

THEOREM 5.5. (Weak Maximum Principle) *Let* L *belong to* $\mathcal{L}(\Omega, N, \nu, \Lambda, P, c_{w,2})$. *Let* $u \in H^1_\lambda(\Omega, w)$ *be a L-supersolution in* Ω. *Then*

$$u(x) \geq \operatorname*{Inf}_{\partial\Omega} u \text{ a.e. in } \Omega,$$

where $\operatorname*{Inf}_{\partial\Omega} u$ *is taken in the* $H^1_\lambda(\Omega, w)$ *sense.*

PROOF. See Theorems 5.5 and 5.7 in [K.S.]. □

THEOREM 5.6. *Let $L \in \mathcal{L}(\Omega, N, \nu, \Lambda, P, c_{w,2})$ and $k > 1$ be as in (4.5a). Assume $f = (f_1, \cdots f_N)$ be such that $|f|/w \in L^p(\Omega, w)$ for $p > 2k/(k-1)$.*

Then if $u \in \overset{0}{\mathrm{H}}{}^1_\lambda(\Omega, w)$ is a solution in Ω of $Lu = -\mathrm{div}_\lambda f$, we have

$$\text{(5.6a)} \qquad \operatorname*{Sup}_{\Omega} |u(x)| \leq C(\Omega) w(Q)^{1/2-1/p-1/2k} \| \, |f|/w \|_{L^p(\Omega, w)}.$$

Where $C(\Omega)$ depends only on the diameter of Ω.

The proof again is like the one of Theorem B.2 of [K.S], replacing the exponent 2^* by $2k$ and the usual gradient by ∇_λ. □

Finally we start with the proofs of Harnack inequality and interior Holder continuity of solutions. To get these results we need the inequalities of Theorems 4.5 and 4.6 (more precisely we will use their corollaries (4.7b) and (4.7d) since it will be more convenient to use the distance δ and the corresponding δ-balls $Q(x, r)$).

Let's start with the following local boundedness theorem that can be proved using (4.7b).

THEOREM 5.7. *Let L belongs to $\mathcal{L}(\Omega, N, \nu, \Lambda, P, c_{w,2})$ and let u be a local subsolution. Then there exists $M > 0$, M independent of r and u, such that for any δ-ball $Q = Q(x, r) \subseteq \Omega$, we have*

$$\text{(5.7a)} \qquad \operatorname*{Sup}_{Q(x,r/2)} u \leq M \Big(\frac{1}{w(Q(x,r))} \int_{Q(x,r)} u^2 w dy \Big)^{1/2}.$$

PROOF. Proceed as in Theorem C.4 of [K.S.], using the cut-off functions defined in Lemma 5.4. The main fact that is needed is: if v is a non-negative local subsolution in Ω and $\alpha \in C_0^\infty(\Omega)$, then

$$\text{(5.7b)} \qquad \int_\Omega \alpha^2 |\nabla_\lambda v|^2 w \; dy \leq C(\nu) \int_\Omega |\nabla_\lambda \alpha|^2 v^2 w \; dy.$$

Now choose α such that Supp $\alpha \subset Q(x, r)$ and use (5.7b) together with Corollary 4.7 to get:

$$\int_{Q(x,r)} \alpha^2 v^2 w \; dy \leq C \; r^2 w(Q(x,r))^{1/k-1} \int_{Q(x,r)} |\nabla_\lambda \alpha|^2 v^2 w \; dy \cdot (w\{x : \alpha \cdot v \neq 0\})^{1/k'},$$

where $1/k + 1/k' = 1$. Now the only change needed is to choose ξ and η so that $\xi + \eta = \Theta \xi$, $\xi = \Theta \eta k'$, with $\Theta > 1$. This can be done with $\eta = 1$, if we choose Θ to be one of the roots of the equations $\Theta^2 - \Theta - 1/k' = 0$. □

From the above theorem it follows that solutions are locally bounded. In fact we have:

COROLLARY 5.8. *Let u be a local solution in Ω of $Lu = 0$, $L \in \mathcal{L}(\Omega, N, \nu, \Lambda, P, c_w)$. Then*

$$\operatorname*{Sup}_{Q(x,r/2)} |u| \leq M\Big(\frac{1}{w(Q(x,r))} \int_{Q(x,r)} u^2 w \ dx\Big)^{1/2}.$$

The following theorem is Harnack's inequality.

THEOREM 5.9. *Let L belong to $\mathcal{L}(\Omega, N, \nu, \Lambda, P, c_{w,2})$ and u be a positive local solution of $Lu = 0$ in Ω. Then there is a positive constant M (depending only on the structural constants of L, but independent of x_0, u, r such that if $8r < \delta(x_0, \partial\Omega)$ then:*

$$\operatorname*{Sup}_{Q(x_0,r)} u \leq M \operatorname*{Inf}_{Q(x_0,r)} u. \tag{5.9a}$$

PROOF. The proof is along the lines of Moser iteration techniques. See for example theorem 8.1 of [Sta]. The main estimates that are used are the following ones: for any $p \neq 1/2$ let $v = u^p$, $\alpha \in C_0^\infty(Q(x_0,\ 2r))$, then:

$$\int_\Omega \alpha^2 |\nabla_\lambda v|^2 w \ dx \leq C(p) \int_\Omega |\nabla_\lambda \alpha|^2 v^2 w \ dx.$$

We also need to estimate $v = \log u$. It is at this point that we use (4.7d). For any $Q(x,\ 2r) \subset \Omega$ we obtain the inequality:

$$\int_{Q(x,\ 2r)} |v - v_Q|^2 w \ dy \leq c\ w(Q(x,\ 2r)) \tag{5.9b}$$

where $v_Q = \int_{Q(x_0,r)} v\ w\ dx \ /\ w(Q(x_0,r))$. Because of the last inequality, using the John-Nirenberg lemma in the setting of the homogeneous space $(R^N, d, w(x)dx)$ (a proof of this lemma in homogeneous spaces can be found in [B]) we obtain:

$$\Big(\int_{Q(x,r)} u^q w \ dy \ /\ w(Q(x,r))\Big)^{1/q} \leq c\Big(\int_{Q(x,r)} u^{-q} w \ dy \ /\ w(Q(x,r))\Big)^{-1/q}, \tag{5.9c}$$

for two positive numbers q and c, both independent of u, x, r.

Now let us fix x_0 and r satisfying the hypotheses, then using Theorem 5.7 applied to $v = u^p$ for $p < 0$ or $p > 1$, Moser's iteration argument gives:

$$\sup_{Q(x_0,r)} u \leq M\Big(\int_{Q(x_0,2r)} u^2 w \ dx \ /\ w(Q(x_0,2r))\Big)^{1/2} \tag{5.9d}$$

and

$$\text{(5.9e)} \qquad \operatorname*{Inf}_{Q(x_0,r)} u \geq M\Big(\int_{Q(x_0,2r)} u^{-q} w \; dx \; / \; w(Q(x_0,2r))\Big)^{-1/q}$$

Combining (5.9c), (5.9d) and (5.9e) the theorem follows if we can show that:

$$\Big(\int_{Q(x_0,2r)} u^2 w \; dx \; / \; w(Q(x_0,2r))\Big)^{1/2} \leq c(q)\Big(\int_{Q(x_0,4r)} u^q w \; dx \; / \; w(Q(x_0,2r))\Big)^{1/q},$$

once more this can be obtained with an iteration argument, similar to the preceeding one. □

We observe here that Theorems 3.1 follows from Theorem 5.9 and the comparability of d and δ. Moreover Holder continuity of solutions, as stated in Theorem 3.3, follows from Harnack's inequality and from (2.7b).

We now wish to study the local behaviour of solutions of

$$Lu = -\mathrm{div}_\lambda f, \;\; f = (f_1, \cdots, f_N), \;\; |f|/w \in L^p(\Omega, w)$$

and p sufficiently large. For the proofs our references here are Theorems 7.1, 7.2, 7.3 in [Sta].

First we obtain the following variant of (5.6a) using (4.5a):

If $u \in \overset{0}{\mathrm{H}}{}^1_\lambda(Q(x,r),w)$ *and* $Lu = -\mathrm{div}_\lambda f$ *where* $|f|/w \in L^p(Q(x,r),w)$ *for* $p > 2k/(k-1)$ *then*:

$$\text{(5.b)} \qquad \operatorname*{Sup}_{Q(x,r)} |u| \leq C \; r \; w(Q(x,r))^{1/2k-1/2} \bullet w(Q(x,r))^{1/2-1/p-1/2k} \cdot \| \; |f|/w \|_{L^p(Q,w)}$$

$$= c \; r \; w(Q(x,r))^{-1/p} \| \; |f|/w \|_{L^p(Q(x,r),w)}.$$

From (5.b) we obtain the following Lemma that is a key step in the proof of Holder continuity:

LEMMA 5.10. *Assume* $Q(x_0,r) \subset Q(x_0,R)$ *for a fixed large* R. *Let* $u \in \overset{0}{\mathrm{H}}{}^1_\lambda(Q(x,r),w)$ *be a solution of* $Lu = -\mathrm{div}_\lambda f$, *with*

$$|f|/w \in L^p(Q(x,r),w), \;\; p > 2 \; k/(k-1)$$

where k *is as in* (4.5a).

Assume also that p *is so large that* $w \in A_{p/\delta}$, *where* $\delta \geq N$ *is such that* $|Q(x_0 r)| \geq r^\delta$.

Then there is $\sigma \in \,]0,1[$ *such that*:

$$\operatorname*{Sup}_{Q(x_0,r)} |u(x)| \leq c(R) \; r^\sigma \| \; |f|/w \|_{L^p(Q(x_0,r),w)}.$$

PROOF. Let q be a number ≥ 1 that will be fixed later.
If $(\frac{1}{w})^{1/(q-1)} \in L^1_{\text{loc}}(R^N)$ it follows that

$$w(Q(x_0, r)) \geq C(R, w)|Q(x_0, r)|^q.$$

Hence

$$w(Q(x_0, r))^{-1/p} \leq c(R)|Q(x_0, r)|^{-q/p} \leq c(R) r^{-\delta q/p}.$$

Since we assume $w \in A_{p/\delta}$ wè have also $w \in A_{(p/\delta)-\varepsilon}$ hence

$$(\frac{1}{w})^{1/(q-1)} \in L^1_{\text{loc}}(R^N) \text{ for } q = (p/\delta) - \varepsilon;$$

with this choice of q we have $\delta q/p < 1$, and the statement follows from (5.b) □

Finally the interior Holder continuity for solutions of $Lu = -\text{div}_\lambda f$ follows in a standard way from the above results:

THEOREM 5.11. *Let L belongs to $\mathcal{L}(\Omega, N, \nu, \Lambda, P, c_{w,2})$. Assume $u \in H^1_\lambda(\Omega, w)$ be a local solution of $Lu = -div_\lambda f$ in Ω, where $p > 2k/(k-1)$, k as in* 4.5a, *and p is such that $w \in A_{p/\delta}$.*

Then u is locally Hölder continuous in Ω, that is there is σ, $0 < \sigma < 1$ such that if

$$x_0 \in \Omega, \ 0 < r < R < \frac{1}{4}\, \delta(x_0, \partial\Omega),$$

then

$$\operatorname*{Sup}_{|x-x_0|<r} |u(x) - u(x_0)| \leq c(R)\Big(\Big(\frac{1}{w(S(x_0,R))} \int_{S(x_0,R)} u^2 w \, dx\Big)^{1/2} + \| \, |f|/w\|_{L^p(S,w)}\Big) \cdot r^\sigma.$$

Section 6.

We now proceed to study the behavior of solutions of $Lu = -\text{div}_\lambda f$ near a portion of the boundary where the solution vanishes. We assume through all this section that

$$L \in \mathcal{L}(\Omega, N, \nu, \Lambda, P, c_{w,2}).$$

Our first result is an L^∞ estimate which is analogous to Theorem 5.3 of [Sta].

LEMMA 6.1. *Let $x_0 \in \partial\Omega$ and let*

$$u \in H^1_\lambda(\Omega \cap Q(x_0, r_0), w)$$

be a subsolution for L in $\Omega \cap Q(x_0, r_0)$. Assume that $u \leq 0$ on $\partial\Omega \cap Q(x_0, r_0)$ in the sense of $H^1_\lambda(\Omega \cap Q(x_0, r_0), w)$, then if $r < r_0$ and $k_0 \geq 0$ we have:

$$\operatorname*{Sup}_{\Omega\cap Q(x_0,r/2)} u \leq k_0 + K\{\frac{1}{w(Q(x_0,r))} \int_{A(k_0,r)} |u-k_0|^2 w \, dx\}^{1/2} \cdot (\frac{w(A(k_0,r))}{w(Q(x_0,r))})^{(\Theta-1)/2}$$

where

$$A(k_0, r_0) = \{x \in Q(x_0, r_0) \cap \Omega : u(x) \geq k_0\}$$

and $\Theta = 1/2 + 1/4 + 1/k' > 1$, $1/k + 1/k' = 1$ *and k as in Theorem* 4.5.

PROOF. As for Theorem 5.7 of the preceeding section the proof follows the same steps as in Theorem 5.3 of [Sta]. More precisely: integrating by parts in the equation and using Theorem 4.5 one gets the following two inequalities.

(i)
$$\int_{A(k_1,r_1)} (u-k_1)^2 w \, dx \leq C \frac{(w(A(k_1,r_1)))^{1/k'}}{w(Q(x_0,r_1))^{1/k'}} \frac{r_1^2}{(r_2-r_1)^2} \int_{A(k_1,r_2)} (u-k_1)^2 w dx, \text{ and}$$

(ii)
$$(k_2-k_1)^2 w(A(k_2,r_1)) \leq \int_{A(k_1,r_1)} (u-k_1)^2 w \, dx,$$

for $k_0 < k_1 < k_2$ and $0 < r_1 < r_2 < r_0$.

From now on the proof has no changes but for the choice of

$$\xi = \Theta k' \text{ and } \Theta = 1/2 + 1/4 + 1/k'.$$

□

The following result follows from Lemma 6.1.

THEOREM 6.2. *Let $x_0 \in \partial\Omega$ and $u \in H^1_\lambda(\Omega \cap Q(x_0, r_0), w)$ be a solution of $Lu = 0$ in $\Omega \cap Q(x_0, r_0)$ such that $u = 0$ on $\partial\Omega \cap Q(x_0, r_0)$ in the H^1_λ sense. Then if $r < r_0$:*

$$\operatorname*{Sup}_{Q(x_0,r/2)\cap\Omega} |u| \leq M\{\frac{1}{w(Q(x_0,r))} \int_{Q(x_0,r)} u^2 w \, dx\}^{1/2}.$$

In order to establish Holder continuity at the boundary, certain smoothness assumptions on the domains to be considered are necessary. We recall here definition 4.8 of domains of class S: (see also [Sta] Def. 6.3) Ω is of class S if there are α, r_0 such that for any $x_0 \in \partial\Omega$ and for any $r < r_0$:

(6.a)
$$|S(x_0, r) - \Omega| \geq \alpha |S(x_0, r)|.$$

From (6.a) and the comparability of the two distances d and δ it follows:

$$|Q(x_0,r)-\Omega| \geq \alpha\ c(A,a)|Q(x_0,r)| \tag{6.b}$$

for any $x_0 \in \partial\Omega$ and $r < r_0$.

We observe also that there are equivalent formulations of (6.a) or (6.b) in terms of the weighted measure. In fact we recall the following A_∞ property of A_p weights (see [C] Lemma 4): if $w \in A_p$ and $E \subset S(x_0,r)$ then

$$(|E|\ /\ |S(x_0,r)|)^p \leq c_{w,p} w(E)\ /\ w(S(x_0,r). \tag{6.c}$$

From (6.c) if Ω is of class S there are $\alpha', r_0 > 0$ such that:

$$w(S(x_0,r)-\Omega) \geq \alpha'\ w(S(x_0,r)). \tag{6.d}$$

A preliminary version of Hölder continuity is given by the following Lemma:

LEMMA 6.3. *Let Ω be of class S, G a bounded open set and u a solution in $G\cap\Omega$ of $Lu=0$. Assume that $u=0$ on $\partial\Omega\cap G$ in the $H^1_\lambda(G\cap\Omega,w)$ sense. Then there is $\eta>1$ such that, for $x_0\in\partial\Omega\cap G$ and r small enough,*

$$\omega(r) \leq \eta\ \omega(4r) \tag{6.3a}$$

where $\omega(r) = \operatorname*{Sup}_{\Omega\cap Q(x_0,r)} u - \operatorname*{Inf}_{\Omega\cap Q(x_0,r)} u.$

PROOF. Once more the proof is similar to the one of Lemma 7.4 of [Sta].

Let $0<h<h'$ and let $v=\min(u,h')-\min(u,h)$ in $S(x_0,r)\cap\Omega$ while $v=0$ in $S(x_0,r)-\Omega$. Define the level sets of u as $A(h,r)=\{x\in S(x_0,r): u(x)\geq h\}$. We recall Lemma 2.9; then $w\in A_2$ implies that $w\in A_p$ for some $p<2$. Finally, since Ω is of class S and $v=0$ on $S(x_0,r)-\Omega$ we can use Theorem 4.1 and we obtain:

$$\begin{aligned}(h'-h)&w(S(x_0,r))^{-1/kp}\ w(A(h',r))^{1/kp}\\ &\leq (w(S(x_0,r))^{-1}\int_{S(x_0,r)} v^{kp} w dx)^{1/kp}\\ &\leq c\ r(w(S(x_0,r))^{-1}\int_{A(h,r)-A(h',r)} |\nabla_\lambda v|^p w dx)^{1/p}.\end{aligned} \tag{6.3b}$$

Finally from Hölder's inequality we have:

$$w(A(h',r))^{2/pk}(h'-h)^2 \leq c\ r^2 w(S(x_0,r))^{-2(1-1/k)/p} \int_{A(h,r)-A(h',r)} |\nabla_\lambda u|^2 w\ dx \cdot (w(A(h,r)-A(h',r))^{2/p-1}. \tag{6.3c}$$

Using (6.3c) the last part of the proof proceeds with almost no changes. □

Our main result on boundary regularity follows from Lemma 6.3 and Lemma 5.10.

THEOREM 6.4. *Let Ω be of class S, G a bounded open set and $u \in H^1_\lambda(G\cap\Omega, w)$ be a solution of $Lu = -\mathrm{div}_\lambda f$. Assume $f = (f_1, \cdots, f_N)$ is such that*

$$|f|/w \in L^p(\Omega, w) \text{ for } p > \mathrm{Max}\{2k/(k-1), 2\delta\}$$

where $k > 1$ is the exponent of (4.5a) *and $\delta \geq N$ is such that $|S(x_0,r)| \geq r^\delta$. If u is 0 on $\partial\Omega\cap G$ in the $H^1_\lambda(G\cap\Omega, w)$ sense, then u is Holder continuous on $G\cap\Omega$, i.e. there are $M > 0$ and $\sigma_1, \sigma_2 \varepsilon\ ]0,1[$ such that:*

$$\omega(r) \leq M\{(\frac{1}{w(S(x,r_0))} \int_{S(x,r_0)\cap\Omega} u^2 w\ dx)^{1/2} + \|\ |f|/w\|_{L^p(\Omega,w)}\} r^{\sigma_1}$$

for $x \in \Omega\cap G$, $0 < r < r_0$ and $S(x,r_0)\cap\Omega \subset G$, and

$$\sup_{|x-y|<r} |u(x)-u(y)| \leq M\{(\frac{1}{w(S(x_0,r_0))} \int_{S(x_0,r_0)} u^2 w\ dx)^{1/2} + \|\ |f|/w\|_{L^p(\Omega,w)}\}|x-y|^{\sigma_2},$$

for $0 < r < r_0$ and $S(x,r_0)\cap\Omega \subset G$.

Section 7.

In this Section, we will prove some technical results we used in the proof of the basic estimates of Section 4. More exactly, we introduce a suitable family of sub-unit curves depending on N real parameters and we prove that the set of points of a given d-ball we can reach along these curves from the centre is as big as we like.

Let us now introduce some notations.

DEFINITION 7.1. Let $\alpha = (\alpha_1, \cdots, \alpha_N)$ and $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_N)$ be given vectors in R^N with $0 < \varepsilon_j < \alpha_j$, $j = 1, \cdots, N$. We shall denote by $\Delta^\alpha_\varepsilon$ the set

$$\{y \in R^N;\ \varepsilon_j \leq y_j \leq \alpha_j,\ j = 1, \cdots, N\}.$$

If $\sigma \in \{-1,1\}^N$, we shall put

$$T_\sigma y = T_\sigma(y_1,\cdots,y_N) = (\sigma_1 y_1,\cdots,\sigma_N y_N)$$

and we shall denote by $\Delta_\varepsilon^\alpha(\sigma)$ the set $T_\sigma(\Delta_\varepsilon^\alpha)$. Moreover, if $x \in R^N$ and $\rho > 0$ we put

$$Q^\sigma(x,\rho) = \{y \in Q(x,\rho);\ \sigma_j(y_j - x_j) \geq 0,\ j = 1,\cdots,N\},$$

if $\sigma = (1,\cdots 1)$, we shall write $Q^+(x,\rho)$ instead of $Q^{(1,\cdots,1)}(x,\rho)$.

DEFINITION 7.2. If $x,y \in R^N$, we shall denote by $H(t,x,y) = (H_1(t,x,y),\cdots)$ the solution at the time t of the Cauchy problem

$$\overset{\bullet}{H}_j(\bullet,x,y) = \lambda_j(H(\bullet,x,y))y_j,\ H_j(0,x,y) = x_j,\ j = 1,\cdots,N.$$

We note explicitly that $H_j(t,x,y)$ depends only on $y_1,\cdots,y_j$.

Let us recall the notation $x* = (|x_1|,\cdots,|x_N|)$.

PROPOSITION 7.3. *Let ε,α be as in Definition 7.1; then, there exist $2N$ real positive constants $C(1),\cdots,C(N)$, $C'(1),\cdots,C'(N)$ such that*

(i) *if $\tau > 0$, $x \in R^N$ and $y \in \Delta_\varepsilon^\alpha$ then there exist m and M, $0 \leq m < M \leq 1$ (depending on τ, x and y) such that*

$$H_j(s,x,y) \geq C(j)(|x_j| + F_j(x*,\tau))$$

for $s \in [m\tau,\ M\tau]$ and for $j = 1,\cdots,N$. Here $C(j)$ depend only on P, ε and α and $M - m \geq 4^{-N}$.

(ii) *if $\tau > 0$ and $y \in \Delta_\varepsilon^\alpha$, we have*

$$|H_j(\tau,x,y)| \leq C'(j)(|x_j| + F_j(x*,\tau)),\ j = 1,\cdots,N,$$

where the constants $C'(j)$ depend only on P,ε and α.

Without loss of generality, we may suppose $C(j) \leq 1$ and $C'(j) \geq 1$.

PROOF. The proof will be carried out by an induction argument on the index j. Without loss of generality, we can suppose $\varepsilon_j \leq 1$, $\alpha_j \geq 1$, $j = 1,\cdots,N$. Moreover, at each step of the proof we will modify our choice of the constants m,M in such a way that the new interval is contained in the old one. More formaly, for each $j \in \{1,\cdots,N\}$ we will find two constants m_j,M_j $0 \leq m_j < M_j \leq 1$ such that

$$m_1 \leq m_2 \leq \cdots \leq m_N < M_N \leq \cdots \leq M_2 \leq M_1 \text{ and } M_j - m_j \geq 4^{-j}.$$

We will put $m = m_N$ and $M = M_N$.

CASE $j = 1$. Here, $H_1(s, x, y) = x_1 + sy_1$. If $x_1 \geq 0$, let us choose $m = 1/2$, $M = 1$; if $s \in [\tau/2, \tau]$, we get:

$$|H_1(s,x,y)| = |x_1 + sy_1| \geq x_1 + s\varepsilon_1 \geq x_1 + \tau\ \varepsilon_1(x_1 + \tau)/2$$
$$= \varepsilon_1(|x_1| + F(x*, \tau))/2.$$

If $x_1 < 0$, then there exists $s_0 \in]0, +\infty[$ such that $x_1 + s_0 y_1 = 0$.

Now, if $s_0 \in [\tau/2, \tau]$, we choose $m = 0$ and $M = 1/4$. Then, if $s \in [0, \tau/4]$, we get:

$$|x_1 + sy_1| = (s_0 - s)y_1 \geq 1/4\ \tau\ y_1 \geq 1/4\ \tau\varepsilon_1 \geq 1/4(1/2\ s_0 + 1/2\ \tau)\varepsilon_1$$
$$= 1/4\ \varepsilon_1(|x_1|/2y_1 + 1/2\ \tau) \geq \varepsilon_1(|x_1|/\alpha_1 + \tau)/8 \geq \varepsilon_1(|x_1| + F_1(x*, \tau))/8\alpha_1$$

If $s_0 \in [\tau, +\infty[$, then $x_1 + \tau\ y_1 \leq 0$. We choose again $m = 0$, $M = 1/4$ and, if $s \in [0, \tau/4]$, we get:

$$|x_1 + sy_1| = -x_1 - sy_1 = |x_1| - sy_1 \geq 1/2|x_1| + (1/2\ \tau - s)y_1$$
$$\geq 1/2|x_1| + 1/4\ \tau\ \varepsilon_1 \geq 1/4\ \varepsilon_1(|x_1| + \tau) = 1/4\ \varepsilon_1(|x_1| + F_1(x*, \tau)).$$

Finally, if $s_0 \in [0, \tau/2]$, we choose $m = 3/4$, $M = 1$. Then, if $s \in [3/4\tau, \tau]$, we get:

$$|x_1 + sy_1| = (s - s_0)y_1 \geq 1/4\ \tau\ \varepsilon_1 \geq 1/4\ \varepsilon_1(s_0 + 1/2\ \tau)$$
$$= 1/4\ \varepsilon_1(|x_1|/y_1 + 1/2\ \tau) \geq \varepsilon_1(|x_1| + \tau)/8\alpha_1$$
$$= \varepsilon_1(|x_1| + F_1(x*, \tau))/8\alpha_1.$$

So, we proved i), by choosing $C(1) = \varepsilon_1/8\alpha_1$. On the other hand, ii) is quite obvious with $C'(1) = \alpha_1$, since

$$|H_1(\tau, x, y)| \leq |x_1| + \tau\ y_1 \leq \alpha_1(|x_1| + \tau) = \alpha_1(|x_1| + F_1(x*, \tau)).$$

CASE $j > 1$. Let us now suppose i) and ii) hold for $1, 2, \cdots, j$ and let us prove the assertions for $j + 1$, possibly with a new choice of the constants m, M, as we said above. For sake of simplicity, let us call m', M' the new constants.

If $x_{j+1} \geq 0$, we choose $m' = 1/2(m + M)$, $M' = M$.

If $s \in [1/2(m + M)\tau, M\tau]$, we get:

$$|H_{j+1}(s, x, y)| = x_{j+1} + y_{j+1} \int_0^s \lambda_{j+1}(H(t, x, y))dt$$

$$\geq x_{j+1} + \varepsilon_{j+1} \int_{m\tau}^{1/2(m+M)\tau} \lambda_{j+1}(\cdots)dt \geq$$

(by the induction hypothesis, (2.c), (2.d) and (2.e))

$$\geq x_{j+1} + \varepsilon_{j+1} \int_{m\tau}^{1/2(m+M)\tau} \lambda_{j+1}(C(1)(|x_1| + F_1(x*,\tau),\cdots)dt \geq \text{ (by (2.f)}')$$
$$\geq x_{j+1} + \varepsilon_{j+1}\frac{M-m}{2}C(1)^{\rho_{j+1,1}}\cdots C(j)^{\rho_{j+1,j}}F_{j+1}(x*,\tau)$$
$$\geq \frac{1}{2}4^{-j}\varepsilon_{j+1}C(1)^{\rho_{j+1,1}}\cdots C(j)^{\rho_{j+1,j}}(|x_{j+1}| + F_{j+1}(x*,\tau)).$$

Let us now suppose $x_{j+1} < 0$; then, there exists $s_0 \in\,]0,+\infty[$ such that

$$x_{j+1} + y_{j+1}\int_0^{s_0} \lambda_{j+1}(H(t,x,y))dt = 0.$$

If $s_0 \in\,]0,\ 1/2(m+M)\tau]$, we choose

$$m' = 1/4(3M+m),\ M' = M.$$

In fact, if $s \in\ [1/4(3M+m)\tau,\ M\tau]$, we get:

$$|x_{j+1} + y_{j+1}\int_0^{s} \cdots dt| = y_{j+1}\int_{s_0}^{s} \lambda_{j+1}(\cdots)dt$$
$$\geq \varepsilon_{j+1}\int_{s_0}^{s}\lambda_{j+1}(\cdots)dt \geq \varepsilon_{j+1}\int_{1/2(m+M)\tau}^{1/4(3M+m)\tau}\lambda_{j+1}(\cdots)dt$$
$\geq$ (by the induction hypothesis, (2.c), (2.d) and (2.e))
$$\varepsilon_{j+1}1/4(M-m)\tau\ \lambda_{j+1}(C(1)(|x_1| + F_1(x*,\tau)),\cdots) \geq \text{(by (2.f)}')$$
$$\varepsilon_{j+1}1/4(M-m)C(1)^{\rho_{j+1,1}}\cdots C(j)^{\rho_{j+1,j}}F_j(x*,\tau).$$

On the other hand,

$$|x_{j+1}| = y_{j+1}\int_0^{s_0}\lambda_{j+1}(\cdots)dt \leq \alpha_{j+1}\int_0^{1/2(M+m)\tau}\lambda_{j+1}(\cdots)dt$$
$$\leq \alpha_{j+1}\ \tau\ \lambda_{j+1}(C'(1)(|x_1| + F_1(x*,\tau)),\cdots)$$
$$\leq \alpha_{j+1}\ C'(1)^{\rho_{j+1,1}}\cdots C'(j)^{\rho_{j+1,j}}F_{j+1}(x*,\tau).$$

Then,

$$(7.3a) \quad |x_{j+1} + y_{j+1} \int_0^s \cdots | \geq$$

$$\geq \frac{M-m}{8} \frac{\varepsilon_{j+1}}{\alpha_{j+1}} (\frac{C(1)}{C'(1)})^{\rho_{j+1,1}} \cdots (\frac{C(j)^{\rho_{j+1,j}}}{C'(j)}) \cdot (|x_{j+1}| + F_{j+1}(*, \tau)).$$

Now, if $s_0 \in [1/2(m+M)\tau, \ M \ \tau]$, we choose

$$m' = m, \ M' = 1/4(3m+M).$$

If $s \in [m\tau, \ 1/4(3m+M)\tau]$, we get:

$$|x_{j+1} + y_{j+1} \int_0^s \cdots dt| = y_{j+1} \int_0^{s_0} \lambda_{j+1}(\cdots)dt$$

$$\geq \varepsilon_{j+1} \int_{\tau(3m+M)/4}^{1/2(m+M)\tau} \lambda_{j+1}(\cdots)dt \geq \text{(by the induction hypothesis,}$$

(2.c), (2.d) and (2.e))

$$\geq \varepsilon_{j+1} \frac{M-m}{4} \tau \ \lambda_{j+1} \ (C(1)(|x_1| + F_1(x*, \tau)), \cdots) \geq \ \text{(by (2.f)}')$$

$$\geq \varepsilon_{j+1} \frac{M-m}{4} C(1)^{\rho_{j+1,1}} \cdots C(j)^{\rho_{j+1,j}} F_j(x*, \tau).$$

Now, since $|x_{j+1}|$ can be estimate as above, we get again the estimate (7.3a).

Finally, let us suppose $s_0 \in [M\tau, +\infty[$. In this case,

$$x_{j+1} + y_{j+1} \int_0^{M\tau} \lambda_{j+1}(\cdots)dt \leq 0.$$

We choose again $m' = m$, $M' = (3m+M)/4$. If

$$s \in [m\tau, \tau(3m+M)/4],$$

we get:

$$|x_{j+1} + y_{j+1} \int_0^s \cdots dt| = -x_{j+1} - y_{j+1} \int_0^s \lambda_{j+1}(\cdots)dt.$$

Now, put

$$\frac{1}{\sigma} = 1 - \frac{M-m}{2} (\frac{\varepsilon_{j+1}}{\alpha_{j+1}})(\frac{C(1)}{C'(1)})^{\rho_{j+1,1}} \cdots (\frac{C(j)}{C'(j)})^{\rho_{j+1,j}} \in [1/2, 1[.$$

We have:

$$- x_{j+1} - y_{j+1} \int_0^s \lambda_{j+1}(\cdots)dt = (1 - \frac{1}{\sigma})|x_{j+1}|$$

$$+ \frac{1}{\sigma}|x_{j+1}| - y_{j+1} \int_0^s \lambda_{j+1}(\cdots)dt$$

$$\geq (1 - \frac{1}{\sigma})|x_{j+1}| + \frac{y_{j+1}}{\sigma} \int_0^{M\tau} \lambda_{j+1}(\cdots)dt - y_{j+1} \int_0^{1/4(3m+M)\tau} \lambda_{j+1}(\cdots)dt$$

$$= (1 - \frac{1}{\sigma})|x_{j+1}| + y_{j+1}(\int_{1/4(3m+M)\tau}^{M\tau} \lambda_{j+1}(\cdots)dt$$

$$+ (\frac{1}{\sigma} - 1) \int_0^{1/4(3m+M)\tau} \lambda_{j+1}(\cdots)dt) \geq \text{(arguing as above)}$$

$$(1 - \frac{1}{\sigma})|x_{j+1}| + \varepsilon_{j+1}\frac{3(M-m)}{4}C(1)^{\rho_{j+1,1}} \cdots C(j)^{\rho_{j+1,j}}F_{j+1}(x*,\tau)$$

$$+ (\frac{1}{\sigma} - 1)\alpha_{j+1}C'(1)^{\rho_{j+1,1}} \cdots C'(j)^{\rho_{j+1,j}}F_{j+1}(x*,\tau)$$

$$= (1 - \frac{1}{\sigma})|x_{j+1}| + \varepsilon_{j+1}\frac{M-m}{4}C(1)^{\rho_{j+1,1}} \cdots C(j)^{\rho_{j+1,j}}F_{j+1}(x*,\tau)$$

(keeping in mind the definition of σ)

$$\frac{M-m}{4}\ \frac{\varepsilon_{j+1}}{\alpha_{j+1}}(\frac{C(1)}{C'(1)})^{\rho_{j+1,1}} \cdots (\frac{C(j)}{C'(j)})^{\rho_{j+1,j}}(|x_{j+1}| + F_{j+1}(x*,\tau))$$

Thus, we proved i), by choosing

$$C(j+1) = \frac{1}{2}\ 4^{-j-1}\ \frac{\varepsilon_{j+1}}{\alpha_{j+1}}(\frac{C(1)}{C'(1)})^{\rho_{j+1,1}} \cdots (\frac{C(j)}{C'(j)})^{\rho_{j+1,j}};$$

in fact $M' - m' \geq \frac{1}{4}(M-m) \geq 4^{-j-1}$.

Finally the proof of ii) follows from (2.c), (2.d) and (2.e). □

PROPOSITION 7.4. *Let $\gamma \in]0,1[$ and $\sigma \in \{-1,1\}^N$ be fixed. Then, there exist ε, $\alpha \in R^N$ as in Definition 7.1 such that, $\forall\ \rho > 0$, $\forall\ x \in R^N$*

$$|H(\rho, x, \Delta_\varepsilon^\alpha(\sigma)) \cap Q^\sigma(x,\rho)| \geq (1-\gamma)|Q^\sigma(x,\rho)|.$$

PROOF. First, we note that

$$T_\sigma(Q^\sigma(x,\rho)) = Q^+(T_\sigma x, \rho)$$

and

$$T_\sigma(H(\rho,x,\Delta_\varepsilon^\alpha(\sigma))) = H(\rho,x,\Delta_\varepsilon^\alpha);$$

so, we can reduce ourselves to the case $\sigma = \{1,\cdots,1\}$.

Put $\Theta = 1-(1-\gamma)^{1/N}$. We will choose ε and α such that

$$[x_j + \Theta F_j(x*,\rho),\ x_j + F_j(x*,\rho)] \subseteq H_j(\rho,x,\Delta_\varepsilon^\alpha),\ j = 1,\cdots N. \tag{7.4a}$$

From (7.4a) the proposition follows easily: in fact

$$\{x : x_j \in\ [x_j + \Theta F_j(x*,\rho),\ x_j + F_j(x*,\rho)]\} \subseteq Q^+(x,\rho),$$

hence

$$|H(\rho,x,\Delta_\varepsilon^\alpha) \cap Q^+(x,\rho)| \geq (1-\Theta)^N \prod_{j=1}^{N} F_j(x*,\rho) = (1-\gamma)|Q^+(x,\rho)|.$$

We will prove (7.4a) by an induction argument:

If $j=1$ we choose $\varepsilon_1 = \Theta,\ \alpha_1 = 1$, then

$$H_1(\rho,x,\Delta_\varepsilon^\alpha) = x_1+\rho[\varepsilon_1,1] = [x_1+\Theta\rho,\ x_1+\rho] = [x_1+\Theta F_1(x*,\rho),\ x_1+F_1(*,\rho)].$$

Now let us assume that (7.4a) holds for $1,2,\cdots,j-1$ and let us prove it for j. Let us choose

$$\varepsilon_j = \Theta/C'(1)^{\rho_{j,1}}\cdots C'(j-1)^{\rho_{j,j-1}}$$
$$\alpha_j = 4^N/C(1)^{\rho_{j,1}}\cdots C(j-1)^{\rho_{j,j-1}}.$$

We note that $C(1)\cdots,C(j-1),C'(1),\cdots,C'(j-1)$ depend only on the previously chosen α_i,ε_i and on P.

In fact, let η belong to $[\Theta F_j(x*,\rho),F_j(x*,\rho)]$ and let us choose $y_j = \eta/\int_0^\rho \lambda_j(H(t,x,y)dt$. We note that, since λ_j depends only on the first $j-1$ variables and H_i depends only on $y_1,\cdots,y_i$, the definition of y_j makes sense for any arbitrary fixed $y_1,\cdots,y_{j-1}$.

We will prove that:

i) $H_j(\rho,x,y_1,\cdots,y_j) = x_j+\eta$;

ii) $\varepsilon_j \leq y_j \leq \alpha_j$.

The first one follows from the definition of $H(\rho,x,y)$. In order to prove ii), we note that, by Proposition 7.3 i),

$$\int_0^\rho \lambda_j(H(t,x,y))dt \geq \int_{m\rho}^{M\rho} \lambda_j(H(t,x,y))dt$$
$$\geq \int_{m\rho}^{M\rho} \lambda_j(C(1)(|x_1|+F_1(x*,\rho)),\cdots)dt$$

(where $C(1), \cdots, C(j-1)$ depend only on P, ε_i and $\alpha_i, i = 1, \cdots, j-1$)

$$\begin{aligned}&\geq 4^{-N} \rho \ \lambda_j(C(1)(|x_1| + F_1(x*, \rho)), \cdots) \\ &\geq 4^{-N} C(1)^{\rho_{j-1}} \cdots C(j-1)^{\rho_{j,j-1}} F_j(x*, \rho).\end{aligned}$$

Hence

$$y_j \leq 4^N / C(1)^{\rho_{j,1}} \cdots C(j-1)^{\rho_{j,j-1}} = \alpha_j.$$

On the other hand, by Proposition 7.3 ii),

$$\begin{aligned}&\int_0^\rho \lambda_j(H(t,x,y))dt \leq \int_0^\rho \lambda_j(C'(1)(|x_1| + F_1(x*, t)), \cdots)dt \\ &\leq \rho \ \lambda_j(C'(1)(|x_1| + F_1(x*, \rho)), \cdots) \\ &\leq C'(1)^{\rho_{j,1}} \cdots C'(j-1)^{\rho_{j,j-1}} F_j(x*, \rho).\end{aligned}$$

Hence

$$y_j \geq \Theta / C'(1)^{\rho_{j,1}} \cdots C'(j-1)^{\rho_{j,j-1}} = \varepsilon_j$$

and the assertion is completely proved. □

PROPOSITION 7.5. *let ε, α be fixed vectors as in Definition 7.1 and let σ belong to $\{-1, 1\}^N$. Then, there exist two constants c_1, c_2 depending only on P, ε and α such that*

$$c_1 |S(x, \rho)| \leq \prod_{j=1}^N \int_0^\rho \lambda_j(H(t,x,y))dt \leq c_2 |S(x, \rho)|$$

for each $x \in R^N$, $\rho > 0$ and $y \in \Delta_\varepsilon^\alpha(\sigma)$.

PROOF. We note that, if

$$y \in \Delta_\varepsilon^\alpha(\sigma), \ |H_j(t,x,y)| = |H_j(t, T_\sigma x, T_\sigma y)|$$

for $j = 1, \cdots, N$ and $T_\sigma y \in \Delta_\varepsilon^\alpha$.

Then, if $j \in \{1, \cdots, N\}$, we have:

$$\begin{aligned}&\int_0^\rho \lambda_j(H(t,x,y))dt = \int_0^\rho \lambda_j(|H_1(t,x,y)|, \cdots)dt \\ &= \int_0^\rho \lambda_j(|H_1(t, T_\sigma x, T_\sigma y)|, \cdots) \leq \rho \ \lambda_j(C'(1)(|x_1| + F_1(x*, \rho)), \cdots) \\ &\leq C'(1)^{\rho_{j,1}} \cdots C'(j-1)^{\rho_{j,j-1}} F_j(x*, \rho).\end{aligned}$$

Moreover, by Proposition 7.3, there exist m, M depending on ρ, x, y, such that:

$$\int_0^\rho \lambda_j(|H_1(t, T_\sigma x, T_\sigma y)|, \cdots)dt \geq \int_{m\rho}^{M\rho} \lambda_j(\cdots)dt$$
$$\geq 4^{-N} C(1)^{\rho_{j,1}} \cdots C(j-1)^{\rho_{j,j-1}} F_j(x*, \rho).$$

Then, the assertion of the Proposition follows by the estimate of $|S(x, \rho)|$ given in Proposition 2.6. □

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