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# Homogeneous $CR$-hypersurface-structures on Spheres

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## Introduction

Let $M$ be a compact $CR$-hypersurface which is homogeneous under the action of a real Lie group $G$ of $CR$-transformations (see §1 for basic definitions). If some additional condition is imposed, a detailed classification is possible. For example in [3] a fine classification is given for the case when the Levi-form is non-degenerate. Instead of the non-degeneracy of the Levi-form a Kähler-condition is imposed in [17].

In this note the additional condition is topological: $M$ is assumed to be a homogeneous $CR$-hypersurface which is homeomorphic to the $(2n+1)$-dimensional sphere. There is a standard (homogeneous) $CR$-structure on $M$ coming from the embedding of $M$ as the boundary of the ball $\mathbb{B}_{n+1}$ in $\mathbb{C}^{n+1}$. We want to answer the following question: Are there homogeneous $CR$-structures on $M$ which are different from the standard $CR$-structure? This is a natural question because in [6] E. Cartan showed that there are non-standard $CR$-structures on $S^3$. These structures are homogeneous and appear as follows: The generic $SU_2$-orbits on the affine quadric $SL_2(\mathbb{C})/\mathbb{C}^*$ (where $\mathbb{C}^* = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} : \lambda \in \mathbb{C}^* \right\}$) are hypersurfaces which are diffeomorphic to $\mathbb{P}_2(\mathbb{R})$. It is known that different orbits have different $CR$-structures (see e.g. [17]). The $CR$-structures on the orbits induce $CR$-structures on the universal covering $S^3$ which is also an $SU_2$-orbit. It can be shown that $S^3$ with one of these structures is never the boundary of a Stein manifold (see e.g. [3]). The situation for $S^3$ is different from the general situation. On one hand, the affine quadric is an affine $\mathbb{C}$-bundle over $\mathbb{P}_1(\mathbb{C})$ which is not equivalent to a holomorphic line bundle. In higher dimensions every affine $\mathbb{C}$-bundle over $\mathbb{P}_n(\mathbb{C})$ is equivalent to a line bundle (see §2.4) and therefore examples don't appear in this way. On the other hand, these non-standard structures come from unit sphere bundles over the symmetric spaces $S^2$ or $\mathbb{P}_2(\mathbb{R})$, while in higher dimensions the universal covering of a unit sphere bundle is never a sphere.

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In fact we have:

**Main Theorem**: *The only CR-hypersurface-structure on* $S^{2n+1} (n \geq 2)$ *which admits a transitive action of a Lie group of CR-transformations is the standard CR-structure.*

Our main tool is the so-called $\mathfrak{g}$-anticanonical fibration of a homogeneous $CR$-hypersurface (cf. [3] or [17]). We use the topological properties of the sphere to show that the $\mathfrak{g}$-anticanonical fibration is principal and the fiber is either finite or the identity component of the fiber is a compact abelian group. In the latter case, using a result of Eckmann, Samuelson and Whitehead ([7]), we show that this group is one-dimensional, and that $M$ is an $S^1$-principal bundle over a projective rational manifold. In the case of a finite $\mathfrak{g}$-anticanonical fibration we consider the so-called Stein-rational fibration. Applying results of Nagano ([16]) we show that the base of this fibration is not trivial. Again we obtain an $S^1$-principal fibration of $M$ over a projective rational manifold.

From the classification of Nagano we also know that in each case the base $X$ of the $S^1$-principal fibration is diffeomorphic to $\mathbb{P}_n(\mathbb{C})$. Using results of Kodaira and Hirzebruch ([10]) we know then that $X$ is biholomorphic to $\mathbb{P}_n(\mathbb{C})$.

The next step is to show that $M$ is an $S^1$-subbundle of a holomorphic $\mathbb{C}^*$-principal bundle over $\mathbb{P}_n(\mathbb{C})$. The principal bundle is shown to be $\mathbb{C}^{n+1}\backslash\{0\}$ and therefore $M = S^{2n+1}$ with the standard structure.

We think that some of the methods should work for the study of simply-connected homogeneous $CR$-hypersurfaces.

The organization of the article is as follows:

As the result might be interesting for someone not familiar with the methods applied, we have explained them in the beginning. Therefore §1 consists of the basic definitions and main theorems about homogeneous $CR$-structures, the $\mathfrak{g}$-anticanonical and the Stein-rational fibration.

In §2 it is shown that there is a $CR$-fibration of $M$ with fiber $S^1$ and base $\mathbb{P}_n(\mathbb{C})$. This lies in a holomorphic line bundle or a $\mathbb{C}^*$-principal bundle over $\mathbb{P}_n(\mathbb{C})$. Finally we prove the uniqueness of the $CR$-structure.

We want to thank Prof. Alan Huckleberry for proposing us the problem and for his help in numerous discussions.

## 1.1 *Basic Definitions and Facts about CR-manifolds*

We begin by giving some basic definitions and facts about $CR$-manifolds. For details we refer to [1] or [2].

Let $M$ be a real $C^\infty$-manifold of dimension $m$. By a *Cauchy-Riemann-* $(CR-)$*-structure of type* $(m, \ell)$ on $M$ we understand a subbundle $HM$ of rank $\ell$ of the complexified tangent bundle $TM \otimes \mathbb{C}$ which satisfies the following two conditions:

(a) $HM \cap \overline{HM} = \{0\}$ (zero section)

(b) $HM$ is involutive, i.e. the Lie product $[\xi, \eta]$ of two sections $\xi, \eta$ in $HM$ is again a section in $HM$.

A *$CR$-manifold of type* $(m, \ell)$ is a pair $(M, HM)$ consisting of a $C^\infty$-manifold $M$ of dimension $m$ and a $CR$-structure $HM$ of type $(m, \ell)$ on $M$. If $2\ell+1 = m$, we will call $(M, HM)$ a *$CR$-hypersurface*. An *analytic $CR$-manifold* $M$ is defined to be a real analytic manifold $M$ with $HM$ locally generated by analytic local sections in $TM \otimes \mathbf{C}$. Let $f$ be a smooth map between two $CR$-manifolds $(M, HM)$ and $(M', HM')$. We call $f$ a *$CR$-map,* if for every $p \in M$ the complexified differential $f_*^{\mathbf{C}}$ carries $HM_p$ into $HM'_{f(p)}$.

If $M' = \mathbf{C}$ and $HM$ is the usual holomorphic tangent bundle, then $f$ is called a *$CR$-function.*

An *embedding* $\tau$ of a *$CR$-manifold $M$ into a complex manifold* $\hat{M}$ is defined to be a smooth embedding $\tau : M \to \hat{M}$, where $\tau$ is a $CR$-map and $\tau(M)$ carries the induced $CR$-structure, i.e. one has $H\ \tau(M) = (T\ \tau(M) \otimes \mathbf{C}) \cap T^{1,0}\hat{M}$. An embedding is called *generic,* if $\dim_{\mathbf{C}} M = m - \ell$ (the smallest possible value), $M$ being of type $(m, \ell)$. In this case $(\hat{M}, \tau)$ is called a *complexification* of $M$.

We recall two basic facts (see e.g. [1], [2]):

THEOREM 1: *Every analytic $CR$-manifold $M$ has a complexification $(\hat{M}, \tau)$. The germ of the complexification is unique.*

THEOREM 2: *Let $M \subset \hat{M}$ and $M' \subset \hat{M}'$ be two analytic $CR$-manifolds with their complexifications and $f : M \to M'$ be an analytic $CR$-map. Then there exist open neighbourhoods $U \subset \hat{M}$ of $M$ (resp. $U' \subset \hat{M}'$ of $M'$) and a holomorphic mapping $\hat{f} : U \to U'$ with $\hat{f}|_M = f$. The germ of the extension $\hat{f}$ of $f$ is unique.*

Defining a *$CR$-vector field $X$* to be a vector field on $M$ inducing local one-parameter groups of $CR$-transformations, one can also show that, for every analytic $CR$-vector field $X$, there exists a neighbourhood $U \subset \hat{M}$ of the analytic $CR$-manifold $M$ and a holomorphic vector field $Z$ on $U$ so that Re $Z = X$ on $M$ (see e.g. [3]). Again the germ of $Z$ on $M$ is unique.

### 1.2 *Homogeneous $CR$-manifolds and the $\mathfrak{g}$-anticanonical Fibration*

The details of what will follow can be found in [3] or [17].

We call a $CR$-manifold $M$ *homogeneous* if there exists a real Lie group $G$ acting transitively on $M$ as a group of $CR$-transformations. We always assume $M$ to be connected. Thus we may also assume that $G$ is connected. Furthermore we assume that $G$ is simply-connected. This can be arranged by going over to the universal covering.

Since $G$ as a Lie group possesses an analytic structure, we can also give $M$ the structure of an analytic manifold so that $G$ acts by analytic transformations. One can show that $HM$ is locally generated by analytic local sections in

$TM \otimes \mathbf{C}$ ([17]). Thus a homogeneous $CR$-manifold $M$ has the structure of an analytic $CR$-manifold. From 1.1.1 we know that $M$ possesses a complexification $(\hat{M}, \tau)$.

We regard the Lie algebra $\mathfrak{g}$ of $G$ as an algebra of $CR$-vector fields on $M$. Since it is finite-dimensional and since individual vector fields can be extended, there is a neighbourhood $U \subset \hat{M}$ and for every analytic $CR$-vector field $X$ a unique holomorphic vector field $Z$ on $U$ such that Re $Z = X$ on $M$. We define $\hat{\mathfrak{g}}$ to be the complex subalgebra of the holomorphic vector fields on $U$ which is generated by $\mathfrak{g}$ and the map $X \mapsto Z$. So we have a Lie algebra homomorphism $\mathfrak{g} \to \hat{\mathfrak{g}}$. We call $(U, \tau)$ a *$\hat{\mathfrak{g}}$-complexification.* After shrinking $\hat{M}$, we may assume that it is a $\hat{\mathfrak{g}}$-complexification.

If there exists a complex Lie group $\hat{G}$ with Lie algebra $\hat{\mathfrak{g}}$ which acts holomorphically and transitively on $\hat{M}$ so that this action induces the $G$-action on $M$, then $\hat{M}$ is called a *$\hat{G}$-complexification* of $M$.

Now let $\hat{M}$ be a $\hat{\mathfrak{g}}$-complexification of $M$, $\dim_{\mathbf{C}} \hat{M} =: n$ and $V_{\hat{\mathfrak{g}}} := \Lambda^n \hat{\mathfrak{g}}$ with $\dim_{\mathbf{C}} V_{\hat{\mathfrak{g}}} =: N+1$. One can take a base $< \sigma_0, \cdots, \sigma_N >$ of $V_{\hat{\mathfrak{g}}}$ and define a map

$$\hat{\phi} : \hat{M} \longrightarrow \mathbb{P}_N(\mathbf{C})$$
$$z \longmapsto [\sigma_0(z) : \cdots : \sigma_N(z)]$$

where $[y_0 : \cdots : y_N]$ denote the homogeneous coordinates on $\mathbb{P}_N(\mathbf{C})$.

Taking $\hat{M}$ small enough, one may assume $\hat{\phi}$ to be holomorphic. The map $\phi$ defined by $\phi := \hat{\phi}|_M$ is called *$\mathfrak{g}$-anticanonical map* of the homogeneous manifold $M$. This map is $G$-equivariant. However $\hat{\phi}$ may not be.

PROPOSITION 1: *A vector field $Z$ in $\hat{\mathfrak{g}}$ vanishing at $p \in M$ vanishes identically on the fiber $\phi^{-1}(\phi(p))$.*

PROPOSITION 2: *Let $\phi : G/H \to G/N$ be the $\mathfrak{g}$-anticanonical map of $M$. Then $N$ normalizes $H^0$.*

So if $H$ is connected, the $\mathfrak{g}$-anticanonical fibration is a $N/H$-principal bundle. Moreover, the right principal action $\lambda : M \times N/H \to M$ is a $CR$-action.

From Proposition 1 one can deduce the following property of the $CR$-structure $HL$ of the fiber $L = N/H$ ([3]): $\Gamma(L, \; HL + \overline{HL})$ is a complex subalgebra of $\Gamma(L, TL \otimes \mathbf{C})$ and therefore generates complex integral submanifolds of $L$.

From now on we assume that $M$ is a $CR$-hypersurface.

Since $\phi$ is a $CR$-map, $HM$ is mapped surjectively onto $H\phi(M)$ (for the general case see [17]). In this case one deduces that the base $G/N$ of the $\mathfrak{g}$-anticanonical fibration is either a $CR$-hypersurface in $\hat{\phi}(\hat{M})$ or is equal to the complex manifold $\hat{\phi}(\hat{M})$.

As the $\mathfrak{g}$-anticanonical fibration is $G$-equivariant, there is a homomorphism $\phi_* : G \to PGL_{N+1}(\mathbf{C})$. Let $\tilde{G}$ be the smallest complex Lie group (not necessarily closed) in $PGL_{N+1}(\mathbf{C})$ containing $\phi_*(G)$. Let $p \in \phi(M)$ and

$\tilde{G}(p) =: \tilde{G}/\tilde{N}$, containing $\hat{\phi}(\hat{M})$. The restriction to $\hat{\phi}(\hat{M})$ of a vector field in the Lie algebra $\tilde{\mathfrak{g}}$ of $\tilde{G}$ is the projection of a vector field on $\hat{M}$ in $\hat{\mathfrak{g}}$, so $\hat{\phi}(\hat{M})$ is open in $\tilde{G}/\tilde{N}$.

If $G/N = \hat{\phi}(\hat{M})$ then $G/N = \tilde{G}/\tilde{N}$. By a theorem of Goto ([8]) we know then that $G/N$ is a projective rational manifold, and it is simply connected. For the special case where the fiber $N/H$ is one-dimensional, it is shown in [3] that $\phi$ realizes $M$ as a principal $S^1$-bundle over the compact rational manifold $Q = G/N$. The natural embedding of $M$ in the associated principal-$\mathbf{C}^*$-bundle $\hat{M}$ gives a $\hat{G}$-complexification of $M$.

In the other case, $G/N \subset \tilde{G}/\tilde{N}$ is a compact hypersurface, $\tilde{G}/\tilde{N}$ being a $\tilde{G}$-complexification of the base $G/N$ of the $\mathfrak{g}$-anticanonical fibration. The fiber $N/H$ is a compact complex manifold. In [3] it is shown, that either $G/N = S^1 \times Q$ with $Q$ projective rational or we are in the situation that $\pi_1(G/N)$ is finite and the semi-simple part $K$ of the maximal compact subgroup of $G$ acts transitively on $G/N = K/L$. In the next section we will consider this situation.

We note that the $\mathfrak{g}$-anticanonical map $\phi$ depends both on the manifold $M$ and on the Lie group $G$ acting on $M$. For example we consider $S^{2n+1}$ with the standard $CR$-structure.

Let be $G_1 := SU_{n+1}$ and $G_2 := SU(n+2,1)$ with associated $\mathfrak{g}$-anticanonical maps $\phi_1$ and $\phi_2$. Then $\phi_1$ is an $S^1$-principal-fibration and $\phi_2$ is an injection.

### 1.3 *The Stein-rational Fibration*

In this section we consider the situation where a $CR$-hypersurface $M$ lies in $\mathbb{P}_N(\mathbf{C})$ (not necessarily as a real hypersurface) as an orbit of a compact semi-simple real subgroup $K$ of $PSL_{N+1}(\mathbf{C})$. Let $\hat{K}$ be the complexification of $K$ and $\Omega = \hat{K}/\hat{L}$ be the $\hat{K}$-orbit of a point in $M = K/L$. By definition $M$ is a real hypersurface in $\Omega$. Note that $\Omega$ is not projective rational, because then it would be simply-connected and a maximal compact subgroup of $\hat{K}$ would act transitively ([14]).

We will see that $\Omega$ fibers over a projective rational manifold. Let $X$ be the algebraic closure of $\Omega$. Applying the Chevalley Constructability theorem ([4]), we see that $\Omega$ is Zariski-open in $X$. Furthermore, $X$ can be realized as a compact almost-homogeneous projective algebraic manifold where the generic $K$-orbit is a real hypersurface. Each of the (at most two) connected components of the exceptional set $E = X \backslash \Omega$ is a $K$-orbit and one can assume that $E$ consists of complex hypersurface orbits of $\hat{K}$.

Then there exists a (minimal) parabolic subgroup $\hat{P} \not\supseteq \hat{L}$ of $\hat{K}$ (possibly $\hat{P} = \hat{K}$) so that either $\Omega$ can be realized as a principal-$\mathbf{C}^*$-bundle over a compact homogeneous rational manifold $Q = \hat{K}/\hat{P}$ (if $E$ has two components) or otherwise there is a $\hat{K}$-equivariant fibration of $\Omega = \hat{K}/\hat{L} \to Q = \hat{K}/\hat{P}$ over a projective rational manifold. This fiber $\hat{P}/\hat{L}$ is either $\mathbf{C}^n$ or the tangent bundle $TN$ of a compact symmetric space of rank one (see [3]). In the first case the fiber of the induced fibration of $M = K/L \subset \Omega$ is $S^{2n-1}$ with the standard structure. In the second case it is the unit sphere bundle in $TN$. The compact symmetric spaces of rank one are known, they are $S^n$, $\mathbb{P}_n(\mathbb{R})$, $\mathbb{P}_n(\mathbb{C})$, $\mathbb{P}_n(\mathbb{H})$ (quaternionic projective space) and $\mathbb{P}_2(\mathcal{O})$ (Cayley projective plane). All cases lead to the following situation

$$\begin{array}{ccccc} M = K/L & \hookrightarrow & \hat{K}/\hat{L} & = & \Omega \\ \downarrow & & \downarrow & & \\ K/P & \simeq & \hat{K}/\hat{P} & = & Q \end{array}$$

with $\hat{P}/\hat{L}$ a Stein manifold and $Q$ a projective rational manifold. The fibration of $\Omega$ is therefore called the *Stein-rational-fibration* ($SR$-fibration). The fiber and the base are uniquely determined, because $\hat{P}$ is chosen minimal. Only if the fiber is $\mathbf{C}^n$, there might be different ways to obtain the fibration ([3]).

### 2.1 *The Fiber of the $\mathfrak{g}$-anticanonical Map of the Sphere*

Now we consider $S^{2n+1} = M$ $(n \geq 2)$ as a homogeneous $CR$-manifold $G/H$. As mentioned above, we can assume $G$ to be connected and simply-connected. Since $M$ is connected and simply-connected, the homotopy sequence of the fibration $H \longrightarrow G \longrightarrow G/H = M$ shows that $H$ is connected. Thus $N$ normalizes $H$ (by prop. 1.2.2) and the fiber $N/H$ of the $\mathfrak{g}$-anticanonical map is a compact Lie group. Furthermore we can prove

PROPOSITION 1: *The fiber $L = N/H$ of the $\mathfrak{g}$-anticanonical fibration is either a* finite group *or $L^0$ is a positive-dimensional* abelian Lie group.

PROOF: The fiber $L$ is a compact Lie group. If it is not discrete, two cases can occur (cf. 1.2).

*Case* 1.

The fiber is a compact complex manifold and the base is a $CR$-hypersurface in complex-projective space. The only connected compact manifolds which are complex Lie groups are tori. So the identity-component $L^0$ of the fiber is a torus and therefore abelian.

*Case* 2.

The fiber is a $CR$-hypersurface and the base is a simply-connected homogeneous projective rational manifold $Q$ (cf. 1.2).

A look at the homotopy sequence of the fibration $L \longrightarrow M \longrightarrow Q$ then shows that $L$ is connected. There exists a complexification $(\hat{L}, \tau)$ of $L$ such that $\hat{L}$ is a complex Lie group (cf. [17]). We denote the Lie algebra of $L$ by $\ell$ (resp. of $\hat{L}$ by $\hat{\ell}$). Since $\Gamma(L, HL + \overline{HL})$ is a complex subalgebra of $\Gamma(L, TL \otimes \mathbf{C})$, we know that there exists a (maximal) complex subalgebra $\mathfrak{s}$ of $\ell$. Then $\exp(\mathfrak{s})$ generates a (not necessarily closed) connected complex subgroup $S$ of $L$ of codimension 1.

If $S$ is not closed, then $\overline{S} = L$. Let $f$ be a $CR$-function on $L$. Since $L$ is compact, $f$ has a maximum at a point $x \in L$. If we look at the $S$-orbit $S(x)$ through $x$, then $f|_{S(x)}$ is a holomorphic function which is constant by the maximum principle. Since $S$ is dense, $f$ is constant on $L$. Thus every $CR$-function on $L$ is constant. We consider the adjoint representation of $L$ in $GL(\hat{\ell})$. Its restriction to $L$ yields the adjoint representation of $L$. It is given by $CR$-functions and is therefore trivial. Hence $L$ is abelian.

If $S$ is closed, then $S$ is a connected compact complex torus. Thus every $CR$-function on $L$ is constant on the $S$-orbits. Considering again the adjoint representation we conclude that $S$ lies in the center of $L$. The factor group $L/S$ is a compact connected 1-dimensional group, i.e. $L/S = S^1$. Looking at the Lie algebra one can easily check that a central extension $L$ of $S^1$ by a connected complex torus is abelian. □

If the $\mathfrak{g}$-anticanonical fibration is not finite, we always have a fibration

$$\begin{array}{ccccc} M = & G/H & \xrightarrow{N/H} & G/N \\ & {\scriptstyle N^0/H} \searrow & & \nearrow {\scriptstyle \text{finite}} \\ & & G/N^0 & \end{array}$$

where $N^0/H$ is a connected real torus, i.e. $N^0/H$ is diffeomorphic to $(S^1)^k$ for some $k$.

The following theorem due to Eckmann, Samuelson and Whitehead shows that such a fibration is only possible for $k = 1$:

THEOREM 2 ([7, p. 437]): *A fiber decomposition of the $n$-sphere $S^n$ with fiber $(S^1)^k$ exists only if $n$ is odd and $k = 1$.*

This theorem has nothing to do with the fact that $M$ is a homogeneous $CR$-hypersurface. It still remains valid if the base is only assumed to be a separable metric space.

COROLLARY 3: *If the $\mathfrak{g}$-anticanonical fibration of $M \cong S^{2n+1}$ is not finite, the fiber is a connected, one-dimensional torus over a projective rational manifold $Q$.*

### 2.2 *Homogeneous Fibrations of Spheres by Spheres*

In this section we consider the possible homogeneous fibrations of spheres by spheres. The classification of these fibrations has been carried out by Nagano ([16]).

We use his results to determine the base space of the above $S^1$-principal fibration of $M$ as well as to handle the case of a finite $\mathfrak{g}$-anticanonical fibration.

Let $E = S/H$ be a homogeneous sphere bundle with fiber $S^k$ and $E$ be homeomorphic to a sphere of dimension $2n + 1$.

If $B = S/I$ is the base of this fibration, then we have the following (see [16, p. 45])

THEOREM 1:(1) *Under the above assumptions there are only the following cases*:

(a) *If* $k = 0$, *then* $E$ *is a double covering space of* $B$ *and* $B$ *is diffeomorphic to* $\mathbb{P}_{2n+1}(\mathbb{R})$.

(b) *If* $k = 1$, *then* $B$ *is diffeomorphic to* $\mathbb{P}_n(\mathbb{C})$.

(c) *If* $k = 3$, *then* $B$ *is diffeomorphic to* $\mathbb{P}_{\frac{n-1}{2}}(\mathbb{H})$ *and* $n$ *is odd.*

(d) *If* $k = 7 \neq 2n+1$, *then* $B$ *is diffeomorphic to* $S^8$.

(e) *If* $k = 2n+1$, *then* $B$ *is a point.*

If $B$ is the base of an $S^1$-principal fibration of a sphere, then $B$ is diffeomorphic to $\mathbb{P}_n(\mathbb{C})$. Suppose we know (as in 2.1) that $B$ is also a projective algebraic manifold.

Then we can apply a result of Hirzebruch and Kodaira ([10]):

If $g$ is the generator of $H^2(B, Z) = \mathbb{Z}$, chosen such that $g$ corresponds to the fundamental class of a Kähler metric on $B$, then there is the following

THEOREM 2: *Let* $B$ *be an* $n$*-dimensional compact Kähler manifold which is diffeomorphic to* $\mathbb{P}_n(\mathbb{C})$. *If* $n$ *is odd, then* $B$ *is biholomorphic to* $\mathbb{P}_n(\mathbb{C})$. *If* $n$ *is even, then* $B$ *is biholomorphic to* $\mathbb{P}_n(\mathbb{C})$ *if the first Chern class* $c_1$ *of* $B$ *is not equal to* $-(n+1)g$.

It was shown by Yau that the additional assumption on $c_1(B)$ is not necessary (because the case $c_1(B) = -(n+1)g$ does not occur) (cf. [20]). In the homogeneous case, Lie group methods yield a direct proof of the fact that $c_1(B)$ is not a negative multiple of $g$.

COROLLARY 3: *If the* $\mathfrak{g}$*-anticanonical fibration of* $M$ *is not finite, then* $M$ *fibers over* $\mathbb{P}_n(\mathbb{C})$ *with fiber* $S^1$.

### 2.3 *The* $SR$*-fibration in the Case of a Finite Fiber*

We now handle the case where the $\mathfrak{g}$-anticanonical fibration is finite. We know from 1.3 that either $\tilde{M} = K/L = S^1 \times Q$ or there exists a $SR$-fibration of $\Omega \supset \tilde{M}$.

The first case can be excluded because $\pi_1(\tilde{M})$ is finite. So we always have a diagram

$$\begin{array}{ccccccccc} M & \xrightarrow{\Phi} & \tilde{M} & = & K/L & \hookrightarrow & \hat{K}/\hat{L} & = & \Omega \\ & & & & & {\scriptstyle P/L}\searrow & \downarrow{\scriptstyle \hat{P}/\hat{L}} & & \\ & & & & & & K/P & = & \hat{K}/\hat{P} = Q \end{array}$$

where $Q$ is a simply-connected homogeneous projective rational manifold.

(1) Indeed it is shown that such a fibration can only exist if $\dim_{\mathbb{R}} E$ and $k$ are odd.

PROPOSITION 1: *If the $\mathfrak{g}$-anticanonical fibration of $M$ is finite, then there are only two possibilities for the Stein-rational fibration*:

(a)
$$\begin{array}{ccccc} \tilde{M} & = & K/L & \longrightarrow & \hat{K}/\hat{L} \\ & & S^1 \big\downarrow & & \big\downarrow \mathbf{C}^* \\ & & K/P & \simeq & \hat{K}/\hat{P} \end{array}$$

(b)
$$\begin{array}{ccccc} \tilde{M} & = & K/L & \longrightarrow & \hat{K}/\hat{L} \\ & & S^1 \big\downarrow & & \big\downarrow \mathbf{C}^* \\ & & K/P & \simeq & \hat{K}/\hat{P} \end{array}$$

PROOF: At first we show that the fibration is not trivial, i.e. $Q$ is not a point. In this case, $\Omega = \mathbf{C}^{n+1}$ or $\Omega$ is the tangent bundle $TN$ of a symmetric space $N$ of rank 1 with $\dim_{\mathbf{R}} N = n+1$.

But $\Omega = \mathbf{C}^{n+1}$ means that a semi-simple group $\hat{K}$ acts transitively on $\mathbf{C}^n$. Then $\hat{L}$ is reductive (see [13, p. 206]). Now $\hat{K}/\hat{L}$ has the same homotopy type as the quotient of the maximal compact subgroups of $\hat{K}$ (resp. $\hat{L}$) (see [15, p. 260]). This quotient is compact and all homology groups vanish. Therefore it is a point. Then the quotient $\hat{K}/\hat{L}$ is also a point, i.e. we have a contradiction.

If $\Omega = TN$, then $\tilde{M}$ is the bundle of unit tangent vectors over $N$.

For simply-connected $N$, the $\mathfrak{g}$-anticanonical fibration is injective, and one can apply 2.2.1 to see that $n = 3$ or $n = 7$. In both cases the base $B$ is diffeomorphic to a sphere ($\mathbb{P}_1(\mathbb{H}) \simeq S^4$). Now the bundle of unit tangent vectors of a sphere $S^{n+1}$ is the Stiefel manifold $V_{n+2,2}$. It is known that $\pi_n(V_{n+2,2}) = \mathbf{Z}_2$ (cf. [18, p. 132]). So $S^7$ (resp. $S^{15}$) is not the tangent sphere bundle of $N = S^4$ (resp. $N = S^8$).

If $N$ is not simply-connected, then $N = \mathbb{P}_{n+1}(\mathbb{R})$ and $\tilde{M} = \mathbb{P}_{2n+1}(\mathbb{R})$ is the unit sphere bundle in $TN = \mathbb{P}_{n+1}(\mathbf{C}) \backslash Q_n$, where $Q_n = \{[z_0 : \cdots : z_{n+1}] : \sum_{i=0}^{n+1} z_i^2 = 0\}$ (cf. [5]). $Q_n$ is a complex $K$-orbit in $\mathbb{P}_{n+1}(\mathbf{C})$. By the "differentiable slice theorem" there is a $K$-equivariant diffeomorphism of a neighbourhood of the zero section in the normal bundle of $Q_n$ onto a neighbourhood of $Q_n$ in $\mathbb{P}_{n+1}(\mathbf{C})$ (see e.g. [12]). In our situation $K$ acts transitively on the unit sphere bundle in the normal bundle (see e.g. [5]). So the $K$-orbit $\tilde{M}$ is diffeomorphic to the unit sphere bundle and we obtain a homogeneous fibration $S^1 \longrightarrow \tilde{M} \longrightarrow Q_n$. This yields a fibration $M \longrightarrow Q_n$. The fiber is connected, so it is again $S^1$. From 2.2.1 and 2.2.2 we conclude then that $Q_n$ is biholomorphic to $\mathbb{P}_n(\mathbf{C})$. It can be shown e.g. by looking at the dimension of the automorphism groups that $Q_n$ and $\mathbb{P}_n(\mathbf{C})$ are not biholomorphic except for $n = 1$.

So $Q$ is a positive-dimensional projective rational manifold. Note that $\pi_1(Q) = 0$ and therefore $P/L$ is connected.

Furthermore $\pi_2(Q) = H_2(Q, \mathbf{Z})$ by the Hurewicz-Isomorphism ([18]), and

$H_2(Q,\mathbf{Z})$ contains an infinite group.

Assume now that $P/L = S$ is the unit sphere bundle in the tangent bundle $\hat{P}/\hat{L} = TN$ of a symmetric space $N$ of rank 1 with $\dim_{\mathbf{R}} N = k$. For $k \geq 2$, one obtains that $\pi_1(S)$ is finite by considering the homotopy sequence of $S^{k-1} \longrightarrow S \longrightarrow N$. But then the homotopy sequence of the fibration $S \longrightarrow \tilde{M} \longrightarrow Q$ shows that $\pi_2(Q)$ is finite and we have a contradiction. If $k = 1$ then $N = S^1$ or $N = \mathbb{P}_1(\mathbb{R})$. Since $S^1$ and $\mathbb{P}_1(\mathbb{R})$ are Lie groups, their tangent bundle is (topologically) trivial and hence $S = 2 \cdot S^1$ (resp. $2 \cdot \mathbb{P}_1(\mathbb{R})$) (two disjoint copies of $S^1$ (resp. $\mathbb{P}_1(\mathbb{R})$)). Therefore the fiber $S$ is not connected, i.e. a contradiction.

If $\hat{P}/\hat{L} = \mathbf{C}^k$, then $P/L = S^{2k-1}$. Assume $k > 1$. Then $\pi_2(Q) = 0$ as above. So the only possibility is $k = 1$ and $P/L = S^1$.

In the case $\hat{P}/\hat{L} = \mathbf{C}^*$ we also have $P/L = S^1$. □

The following is an immediate consequence of the above proposition together with 2.2.1 and 2.2.2.

COROLLARY 2: *If the* $\mathfrak{g}$*-anticanonical fibration of* $M$ *is finite, then* $M$ *fibers over* $\mathbb{P}_n(\mathbf{C})$ *with fiber* $S^1$. □

The next point in the proof of the main theorem is to show that the situation where the non-standard structures on $S^3$ appear is impossible for higher dimensions. These appear as $S^1$-bundles over $\mathbb{P}_1(\mathbf{C})$ lying in a $\mathbf{C}$-bundle over $\mathbb{P}_1(\mathbf{C})$ which is not holomorphically equivalent to a line bundle. For higher dimensions this situation is not possible.

LEMMA 3: *For* $n \geq 2$, *every affine* $\mathbf{C}$*-bundle on* $\mathbb{P}_n(\mathbf{C})$ *is holomorphically equivalent to a line bundle.*

PROOF: A locally trivial affine $\mathbf{C}$-bundle over a complex manifold $X$ is given by transition matrices

$$A_{ij} := \begin{pmatrix} \lambda_{ij} & \mu_{ij} \\ 0 & 1 \end{pmatrix} : U_{ij} \longrightarrow GL_2(\mathbf{C}).$$

Thus it defines a holomorphic line bundle $L$ on $X$ with transition functions $\lambda_{ij}$ and a rank-two vector bundle $E$ on $X$ which contains a trivial subbundle of rank 1, i.e. we have a sequence $0 \longrightarrow \mathcal{O} \longrightarrow E \longrightarrow L \longrightarrow 0$. This is a holomorphically trivial extension if and only if the affine bundle is equivalent to a line bundle. It is well known that for vector bundles $E'$, $E''$

$$\begin{aligned} \mathrm{Ext}^1_{\mathcal{O}}(E'',E') &\doteq \check{H}^1(X,\mathrm{Hom}(E'',E')) \\ &\doteq \check{H}^1(X,(E'')^* \otimes E') \qquad \text{(cf. [9])} \end{aligned}$$

It follows that the affine bundle is holomorphically equivalent to the line bundle $L$, if $\mathrm{Ext}^1_{\mathcal{O}}(L,\mathcal{O}) = \check{H}^1(X,L^*) = 0$.

Now every line bundle on $\mathbb{P}_n(\mathbf{C})$ is of the form $L = H^{\ell}$ ($\ell \in \mathbf{Z}$), where $H$ is the hyperplane bundle, and the canonical bundle $K$ on $\mathbb{P}_n(\mathbf{C})$ is

$H^{-(n+1)}$. If $\ell > -(n+1)$, we have by the Kodaira-Vanishing-Theorem ([19, p. 219]) $H^q(X,L) = 0$ for every $q > 0$. If $L$ is a negative bundle, we apply again the Kodaira-Vanishing-Theorem and make use of Serre-Duality to obtain $H^{n-1}(X, L^* \otimes K) = H^1(X,L) = 0$. This completes the proof. □

## 3. Uniqueness of the $CR$-structure of $M$

For the proof of the main theorem we introduce the following notation: As usual we denote the hyperplane bundle on $\mathbb{P}_n(\mathbb{C})$ by $H$ and the associated $\mathbb{C}^*$-principal bundle to $H^m$ $(m \in \mathbb{Z})$ by $P^{(m)}$. $P^{(m)}$ and $P^{(-m)}$ are biholomorphic. Of course

$$\mathbb{C}^{n+1}\backslash\{0\} \xrightarrow{\mathbb{C}^*} \mathbb{P}_n(\mathbb{C})$$

is $P^{(-1)}$ and one has an associated $S^1$-principal bundle by the inclusion

$$\begin{array}{ccc} S^{2n+1} & \longrightarrow & \mathbb{C}^{n+1}\backslash\{0\} \\ \big\downarrow S^1 & & \big\downarrow \mathbb{C}^* \\ \mathbb{P}_n(\mathbb{C}) & \simeq & \mathbb{P}_n(\mathbb{C}) \end{array}$$

(the first fibration simply being the Hopf-fibration). This $S^1$-bundle is denoted by $\mathcal{A}^{(-1)}$. There exists a $(m:1)$-covering map

$$\phi_m : P^{(+1)} \longrightarrow P^{(m)},$$

where $\mathcal{A}^{(m)} := \phi_m(\mathcal{A}^{(+1)})$ is the $S^1$-principal bundle associated to $P^{(m)}$. In particular $\pi_1(\mathcal{A}^{(m)}) = \mathbb{Z}_m$.

Now we can prove the

**Main Theorem**: *Let $M = G/H$ be a homogeneous $CR$-hypersurface, which is homeomorphic to $S^{2n+1} (n > 1)$. Then $M$ carries the standard $CR$-structure (which comes from the embedding $S^{2n+1} \hookrightarrow \mathbb{C}^{n+1}$).*

PROOF: By considering the $\mathfrak{g}$-anticanonical and $SR$-fibration of $M$, we have shown that only the two following cases can occur.

$$(1) \qquad \begin{array}{cccc} M = & G/H & \longrightarrow & \hat{G}/\hat{H} \\ & \big\downarrow S^1 & & \big\downarrow \mathbb{C}^* \\ & \mathbb{P}_n & \simeq & \mathbb{P}_n \end{array}$$

$$(2) \qquad \begin{array}{cccc} M \xrightarrow{t:1} & \tilde{M} = K/L & \longrightarrow & \hat{K}/\hat{L} \\ & \big\downarrow S^1 & & \big\downarrow \mathbb{C}, \mathbb{C}^* \\ & \mathbb{P}_n & \simeq & \mathbb{P}_n \end{array}$$

In the first case we know that $M = \mathcal{A}^{(m)}$ $(m \in \mathbf{Z})$. But $\pi_1(M) = 0$ and so $m = 1$ and

$$M = S^{2n+1} \hookrightarrow \mathbf{C}^{n+1} \backslash \{0\} = \hat{G}/\hat{H}.$$

Therefore the statement is proved in this case.

In the other case $\hat{K}/\hat{L}$ is either a $\mathbf{C}^*$-principal bundle or the fiber is $\mathbf{C}$. Since we assumed $n > 1$, we know by lemma 2.3.3 that the $\mathbf{C}$-bundle $\hat{K}/\hat{L}$ must be holomorphically equivalent to a holomorphic line bundle over $\mathbf{P}_n$. The $S^1$-bundle $K/L$ is then $CR$-diffeomorphic to an $S^1$-bundle in $H^m$ for some $m$ and thus $CR$-diffeomorphic to $\mathcal{A}^{(m)}$. This situation is the same as in the case when $\hat{P}/\hat{L} = \mathbf{C}^*$. By the above remarks we may assume $m > 0$ and $m = t$ (since $\pi_1(\tilde{M}) = \mathbf{Z}_t = \pi_1(\mathcal{A}^{(t)})$).

We know then that there is a holomorphic $(t : 1)$-covering of $P^{(t)}$ by $\mathbf{C}^{n+1} \backslash \{0\}$, which induces a $(t : 1)$-covering of $\mathcal{A}^{(t)} \doteq \tilde{M}$ by the sphere $S^{2n+1}$ equipped with the standard structure.

Now we have two universal $CR$-coverings $M$ and $S^{2n+1}$ of the $CR$-hypersurface $\tilde{M}$. Then $M$ is $CR$-equivalent to $S^{2n+1}$ with the standard $CR$-structure. □

FINAL REMARK: For $S^3$ all possible homogeneous $CR$-hypersurface-structures are classified (e.g. [11]). Together with our main theorem one obtains a complete classification of all homogeneous $CR$-hypersurface-structures on spheres.

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