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Regularity of the Solutions of Second Order Evolution Equations and Their Attractors

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Introduction.

In this article we consider linear and nonlinear evolution equations of the second order in time of the form

$$\frac{d^2u}{dt^2}(t) + \alpha\frac{du}{dt}(t) + Au(t) + g(t, u(t)) = f(t), \tag{0.1}$$

$$u(0) = u_0,\ \frac{du}{dt}(0) = u_1, \tag{0.2}$$

and we are interested in questions related to the regularity of the solutions in time and space (when (0.1) corresponds to a partial differential equation in x and t). Two types of regularity problems are addressed. The first one is that of the regularity of the solutions of (0.1) (0.2) for all $t \in \mathbb{R}$. A scale of Hilbert spaces F_m is considered

$$F_{m+1} \subset F_m \subset \ldots \subset F_0,$$

where $F_1 \times F_0 \supset D(A^{1/2}) \times H$ and H is the basic Hilbert space for (0.1), and we seek necessary and sufficient conditions for the data (A, g, f) and the initial values u_0, u_1, which ensure that the solution $\{u, \dot{u} = \frac{du}{dt}\}$ of (0.1) belongs to $F_{m+1} \times F_m$ for all $t \in \mathbb{R}$. An obvious necessary condition is that $\{u_0, u_1\} \in F_{m+1} \times F_m$: the other conditions are related to the so-called *compatibility conditions* which have been investigated in R. Temam [23] for the first order parabolic equations. Let us point out that the situation is definitely different for second order and first order equations: indeed for second order equations there is no regularizing effect as one can reverse the time and solve as well the equations backward in t. However some of the technics of [23] are used here and, as in this last reference, we produce *very simple* necessary and sufficient conditions of regularity. For other results on the regularity in

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relation with the compatibility conditions the reader is referred besides [23] to O.A. Ladyzhenskaya, V.A. Solonnikov and N.N. Uralceva [13], J.B. Rauch and F. Massey [19], S. Smale [22] and J. Sather [20].

The second type of regularity results which is addressed here is related to the attractors and the long time behavior of the solutions of (0.1) (0.2). The existence and the properties of the attractors for (0.1) have been studied in a related paper [8] to which the reader is referred for more details and for some points which are recalled without proofs, but the reading of this article is essentially independent of [8]. In [8] and in previous related works of A. Babin and M.I. Vishik [2,3], A. Haraux [12] and J. Hale [10] the existence of a maximal attractor $\mathbf{A}$ for (0.1) was proved, and in [8] it was shown that its Hausdorff and fractal dimensions in $D(A^{1/2}) \times H$ were finite; in particular $\mathbf{A}$ is compact in $D(A^{1/2}) \times H$ and attracts in the norm of $D(A^{1/2}) \times H$ any bounded set of $D(A^{1/2}) \times H$. Here we are interested in determining under what conditions on f, g, A, the attractor $\mathbf{A}$ lies in a subspace $F_{m+1} \times F_m$ of $F_1 \times F_0$ (corresponding to more regular functions in the space variables); and when $\mathbf{A} \subset F_{m+1} \times F_m$, under what conditions a bounded set B_m of $F_{m+1} \times F_m$ is attracted by $\mathbf{A}$ in the norm of $F_{m+1} \times F_m$. As far as we know the problem of the regularity of $\mathbf{A}(\mathbf{A} \subset F_{m+1} \times F_m)$ has been only investigated, in a direct framework, by J. Hale and J. Scheurle [11] but they make the assumption that the nonlinear term g is small in an approppriate sense, a smallness assumption that is not needed here.

We now describe how this article is organized: the compatibility conditions and the smoothness at finite t are studied in §1, the regularity results for the attractors are studied in §2. In §1.1 we describe the abstract setting for (0.1) in the linear case ($g = 0$) and in §1.2a we show how the general setting applies to the linear wave equation. The compatibility results for the linear wave equation are proved in §1.2b and §1.2c for the nonlinear case; the assumption on the nonlinear term g are the same as in [8] and this allows for non gradient systems, non local nonlinear terms, linear self-adjoint operators other then $-\Delta$ and general boundary conditions. Finally the application of the results to the Sine-Gordon equation and the nonlinear wave equation of quantum mechanics are described in §1.2d. Regularity results for solutions defined on the real line are given in §1.3. In §2.1 we investigate the regularity result for $\mathbf{A}$ (conditions insuring that $\mathbf{A} \subset F_{m+1} \times F_m$); and finally in §2.2 we study the convergence of bounded sets to $\mathbf{A}$ in the norm of $F_{m+1} \times F_m$.

A few words about notations. Let Ω be a domain in $\mathbb{R}^n$ and let $p \in [1, \infty]$. We denote by $L^p(\Omega)$ the class of all measurable funtions u, defined on Ω, for which

$$|u|_{L^p(\Omega)} = \left(\int_\Omega |u(x)|^p dx \right)^{1/p} < \infty$$

for $1 \leq p < +\infty$, and for $p = \infty$ those which are essentially bounded on Ω:

$$|u|_{L^\infty(\Omega)} = \operatorname*{ess\,sup}_{x \in \Omega} |u(x)| < \infty.$$

We denote by $H^m(\Omega)$ the Sobolev L^2-space of order m and by $H_0^m(\Omega)$ the closure of $\mathcal{D}(\Omega)$ (the space of C^∞ functions with compact support in Ω).

Let I be and interval of $\mathbb{R}$, $p \in [1,\infty]$ and X be a Banach space. We denote by $L^p(I;X)$ the space of classes of measurable functions f from I into X such that $\|f\|_X \in L^p(I)$. This space is a Banach space endowed with the norm

$$\|f\|_{L^p(I;X)} = |\|f\|_X|_{L^p(I)}.$$

The space $C(I;X)$ is the space of continuous functions from I into X and we shall denote by $C_b(I;X) = C(I,X) \cap L^\infty(I,X)$ the space of continuous bounded functions from I into X.

TABLE OF CONTENT

1. - Smoothness and compatibility conditions.

1.1. - *Abstract setting.*

We are given on a real Hilbert space H, with scalar product $(\cdot,\cdot)$ and norm $|\cdot|$, a linear self-adjoint unbounded operator A with domain

$$D(A) = \{v \in H, Av \in H\}$$

dense in H. We assume that A is an isomorphism from $D(A)$ (equipped with the graph norm) onto H and that A is positive:

$$(Av, v) > 0, \ \forall v \in D(A), \ v \neq 0. \tag{1.1}$$

We recall that under these hypotheses, one can define the powers A^s, $s \in \mathbf{R}$ (see [6]) and that the space

$$V_{2s} = D(A^s), \quad s \in \mathbf{R}, \tag{1.2}$$

is a Hilbert space for the scalar product and norm

$$(u,v)_{2s} = (A^s u, A^s v), \ |u|_{2s} = (u,u)_{2s}^{1/2}, \ \forall u, \ v \in D(A^s).$$

It is well known that, given T with $0 < T < \infty$, and f, u_0, u_1 satisfying

$$f \in L^2(0,T;H), \ u_0 \in V_1, \ u_1 \in H, \tag{1.3}$$

the initial value problem on $[0,T]$:

$$\ddot{u}(t) + \alpha\dot{u}(t) + Au(t) = f(t), \quad 0 < t < T, \tag{1.4}$$

$$u(0) = u_0, \ \dot{u}(0) = u_1. \tag{1.5}$$

($\alpha \in \mathbf{R}$) possesses a unique solution $\{u, \dot{u}\} \in C([0,T]; V_1 \times H)$; while if $f, \dot{f} \in L^2(0,T;H)$, $u_0 \in V_2$ and $u_1 \in V_1$, $\{u, \dot{u}, \ddot{u}\} \in C([0,T]; \ V_2 \times V_1 \times H)$. We have denoted by a dot the differentiation with respect to the ("time") variable t. The proof of this result can be found for instance in J.L. Lions and E. Magenes [16]. The reader is also referred to [8] where some complements are given.

1.2. - *Compatibility conditions.*

a) An example: the wave equation.

Let Ω be an open bounded set in $\mathbb{R}^n$ with smooth C^∞ boundary $\partial\Omega$. The wave equation in $\overline{\Omega} \times \mathbb{R}_+$ reads

$$
\begin{aligned}
&\frac{\partial^2 u}{\partial t^2}(x,t) && - \Delta u(x,t) = f(x,t) \text{ for } x \in \Omega,\ t > 0,\\
&u(x,t) && = 0 \text{ for } x \in \partial\Omega,\ t \geq 0, \qquad (1.6)\\
&u(x,0) && = u_0(x),\ \frac{\partial u}{\partial t}(x,0) = u_1(x) \text{ for } x \in \Omega.
\end{aligned}
$$

This set of equations can be written in the form (1.4)-(1.5) with $\alpha = 0$ and

$$H = L^2(\Omega),\ D(A) = H^2(\Omega) \cap H_0^1(\Omega),\ Au = -\Delta u.$$

We seek a condition on the data f, u_0 and u_1 such that the function $\{u(.,t),\ \frac{\partial u}{\partial t}(.,t)\}$ belongs to $H^{m+1}(\Omega) \times H^m(\Omega)$ for every $t \geq 0$, where m is a given integer. This problem is naturally connected with the regularity properties of the operator A (here the Dirichlet problem for the Laplacian on Ω). If we introduce the spaces $F_m = H^m(\Omega)$, $m \geq 0$, the question can be rephrased as

$$
\begin{aligned}
&\text{Find a condition on the data } f, u_0, u_1\\
&\text{such that the solution of (1.4)-(1.5) lies in} \qquad (1.7)\\
&F_{m+1} \times F_m \text{ i.e. } \{u, \dot{u}\} \in C([0,T];\ F_{m+1} \times F_m).
\end{aligned}
$$

As already noticed the well known regularity results on the operator A (see e.g. S. Agmon, A. Douglis and L. Nirenberg [1] or J.L. Lions and E. Magenes [16]) are related to (1.7). Indeed the (trivial) case where f does not depend on time t, u_0 is such that $-\Delta u_0 = f$, $u_0 \in V_1 = H_0^1(\Omega)$ and $u_1 \equiv 0$ (i.e. the stationary case) shows that (1.7) is obtained if and only if $f \in H^{m-1}(\Omega) = F_{m-1}$. This is an immediate consequence of the fact ($\mathcal{A}$ denotes the p.d.e. operator $-\Delta$) that

$$
\begin{aligned}
&\mathcal{A} \text{ is continuous from } F_{m+2} \text{ into } F_m \text{ and}\\
&\mathcal{A} \text{ is an isomorphism from } V_2 \cap F_{m+2} \text{ onto } F_m,\ \forall m \in \mathbf{N}, \qquad (1.8)\\
&\mathcal{A}u = Au,\ \forall u \in V_m,\ m \geq 2.
\end{aligned}
$$

REMARK 1.1. The trivial case of stationary solutions to (1.6) shows that if we take instead of $F_m = H^m(\Omega)$, $F_m = V_m = D(A^{m/2})$, which is a closed proper subspace of $H^m(\Omega)$ when $m \geq 1$, the answer to (1.7) is $u_0 \in V_{m+1}$ and $f \in V_{m-1}$ since (1.6) reads $Au_0 = f$. But these conditions ($u_0 \in V_{m+1}$, $f \in V_{m-1}$) are much more restrictive for $m \geq 2$. They include boundary conditions on f and u_0 which are not present when $F_m = H^m(\Omega)$. Indeed for, e.g. $m = 4$, the first condition is $f \in H^3(\Omega)$ whereas the second is $f \in V_3 = \{v \in H^3(\Omega),\ v = \Delta v = 0 \text{ on } \partial\Omega\}$. To conclude we say that taking

$F_m = H^m(\Omega)$ leads to simpler conditions than taking $F_m = V_m$ (also note that the constraints in the last case are not physically relevant). However the results in the second case are stronger since $V_{m+1} \times V_m$ is a proper subspace of $H^{m+1}(\Omega) \times H^m(\Omega)$.

b) The linear equations.

In accordance with the previous example, we assume that besides the scale of Hilbert spaces $\{V_s\}_{s\in\mathbb{R}}$, we are given a family of Hilbert spaces $\{F_m\}_{m\in\mathbb{N}}$ and an operator $\mathcal{A}$ that satisfy (1.8) with ([1])

$$F_{m-1} \subset F_m \text{ the injection being continuous, } V_m \text{ is a closed subspace of } F_m, \text{ the norm induced by } F_m \text{ on } V_m \text{ being equivalent to } |\cdot|_m;\ \forall m \in \mathbb{N};\ \text{finally } F_0 = H. \tag{1.9}$$

Our goal in this paragraph is to answer (1.7). In the previous example of the wave equation, this problem is clearly a regularity result. Problems of this kind have been investigated by various authors in case of linear equations ([4], [19], [22], ...) and by one of the authors ([23]) in the context of semi-linear evolution equations of parabolic type. Let us recall that, due to the well-known smoothing effect of parabolic equations the analogue of Problem (1.7) reduces in that case to the behavior at time $t = 0$. In the case of the second order in time problem (1.4) (which includes hyperbolic problems) the situation is definitely different. Indeed, this equation has no smoothing effect since one can reverse time because the change of t in $-t$ does not affect the type of the equation. However the results and technics for solving (1.7) are very similar to those of [23] and produce a very simple necessary and sufficient condition that answers (1.7), see (1.17).

In accordance with the technics of [16] and [23], we introduce the Banach space

$$W_m(I) = \{v \in \mathcal{C}_b(I, F_{m+1}),\ v^{(j)} \in \mathcal{C}_b(I; F_{m+1-j}),\ j = 1,\ \ldots, m+1\}, \tag{1.10}$$

endowed with the natural norm

$$\|v\|_m = \operatorname*{Sup}_{0\leq j\leq m+1} |v^{(j)}|_{L^\infty(I;F_{m+1-j})}, \tag{1.11}$$

where I denotes a closed (not necessarily bounded) interval of $\mathbb{R}$ and m is an integer ≥ -1. We shall write $W_m(0,T)$ instead of $W_m([0,T])$ and W_m instead of $W_m(\mathbb{R})$.

First, we assume that the conclusion of (1.7) holds, and we seek a necessary condition. We suppose that for some $m \geq 2$,

$$f \in W_{m-2}(0,T);\ \{u, \dot{u}\} \in \mathcal{C}([0,T];\ F_{m+1} \times F_m). \tag{1.12}$$

By successive differentiations of (1.4), it follows that

$$u \in W_m(0,T). \tag{1.13}$$

Indeed, let j, $0 \leq j \leq m+1$ and

$$u^{(k)} \in C([0,T];\ F_{m-k+1}),\ k=0,\ \ldots,\ j. \tag{1.14$_j$}$$

We obtain by (1.12) that $(1.14)_0$ and $(1.14)_1$ hold true. Assuming that $(1.14)_j$ holds with $1 \leq j \leq m$, we differentiate (1.4) $j-1$ times with respect to t. Hence

$$\frac{d^{j+1}u}{dt^{j+1}} \equiv u^{(j+1)} = -\alpha u^{(j)} - \mathcal{A}u^{(j-1)} + f^{(j-1)}, \tag{1.15}$$

and using (1.8)-(1.9), (1.12) and $(1.14)_j$ we find that

$$u^{(j+1)} \in C([0,T];\ F_{m-j}).$$

This proves $(1.14)_{j+1}$ and by induction on j we obtain $(1.14)_{m+1}$ which is exactly (1.13).

□

Now, if a solution u to (1.4)-(1.5) satisfies (1.13), thanks to (1.15) we can easily compute the successive derivatives of u at time $t=0$. These values are determined by the following recurrent formula:

$$\begin{aligned} &u(0) = u_0,\ \dot{u}(0) = u^{(1)}(0) = u_1, \\ &u^{(j+1)}(0) = f^{(j-1)}(0) - \alpha u^{(j)}(0) - \mathcal{A}u^{(j-1)}(0),\ 0 \leq j \leq m. \end{aligned} \tag{1.16}$$

We have $\mathcal{A}u^{(j)} = f^{(j)} - \alpha u^{(j+1)} - u^{(j+2)}$, thus according to (1.13), $\mathcal{A}u^{(j)} \in C([0,T];\ F_{m-1-j})$ for $0 \leq j \leq m-1$. Since $F_{m-1-j} \subset F_0$ it follows then from (1.8) that $u^{(j)} \in C([0,T];\ V_2)$ hence $u^{(j)}(0) \in V_2$, $0 \leq j \leq m-1$. On the other hand by (1.13), $u^{(m)} \in C([0,T];\ F_1)$; since V_1 is closed in F_1 and $u \in C([0,T];\ V_1)$ we deduce that $u^{(m)} \in C([0,T];\ V_1)$, hence $u^{(m)}(0) \in V_1$. We have shown that

$$\begin{cases} u^{(j)}(0) \in V_2 = D(A) \text{ for } j = 0, \ldots,\ m-1, \\ u^{(m)}(0) \in V_1. \end{cases} \tag{1.17}$$

REMARK 1.2. (i) These conditions are compatibility conditions between the data f, u_0 and u_1 since the $u^{(j)}(0)$ can be computed in term of these quantities using (1.16).

(ii) Returning to the wave equation (1.6), (1.17) simply means that the $u^{(j)}(0)$ (which are elements of $F_1 = H^1(\Omega)$) vanish in the sense of trace on $\partial\Omega : u^{(j)}(0)|_{\partial\Omega} = 0$ (see also Corollary 1.1).

□

We are going to prove that (1.17) is in fact a necessary and sufficient condition. More precisely we have

THEOREM 1.1. *Assume that the data f, u_0 and u_1 are such that*

$$\begin{aligned}&f \in W_{m-2}(0,T),\ f^{(m)} \in L^2(0,T;H),\\ &\{u_0,u_1\} \in F_{m+1} \times F_m \text{ where } 0 < T < \infty \text{ and } m \geq 1.\end{aligned} \tag{1.18}$$

The solution u of (1.4)-(1.5) *belongs to $W_m(0,T)$ if and only if the $u^{(j)}(0)$ given by* (1.16) *satisfy the compatibility conditions*

$$\begin{cases} u^{(j)}(0) \in V_2 \text{ for } j = 0,\dots,\ m-1,\\ u^{(m)}(0) \in V_1.\end{cases}$$

PROOF. We have already noticed that this condition is necessary. For the sufficient part, we denote by v the solution to the Cauchy problem

$$\ddot{v} + \alpha\dot{v} + Av = f^{(m)}, \tag{1.19}$$

$$v(0) = u^{(m)}(0),\ \dot{v}(0) = u^{(m+1)}(0), \tag{1.20}$$

where $u^{(m)}(0)$ and $u^{(m+1)}(0)$ are obtained through (1.16). Thanks to (1.17) and (1.18) the problem (1.19)-(1.20) is identical to (1.4)-(1.5) and $\{v,\dot{v}\} \in C([0,T];V_1 \times H)$. Since $u^{(m)} = v$, it follows that

$$\{u^{(m)},\ u^{(m+1)}\} \in C([0,T];\ V_1 \times H).$$

For $0 \leq k \leq m-1$, the function $w = u^{(k)}$ is also solution of a problem (1.4)-(1.5) with right hand side $g = f^{(k)}$ and initial conditions $w_0 = u^{(k)}(0)$, $w_1 = u^{(k+1)}(0)$. Since by (1.17) $w_0 \in V_2$, $w_1 \in V_1$ and by (1.18), $g, \dot{g} \in L^2(0,T;H)$, we have $\{u^{(k)},\ u^{(k+1)}\} \in C([0,T];V_2 \times V_1)$. Hence

$$\begin{aligned}&u^{(k)} \in C([0,T];V_2),\ 0 \leq k \leq m-1,\\ &\{u^{(m)},u^{(m+1)}\} \in C([0,T];V_1 \times H).\end{aligned} \tag{1.21}$$

The proof proceeds by induction. We assume that

$$u^{(m+k-1)} \in C([0,T];F_k),\ 0 \leq k \leq j, \tag*{(1.22)$_j$}$$

holds for some k, $1 \leq k \leq m$. We differentiate equation (1.4) $m-j$ times with respect to t; it follows that

$$Au^{(m-j)} = f^{(m-j)} - \alpha u^{(m-j+1)} - u^{(m+2-j)} \in C([0,T];F_{j-1}).$$

Thanks to (1.21), $u^{(m-j)} \in V_2$ hence by (1.8), $Au^{(m-j)} = \mathcal{A}u^{(m-j)}$ and then $u^{(m-j)} \in C([0,T];\ F_{(j+1)}) : (1.22)_{j+1}$ is proved.

According to (1.21), $(1.22)_0$ and $(1.22)_1$ hold and $(1.22)_{m+1}$ follows by induction. Therefore (1.13) and the Theorem are proved.

c) The nonlinear equations.

In this section we generalize Theorem 1.1 to a class of nonlinear equations which includes in particular nonlinear wave equations. Although the method is very general we will restrict ourselves for the sake of simplicity to the class of second order in time nonlinear evolution equation:

$$\ddot{u} + \alpha\dot{u} + Au + g(u) = f, \tag{1.23}$$

$$u(0) = u_0, \ \ \dot{u}(0) = u_1. \tag{1.24}$$

We assume that the nonlinear operator g maps continuously V_1 into H and V_2 into V_1. Moreover we assume that the problem (1.23)-(1.24) is well-posed (2) in $V_1 \times H$ and $V_2 \times V_1$, i.e. if f, u_0 and u_1 are given such that

$$f \in C([-T,T];\ H), \ \ u_0 \in V_1 \text{ and } u_1 \in H, \tag{1.25}$$

(1.23)-(1.24) possesses on $[-T,T]$ a unique solution satisfying

$$\{u, \dot{u}\} \in C([-T,T];\ V_1 \times H). \tag{1.26}$$

While, if f, u_0 and u_1 satisfy

$$f, \dot{f} \in C([-T,T];\ H), \ \ u_0 \in V_2 \text{ and } u_1 \in V_1, \tag{1.27}$$

the previous solution satisfies

$$\{u, \dot{u}\} \in C([-T,T];\ V_2 \times V_1). \tag{1.28}$$

In both cases, for t fixed, the solution $\{u(t), \dot{u}(t)\}$ depends continuously on $\{u_0, u_1\}$ with respect to the corresponding topology.

We introduce a mapping G_k on $F_{k+1} \times \cdots \times F_0$ as follows. For $\{u^{(0)}(0), u^{(1)}(0), \ldots, u^{(k+1)}(0)\} \in F_{k+1} \times \cdots \times F_0$, we set

$$G_k\{u^{(0)}(0), u^{(1)}(0), \ldots, u^{(k+1)}(0)\} = \left\{ \frac{d^j}{dt^j} g(u(t)) \Big|_{t=0} \right\}_{0 \le j \le k+1}.$$

We shall assume that for $k \geq 1$,

$(1.29)_k$ G_k is a continuous bounded map from $F_{k+1} \times \cdots \times F_0$ into itself.

It follows from this hypothesis that for every interval $I \subset \mathbf{R}$, g is a continuous and bounded map from $W_k(I)$ into itself. This is shown by observing that for any τ, the formula above is verified:

$$G_k\{u^{(0)}(\tau), \ldots, u^{(k+1)}(\tau)\} = \left\{ \frac{d_j}{dt^j} g(u(t)) \Big|_{t=\tau} \right\}_{0 \leq j \leq k+1}$$

($\tau = 0$ does not play a particular role) and then we apply $(1.29)_k$.

Let there be given $m \geq 2$. Assuming that $(1.29)_k$ holds for $1 \leq k \leq m-1$, by the same kind of method as in the linear case it follows that if (1.12) holds then u satisfies (1.13) and the $u^{(j)}(0)$ are determined by

$$\begin{aligned} & u(0)) = u_0, \ \dot{u}(0) = u^{(1)}(0) = u_1, \\ (1.30) \qquad & u^{(j+1)}(0) = f^{(j-1)}(0) - \frac{d^{j-1}}{dt^{j-1}} g(u(t)) \Big|_{t=0} - \alpha u^{(j)}(0) \\ & \qquad\qquad - Au^{(j-1)}(0), \ 1 \leq j \leq m. \end{aligned}$$

We can now state the analogue of Theorem 1.1:

THEOREM 1.2. *Let $m \geq 2$ and assume that $(1.29)_k$ holds for $1 \leq k \leq m-1$. If the data f, u_0 and u_1 are such that*

$$(1.31)_m \qquad \begin{aligned} & f \in W_{m-2}(0,T), \ f^{(m)} \in L^2(0,T;H), \\ & \{u_0, u_1\} \in F_{m+1} \times F_m \ \text{where} \ 0 < T < \infty. \end{aligned}$$

Then the solution u of (1.23)-(1.24) belongs to $W_m(0,T)$ if and only if the $u^{(j)}(0)$ given by (1.30) satisfy the compatibility conditions:

$$(1.32)_m \qquad \begin{cases} u^{(j)}(0) \in V_2 \ \text{for} \ j = 0, \ldots, m-1, \\ u^{(m)}(0) \in V_1. \end{cases}$$

PROOF. The proof of the necessary part is parallel to that in Theorem 1.1. We are going to establish the sufficient part by induction on m. For $m = 2$, it follows from (1.31)-(1.32) that (1.27) is fulfilled. Hence according to (1.28) and (1.23), we have: $u \in W_1(0,T)$. Thanks to $(1.29)_1$,

$$(1.33) \qquad g(u) \in W_1(0,T).$$

We write (1.23) as

$$(1.34) \qquad \ddot{u} + \alpha\dot{u} + Au = f - g(u);$$

According to (1.31), (1.32) and (1.33), Theorem 1.1 (with $m = 2$) applies to (1.34) and we obtain that $u \in W_2(0,T)$. Theorem 1.2 is proved for $m = 2$.

Suppose that this theorem is true for some $m \geq 2$ and that hypotheses $(1.29)_m$, $(1.31)_m$ and $(1.32)_{m+1}$ are satisfied. Applying Theorem 1.2 at rank m, we deduce that

$$u \in W_m(0,T). \tag{1.35}$$

From $(1.29)_m$ it follows that

$$g(u) \in W_m(0,T) \tag{1.36}$$

and according to $(1.31)_{m+1}$, $(1.32)_{m+1}$ and (1.36) we can apply Theorem 1.1 at rank $m+1$ and it follows that $u \in W_{m+1}(0,T)$.

REMARK 1.2. (i) In the proof of this Thorem we have only used the fact that g is a continuous and bounded map from $W_k(I)$ into $W_{k-1}(I)$ for $k \leq m-1$.

(ii) The case where g depends on time t can also be considered. In that case $(1.29)_k$ is replaced by the assumption that g is a continuous and bounded map from $W_k(I)$ into itself.

d) Example: a nonlinear wave equation.

We return to the notation of Section 1.2.a on the linear wave equation in $\overline{\Omega} \times \mathbb{R}_+$ and introduce g a C^∞ mapping from the real line $\mathbb{R}$ into itself ([3]). The nonlinear wave equation in $\overline{\Omega} \times \mathbb{R}_+$ reads

$$\frac{\partial^2 u}{\partial t^2} + \alpha \frac{\partial u}{\partial t} - \Delta u + g(u) = f \text{ in } \Omega \times \mathbb{R}_+^\star, \tag{1.37}$$

together with $(1.6)_2$ and $(1.6)_3$.

We make the following assumptions on g $\left(G(s) = \int_0^s g(\sigma)d\sigma\right)$

$$\liminf_{|s| \to \infty} \frac{G(s)}{s^2} \geq 0, \tag{1.38}$$

there exists $C_1 > 0$ such that

$$\liminf_{|s| \to \infty} \frac{sg(s) - C_1 G(s)}{s^2} \geq 0, \tag{1.39}$$

and (when $n \geq 2$)

$$|g'(s)| \leq C(1+|s|^\gamma) \tag{1.40}$$
$$\text{with} \begin{cases} 0 \leq \gamma < \infty & \text{when } n = 2 \\ 0 \leq \gamma \leq 2 & \text{when } n = 3 \\ \gamma = 0 \text{ (i.e. } g' \text{ is bounded)} & \text{when } n \geq 4. \end{cases}$$

The two main types of applications we have in view are

(i) $g(u) = \sin u$ in which (1.37) is the well known Sine-Gordon equation,
(ii) $g(u) = u^{2p+1}$, $p \in \mathbf{N}$, and $\gamma = 2p$ satisfying (1.40) (here for physical reasons $\Omega \subset \mathbb{R}^n$ with $n \leq 3$).

These examples satisfy obviously that g is C^∞ and (1.38)-(1.40).

□

Under the previous hypotheses, it is well known (see [2], [8]) that Equations (1.37), $(1.6)_2$, $(1.6)_3$ are well-posed in $V_1 \times H$ and $V_2 \times V_1$.

In order to apply Theorem 1.2 to (1.37), we observe that for $\Omega \subset \mathbb{R}^n$, $n \leq 3$ (the physical case) it follows from Sobolev imbeddings Theorems (in particular since $H^2(\Omega) \subset L^\infty(\Omega)$) and from Faà di Bruno formula (see L. Comtet [5], p. 137) which gives the derivatives $\frac{d^k}{dt^k}(g(u(t)))$ as functions of those of g and u, that $(1.29)_k$ holds for every $k \geq 1$ (see the Appendix). Therefore under the previous hypotheses, Theorem 1.2 reads

COROLLARY 1.1. *We assume that Ω is a smooth (C^∞) bounded open set in $\mathbb{R}^n$, $n \leq 3$. If the data f, u_0 and u_1 are such that* ($m \geq 2$)

$$f^{(j)} \in C([0,T]; H^{m-1-j}(\Omega)),\ 0 \leq j \leq m-1,\ f^{(m)} \in L^2(0,T; L^2(\Omega)),$$
$$u_0 \in H^{m+1}(\Omega),\ u_1 \in H^m(\Omega),$$

then the solution u of (1.37), $(1.6)_2$, $(1.6)_3$ *satisfies*

$$u^{(j)} \in C([0,T] : H^{m+1-j}(\Omega)),\ 0 \leq j \leq m+1,$$

if and only if the $u^{(j)}(0)$ computed recursively by

$$u^{(0)}(0) = u_0,\ u^{(1)}(0) = u_1,$$
$$u^{(j+1)}(0) = f^{(j-1)}(0) - \frac{\partial^{j-1}}{\partial t^{j-1}} g(u(t))\Big|_{t=0} - \alpha u^{(j)}(0) - \Delta u^{(j-1)}(0),$$
$$0 \leq j \leq m,$$

are such that

$$u^{(j)}(x,0) = \frac{\partial^j u}{\partial t^j}(x,0) = 0 \text{ for } x \in \partial\Omega,\ j = 0, \ldots, m.$$

□

1.3. - *Bounded trajectories on the real line.*

This section is devoted to the study in the *dissipative case* (i.e. $\alpha > 0$) of solutions of the *linear* equation

$$\ddot{u}(t) + \alpha\dot{u}(t) + Au(t) = h(t), \quad t \in \mathbb{R}, \tag{1.41}$$

satisfying the boundary condition at $-\infty$

$$\limsup_{t\to-\infty} (|u(t)|_1^2 + |\dot{u}(t)|^2) < \infty, \tag{1.42}$$

where $h \in C_b(\mathbb{R}, H)$.

We write (1.41) as a first order system by introducing $\varphi = \{u, v\}$, $\mathcal{F} = \{0, h\}$, $\Lambda \in \mathcal{L}(V_1 \times H, H \times V_{-1})$:

$$\Lambda = \begin{pmatrix} \epsilon & -I \\ \epsilon(\epsilon-\alpha) + A & \alpha - \epsilon \end{pmatrix}$$

where ϵ is chosen in (1.45).

With these notations (1.41) reads

$$\dot{\varphi}(t) + \Lambda\varphi(t) = \mathcal{F}(t), \quad t \in \mathbb{R} \tag{1.43}$$

and we introduce the *linear group* $\{\Sigma(t)\}_{t\in\mathbb{R}}$ which acts on $V_1 \times H$ by setting $\Sigma(t)\varphi_0 = \varphi(t)$ where φ is the solution to (1.43) with this time $\mathcal{F} \equiv 0$ and

$$\varphi(0) = \varphi_0. \tag{1.44}$$

Concerning the choice of ϵ we denote by κ the norm of the injection from V_1 into $H(|u| \le \kappa|u|_1,\ \forall u \in V_1)$ and we compute for $\varphi \in V_2 \times V_1$, $\varphi = \{u, v\}$,

$$\begin{aligned}
(\Lambda\varphi, \varphi)_{V_1\times H} &= (\epsilon u - v, u)_1 + (\epsilon(\epsilon-\alpha)u + Au + (\alpha-\epsilon)v, v) \\
&= \epsilon|u|_1^2 + (\alpha-\epsilon)|v|^2 - \epsilon(\alpha-\epsilon)(u, v), \\
(\Lambda\varphi, \varphi)_{V_1\times H} &\ge \epsilon|u|_1^2 + (\alpha-\epsilon)|v|^2 - \epsilon|\alpha-\epsilon|\kappa|u|_1|v|.
\end{aligned}$$

Now we take

$$\epsilon = \min\left(\frac{\alpha}{4}, \frac{1}{2\alpha\kappa^2}\right); \tag{1.45}$$

hence

$$(\Lambda\varphi, \varphi)_{V_1\times H} \ge \frac{\epsilon}{2}(|u|_1^2 + |v|^2) \equiv \frac{\epsilon}{2}|\varphi|_{V_1\times H}^2. \tag{1.46}$$

From (1.46) it follows that for $\varphi_0 \in V_2 \times V_1$,

$$|\Sigma(t)\varphi_0|_{V_1\times H}^2 \le e^{-\epsilon t}|\varphi_0|_{V_1\times H}^2, \quad t \ge 0, \tag{1.47}$$

and since $\Sigma(t)$ is linear and continuous on $V_1 \times H$ by density of $V_2 \times V_1$ in $V_1 \times H$ we deduce (1.47) for every $\varphi_0 \in V_1 \times H$. Hence

$$|\Sigma(t)|_{\mathcal{L}(V_1\times H)} \le e^{-\epsilon t/2}, \quad t \ge 0. \tag{1.48}$$

The goal of this section is the following result

PROPOSITION 1.1. *Let k be a positive integer and let h be such that*

$$h^{(j)} \in C_b(\mathbb{R}; F_{k-j-1}),\ j = 0, \ldots, k-1, \quad h^{(k)} \in C_b(\mathbb{R}; H). \tag{1.49}$$

Then equation (1.41) *possesses a unique solution u which satisfies*

$$\{u, \dot{u}\} \in C_b(\mathbb{R}; V_1 \times H). \tag{1.50}$$

Moreover $u \in W_k$ and there exists a constant c_k such that

$$\|u\|_k \leq c_k(|h^{(k)}|_{L^\infty(\mathbb{R},H)} + \operatorname*{Sup}_{0 \leq j \leq k-1} |h^{(j)}|_{L^\infty(\mathbb{R};F_{k-j-1})}). \tag{1.51}$$

PROOF. Let us first prove this Proposition for $k = 1$. For the uniqueness we have to prove that if u satisfies (1.50) and (1.41) with $h = 0$, then $u \equiv 0$. Let $t \geq s$ then ($\varphi = \{u, v\}$) by (1.43) with $\mathcal{F} \equiv 0$,

$$\varphi(t) = \Sigma(t-s)\ \varphi(s),$$

and by (1.48),

$$|\varphi(t)|_{V_1 \times H} \leq \exp(\frac{-\epsilon(t-s)}{2})|\varphi(s)|_{V_1 \times H}. \tag{1.52}$$

But by (1.50), since $v = \dot{u} + \epsilon u$, $|\varphi(s)|_{V_1 \times H}$ is bounded when $s \to -\infty$. Hence letting $s \to -\infty$ in (1.52),

$$|\varphi(t)|_{V_1 \times H} \leq 0,$$

i.e. $\varphi(t) = 0$ which implies $u \equiv 0$. The existence is obtained by the variation of constants formula. Indeed we set $\psi(t, \tau) = \Sigma(t-\tau)\{0, h(\tau)\}$; from (1.48) we have $\psi(t, .) \in L^1(-\infty; t; V_1 \times H)$ and

$$|\psi(t, \tau)|_{V_1 \times H} \leq \exp(\frac{\epsilon(t-\tau)}{2})|h|_{L^\infty(\mathbb{R},H)}, \quad \tau \leq t.$$

Therefore we can define $\varphi \in C(\mathbb{R}, V_1 \times H)$ by

$$\varphi(t) = \int_{-\infty}^{t} \psi(t, \tau) d\tau, \tag{1.53}$$

and

$$|\varphi|_{L^\infty(\mathbb{R},V_1 \times H)} \leq \left(\int_{-\infty}^{t} \exp(-\frac{\epsilon(t-\tau)}{2}) d\tau \right) |h|_{L^\infty(\mathbb{R},H)} = \frac{2}{\epsilon}|h|_{L^\infty(\mathbb{R},H)}. \tag{1.54}$$

It is easily seen that φ defined by (1.53) satisfies (1.43) and since $\varphi = \{u, v\}$ with $v = \dot{u} + \epsilon u$, u is solution to (1.41) with (1.50) thanks to (1.54) which shows also (1.51). In fact we have obtained that the solution is given by

$$(1.55) \qquad \{u(t), \dot{u}(t) + \epsilon u(t)\} = \int_{-\infty}^{t} \Sigma_\epsilon(t-\tau)\{0, h(\tau)\}d\tau,$$
$$= \int_{t}^{+\infty} \Sigma_\epsilon(\tau)\{0, h(t-\tau)\}d\tau.$$

For $k \geq 2$, we first differentiate (1.55) k times with respect to t. Since $h^{(k)} \in C_b(\mathbb{R}, H)$ we find that $\{u^{(k)}, u^{(k+1)}\} \in C_b(\mathbb{R}; V_1 \times H)$. Let us prove by induction on j, $0 \leq j \leq k+1$, that

$$(1.56)_j \qquad u^{(k+1-j)} \in C_b(\mathbb{R}; F_j).$$

For $j = 0$ and $j = 1$ it has just been shown. We assume that $(1.56)_j$ holds for some $j \geq 1$. According to (1.41),

$$u^{(k-j+2)} + \alpha u^{(k-j+1)} + \mathcal{A} u^{(k-j)} = h^{(k-j)},$$

and by $(1.56)_{j-1}$, $(1.56)_j$ and (1.49),

$$\mathcal{A} u^{(k-j)} \in C_b(\mathbb{R}; F_{j-1}).$$

Since $\mathcal{A}$ is an isomorphism from $V_2 \cap F_{j+1}$ into F_{j-1} (by (1.18)) we deduce $(1.56)_{m+1}$.

Hence $(1.56)_{k+1}$, i.e. $u \in W_k$, follows by induction and (1.51) by inspection of the proof.

2. - Application to the attractors.

In this paragraph we derive some properties concerning the long time behavior of the infinite dimensional dynamical system generated by the equation

$$(2.1) \qquad \ddot{u}(t) + \alpha \dot{u}(t) + Au(t) + g(u(t)) = f \text{ for } t \geq 0,$$
$$(2.2) \qquad u(0) = u_0, \ \dot{u}(0) = u_1,$$

where α is positive. We assume that the right-hand side, f, of (2.1) is independent of t and belongs to H, so that (2.1) is an *autonomous* ([4]) dynamical system.

The hypotheses are those of Section 1.2.c, in particular those concerning the well-posedness of (2.1)-(2.2) in $V_1 \times H$ and $V_2 \times V_1$; we assume further that the injection from V_1 into H is *compact*. Equation (2.1) can be for instance the damped nonlinear wave equation of Section 1.2.d (see (1.37)).

The previous hypotheses are such that the mapping $S(t)$ from $V_1 \times H$ or $V_2 \times V_1$ into itself defined by

$$S(t)\{u_0, u_1\} = \{u(t), \dot{u}(t)\}, \quad \forall t \in \mathbb{R}, \tag{2.3}$$

is continuous and since (2.1) is autonomous, $\{S(t)\}_{t\in\mathbb{R}}$ is a *group* which acts on both spaces. We recall that a subset $\mathcal{X}$ of $V_1 \times H$ is a *functional invariant set* for $S(t)$ if

$$S(t)\mathcal{X} = \mathcal{X}, \quad \forall t \in \mathbb{R}. \tag{2.4}$$

Given $\{S(t)\}_{t\in\mathbb{R}}$ which acts continuously on a metric space $\mathcal{E}$, we recall that B_a is a *bounded absorbing set* in $\mathcal{E}$ if B_a is bounded in $\mathcal{E}$ and for every bounded subset B in $\mathcal{E}$, there exists $T(B) \in \mathbb{R}$ such that $S(t)B \subset B_a$, for every $t \geq T(B)$.

Concerning the long time behavior of (2.1) we assume that there exist a bounded absorbing set B_0 (respectively B_1) in $V_1 \times H$ (resp. $V_2 \times V_1$) and a compact set in $V_1 \times H$, $\mathbf{A}$, which is bounded in $V_2 \times V_1$, functional invariant and attracts bounded sets in $V_1 \times H$ i.e. for every bounded set B in $V_1 \times H$,

$$\lim_{t\to+\infty} \operatorname*{Sup}_{\{u_0,u_1\}\in B} d(S(t)\{u_0, u_1\}, \mathbf{A}) = 0, \tag{2.5}$$

where we have denoted

$$d(y, \mathbf{A}) = \operatorname*{Min}_{a\in\mathbf{A}} \|y - a\|_{V_1\times H}. \tag{2.6}$$

This set, $\mathbf{A}$, which is necessarily unique is called the *universal attractor* for the flow (2.1) in $V_1 \times H$.

Under the assumptions of Section 1.2.d., it was shown in [8] that the maximal attractor $\mathbf{A}$ exists for the nonlinear wave equation of Sec. 1.2.d (provided $\gamma < 2$ if $n = 3$); cf. also related results of A.V. Babin and M.I. Vishik [8,3], A. Haraux [12] and J.K. Hale [10]. The more general framework considered in [8] includes non gradient systems, non local nonlinear terms, linear self-adjoint elliptic differential operators other than $-\Delta$ and other boundary conditions.

For f given in $F_0 = H$, according to the previous assumptions and remarks, the universal attractor is included and bounded in $V_2 \times V_1$. In the following section 2.1 we address the natural question whether $\mathbf{A}$ is more regular, i.e. included in $F_{p+1} \times F_p$ for some p, provided f is more regular. We will give a positive (and optimal) answer in Theorem 2.1 which shows that if $f \in F_m$, we have $\mathbf{A} \subset F_{m+2} \times F_{m+1}$. Then in Section 2.2 we study whether (when $f \in F_m$) the convergence in (2.5) is achieved in a better norm than that of $V_1 \times H$. Finally we give some generalizations to the time periodic case.

2.1. - *Attractors are made of smooth functions.*

In this section we prove

THEOREM 2.1. *Let m be a non negative integer and assume that $f \in F_m$ and g satisfies $(1.29)_k$ for $1 \leq k \leq m$. Then every functional invariant set $\mathcal{X}$ bounded in $V_1 \times H$ is included and bounded in $F_{m+2} \times F_{m+1}$.*

PROOF. Let us first observe that if $\mathcal{X}$ satisfies the hypotheses of the Theorem then $\mathcal{X} \subset \mathbf{A}$. Indeed since $S(t)\mathcal{X} = \mathcal{X}$ and $\mathcal{X}$ is bounded in $V_1 \times H$ by (2.5) we have $d(\{u_0, u_1\}, \mathbf{A}) = 0$ for every $\{u_0, u_1\} \in \mathcal{X}$. Hence the Theorem is proved for $m = 0$. In fact according to what precedes it is sufficient to show that if $f \in F_m$,

$$\mathbf{A} \subset F_{m+2} \times F_{m+1}. \tag{2.7}$$

Let there be given $\{u_0, u_1\} \in \mathbf{A}$. Since $\mathbf{A}$ is invariant by $S(t)$ and bounded in $V_1 \times V_2$ we know that the trajectory $\{u(t), \dot{u}(t)\} = S(t)\{u_0, u_1\}$ lies in $\mathbf{A}$ for every $t \in \mathbb{R}$ and therefore is bounded in $V_2 \times V_1$, i.e. we have

$$\ddot{u}(t) + \alpha\dot{u}(t) + Au(t) = h(t) \tag{2.8}$$

where

$$h(t) = f - g(u(t)) \tag{2.9}$$

and

$$\{u, \dot{u}\} \in C_b(\mathbb{R}; V_2 \times V_1). \tag{2.10}$$

We are going to prove by induction on m that

$$\text{if } f \in F_m, \text{ then } u \in W_{m+1} \text{ and } \|u\|_{m+1} \leq \rho_m < +\infty, \tag*{$(2.11)_m$}$$

where ρ_m is independent of $\{u_0, u_1\} \in \mathbf{A}$.

The case $m = 0$ follows immediately from (2.10). Indeed since g is bounded from V_2 into H we have $\ddot{u} = f - g(u) - Au - \alpha\dot{u} \in C_b(\mathbb{R}, H)$ and the norm of u in W_1 is bounded by a constant which only depends on $\mathbf{A}$.

We assume that $(2.11)_m$ is proved for some $m \geq 0$. Let $f \in F_{m+1}$; since $F_{m+1} \subset F_m$ by $(2.11)_m$ we deduce that $u \in W_{m+1}$ and since g satisfies $(1.29)_{m+1}$ we know that

$$g(u) \in W_{m+1} \text{ and } \|g(u)\|_{m+1} \leq \rho'_m < +\infty, \tag{2.12}$$

where ρ'_m does not depend on $\{u_0, u_1\} \in \mathbf{A}$. Now it follows from (2.12) and $f \in F_{m+1}$, that h, given by (2.9), satisfies

$$h^{(j)} \in C_b(\mathbb{R}; F_{m+1-j}), \ j = 0, \dots, m,$$
$$h^{(m+2)} \in C_b(\mathbb{R}; H),$$

and by Proposition 1.1 with $k = m + 1$, we deduce $(2.11)_{m+1}$.

Now Theorem 2.1 follows from $(2.11)_m$ since from $u \in W_{m+1}$ we deduce that $\{u, \dot{u}\} \in C_b(\mathbb{R}, F_{m+2} \times F_{m+1})$ hence at time $t = 0$, $\{u_0, u_1\} \in F_{m+2} \times F_{m+1}$, and (2.7) follows. The boundedness of $\mathbb{A}$ in $F_{m+2} \times F_{m+1}$ is a consequence of the estimate in $(2.11)_m$.

□

In particular, the application of Theorem 2.1 to the nonlinear wave equation of Sec. 1.2.d. gives the

COROLLARY 2.1. *We assume that Ω is a bounded domain in $\mathbb{R}^n$, $n \leq 3$, with C^∞ boundary and that $f \in H^m(\Omega)$. The universal attractor $\mathbb{A}$ for (1.37)-$(1.6)_2$ is included and bounded in $H^{m+2}(\Omega) \times H^{m+1}(\Omega)$. When $f \in C^\infty(\overline{\Omega})$, $\mathbb{A}$ is included in $C^\infty(\overline{\Omega})^2$.*

2.2. - *Convergence to the universal attractor in stronger norms.*

According to (2.5), we know that for $\{u_0, u_1\} \in V_1 \times H$, the trajectory starting at time $t = 0$ from this point converges to $\mathbb{A}$ in the norm of $V_1 \times H$. But when f belongs to F_m, Theorem 2.1 shows that $\mathbb{A}$ is included in $F_{m+2} \times F_{m+1}$, therefore a natural question is whether the convergence to $\mathbb{A}$ of the trajectory is achieved in a norm stronger than that of $V_1 \times H$. This problem is by nature connected with the question of smoothness of the trajectory i.e. the analogue of (1.7) for the nonlinear equation (2.1)-(2.2). Let us emphazise the fact that when the semi-group $\{S(t)\}_{t \geq 0}$ possesses a smoothing effect (e.g. in the case of parabolic equations) the problem of the convergence of the trajectories with respect to stronger norms is totally different. Indeed, starting from any point the trajectory becomes regular for $t > 0$, and since it is only the long time behavior that is concerned the convergence is automatically achieved in stronger norms.

In our case, since we work with a group (i.e. the Cauchy problem is well-posed forward and backward), the smoothness of the points of the trajectory is constant in time. Hence in order to insure that the trajectory $\{u(t), \dot{u}(t)\}_{t \geq 0}$ lies in some space $F_{p+1} \times F_p$ it is necessary and sufficient that the initial conditions $\{u_0, u_1\}$ satisfy the compatibility conditions $(1.32)_p$.

a) A family of invariant nonlinear manifolds.

In this Section we study the set of initial conditions that produce regular trajectories.

Let there be given $m \geq 2$ and f and g such that

(2.13) $\quad f \in F_m$ and G_k is bounded and continuous from $F_{k+1} \times \cdots \times F_0$ into itself (see $(1.29)_k$) for $1 \leq k \leq m$.

For $\{u_0, u_1\} \in F_{m+1} \times F_m$, we introduce the finite sequence $\{u^{(j)}(0)\}_{j=0}^{j=m}$:

$$u^{(0)}(0) = u_0,\ u^{(1)}(0) = u_1,\ u^{(2)}(0) = f - g(u_0) - \alpha u_1 - \mathcal{A} u_0, \tag{2.14}$$
$$u^{(j+2)}(0) + \alpha u^{(j+1)}(0) + \mathcal{A} u^{(j+1)}(0) + \frac{d^j}{dt^j} g(u(t))\Big|_{t=0} = 0, \quad 1 \leq j \leq m-2.$$

Now for f and g satisfying (2.13), we set

(2.15) $\quad \varepsilon_m(f) = \{\ \{u_0, u_1\} \in F_{m+1} \times F_m$ such that the $\{u^{(j)}(0)\}_{j=0}^m$ computed by (2.14) satisfy $(1.32)_m\}$,

(2.16) $\quad \varepsilon_0(f) = V_1 \times H,\ \varepsilon_1(f) = V_2 \times V_1.$

PROPOSITION 2.1. *For every $m \in \mathbf{N}$, f and g satisfying* (2.13) *when* $m \geq 2$, the set *$\varepsilon_m(f)$ is a closed subset of $F_{m+1} \times F_m$ which satisfies*

$$\varepsilon_0(f) \supset \varepsilon_1(f) \supset \ldots \supset \varepsilon_m(f) \supset \ldots, \tag{2.17}$$
$$\mathbf{A} \subset \varepsilon_m(f), \tag{2.18}$$
$$S(t)\varepsilon_m(f) = \varepsilon_m(f), \quad \forall t \in \mathbb{R}. \tag{2.19}$$

REMARKS 2.2. (i) From (2.18), it follows that $\varepsilon_m(f)$ is not empty.
(ii) The $\varepsilon_m(f)$ are, in general, unbounded functional invariant sets.

□

PROOF. The fact that $\varepsilon_m(f)$ is closed is a consequence of the continuity of G_m from $F_{m+1} \times \cdots \times F_0$ into itself, (see $(1.29)_m$)). Let $\{u_0, u_1\} \in \mathbf{A}$; thanks to Theorem 2.1 we have $\{u(t), \dot{u}(t)\} = S(t)\ \{u_0, u_1\} \in C_b(\mathbb{R}; F_{m+1} \times F_m)$ and by (1.12)-(1.13), which is also valid in the nonlinear case, we deduce that $u \in W_m$. Now from the necessary part of Theorem 1.2 we deduce that the $\{u^{(j)}(0)\}_{j=0}^m$ satisfy $(1.32)_m$, hence $\{u_0, u_1\} \in \varepsilon_m(f)$. This shows (2.18). Concerning (2.19), we notice that for $m = 0$ and $m = 1$ the invariance follows from the fact that $\{S(t)\}_{t \in \mathbb{R}}$ is a group on $V_1 \times H$ and $V_2 \times V_1$. Let m be greater or equal to 2; if we take $\{u_0, u_1\} \in \varepsilon_m(f)$, by the sufficient part of Theorem 1.2, the corresponding trajectory u belongs to $W_m(0, t)$ for every $t \in \mathbb{R}$ and therefore by the necessary part $\{u(t), \dot{u}(t)\} \in \varepsilon_m(f)$:

$$S(t)\varepsilon_m(f) \subset \varepsilon_m(f), \quad t \in \mathbb{R}. \tag{2.20}$$

Applying $S(-t)$ to both sides of (2.20) we obtain (2.19). Finally (2.17) follows by construction.

b) The dynamics on the nonlinear manifolds.

PROPOSITION 2.2. *The hypotheses on f and g are those in Proposition 2.1. The group $\{S(t)\}_{t\in\mathbb{R}}$ possesses a bounded absorbing set B_m in $\varepsilon_m(f)$.*

PROOF. For $m=0$ and 1, the conclusion of this proposition is included in the assumptions we have made at the beginning of Section 2. We then proceed by induction on m. Let $m \geq 2$ and $f \in F_m$. Since Proposition 2.2 is true at rank $m-1$, there exists B_{m-1} bounded in $\varepsilon_{m-1}(f)$ i.e. in $F_m \times F_{m-1}$ such that for every bounded set B in $\varepsilon_{m-1}(f)$, there exists $T_{m-1}(B)$ such that

$$S(t)B \subset B_{m-1} \text{ for } t \geq T_{m-1}(B). \tag{2.21}$$

Now we take a bounded set B in $\varepsilon_m(f)$, since B is also bounded in $\varepsilon_{m-1}(f)$, (2.21) holds. It follows that the trajectory u belongs to a bounded set in $W_{m-1}(T_{m-1}(B),+\infty)$ and since $f \in F_{m-1}$ and $(1.29)_{m-1}$ holds,

$$f - g(u) \text{ belongs to a bounded set in } W_{m-1}(T_{m-1}(B),+\infty) \tag{2.22}$$

with a bound independent of B.

We write now (2.1) as

$$\ddot{u} + \alpha\dot{u} + Au = f - g(u),$$

and from (2.22) and the fact that α is positive, it follows that there exists $T_m(B) \geq T_{m-1}(B)$ and B_m which is bounded in $\varepsilon_m(f)$ and does not depend on B such that

$$\{u(t),\dot{u}(t)\} \in B_m, \text{ for } t \geq T_m(B).$$

Hence Proposition 2.2 is proved at rank m.

□

Now we can state the main result of this section. Before that, we denote (when $f \in F_m$)

$$d_m(y,\mathbf{A}) = \operatorname*{Min}_{a\in\mathbf{A}} \|y-a\|_{F_{m+1}\times F_m}. \tag{2.23}$$

THEOREM 2.2. *Let there be given $f \in F_m$ and assume that G_k satisfies $(1.29)_k$ for $1 \leq k \leq m$. For every bounded set B in $\varepsilon_m(f)$,*

$$\lim_{t\to+\infty} \operatorname*{Sup}_{\{u_0,u_1\}\in B} d_m(S(t)\{u_0,u_1\},\mathbf{A}) = 0, \tag{2.24}$$

i.e. $\mathbf{A}$ is the universal attractor for $S(t)$ in $\varepsilon_m(f)$.

Before giving the proof of this Theorem we recall some definitions and a general result on abstract groups.

Let $\mathcal{E}$ be a metric space and $\{S_t\}_{t\in\mathbb{R}}$ a group which acts on $\mathcal{E}$ and such that for every $t \in \mathbb{R}$, S_t is continuous on $\mathcal{E}$. We recall that the *ω-limit set* of a subset B in $\mathcal{E}$ is

$$\omega(B) = \bigcap_{s\geq 0} \overline{\bigcup_{t\geq s} S_t\, B}^{\mathcal{E}}. \tag{2.25}$$

We have the following result (see for instance [8]):

PROPOSITION 2.3. *We assume that*

(i) *S_t possesses a bounded absorbing set B_a in $\mathcal{E}$,*
(ii) *for every bounded set B in $\mathcal{E}$, there exists a compact set K in $\mathcal{E}$ such that*

$$\lim_{t\to+\infty} \operatorname{Sup}_{\Phi\in B} d(S_t\Phi, K) = 0. \tag{2.26}$$

Then $\omega(B_a)$ is the universal attractor for S_t in $\mathcal{E}$.

□

PROOF of THEOREM 2.2. Let there be given $m \geq 1$, according to Proposition 2.2, the point (i) of Proposition 2.1 is satisfied with $\mathcal{E} = \varepsilon_m(f)$ and $B_a = B_m$. We prove now the point (ii). Let there be given B a bounded set in $\varepsilon_m(f)$, and let $\{u_0, u_1\} \in B$. We set $\{u(t), \dot{u}(t)\} = S(t)\{u_0, u_1\}$; this trajectory satisfies $u \in W_m(0,+\infty)$ and by the hypotheses on g (namely $(1.29)_m$) we find that $g(u) \in W_m(0,+\infty)$ and $\|g(u)\|_m \leq r_m$. Since $f \in F_m$, we have

$$(f - g(u))^{(j)} \in C_b(\mathbb{R}_+; F_{m-j}),\ j = 0,\ldots,m,$$
$$(f - g(u))^{(m+1)} \in C_b(\mathbb{R}_+; H).$$

By reflexion around the origin $t = 0$, we can construct h with

$$\begin{aligned} &h^{(j)} \in C_b(\mathbb{R}; F_{m-j}),\ j = 0,\ldots,m,\\ &h^{(m+1)} \in C_b(\mathbb{R}; H), \end{aligned} \tag{2.27}$$

$$\begin{aligned} &|h^{(m+1)}|_{L^\infty(\mathbb{R},H)} + \operatorname{Sup}_{0\leq j\leq m} |h^{(j)}|_{L^\infty(\mathbb{R};F_{m-j})}\\ &\leq C_m(|f|_{F_m} + \|g(u)\|_m), \end{aligned} \tag{2.28}$$

and

$$h(t) = f - g(u(t)) \text{ for } t \geq 0, \tag{2.29}$$

where C_m is a constant which does not depend on f and u.

According to Proposition 1.1, with $k = m+1$, the equation

$$v + \alpha \dot{v} + Av = h,$$
$$v \in W_{m+1},$$

possesses a unique solution and

$$\|v\|_{m+1} \leq r'_m \tag{2.30}$$

where r'_m depends only on r_m, C_m and $|f|_{F_m}$ by (2.28). If we study the difference between u and v, $\delta = u - v$, according to (2.29) we have

$$\begin{aligned} &\ddot{\delta} + \alpha\dot{\delta} + A\delta = 0, \text{ for } t \geq 0, \\ &\delta(0) = u_0 - v(0), \\ &\dot{\delta}(0) = u_1 - \dot{v}(0). \end{aligned} \tag{2.31}$$

Since $\{u, \dot{u}\} \in C_b(\mathbb{R}_+; F_{m+1} \times F_m)$ and $\{v, \dot{v}\} \in C_b(\mathbb{R}_+; F_{m+1} \times F_m)$ we have $\{\delta, \dot{\delta}\} \in C_b(\mathbb{R}_+; F_{m+1} \times F_m)$ and using (2.31) and (1.48) it follows that $\{\delta, \dot{\delta}\}$ goes to $\{0, 0\}$ exponentially in $F_{m+1} \times F_m$, uniformly with respect to $\{u_0, u_1\}$ in B.

Thanks to (2.30),

$$K = \bigcup_{\substack{\{u_0, u_1\} \in B \\ t \geq 0}} \overline{\{v(t), \dot{v}(t)\}}^{F_{m+1} \times F_m}$$

is bounded in $F_{m+2} \times F_{m+1}$, it is compact in $F_{m+1} \times F_m$ and by

$$d_m(S(t)\{u_0, u_1\}, K) \leq |\{\delta(t), \dot{\delta}(t)\}|_{F_{m+1} \times F_m}$$

we deduce the point (ii) of Proposition 2.3. According to this result, $\mathbf{A}_m = \omega(B_m)$ is the universal attractor for $S(t)$ in $\varepsilon_m(f)$. Now since $f \in F_m$, we know by Theorem 2.1 that $\mathbf{A}$ is included and bounded in $F_{m+2} \times F_{m+1}$; on the other hand $S(t)\mathbf{A} = \mathbf{A}$, $\forall t \in \mathbb{R}$; it follows that $\mathbf{A} \subset \mathbf{A}_m$. Conversely $\mathbf{A}_m$ is included in $V_1 \times H$ and $S(t)\mathbf{A}_m = \mathbf{A}_m$ therefore $\mathbf{A}_m \subset \mathbf{A}$.

□

c) Applications.

In particular, the application of Theorem 2.2 to the nonlinear wave equations of Section 1.2.d gives the

COROLLARY 2.2. *Let Ω be a bounded domain in $\mathbb{R}^n$, $n \leq 3$, with C^∞ boundary and $f \in H^m(\Omega)$. For every bounded set B in $\varepsilon_m(f)$,*

$$\lim_{t \to +\infty} \operatorname{Sup}_{\{u_0, u_1\} \in B} d_{H^{m+1}(\Omega) \times H^m(\Omega)}(S(t)\{u_0, u_1\}, \mathbf{A}) = 0.$$

□

In this Section (Sec. 2) we have only considered the autonomous case. In fact Theorems 2.1 and 2.2 can be extended to the time periodic case, i.e. the case where the right-hand side of (2.1) depends on t and satisfies

$$f(t+T) = f(t), \quad \forall t \in \mathbb{R}, \tag{2.32}$$

for some positive T (a period). This case occurs naturally in physics in e.g. Sine-Gordon equation. Concerning the existence of an universal attractor and related results the reader is referred to [8]. Let us only mention that the hypotheses $f \in F_m$ in Theorems 2.1, 2.2 must be replaced by

$$f^{(j)} \in C(\mathbb{R}; F_{m-j}), \quad 0 \leq j \leq m. \tag{2.33}$$

Then the universal attractor $\mathbb{A}$ is this time invariant under the action of the discrete group $\{S(mT)\}_{m\in\mathbf{Z}}$. The conclusions are the same, under hypotheses (2.33) and $(1.29)_{1\leq l\leq m}$ we obtain that $\mathbb{A}$ is included and bounded in $F_{m+2} \times F_{m+1}$ and provided that the compatibility conditions of order m are satisfied, the convergence to the attractor is achieved with respect to the norm induced by $F_{m+1} \times F_m$.

Appendix

We aim to show that the continuity and boundedness properties $(1.27)_k$ hold in the case of the nonlinear wave equation (1.37). We are given a function $g \in C^\infty(\mathbb{R},\mathbb{R})$ and we want to prove that the mapping G_k, $k \geq 1$:

$$G_k\{u^{(0)}(0),\ldots,u^{(k+1)}(0)\} = \left\{\frac{d^j}{dt^j}\{g(u(t))\}\Big|_{t=0}\right\}_{0\leq j\leq k+1} \tag{A.1}$$

is continuous and bounded from $F_{k+1} \times \ldots \times F_0$ into itself:

$$G_k : F_{k+1} \times \ldots \times \ F_0 \supset, \ k \geq 1. \tag{A.2}$$

We recall that in the present case

$$F_k = H^k(\Omega), \tag{A.3}$$

and since Ω is a regular bounded open set in $\mathbb{R}^n$, $n = 1,2$ or 3,

$$F_k \text{ is an algebra for every } k \geq 2. \tag{A.4}$$

More generally, when Ω is a regular bounded open set in $\mathbb{R}^d$ (d arbitrary) and $h \in C^\infty(\mathbb{R},\mathbb{R})$, the composition mapping: $u \to h \ o \ u$ is continuous and bounded from $H^s(\Omega)$ into itself as soon as $s > d/2$, $s \in \mathbf{R}$. Although a direct proof when $s \in \mathbf{N}$ ($s > d/2$) can be made possible using Faà di Bruno formula (A.7) below and Sobolev-Gagliardo-Nirenberg inequalities, the general result i.e. for $s > d/2$, $s \in \mathbf{R}$ is proved in [24] using the paradifferential calculus. Therefore when $h \in C^\infty(\mathbf{R},\mathbf{R})$,

(A.5) $\qquad u \to h \circ u$ is continuous and bounded in F_k, $k \geq 2$.

In order to prove (A.2), we have to show that ($u_j \equiv u^{(j)}(0)$,

$$(\text{A.6})_j \qquad \begin{cases} \{u_0,\ldots,u_{k+1}\} \to \dfrac{d^j}{dt^j}\{g(u(t))\}\Big|_{t=0} \\ \text{is continuous and bounded from } F_{k+1}\times\ldots\times F_0 \text{ into } F_{k+1-j} \end{cases}$$

holds for $j = 0,\ldots,k+1$. We first notice that (A.5) shows $(\text{A.6})_0$.
Concerning the cases $j \geq 1$, we recall that the Faà di Bruno formula (cf. [5] p. 137) yields

$$(\text{A.7}) \qquad \frac{d^j}{dt^j}\,\{g(u(t))\}\Big|_{t=0} = \sum_{p=1}^{j}\sum_{C_1} g^{(p)}(u_0)\; u_1^{C_1}\ldots u_{j-p+1}^{C_{j-p+1}}$$

where the $C_1 \in \mathbf{N}$ are such that

$$(\text{A.8}) \qquad \sum_{\ell=1}^{j-p+1} \ell C_\ell = j;\; \sum_{\ell=1}^{j-p+1} C_\ell = p$$

and we have set all the multiplicative constants appearing in the terms in the right-hand side of (A.7), equal to 1. We have

$$u_{j-p+1} \in F_{k+p-j},$$

since $p \geq 1$, $k+p-j \geq k+1-j$ and then according to (A.4) and (A.5), regarding (A.7) we obtain $(\text{A.6})_j$ for j, $1 \leq j \leq k-1$. It remains to study the two cases $j = k$ and $j = k+1$. In the former case, (A.7) reads

$$\sum_{p=1}^{k}\sum_{C_1} g^{(p)}(u_0)\; u_1^{C_1}\ldots u_{k-p+1}^{C_{k-p+1}}.$$

The question is whether this function is continuous and bounded with values in $F_1 = H^1(\Omega)$. The terms corresponding to $p \geq 2$ are easy since the u_{k-p+1} belong to $H^p(\Omega)$ which is an algebra. It remains to study the case $p = 1$:

$$g'(u_0)u_1^{C_1}\ldots u_k^{C_k}$$

with (A.8). We have $C_k \in \{0,1\}$. When $C_k = 0$, all the terms are in $H^2(\Omega)$. When $C_k = 1$, then by (A.8), $C_1 = C_2 = \ldots = C_{k-1} = 0$ and the term reduces to $g'(u_0)u_k$ which obviously define a continuous and bounded function from $H^k(\Omega) \times H^1(\Omega)$ into $H^1(\Omega)$. We have shown $(A.6)_k$. Finally we consider the case $j = k+1$, then (A.7) reads

$$\sum_{p=1}^{k+1} \sum_{C_1} g^{(p)}(u_0)\ u_1^{C_1} \ldots u_{k+2-p}^{C_{k+2-p}}.$$

We want to check that this defines a continuous and bounded function with values in $F_0 = L^2(\Omega)$. As before the $p \geq 3$ terms are easy. When $p = 1$, we have to consider

$$g'(u_0)\ u_1^{C_1} \ldots u_{k+1}^{C_{k+1}}$$

and the worst case is again $C_{k+1} = 1 : g'(u_0)u_{k+1}$, which belongs to $L^2(\Omega)$ since $u_{k+1} \in L^2(\Omega)$ and $g'(u_0) \in H^{k+1}(\Omega) \subset H^2(\Omega) \subset L^\infty(\Omega)$. When $p = 2$, we study

$$g''(u_0)\ u_1^{C_1} \ldots u_k^{C_k},$$

with

$$\sum_{\ell=1}^{k} \ell C_\ell = k+1,\ \sum_{\ell=1}^{k} C_k = 2.$$

Here the worst case is $C_1 = C_2 = \ldots = C_{k-1} = 0,\ C_k = 2$ i.e.

$$g''(u_0)\ u_k^2.$$

But $u_k \in H^1(\Omega) \subset L^4(\Omega)$, hence $\{u_0, u_k\} \to g''(u_0)u_k^2$ is continuous and bounded from $H^k(\Omega) \times H^1(\Omega)$ into $L^2(\Omega)$.

□

Notes.

(1) The following properties are well-known in the previous example.

(2) In Section 1.2.d we give, in the particular case of nonlinear wave equations, sufficient conditions which guarantee the well-posedness. More general conditions are given in details in [8].

(3) The smoothness assumption on g can be considerably weakened. In fact it is sufficient that g satisfies $(1.29)_k$ for $1 \leq k \leq m-1$ as in Theorem 1.2.

(4) See Section 2.2.c for the time periodic case.

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