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1. - Introduction and statement of the problem

This paper aims at studying wave propagation and uniqueness of regular solutions for the system governing the motion of a nonviscous fluid filling an *unbounded* region $\Omega \subseteq \mathbb{R}^3$, whose complement $\Omega_0 = \mathbb{R}^3 - \Omega$ is thought to be occupied by a rigid body. Because of the deep difference between the compressible and incompressible cases, we have deemed it suitable to distinguish them and, accordingly, to divide the paper in two separate parts.

First, we consider a barotropic nonviscous fluid, whose motions, as is well known, obey the system [1](1)

$$\begin{cases} \rho \partial_t \mathbf{v} = -\rho (\nabla \mathbf{v}) \mathbf{v} - \nabla p + \mathbf{b} \\ \partial_t \rho = -\mathbf{v} \cdot \nabla \rho - \rho \operatorname{div} \mathbf{v}, \qquad \text{on } Q = \Omega \times (0, +\infty) \\ p = p(\rho) \end{cases} \tag{1.1}$$

where $\rho = \rho(\mathbf{x}, t)$, $\mathbf{v} = \mathbf{v}(\mathbf{x}, t)$, p are respectively the (unknown) mass density, velocity and (thermodynamical) pressure fields and $\mathbf{b} = \mathbf{b}(\rho; \mathbf{x}, t)$ is the body force (per unit volume) field.

We assume that Ω is so regular as required for the validity of the divergence theorem, that the function p: $\mathbb{R}^+ \to \mathbb{R}^+$ is twice continuously differentiable and that $\mathbf{b}$ is a continuously differentiable function of all its arguments.

We shall merely consider *classical* solutions to System (1.1), i.e. couples.

(1) Light-face letters denote scalars, bold-face letters different from $\mathbf{x}, \mathbf{y}$ and $\mathbf{o}$ denote vectors (on $\mathbf{R}^3$), while $(\mathbf{x}, \mathbf{y})$ and $\mathbf{o}$ stand respectively for generic points of $\mathbf{R}^3$ and the origin of an assigned reference frame $R = \{\mathbf{o}, e_i\}$ in which Ω_0 is at rest. The symbol ∂_t means partial differentiation with respect to time t, $\nabla\rho$ is the vector with components $\partial\rho/\partial x_i$, $\nabla\mathbf{u}$ is the second-order tensor (linear transformation from $\mathbf{R}^3$ to $\mathbf{R}^3$) with components $(\nabla\mathbf{u})_{ij} = \partial u_i/\partial x_j$ and div $\mathbf{u}$ is the scalar $\sum_i (\partial u_i/\partial x_i)$. Here all the indexes run from 1 to 3.

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$(\rho, \mathbf{v})$ such that ρ and the components of $\mathbf{v}$ are continuously differentiable on $\overline{\Omega} \times [0, +\infty)$, where $\overline{\Omega} = \Omega \cup \partial\Omega$ with $\partial\Omega$ boundary of Ω.

We are mainly interested in comparing the behaviour at each instant of two solutions $(\rho, \mathbf{v})$, $(\tilde{\rho}, \tilde{\mathbf{v}})$ to System (1.1), corresponding to the body force fields $\mathbf{b}$ and $\tilde{\mathbf{b}}$ respectively. To this aim, we write the system which is to be satisfied by the *perturbation* $(\sigma = \tilde{\rho} - \rho, \mathbf{u} = \tilde{\mathbf{v}} - \mathbf{v})$:

$$\text{(1.2)} \quad \begin{cases} \tilde{\rho}\partial_t \mathbf{u} = -\sigma\partial_t \mathbf{v} - \tilde{\rho}\,(\nabla\mathbf{u})\,\tilde{\mathbf{v}} - \sigma\,(\nabla\mathbf{v})\,\tilde{\mathbf{v}} - \rho\,(\nabla\mathbf{v})\,\mathbf{u} - \pi_\rho \nabla\rho + p'\,(\tilde{\rho})\,\nabla\sigma + \mathbf{f} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{on } Q \\ \partial_t \sigma = -\tilde{\mathbf{v}} \cdot \nabla\sigma - \mathbf{u} \cdot \nabla\rho - \sigma \text{ div } \mathbf{v} - \tilde{\rho} \text{ div } \mathbf{u} \end{cases}$$

where $\pi_\rho = p'(\rho) - p'(\tilde{\rho})$, $\mathbf{f} = \tilde{\mathbf{f}}_\sigma + \mathbf{f}_\rho$, with $\tilde{\mathbf{f}}_\sigma = \tilde{\mathbf{b}}(\tilde{\rho}; \mathbf{x}, t) - \tilde{\mathbf{b}}(\rho; \mathbf{x}, t)$ and $\mathbf{f}_p = \tilde{\mathbf{b}}(\rho; \mathbf{x}, t) - \mathbf{b}(\rho; \mathbf{x}, t)$.

The most interesting property we shall point out for a quite general class of solutions to System (1.1) is that any *perturbation* $(\mathbf{u}, \sigma)$ which is confined in a bounded region at instant $t = 0$, identically vanishes outside a suitable bounded region at each instant $t > 0$. This result will be a consequence of a general *domain of influence theorem* we shall prove for System (1.2), under the following assumption:

A smooth, positive, increasing and convex function q *on* $[0, +\infty)$ *exists such that* $\lim_{\varepsilon \to \infty} q(\varepsilon) = +\infty$ *and*

$$\text{(1.3)} \qquad \left\{ |\tilde{\mathbf{v}}| + |\sqrt{p'(\tilde{\rho})}| \right\} q'(r) \le c,$$

where c is a *reference velocity* and, $\forall\, \mathbf{x} \in \mathbb{R}^3$, $r = |\mathbf{x} - \mathbf{o}| = (\sum_i (x^i)^2)^{\frac{1}{2}}$.

REMARK 1. It should be mentioned that, in a celebrated paper of 1959 [2], as a consequence of a general uniqueness theorem for regular solutions to System (1.1) in *bounded domains*, J. Serrin proved ([2], p.280, Corollary) a result which may be viewed as concerning the propagation of perturbations. The statement of Serrin's result and the discussion of its link with our domain of influence theorem will be carried out at the end of Section 3 (Remark 5).

REMARK 2. It is worth noting that condition (1.3) involves $\tilde{\mathbf{v}}$ and $p'(\tilde{\rho})$ because we chose $(\tilde{\rho}, \tilde{\mathbf{v}})$ as the basic flow: of course, it may be replaced by the symmetric assumption

$$\text{(1.4)} \qquad \left\{ |\mathbf{v}| + |\sqrt{p'(\rho)}| \right\} q'(r) \le c,$$

which turns out to be completely equivalent to (1.3). This equivalence expresses the obvious fact that $(\rho, \mathbf{v})$ may be taken as the unperturbed motion in place of $(\tilde{\rho}, \tilde{\mathbf{v}})$. The physical situation expressed by this mathematical symmetry may be described as follows: since $\tilde{\mathbf{v}}(\mathbf{x}, t)$ [respectively $\mathbf{v}(\mathbf{x}, t)$] represents the *local velocity*(2) of the fluid in R, and $\sqrt{p'(\tilde{\rho})}$ [respectively $\sqrt{p'(\rho)}$] is the *local sound*

(2) By this phrase we mean the velocity of the particle which occupies the position $\mathbf{x}$ at instant t.

velocity[3] in a reference frame "passing through" the point $\mathbf{x}$ at instant t with velocity $\tilde{\mathbf{v}}(\mathbf{x},t)$ [respectively $\mathbf{v}(\mathbf{x},t)$], then (1.3) [respectively (1.4)] gives an upper bound on the sound velocity in R. This bound, as will be clear in the sequel, seems to be the most natural one.

For incompressibile (homogeneous) fluids, System (1.1) takes the form

$$\text{(1.5)} \qquad \begin{cases} \rho\partial_t \mathbf{v} = -\rho(\nabla \mathbf{v})\,\mathbf{v} - \nabla p + \mathbf{b} \\ \\ \operatorname{div}\,\mathbf{v} = \mathbf{0} \end{cases} \qquad \text{on } Q$$

where ρ is now a constant and p is the (unknown) dynamical pressure. Of course, from (1.5) it follows that p is defined up to an arbitrary function of time.

A *classical* solution to System (1.5) is a couple $(\mathbf{v}, p)$ satisfying (1.5) and such that p and the components of $\mathbf{v}$ are continuously differentiable on $\overline{Q}$. If $(\mathbf{v}+\mathbf{u}, p+\pi)$ is another solution to (1.5) corresponding to the body force field $\mathbf{b}+\mathbf{f}$, then the perturbation $(\mathbf{u},\pi)$ satisfies the system

$$\text{(1.6)} \qquad \begin{cases} \rho\partial_t \mathbf{u} = -\rho(\nabla \mathbf{u})(\mathbf{v}+\mathbf{u}) - \rho(\nabla \mathbf{v})\,\mathbf{u} - \nabla\pi + \mathbf{f} \\ \\ \operatorname{div}\,\mathbf{u} = 0 \end{cases} \qquad \text{on } Q$$

For System (1.6), under mild restrictions upon the behaviour of solutions at infinity, we shall prove the work and energy theorem and obtain some uniqueness results.

It should be noted that our choice of the reference frame R implies that the normal component at the points of $\partial\Omega$ of the kinetic field of the fluid must be zero. In other words, Systems (1.1), (1.5) and (1.2), (1.6) must be considered as implicitly completed by the conditions

$$\begin{cases} \mathbf{v}\cdot\mathbf{N} = 0 \\ \\ \mathbf{u}\cdot\mathbf{N} = 0 \end{cases} \qquad \text{on } \partial\Omega \times (0,+\infty)$$

respectively, where $\mathbf{N}$ is the outward unit normal to $\partial\Omega$.

The plan of the work is as follows: Section 2 is devoted to the statement and proof of our main result for the compressible case, i.e. the *domain of dependence inequality*, while in Section 3 its consequences are derived and discussed. Finally, Section 4 deals with the results for the incompressible case.

(3) This is the sound velocity at the place $\mathbf{x}$ and at time t.

2. - The domain of dependence inequality for compressible fluids

It is the purpose of this section to obtain the most expressive result of this paper on compressible nonviscous fluids, namely the so-called *domain of dependence inequality* for System (1.1). Inequalities of this type have already proved to be useful in several problems related to linear hyperbolic systems. Indeed, as we shall subsequently show, such a relation gives, among other things, the *domain of influence theorem*, whence it follows that, if the data have a compact support, then so does the solution at each instant $t > 0$.

For the sake of formal simplicity, we set

$$\eta(\mathbf{x},s) = \frac{1}{2}\left(\tilde{\rho}\mathbf{u}^2 + \tilde{\rho}^{-1}p'(\tilde{\rho})\sigma^2\right)(\mathbf{x},s), \ \forall(\mathbf{x},s)\in\overline{Q},$$

$$S(\mathbf{x},R) = \{\mathbf{y}\in\mathbb{R}^3 : |\mathbf{y}-\mathbf{x}| < R\}, \ \Omega(\mathbf{x},R) = \Omega\cap S(\mathbf{x},R),$$

$$S_R = S(\mathbf{o},R), \ \Omega_R = \Omega\cap S_R, \ \Sigma_R = \Omega\cap\partial S_R, \ \forall R > 0.$$

THEOREM 2.1 (Domain of dependence inequality). *Let* $(\tilde{\rho},\tilde{\mathbf{v}})$, $(\rho,\mathbf{v})$ *be two solutions to System* (1.1) *corresponding to the body force fields* $\tilde{\mathbf{b}}$ *and* $\mathbf{b}$ *respectively, and assume that* (1.3) *holds. Then,* $\forall(\mathbf{x}_0,t)\in Q, \forall R > 0$,

$$\int_{\Omega(\mathbf{x}_0,R)} \eta(\mathbf{x},t)\,dv \le \exp[h_R(t)]\left\{\int_{\Omega(\mathbf{x}_0,R_0)} \eta(\mathbf{x},0)\,dv + t_0\int_0^t ds \int_{\Omega(\mathbf{x}_0,R_s)} \left(\tilde{\rho}^{-1}\mathbf{f}_\rho^2\right)(\mathbf{x},s)\,dv\right\}, \tag{2.1}$$

where $R_s = q^{-1}[q(R+r_0)+c(t-s)] - r_0$, $r_0 = |\mathbf{x}_0 - \mathbf{o}|$ *and* $h_R(t)$ *is a positive increasing function of* R.

PROOF. Consider the piecewise smooth function on $\mathbb{R}^3\times[0,t]$ [2]

$$g(\mathbf{x},s) = w\left((c\delta)^{-1}\left(q(R+r_0)+c(t-s)-q(r_0+|\mathbf{x}-\mathbf{x}_0|)\right)\right), \ (\delta>0)$$

where w is a smooth and increasing function on $\mathbb{R}$, identically vanishing on $(-\infty,0]$ and equal to 1 on $[1+\infty)$. It should be observed that the support of g is the set

$$\mathcal{L} = \cup_{s=0}^{t} S(\mathbf{x}_0, R-s)$$

and that g is everywhere smooth even if ∇g is not defined along the axis $\mathbf{x}=\mathbf{x}_0$. Indeed, if δ is chosen sufficiently small, then $g\equiv 1$ in the set $\cup_{s=0}^{t} S(\mathbf{x}_0, R_{s+\delta})$. Moreover, as $\delta\to 0$, g tends boundedly to the characteristic function of $\mathcal{L}$.

Multiply both sides of $(1.2)_1$ by $g\mathbf{u}$ and integrate over $\Omega \times (0,t)$. Since, by virtue of $(1.2)_2$

$$-g\tilde{\rho}\tilde{\mathbf{v}} \bullet \nabla \mathbf{u}^2 = - \operatorname{div} \{g\tilde{\rho}\mathbf{u}^2\tilde{\mathbf{v}}\} + \tilde{\rho}\mathbf{u}^2\tilde{\mathbf{v}} \bullet \nabla g + g\mathbf{u}^2\tilde{\mathbf{v}} \bullet \nabla\tilde{\rho} + g\tilde{\rho}\mathbf{u}^2 \operatorname{div} \tilde{\mathbf{v}},$$
$$-gp'(\tilde{\rho})\mathbf{u} \bullet \nabla\sigma = - \operatorname{div} \{gp'(\tilde{\rho})\sigma\mathbf{u}\} + p'(\tilde{\rho})\sigma\mathbf{u} \bullet \nabla g + gp''(\tilde{\rho})\sigma\mathbf{u} \bullet \nabla\tilde{\rho}$$
$$+ gp'(\tilde{\rho})\sigma \operatorname{div} \mathbf{u} = - \operatorname{div} \{gp'(\tilde{\rho})\sigma\mathbf{u}\} + p'(\tilde{\rho})\sigma\mathbf{u} \bullet \nabla g + gp''(\tilde{\rho})\sigma\mathbf{u} \bullet \nabla\tilde{\rho}$$
$$- g\tilde{\rho}^{-1}p'(\tilde{\rho})\sigma(\partial_t\sigma + \tilde{\mathbf{v}} \bullet \nabla\sigma + \mathbf{u} \bullet \nabla\rho + \sigma \operatorname{div} \mathbf{v}),$$
$$-g\tilde{\rho}^{-1}p'(\tilde{\rho})\tilde{\mathbf{v}} \bullet \nabla\sigma^2 = - \operatorname{div} \{g\tilde{\rho}^{-1}p'(\tilde{\rho})\sigma^2\tilde{\mathbf{v}}\} + \tilde{\rho}^{-1}p'(\tilde{\rho})\sigma^2\tilde{\mathbf{v}} \bullet \nabla g$$
$$+ g\sigma^2\tilde{\mathbf{v}} \bullet \nabla(\tilde{\rho}^{-1}p'(\tilde{\rho})) + g\tilde{\rho}^{-1}p'(\tilde{\rho})\sigma^2 \operatorname{div} \tilde{\mathbf{v}},$$

by making use of the divergence theorem, we have

$$\begin{aligned}
\int_\Omega (g\eta)(\mathbf{x},t)\,dv &= \int_\Omega (g\eta)(\mathbf{x},0)\,dv + \int_0^t ds \int_\Omega (\eta\partial_t g)(\mathbf{x},s)\,dv \\
&- (c\delta)^{-1} \int_0^t ds \int_\Omega \left\{ w'q' \left(\frac{1}{2}\tilde{\rho}\mathbf{u}^2\tilde{\mathbf{v}} + p'(\tilde{\rho})\sigma\mathbf{u} + \right.\right. \\
&\qquad\qquad \left.\left. \frac{1}{2}\tilde{\rho}^{-1}p'(\tilde{\rho})\sigma^2\tilde{\mathbf{v}} \right) \right\} \bullet \mathbf{e}_r\ (\mathbf{x},s)\,dv \\
&- \int_0^t ds \int_\Omega \{g[\sigma\mathbf{u} \bullet \partial_t\mathbf{v} + \sigma\mathbf{u} \bullet ((\nabla\mathbf{v})\tilde{\mathbf{v}}) \\
&- \pi_\rho\mathbf{u} \bullet \nabla\rho - \mathbf{f} \bullet \mathbf{u} - p''(\tilde{\rho})\sigma\mathbf{u} \bullet \nabla\tilde{\rho} - \frac{1}{2}\sigma^2\partial_t(\tilde{\rho}^{-1}p'(\tilde{\rho})) \\
&- \frac{1}{2}\sigma^2\tilde{\mathbf{v}} \bullet \nabla(\tilde{\rho}^{-1}p'(\tilde{\rho})) - \frac{1}{2}\tilde{\rho}^{-1}p'(\tilde{\rho})\sigma\mathbf{u} \bullet \nabla\rho \\
&+ \tilde{\rho}^{-1}p'(\tilde{\rho})\sigma^2 \operatorname{div} \mathbf{v}]\}(\mathbf{x},s)\,dv.
\end{aligned} \tag{2.3}$$

By virtue of the inequality[4]

$$|p'(\tilde{\rho})\sigma\mathbf{u}| = [p'(\tilde{\rho})]^{\frac{1}{2}} \left(\tilde{\rho}^{\frac{1}{2}}|\mathbf{u}|\right) \left\{[\tilde{\rho}^{-1}p'(\tilde{\rho})]^{\frac{1}{2}}\sigma\right\}$$
$$\le \frac{1}{2}[p'(\tilde{\rho})]^{\frac{1}{2}} \{\tilde{\rho}\mathbf{u}^2 + \tilde{\rho}^{-1}p'(\tilde{\rho})\sigma^2\} = |p'(\tilde{\rho})|^{\frac{1}{2}}\eta,\text{[5]}$$

as well as of hypothesis (1.3), we see that the quantity

$$J = \eta\partial_t g + \left\{\frac{1}{2}\tilde{\rho}\mathbf{u}^2\tilde{\mathbf{v}} + p'(\tilde{\rho})\sigma\mathbf{u} + \frac{1}{2}\tilde{\rho}^{-1}p'(\tilde{\rho})\sigma^2\tilde{\mathbf{v}}\right\} \bullet \nabla g$$

(4) From now on we shall repeatedly use the arithmetic-geometric mean inequality: $2ab \le \xi^{-1}a^2 + \xi b^2$, $\forall a,b \in \mathbf{R}$, $\forall \xi > 0$.

(5) Of course, by symbol $\xi^{\frac{1}{2}}$ $(\xi > 0)$ we denote the positive square root of ξ.

is certainly not positive. Indeed, setting, $\forall \mathbf{x} \neq \mathbf{x}_0$, $\mathbf{e}_r^0 = |\mathbf{x}-\mathbf{x}_0|^{-1}(\mathbf{x}-\mathbf{x}_0)$ and noting that $\partial_t g = -\delta^{-1}w'$, $\nabla g = -(c\delta)^{-1}w'q'\mathbf{e}_r^0$,

$$\begin{aligned} J = & -\delta^{-1}w'\left\{\eta + c^{-1}q'\left(|\mathbf{x}-\mathbf{x}_0|+r_0\right)\left(\frac{1}{2}\tilde{\rho}\mathbf{u}^2\tilde{\mathbf{v}}\right.\right. \\ & \left.\left. + p'(\tilde{\rho})\sigma\mathbf{u} + \frac{1}{2}\tilde{\rho}^{-1}p'(\tilde{\rho})\sigma^2\tilde{\mathbf{v}} \bullet \mathbf{e}_r^0\right)\right\} \\ & \leq -\delta^{-1}w'\left\{\eta + c^{-1}q'(r)(|\tilde{\mathbf{v}}| + [p'(\tilde{\rho})]^{\frac{1}{2}})\eta\right\} \leq 0 \end{aligned}$$

where we have used the convexity of q.

Take now into account the inequalities

$$-\sigma\mathbf{u} \bullet \partial_t\mathbf{v} \leq |\partial_t\mathbf{v}|\,[p'(\tilde{\rho})]^{-\frac{1}{2}}\,\eta,$$

$$-\sigma\mathbf{u} \bullet [(\nabla\mathbf{v})\mathbf{v}] \leq |(\nabla\mathbf{v})\mathbf{v}|\,[p'(\tilde{\rho})]^{-\frac{1}{2}}\,\eta,$$

$$\mathbf{f}_\rho \bullet \mathbf{u} \leq \tfrac{1}{2}t_0\tilde{\rho}^{-1}\mathbf{f}_\rho^2 + \tfrac{1}{2}t_0^{-1}\tilde{\rho}\mathbf{u}^2 \leq t_0\tilde{\rho}^{-1}\mathbf{f}_\rho^2 + t_0^{-1}\eta,$$

$$\begin{aligned} \tilde{\mathbf{f}}_\sigma \bullet \mathbf{u} = \tilde{f}_{\sigma_j}u_j = \tilde{b}_j'\left(\rho+\theta_j\sigma;\mathbf{x},t\right)\sigma u_j \\ \leq \tfrac{1}{2}m(\mathbf{x},t)\,[p'(\tilde{\rho})]^{-\frac{1}{2}}\left\{\tilde{\rho}\mathbf{u}^2 + \tilde{\rho}^{-1}p'(\tilde{\rho})\sigma^2\right\} \leq m(\mathbf{x},t)\,[p'(\tilde{\rho})]^{-\frac{1}{2}}\,\eta, \end{aligned}$$

$$\begin{aligned} -\pi_\rho\mathbf{u} \bullet \nabla\rho = -p''\left(\rho+\theta\sigma\right)\sigma\mathbf{u} \bullet \nabla\rho \leq \tfrac{1}{2}\left|p''\left(\rho+\theta\sigma\right)\nabla\rho\right|[p'(\tilde{\rho})]^{-\frac{1}{2}}\left\{\tilde{\rho}\mathbf{u}^2\right. \\ \left. + \tilde{\rho}^{-1}p'(\tilde{\rho})\sigma^2\right\} \leq \left|p''\left(\rho+\theta\sigma\right)\nabla\rho\right|[p'(\tilde{\rho})]^{-\frac{1}{2}}\,\eta, \end{aligned}$$

$$\{p''(\tilde{\rho})\nabla\tilde{\rho} - \tilde{\rho}^{-1}p'(\tilde{\rho})\nabla\rho\} \bullet \sigma\mathbf{u} \leq |p''(\tilde{\rho})\nabla\tilde{\rho} - \tilde{\rho}^{-1}p'(\tilde{\rho})\nabla\rho|\,[p'(\tilde{\rho})]^{-\frac{1}{2}}\,\eta,$$

where

$$m(\mathbf{x},t) = \max_j\left\{\tilde{b}_j'(\rho+\theta_j\sigma;\mathbf{x},t)\right\}$$

and θ, $\theta_j \in (0,1)$. Furthermore, set

$$\begin{aligned} M_R(s) = T_0^{-1} + \sup_{S(\mathbf{x}_0,R_s)} & \left\{[p'(\tilde{\rho})]^{-\frac{1}{2}}\left(|\partial_t\mathbf{v}| + m(\mathbf{x},t) + |(\nabla\mathbf{v})\mathbf{v}|\right.\right. \\ & + \left.|p''(\rho+\theta\sigma)\nabla\rho| + |p''(\tilde{\rho})\nabla\tilde{\rho} - \tilde{\rho}^{-1}p'(\tilde{\rho})\nabla\rho|\right) \\ & + \tilde{\rho}\,[p'(\tilde{\rho})]^{-1}\left(|\partial_t(\tilde{\rho}^{-1}p'(\tilde{\rho}))| + |\tilde{\mathbf{v}} \bullet \nabla(\tilde{\rho}^{-1}p'(\tilde{\rho}))|\right) \\ & \left. + |\operatorname{div}\tilde{\mathbf{v}}| + |\operatorname{div}\mathbf{v}|\right\}. \end{aligned}$$

Then (2.3) yields

$$\begin{aligned} \int_\Omega (g\eta)(\mathbf{x},t)dv \leq \int_\Omega (g\eta)(\mathbf{x},0)dv + \int_0^t M_R(s)ds\int_\Omega (g\eta)(\mathbf{x},s)dv \\ + t_0\int_0^t ds\int_\Omega (g\tilde{\rho}^{-1}\mathbf{f}_\rho^2)(\mathbf{x},s)dv. \end{aligned}$$

So, putting

$$h_R(t) = \int_0^t M_R(s)ds$$

and using Grönwall's lemma, it follows that

$$\int_\Omega (g\eta)(\mathbf{x},t)dv \le \exp[h_R(t)] \left\{ \int_\Omega (g\eta)(\mathbf{x},0)dv + \int_0^t ds \int_\Omega (g\tilde{\rho}^{-1}\mathbf{f}_\rho^2)(\mathbf{x},s)dv \right\} \tag{2.4}$$

Thanks to the properties of the function g, (2.4) may be rewritten as

$$\int_{\Omega_R} (g\eta)(\mathbf{x},t)dv \le \exp[h_R(t)] \left\{ \int_{\Omega_{R_0}} (g\eta)(\mathbf{x},0)dv + \int_0^t ds \int_{\Omega_{R_S}} (g\tilde{\rho}^{-1}\mathbf{f}_\rho^2)(\mathbf{x},s)dv \right\},$$

so that, since $g\eta \le \eta$ for any $\delta > 0$, letting $\delta \to 0$ and making use of Lebesgue's dominated convergence theorem lead to the desired result.

3. - Some results in the compressible case

Let $(\tilde{\rho},\tilde{\mathbf{v}})$, $(\rho,\mathbf{v})$ be two solutions to System (1.1) corresponding to the body force fields $\tilde{\mathbf{b}}$ and $\mathbf{b}$ respectively, and let $\Omega^*(t)$ be the set of the points $\mathbf{x} \in \Omega$ such that

$$\exists s \in [0,t] : \eta(\mathbf{x},0) \ne 0 \text{ or } \mathbf{f}_\rho(\mathbf{x},s) \ne \mathbf{0}.$$

We define the *domain of influence of the initial perturbation* $(\sigma(\mathbf{x},0),\ \mathbf{u}(\mathbf{x},0))$ *and the difference body force field* $\tilde{\mathbf{b}}(\rho;\mathbf{x},t) - \mathbf{b}(\rho;\mathbf{x},t)$ *at instant* t, as the set

$$\Omega_q(t) = \left\{\mathbf{x}_0 \in \Omega : \Omega^*(t) \cap S(\mathbf{x}_0, q^{-1}[q(r_0) + ct] - r_0) \ne \emptyset\right\}.$$

The domain of dependence inequality enables us to prove the following

THEOREM 3.1 (Domain of influence theorem). *Let the assumptions of Theorem* 2.1 *be met. Then*

$$\mathbf{u} = \mathbf{0}, \quad \sigma = 0 \ \text{ on } \ \left\{\overline{\Omega - \overline{\Omega_q(t)}}\right\} \times [0,t]. \tag{3.1}$$

PROOF. Let $(\mathbf{x}_0, \tau) \in \{\Omega - \overline{\Omega_q(t)}\} \times (0,t)$. Apply (2.1) with $t = \tau$ and $R = q^{-1}[q(r_0) + c(t-\tau)] - r_0$. Then

$$\int_{\Omega(\mathbf{x}_0,R)} \eta(\mathbf{x},\tau)dv \le \exp[h_R(\tau)] \left\{ \int_{\Omega(\mathbf{x}_0,q^{-1}[q(r_0)+ct]-r_0)} \eta(\mathbf{x},0)dv \right.$$
$$\left. + t_0 \int_0^t ds \int_{\Omega(\mathbf{x}_0,q^{-1}[q(r_0)+c(t-s)]-r_0)} \left(\tilde{\rho}^{-1}\mathbf{f}_\rho^2\right)(\mathbf{x},s)dv \right\}. \tag{3.2}$$

Since the integrals at RHS of (3.2) vanish and $\eta(\mathbf{x},\tau)$ is continuous and nonnegative, we have that $\eta(\mathbf{x},\tau) = 0$ on $\Omega(\mathbf{x}_0, R)$. On the other hand, since $\mathbf{x}_0$ and τ are arbitrarily chosen in $\{\Omega - \Omega_q(t)\}$ and in $(0,t)$ respectively, (3.1) immediately follows.

REMARK 3. It is worth noting explicitly that theorem 3.1 becomes particularly meaningful when $\sigma(\mathbf{x},0)$, $\mathbf{u}(\mathbf{x},0)$ and $\mathbf{f}_\rho(\mathbf{x},t)$ have compact supports. Indeed, in this case it assures that $\sigma(\mathbf{x},t)$ and $\mathbf{u}(\mathbf{x},t)$ have compact supports at each instant $t > 0$.

Of course, Theorem 3.1 implies the following.

THEOREM 3.2 (Uniqueness theorem). *Let* $(\tilde{\rho},\tilde{\mathbf{v}})$ *and* $(\rho,\mathbf{v})$ *be two solutions to system* (1.1) *corresponding to the same body force field. If*

$$\tilde{\mathbf{v}} = \mathbf{v}, \quad \tilde{\rho} = \rho \quad \text{on } \Omega \times \{0\},$$

then

$$\tilde{\mathbf{v}} = \mathbf{v}, \quad \tilde{\rho} = \rho \quad \text{on } \overline{Q}.$$

REMARK 4. The domain of influence theorem (Theorem 3.1) suggests to us that hypothesis (1.3) may be considered as the best possible. Indeed the requirement that $q(r)$ diverges at infinity is the sole condition which really bounds the choice of the function $q(r)$. Dropping this assumption would allow the domain of influence of data to invade the whole space in a finite time, since a finite instant τ would certainly exist such that $\lim_{t\to\tau} q^{-1}[q(r_0)+ct] = +\infty$. On the other hand, if this assumption were not satisfied, then a signal propagating with speed $c[q'(r)]^{-1}$ would reach any point of the space at a time t which may be at most equal to $\tau = \lim_{r\to\infty}\{q(r) - q(r_0)\} < +\infty$. This would exclude the possibility of wave propagation in the whole interval $[0,+\infty)$, and it is a matter of doubt whether the mathematical model of compressible nonviscous fluids could be applied to physical problems related to this phenomenon.

Finally, in order to confirm our conjecture, we may recall an analogous feature of the evolution of an unbounded elastic continuum: denoting by A the acoustic tensor [4], if $(|\mathbf{A}|)^{\frac{1}{2}}$, which for solids has the same meaning that $|p'(\rho)|^{\frac{1}{2}}$ has for fluids, grows at infinity more rapidly than $c[q'(r)]^{-1}$ does (with $q(r)$ as in (1.3)), then all the classical properties of the solutions to the system of linear elastodynamics (e.g. uniqueness, wave propagation, etc. ...) are lost [5,6](6) .

REMARK 5. We are now in a position to recall Serrin's result quoted in Remark 1, and to point out its links with ours. To this aim, let Ω be bounded and let $f(\mathbf{x},t)$ be a continuously differentiable function on Q. Assume that

a) the equation

$$f(\mathbf{x},t) = k$$

defines a *closed real surface*(7) $S_k \subset Q$ at least for some $k \in \mathbb{R}$;

b) if $(\tilde{\rho},\tilde{\mathbf{v}})$ is a solution to System (1.1), with $\mathbf{b} = \rho\mathbf{f}(\mathbf{x},t)$, the set

$$\mathcal{I} = \left\{(\mathbf{x},t) \in Q : \tilde{\mathbf{v}}(\mathbf{x},t)\cdot\mathbf{m}(\mathbf{x},t) - G(\mathbf{x},t) \geq \sqrt{p'(\tilde{\rho}(\mathbf{x},t))}\right\}$$

where, $\forall(\mathbf{x},t) \in Q$, $\mathbf{m}(\mathbf{x},t) = |\nabla f|^{-1}\nabla f$ and $G(\mathbf{x},t) = \partial_t f$, is not empty.

Denote by R_k the (space-time) region bounded by S_k. The Corollary to Theorem 2 of [2] states that, if $\overline{\mathrm{R}}_k \subset \mathcal{I}^0$, and $(\rho,\mathbf{v})$ is another solution to System (1.1) (with $\mathbf{b} = \rho\mathbf{f}$) such that

$$\mathbf{v} = \tilde{\mathbf{v}}, \quad \rho = \tilde{\rho} \quad \text{on } R_k \cap (\Omega\times\{0\})$$

then

$$\mathbf{v} = \tilde{\mathbf{v}} \quad \rho = \tilde{\rho} \quad \text{on } R_k.$$

The region R_k is called by Serrin *domain of determinacy* for the region $\mathrm{R}_k \cap (\Omega\times\{0\})$. It is obvious that this result is related to the propagation of perturbations in the fluid in that it asserts that any perturbation initially confined in $\Omega - (\mathrm{R}_k \cap (\Omega\times\{0\}))$, is certainly confined in $\Omega - (\mathrm{R}_k \cap (\Omega\times\{t\}))$ at each instant t.

We observe that the above result is easily obtained as a consequence of (2.1) with $q(\xi) = \xi$, by considering the region R_k as a join of cones of axes $\mathbf{x} = \mathbf{x}_0$ ($\mathbf{x}_0 \in \mathrm{R}_k \cap (\Omega\times\{0\})$). It should be stressed that condition (1.3) assures that, if $G = -c$ and $\mathbf{m} = \mathbf{e}_r^0$, then $\mathcal{I} = Q$, so that we are allowed to apply (2.1) to the cone of axis $\mathbf{x}_0$, for any $\mathbf{x}_0 \in \Omega$.

As a further simple consequence of the domain of dependence inequality, we have:

(6) Nevertheless, it should be observed that all the results stated above hold unchanged in any interval $[0,T]$, with $T<\tau$.

(7) The assumption that S_k is closed is made only to avoid formal complications. Indeed, in Serrin's paper [2] it is sufficient that $S_k \cap Q \neq \emptyset$.

THEOREM 3.3. *Let the assumptions of Theorem* 2.1 *be satisfied and assume that*

$$\lim_{R\to+\infty} h_R(t) = h(t) < +\infty, \qquad \forall t \in [0,+\infty) \tag{3.3}$$

and

$$\int_\Omega \eta(\mathbf{x},0)\,dv < +\infty, \qquad \int_\Omega \left(\tilde{\rho}^{-1}\mathbf{f}^2\right)(\mathbf{x},t)\,dv < +\infty \quad \forall t \in [0,+\infty) \tag{3.4}$$

then

$$\int_\Omega \eta(\mathbf{x},t)dv \le \exp[h(t)]\left\{\int_\Omega \eta(\mathbf{x},0)dv + t_0\int_0^t ds\int_\Omega (\tilde{\rho}^{-1}\mathbf{f}_\rho^2)(\mathbf{x},s)dv\right\}. \tag{3.5}$$

PROOF. (3.5) is immediately obtained by letting $R \to +\infty$ in (2.1) and bearing in mind (3.4).

We conclude this section by pointing out that relation (3.5) may be usefully applied to obtain continuous data dependence results for solutions to System (1.1) satisfying (3.3).

4. - Theorems concerning the incompressible case

This last section is devoted to a discussion about some classical theorems concerning regular solutions to System (1.5). We start with establishing a basic *a priori* estimate for solutions to System (1.6).

THEOREM 4.1. *Let* $(\mathbf{v},p)$ *and* $(\mathbf{v}+\mathbf{u},p+\pi)$ *be two solutions to System* (1.5) *corresponding to the body force fields* $\mathbf{b}$ *and* $\mathbf{b}+\mathbf{f}$ *respectively, and assume that a smooth, positive and increasing function* $q(r)$ *exists such that* $\lim_{r\to+\infty} q(r) = +\infty$ *and*

$$\begin{aligned}
&\forall t > 0, \exists m_1(t), m_2(t) > 0 : q'(r)|\mathbf{v}+\mathbf{u}| \le m_1(t), |\nabla\mathbf{v}| \\
&\qquad\qquad \le \frac{1}{2}m_2(t) \text{ on } \Omega\times\{t\}, \\
&\lim_{r\to+\infty} r^{n-1}q'(r)\pi^2 = 0, \\
&\int_\Omega \rho\mathbf{u}^2(\mathbf{x},0)dv < +\infty, \int_\Omega \mathbf{f}^2(\mathbf{x},t)dv < +\infty, \ t\in[0,+\infty).
\end{aligned} \tag{4.1}$$

Then $\exists k(t) > 0$ *such that,* $\forall t \geq 0$,

$$\int_\Omega \rho \mathbf{u}^2(\mathbf{x},t)dv \leq \exp[k(t)] \left\{ \int_\Omega \rho \mathbf{u}^2(\mathbf{x},0)dv + t_0 \rho^{-1} \int_0^t ds \int_\Omega \mathbf{f}^2(\mathbf{x},s)dv \right\}. \tag{4.2}$$

PROOF. Multiply both sides of $(1.6)_1$ by $g\mathbf{u}$, where g is given by (2.2) but $\mathbf{x}_0$ is fixed once and for all as the origin of $\mathbb{R}^3$, and integrate over $\Omega \times (0,t)$. Then, an integration by parts gives

$$\begin{aligned} \frac{1}{2}\int_\Omega \left(g\rho\mathbf{u}^2\right)(\mathbf{x},t)dv = {} & \frac{1}{2}\int_\Omega \left(g\rho\mathbf{u}^2\right)(\mathbf{x},0)dv + (2\delta)^{-1}\int_0^t ds \int_\Omega \left(w'\rho\mathbf{u}^2\right)(\mathbf{x},s)dv \\ & + (2c\delta)^{-1}\int_0^t ds \int_\Omega \left(\rho q' w' \mathbf{u}^2(\mathbf{v}+\mathbf{u}) \bullet \mathbf{e}_r\right)(\mathbf{x},s)dv \\ & - \int_0^t ds \int_\Omega \{g\rho\mathbf{u} \bullet [(\nabla\mathbf{u})\mathbf{u}]\}(\mathbf{x},s)dv \\ & + (c\delta)^{-1}\int_0^t ds \int_\Omega (q'w'\pi\mathbf{u} \bullet \mathbf{e}_r)(\mathbf{x},s)dv \\ & + \int_0^t ds \int_\Omega (g\mathbf{f} \bullet \mathbf{u})(\mathbf{x},s)dv. \end{aligned} \tag{4.3}$$

By means of the inequalities:

$$(c\delta)^{-1} q'w'\pi\mathbf{u} \bullet \mathbf{e}_r \leq (2c\delta)^{-1} w' \left\{ (\xi\rho)^{-1} [q'(r)]^2 \pi^2 + \xi\rho C^2 \right\}, (\xi > 0)$$

$$\mathbf{f} \bullet \mathbf{u} \leq \frac{1}{2}\left(t_0^{-1}\rho\mathbf{u}^2 + t_0\rho^{-1}\mathbf{f}^2\right),$$

(4.3) implies:

$$\begin{aligned} \int_\Omega \left(g\rho\mathbf{u}^2\right)(\mathbf{x},t)\,dv \leq {} & \int_\Omega \left(g\rho\mathbf{u}^2\right)(\mathbf{x},0)\,dv \\ & + \rho^{-1}\int_0^t \left\{c^{-1}[m_1(s)+\xi]-1\right\} ds \int_\Omega \rho\mathbf{u}^2\,(\mathbf{x},s)\,dv \\ & + \int_0^t \left\{m_2(s)+t_0^{-1}\right\} ds \int_\Omega \left(g\rho\mathbf{u}^2\right)(\mathbf{x},s)\,dv \\ & + (c\delta\xi\rho)^{-1}\int_0^t ds \int_\Omega \left(w'q'^2\pi^2\right)(\mathbf{x},s)\,dv + t_0\rho^{-1}\int_0^t ds \int_\Omega \mathbf{f}^2(\mathbf{x},s)dv. \end{aligned}$$

We now choose c and ξ such that $\sup_{(0,t)}\{m_1(s)\} + \xi < c$, so that

$$\int_\Omega \left(g\rho\mathbf{u}^2\right)(\mathbf{x},t)dv \le \int_\Omega \left(g\rho\mathbf{u}^2\right)(\mathbf{x},0)dv + \int_0^t \left\{m_2(s) + t_0^{-1}\right\} ds \int_\Omega \left(g\rho\mathbf{u}^2\right)(\mathbf{x},s)dv$$
$$+ (c\delta\xi\rho)^{-1} \int_0^t ds \left(w' q'^2 \pi^2\right)(\mathbf{x},s)dv + t_0\rho^{-1} \int_0^t ds \int_\Omega \mathbf{f}^2(\mathbf{x},s)dv,$$

whence, using Grönwall's lemma and setting $k(t) = \int_0^t \{m_2(s) + t_0^{-1}\}ds$, it follows that

$$(4.4)\quad \int_\Omega \left(g\rho\mathbf{u}^2\right)(\mathbf{x},t)dv \le \exp\left[k(t)\right] \left\{ \int_\Omega \left(g\rho\mathbf{u}^2\right)(\mathbf{x},0)dv \right.$$
$$\left. + t_0\rho^{-1} \int_0^t ds \int_\Omega \mathbf{f}^2(\mathbf{x},s)dv + (c\delta\xi\rho)^{-1} \int_0^t ds \int_\Omega \left(w' q'^2 \pi^2\right)(\mathbf{x},s)dv. \right.$$

Observe now that, setting $\Omega_{s,\delta} = \Omega \cap (S_{R_s} - S_{R_s+\delta})$,

$$(c\delta)^{-1} \int_\Omega w' q'^2 \pi^2 dv = - \int_{\Omega_{s,\delta}} q'\pi^2 \nabla g \cdot \mathbf{e}_r dv = - \int_{\Omega_{s,\delta}} \operatorname{div}\left\{q'\pi^2 g\mathbf{e}_r\right\} dv$$
$$+ \int_{\Omega_{s,\delta}} g \operatorname{div}\left\{q'\pi^2 \mathbf{e}_r\right\} dv = \int_{\Sigma_{R_s+\delta}} q'\pi^2 da + \int_{\Omega_{s,\delta}} g \operatorname{div}\left\{q'\pi^2 \mathbf{e}_r\right\} dv.$$

As a consequence, since g div $\{q'\pi^2\mathbf{e}_r\}$ is regular,

$$\lim_{\delta\to 0} \left\{ (c\delta)^{-1} \int_\Omega w' q'^2 \pi^2 dv \right\} = \int_{\Sigma_{R_s}} q'\pi^2 da.$$

Thus, letting $\delta \to 0$ in (4.4) yields

$$\int_{\Omega_R} \rho\mathbf{u}^2(\mathbf{x},t)\, dv \le \exp\left[k(t)\right] \left\{ \int_{\Omega_{R_0}} \rho\mathbf{u}^2(\mathbf{x},0)dv \right.$$
$$\left. + \; t_0\rho^{-1} \int_0^t ds \int_{\Omega_{R_s}} \mathbf{f}^2(\mathbf{x},s)\, dv + (\xi\rho)^{-1} \int_0^t ds \int_{\Sigma_{R_s}} \left(q'\pi^2\right)(\mathbf{x},s)\, da \right\}.$$

Hence (4.2) follows by letting $R \to +\infty$ and taking into account $(4.1)_{2,3}$.

As an immediate consequence of Theorem 4.1, we have the following uniqueness theorem.

THEOREM 4.2 *let* $(\mathbf{v}, p)$, $(\mathbf{v}+\mathbf{u}, p+\pi)$ *be two solutions to System* (1.5) *such that* (4.1) *holds. If* $\mathbf{u} = \mathbf{0}$ *on* $\overline{\Omega} \times \{0\}$, *then* $\mathbf{u} = \mathbf{0}$ *on* $\overline{Q}$.

It should be pointed out that the above uniqueness theorem is proved without growth-at-infinity assumptions on $\nabla \mathbf{u}$ and that hypotheses on $\mathbf{u}$, $\mathbf{v}$ and π appear sufficiently weak. Indeed, choosing $g = \log(1 + \ell^{-1} r)$, conditions $(4.1)_{1,2}$ read

$$|\mathbf{v} + \tilde{u}| \leq m_1(t)\left(1 + \ell^{-1} r\right),$$

$$\lim_{r \to +\infty} r\pi^2 = 0,$$

where ℓ is a reference length.

On the other hand, for homogeneous data, System (1.5) admits nontrivial solutions when the velocity is supposed to be bounded and the pressure is allowed to grow as the distance r does. As an example of such nontrivial solutions on $\Omega = \mathbb{R}^3$ corresponding to vanishing initial data and body forces, we may exhibit the couple $(\mathbf{v} = t\Sigma_i \mathbf{e}_i, p = \Sigma_i x_i)$ [7].

By starting from Theorem 4.1, one may easily obtain a result of continuous dependence upon data. Nevertheless, we shall only prove a result concerning the stability of the rest solution. To this aim, we recall that the null solution $(\mathbf{v} = \mathbf{0}, p = p(t))$ of System (1.5) is *energy stable* with respect to perturbations $(\mathbf{u}, \pi)$ belonging to a class $\mathcal{Q}$(8) if

$$\forall \varepsilon > 0,\ \exists \delta > 0 : \int_\Omega \mathbf{u}^2(\mathbf{x}, 0) dv < \delta \Longrightarrow \int_\Omega \mathbf{u}^2(\mathbf{x}, t) dv < \varepsilon,\ \ \forall t > 0,\ \forall \mathbf{u} \in \mathcal{Q}.$$

The following theorem holds.

THEOREM 4.3. *The rest solution* $(\mathbf{0}, p(t))$ *of System* (1.5) *is energy stable with respect to perturbations* $(\mathbf{u}, \pi)$ *such that* (4.1) *holds.*

PROOF. By tracing the steps in the proof of Theorem 4.1, and bearing in mind that now $\mathbf{v} = \mathbf{0}$, we are led to

$$\int_\Omega \left(g\mathbf{u}^2\right) dv \leq \int_\Omega \left(g\mathbf{u}^2\right)(\mathbf{x}, 0)\, dv + \left(c\xi\rho^2\delta\right)^{-1} \int_0^t ds \int_\Omega \left(w' q'^2 \pi^2\right)(\mathbf{x}, s)\, dv$$

(8) Of course, we are referring to solutions $(\mathbf{u}, \pi)$ to System (1.5) corresponding to $\mathbf{f} \equiv \mathbf{0}$.

Letting $\delta \to 0$ and $R \to +\infty$, the above relation yields

$$\int_\Omega \mathbf{u}^2 (\mathbf{x}, t)\, dv \leq \int_\Omega \mathbf{u}^2 (\mathbf{x}, 0)\, dv$$

which proves the theorem.

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