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GIOVANNI ALESSANDRINI

Introduction

In this paper we consider solutions u to the two-dimensional elliptic equation

$$\sum_{i,j=1}^{2} a_{ij} u_{x_i x_j} + \sum_{i=1}^{2} b_i u_{x_i} = 0 \text{ in } \Omega, \tag{1}$$

where Ω is a bounded simply connected domain of $\mathbb{R}^2$.

It was proven by Hartman and Wintner [7] that if the coefficients a_{ij}, b_i are sufficiently smooth, then the zeros of the gradient of u (critical points) are isolated and have a finite integral multiplicity.

Here we consider the solution u of (1) subject to the Dirichlet condition

$$u = g \text{ on } \partial\Omega \tag{2}$$

and we are interested in evaluating how many are the interior critical points and how large are the respective multiplicities.

The main result of this paper, see *Section 1*, states the following. If the set of points of relative maximum of g on $\partial\Omega$ is made of N connected components, then the interior critical points of u are finite in number, and, denoting by $m_1, \ldots, m_K$ the respective multiplicities, the following estimate holds

$$\sum_{i=1}^{K} m_i \leq N - 1. \tag{3}$$

Moreover, see *Section 2*, if $\partial\Omega$ and g are sufficiently smooth a lower bound on $|Du|$ can be obtained. Namely, for every $\Omega' \subset\subset \Omega$ there exists a positive constant C depending only on Ω, Ω', on the coefficients in (1) and on g, such

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that, for every $x \in \Omega'$,

$$|Du(x)| \geq C \prod_{i=1}^{K} |x - x_i|^{m_i}, \tag{4}$$

where $x_1, \ldots, x_K$ are the interior critical points of u and $m_1, \ldots, m_K$ are the respective multiplicities.

In *Section 3* we consider the case when g has relative maxima and minima at two levels only and it has non-zero tangential derivative out of the extremal points. With this additional assumption the equality sign hold in (3), that is the inequality (3) is the best possible.

Another consequence of such an assumption is that the estimate (4) holds up to the boundary.

In *Section 4* we also treat the case when no smoothness assumption on the coefficients is made. In such a case the estimate (3) does not necessarily hold. However we prove that

$$K \leq N - 1, \tag{5}$$

where K stands for the number of critical points, and N is, as above, the number of maxima of g on Ω.

Results like those of Sections 1 and 2 were originally proven by the author for the isotropic divergence structure equation

$$\operatorname{div}\,(a \text{ grad } u) = 0,$$

and were applied to an inverse problem of identification in [2].

Basic assumptions and definitions

We assume that the coefficients a_{ij} satisfy: $a_{ij} = a_{ji}$, and a uniform ellipticity condition:

$$\lambda^{-1}|\xi|^2 \leq \sum_{i,j=1}^{2} a_{ij}(x)\xi_i\xi_j \leq \lambda|\xi|^2, \text{ for every } x \in \Omega, \xi \in \mathbb{R}^2, \tag{6a}$$

furthemore it is convenient to impose the following normalization condition:

$$a_{11}(x)a_{22}(x) - a_{12}^2(x) = 1, \text{ for every } x \in \Omega. \tag{6b}$$

For the sake of brevity we will use the following convention. Given a continuous function f on a set $S \subset \mathbb{R}^2$, we will refer to the number of maxima (minima) of f as the number of connected components of the set of points of relative maximum (minimum) of f in S. Note that if S is a simple closed curve, then f has the same number of maxima and minima.

We assume that $\Omega \subset \mathbb{R}^2$ is the interior of a simple closed curve. We will denote:

$$\operatorname{diam}\Omega = \text{ diameter of } \Omega; \quad |\Omega| = \text{ measure of } \Omega;$$
$$\Omega_d = \{x \in \Omega |\ \operatorname{dist}(x, \partial\Omega) > \mathrm{d}\}.$$

We will denote by: $B_r(x)$, the disk of radius r centered at x.

1. - Estimate on the multiplicities of interior critical points.

We start stating a theorem which can be found in Hartman and Wintner [7].

THEOREM (H. - W.) *Let $u \in W^{2,2}_{\mathrm{loc}}(\Omega)$ be a non constant solution of* (1), *where $a_{ij} \in C^1(\Omega)$, $b_i \in C(\Omega)$, $i, j = 1, 2$.*

For every $x^0 \in \Omega$ there exist an integer $n \geq 1$ and a homogeneous harmonic polynomial H_n, of degree n, such that u satisfies, as $x \to x^0$,

$$\left\{\begin{array}{l} u(x) = u(x^0) + H_n(J(x - x^0)) + 0(|x - x^0|^n), \\ Du(x) = DH_n(J(x - x^0)) + 0(|x - x^0|^{n-1}). \end{array}\right. \tag{1.1}$$

Here J is the 2×2 matrix defined as follows:

$$J = \sqrt{\begin{pmatrix} a_{11}(x^0) & a_{12}(x^0) \\ a_{12}(x^0) & a_{22}(x^0) \end{pmatrix}^{-1}}$$

REMARK 1.1. The smoothness assumptions on the coefficients a_{ij}, b_i, could be slightly weakened. For instance the above Theorem and, consequently, all the results of Sections 1 to 3, are valid provided the a_{ij} 's are Lipschitz continuous and the b_i 's are L^∞. In such a case, a proof of Theorem (H. - W.) still can be obtained adapting the method developed in [9].

See Section 4 for counter-examples in the case $a_{ij} \in L^\infty(\Omega)$ or $a_{ij} \in C(\Omega)$.

REMARK 1.2. Let us stress some useful consequences of the above Theorem.

(I) The interior critical points of u are isolated.

(II) Every interior critical point x^0 of u has a finite multiplicity, that is, for every x in a neighbourhood of x^0,

$$c_1|x - x^0|^m \leq |Du(x)| \leq c_2|x - x^0|^m,$$

where c_1, c_2 are positive constants, $m = n - 1$ and n is the integer appearing in Theorem (H. - W.).

(III) If x^0 is an interior critical point of multiplicity m, then, in a neighbourhood of x^0, the level line $\{x \in \Omega | u(x) = u(x^0)\}$ is made of $m + 1$ simple arcs interecting at x^0.

THEOREM 1.1 *Let $a_{ij} \in C^1(\Omega)$, $b_i \in C(\Omega)$, $i, j = 1, 2$ and let $g \in C(\overline{\Omega})$. Let $u \in W^2_{\mathrm{loc}}(\Omega) \cap C(\overline{\Omega})$ satisfy* (1), (2).

If $g_{|\partial\Omega}$ has N maxima (and N minima), then the interior critical points of u are finite in number and, denoting by $m_1, \ldots, m_K$ their multiplicities, the following estimate holds

$$\sum_{i=1}^{K} m_i \leq N - 1. \tag{1.2}$$

REMARK 1.3 In the above Theorem, the existence of the solution u of (1), (2) in the function space $W^{2,2}_{\mathrm{loc}}(\Omega) \cap C(\overline{\Omega})$, plays the role of an hypothesis. As is well known, a sufficient condition for such a u to exist, is that $\partial\Omega$ satisfies some smoothness assumption, for instance: Lipschitz smoothness suffices.

REMARK 1.4 It is worthwile to stress the analogy between the above Theorem and a result due to Radò [12] with states that if u is a non-constant harmonic function in a disk such that $|Du(x^0)| = 0$ for some x^0 in the interior, then $u = u(x^0)$ on at least four distinct points of the boundary.

We premise the proof with three lemmas.

LEMMA 1.1 *Let the hypotheses of Theorem 1.2 be satisfied. The set of interior critical points of u is finite.*

PROOF. By contradiction, let us assume that u has an infinite number of interior critical points. We may distinguish two cases.

(i) There exists a critical point $x^0 \in \Omega$ such that an infinite number of critical points of u is contained in the level line $\{x \in \Omega | u(x) = u(x^0)\}$.

(ii) There exist a sequence $\{x_n\}$ of critical points in Ω, in such that for every $n, m, n \neq m, u(x_n) \neq u(x_m)$.

Case (i). Note that $\Omega \setminus \{x \in \Omega | u(x) = u(x^0)\}$ is made of an infinite number of connected components $A_1, A_2, \ldots, A_n, \ldots$ such that, for every i, on $\partial A_i \cap \Omega$, $u = u(x^0)$ and, on A_i, either $u > u(x^0)$ or $u < u(x^0)$.

Case (ii). By induction we may find a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and two sequences of non-empty open subsets of Ω, $\{C_k\}$, $\{B_k\}$ having the following properties

(a) $C_k \supset C_{k+1}$, for every $k = 1, 2, \ldots,$

(b) $u_{|\partial C_k \cap \Omega} = u(x_{n_k})$, for every $k = 1, 2, \ldots$, and: $x_{n_j} \in C_k$, for every $j \geq k + 1$.

(c) $B_k \subset C_k \setminus C_{k+1}$, $u_{|\partial B_k \cap \Omega} = u(x_{n_{k+1}})$; and: $u_{|B_k} > u(x_{n_{k+1}})$, or: $u_{|B_k} < u(x_{n_{k+1}})$; for every $k = 1, 2, \ldots$.

Therefore, in both cases there exists an infinite sequence, $\{A_n\}$ or $\{B_k\}$, of disjoint open subsets of Ω each containing a point of relative maximum or minimum for u. Thus, by the maximum principle, $u_{|\partial\Omega} = g$ has an infinite number of maxima and minima.

LEMMA 1.2 *Let the hypotheses of Theorem 1.2 be satisfied. Let us assume that u has a unique interior critical point x^0, and let its multiplicity be m.*

The number of connected components of the level set $\{x \in \Omega | u(x) > u(x^0)\}$ *is at least* $m + 1$.

PROOF. Straightforward. it suffices to recall (III) and to note that, by the maximum principle, the level lines of u cannot contain closed curves.

LEMMA 1.3 *Let the hypotheses of Theorem 1.2 be satisfied. Let* $x_1, \dots, x_K$ *and* $m_1, \dots, m_K$, *be the interior critical points of* u *and their respective multiplicities. Let us assume that all the points* $x_1, \dots, x_K$ *belong to the same connected component of the level line* $\{x \in \Omega | u(x) = t\}$ *for some* $t \in \mathbb{R}$.

The level set $\{x \in \Omega | u(x) > t\}$ *has at least* $\sum_{i=1}^{K} m_i + 1$ *connected components.*

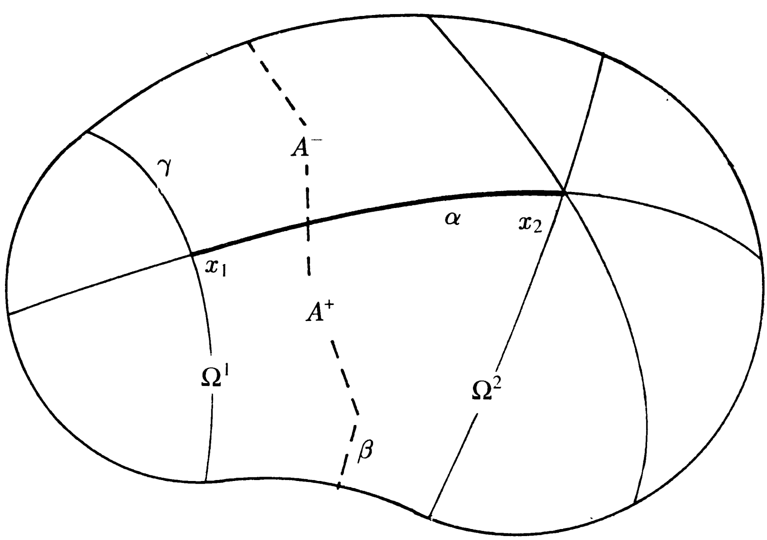

Fig. 1 ($k = 2$)

PROOF. (See: Fig.1). By the induction on the number K of critical points. If $K = 1$ then the result holds by Lemma 1.2.

Let us assume, as induction hypothesis, that if $K \leq \overline{K}$, then we have

$$\sum_{i=1}^{K} m_i + 1 \leq L, \tag{1.4}$$

where L is the number of connected components of the set $\{x \in \Omega | u(x) > t\}$. Let $K = \overline{K}+1$. Let γ be the connected component of $\{x \in \Omega | u(x) = t\}$ containing $x_1, \ldots, x_{\overline{K}+1}$. By the maximum principle, no closed curve is contained in γ, thus we can always find a critical point such that there exists only one arc in γ, connecting it to another critical point. Up to a renumbering, we may denote these two critical points $x_{\overline{K}+1}, x_{\overline{K}}$. Let α be the arc in γ which connects them.

Now we see that there exist exactly two regions A^+, A^- in Ω, which are connected components of the level sets $\{x \in \Omega | u(x) > t\}$, $\{x \in \Omega | u(x) < t\}$ respectively and which satisfy the condition

$$\partial A^+ \cap \partial A^- = \alpha.$$

Let us pick a simple arc β in $A^+ \cup A^- \cup \alpha$, having endpoints on $\partial A^+ \cap \partial\Omega$ and on $\partial A^- \cap \partial\Omega$.

Note that β splits Ω into two simply connected domains Ω^1, Ω^2 such that $x_1, \ldots, x_{\overline{K}} \in \Omega^1$, $x_{\overline{K}+1} \in \Omega^2$.

Let L_i be the number of connected components of the level set $\{x \in \Omega^i | u(x) > t\}$, $i = 1, 2$. Clearly

$$L_1 + L_2 = L + 1$$

in fact β splits A^+ into two connected parts one contained in Ω^1 and the other in Ω^2. Now by the induction hypothesis

$$\sum_{i=1}^{\overline{K}} m_i + 1 \leq L_1,$$
$$m_{\overline{K}+1} + 1 \leq L_2,$$

and (1.4) follows.

PROOF of THEOREM 1.1 (See Fig. 2). Let $x_1, \ldots, x_K$ be the critical points, let us denote:

$$S = \bigcup_{i=1}^{K} \{x \in \Omega | u(x) = u(x_i)\}.$$

Let $\gamma_1, \ldots, \gamma_\ell$ be the connected components of S which contain at least one of the critical points. Clearly $\ell \leq K$. We proceed by induction on ℓ. If $\ell = 1$ then Lemma 1.3 and the maximum principle yields (1.2).

Assume that (1.2) holds when $\ell \leq \overline{\ell}$. Let $\ell = \overline{\ell} + 1$. Up to a renumbering, we choose $\gamma_{\overline{\ell}+1}$ in such a way that $\gamma_1, \ldots, \gamma_{\overline{\ell}}$ all lie in the same connected component of $\Omega \setminus \gamma_{\overline{\ell}+1}$. Let A be such component. Up to a change of sign, setting $t = u_{|\gamma_{\overline{\ell}+1}}$, we assume $u > t$ ona neighbourhood of $\Gamma_{\overline{\ell}+1} \cap \partial A$ in A.

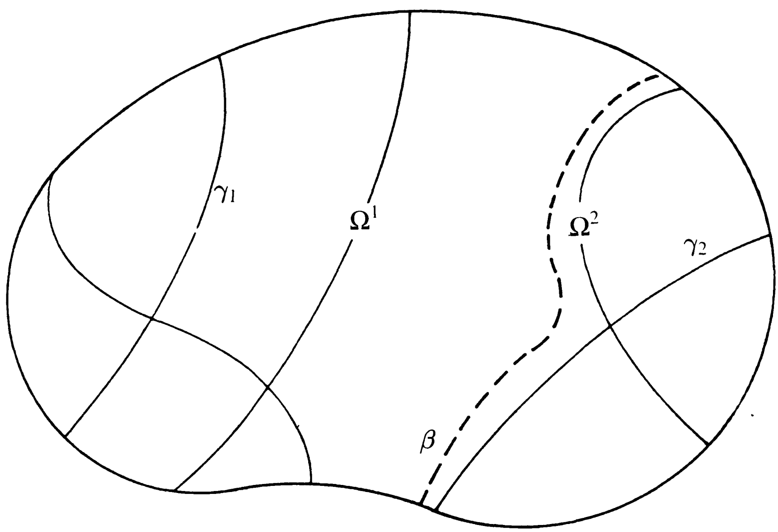

Fig. 2 ($\ell = 2$)

Now it is immediately seen that there exist $\varepsilon > 0$ and a simple arc β in A, with endponts on $\partial\Omega$, which is contained in the level line $\{x \in \Omega | u(x) = t + \varepsilon\}$ and which separates $\gamma_{\bar{\ell}+1}$ from $\gamma_1, \ldots, \gamma_{\bar{\ell}}$. Let Ω^1, Ω^2 be the components of $\Omega \backslash \beta$, let $\gamma_1, \ldots, \gamma_{\bar{\ell}} \subset \Omega^1$, and $\gamma_{\bar{\ell}+1} \subset \Omega^2$. Now, if N_1, N_2 are the numbers of maxima of $u_{|\partial\Omega^1}, u_{|\partial\Omega^2}$ respectively, then we find that

$$N_1 + N_2 = N + 1,$$

in fact β is a connected set of points of relative maximum for Ω^2 and of relative minimum for Ω^1.

Therefore, by the induction hypothesis,

$$\sum_{i=1}^{K} m_i = \sum_{x_i \in \Omega^1} m_i + \sum_{x_i \in \Omega^2} m_i \leq (N_1 - 1) + (N_2 - 1) = N - 1.$$

2. - The interior lower bound on $|Du|$.

In this section we will assume that $\partial\Omega$ is C^2. In such a case Ω satisfies an interior and exterior sphere condition. We define d_0 as the largest positive number such that, for every $x \in \mathbb{R}^2 \setminus \partial\Omega$, there exists a disk of radius d_0, which contains x and does not intersect $\partial\Omega$.

We will also assume the following bound on the coefficients in (1):

$$\sum_{ijh=1}^{2} \left(\frac{\partial}{\partial x_h} a_{ij}(x) \right)^2 + \sum_{i=1}^{2} b_i^2(x) \leq E^2, \text{ for every } x \in \Omega. \tag{2.1}$$

THEOREM 2.1 *Let $\partial\Omega$ be C^2, let $a_{ij} \in C^1(\Omega)$, $b_i \in C(\Omega)$, $i, j = 1, 2$ and let (2.1) be satisfied. Let $g \in C^2(\overline{\Omega})$ be such that $g_{|\partial\Omega}$ has N maxima. Let u be the solution of the Dirichlet problem (1), (2) belonging to the function space $W^{2,2}(\Omega)$.*

Let $x_1, \ldots, x_K$ and $m_1, \ldots, m_K$ be the interior critical points of u and the respective multiplicities.

The following estimate holds for every $x \in \Omega_d$, $d > 0$

$$|Du(x)| \geq C_1 \prod_{i=1}^{K} (C_2 |x - x_i|)^{m_i}, \tag{2.2}$$

here C_1, C_2 are positive constants: C_1 depends only on $d, |\Omega|, d_0, \lambda$, E osc $_{\partial\Omega} g$, *and $\|g\|_{C^2(\overline{\Omega})}$, and C_2 depends only on diam Ω, λ and E.*

We premise the proof of Theorem 2.1 with some notation, definitions and with three lemmas. First let us denote:

$$A(x) = \begin{pmatrix} a_{11}(x) & a_{12}(x) \\ a_{12}(x) & a_{22}(x) \end{pmatrix}, x \in \Omega,$$
$$B(x) = (b_1(x), b_2(x)), \quad x \in \Omega.$$

We define functions v, ϕ, h vector fields X, F, and the divergence structure elliptic operator L as follows:

$$v = (ADu) \cdot Du, \tag{2.3}$$
$$\phi = \log v, \tag{2.4}$$
$$X_i = v^{-1} \left[\left(\frac{\partial}{\partial x_i} A \right) Du \right] \cdot Du, \quad i = 1, 2, \tag{2.5}$$
$$F = A(x - 2v^{-1}(B \cdot Du)Du), \tag{2.6}$$
$$h = (AX) \cdot X, \tag{2.7}$$
$$L\psi = \text{ div } (AD\psi) + (AX) \cdot D\psi. \tag{2.8}$$

LEMMA 2.1 *Let the hypotheses of Theorem 2.1 be satisfied. The function ϕ defined by (2.4) is a $W^{1,p}_{\text{loc}}(\Omega \setminus \{x_1, \ldots, x_K\})$ solution of*

$$L\phi = \text{div } (F) + h, \tag{2.9}$$

for every $p < \infty$.

REMARK 2.1 The idea of deriving an equation for the modulus of the gradient of solutions of elliptic equations can be traced back to Bernstein [3], and it has shown to be useful in proving upper bounds on the gradient (see for instance Peletier-Serrin [11], Sperb [14]). Here equation (2.9) is used to get a lower bound, this possibility seems however to be confined to the two-dimensional case.

PROOF. Let us denote $\Omega' = \Omega \setminus \{x_1, \ldots, x_K\}$, and let us temporarily assume $a_{ij}, b_i \in C^\infty(\Omega)$.

The following identity holds in Ω'

$$\text{div}(AD\phi) = v^{-1}\ \text{div}(A\Gamma) - v^{-2}(A\Gamma)\cdot\Gamma - v^{-1}(AX)\cdot\Gamma + \ \text{div}\ (AX) \tag{2.10}$$

where

$$\Gamma = 2A(D^2uDu). \tag{2.11}$$

Note that, on the right hand side of (2.10), only the first summand contains third order derivates of u. In fact they can be eliminated. We may use equation (1), or, as is the same,

$$\text{tr}(AD^2u) = f, \tag{2.12}$$

where

$$f = -B\cdot Du. \tag{2.13}$$

Thus, with the aid of (2.12), (6b), we are led, by lenghty but not difficult calculations, to the following formula:

$$\text{div}(A\Gamma) = 2\ \text{div}\ (fADu) + 4\ \det\ (D^2u). \tag{2.14}$$

We may consider (2.11) (2.12) as a linear system of 3 equations in the 3 unknowns: $u_{x_1x_1}, u_{x_1x_2}, u_{x_2x_2}$. The determinant of this system is $4v$, that is, it is non-singular on Ω'. Solving this system we eventually obtain

$$\det(D^2u) = (4v)^{-1}(A\Gamma)\cdot\Gamma - (2v)^{-1}f(A\Gamma)\cdot Du, \tag{2.15}$$

hence, combining (2.10), (2.14), (2.15) and making use of the identity

$$\Gamma = v(D\phi - X)$$

we obtain (2.9). If the coefficients are not C^∞ such equation has to be meant in the weak sense, that is

$$\int_{\Omega'} [(AD\phi)\cdot D\varsigma - (AX)\cdot(D\phi)\varsigma]\mathrm{d}x = \tag{2.16}$$

$$= \int_{\Omega'} (F\cdot D\varsigma - h\varsigma)\mathrm{d}x, \text{ for every } \varsigma \in C_0^\infty(\Omega').$$

This result may be achieved as follows.

Let us fix $\varsigma \in C_0^\infty(\Omega')$ and let Ω'' be such that: supp $\varsigma \subset \Omega'' \subset\subset \Omega'$. Let $\{A_n\}$, $\{B_n\}$ be sequences of $C^\infty(\Omega)$ matrices, respectively vectors, such that A_n satisfies (6) for every n, and also

$$\text{(i) } A_n \to A \text{ in } W^{1,p}(\Omega),$$
$$\text{(ii) } B_n \to B \text{ in } L^p(\Omega),$$

as $n \to +\infty$ for every $p < \infty$.

Let $u_n \in W^{2,2}(\Omega)$ be the solution of

$$\begin{cases} \operatorname{tr}(A_n D^2 u_n) + B_n \cdot Du_n = 0, & \text{in } \Omega, \\ u_n = u, & \text{on } \partial\Omega. \end{cases}$$

By well-known regularity theorems, see for instance Agmon-Douglis-Nirenberg [1], we get that

$$\text{(iii) } u_n \in C^\infty(\Omega) \quad \text{for every } n,$$
$$\text{(iv) } u_n \to u \text{ in } W^{2,p}(\Omega) \text{ as } n \to \infty, \text{ for every } p < \infty.$$

Consequently, since $|Du| \geq$ const.′ > 0 in Ω'', there exists $\overline{n} > 0$ such that for every $n \geq \overline{n}$

$$|Du_n| \geq \text{ constant } > 0 \text{ in } \Omega''.$$

Therefore (2.16) holds if we replace A, B, u respectively with A_n, B_n, u_n in (2.3) - (2.8), (2.16). The convergence properties (i), (ii), (iv) enable us to pass to the limit and obtain that $\phi \in W^{1,p}(\Omega'')$ for every $p < \infty$ and it satisfies (2.16).

Since the choice of $\varsigma \in C_0^\infty(\Omega')$ was arbitrary, the proof is complete.

LEMMA 2.2 *Let the hypotheses of Theorem 2.1 be satisfied. Let $\psi \in W^{1,2}_{\mathrm{loc}}(\Omega \setminus \{x_1, \ldots, x_K\})$ be a weak solution of*

$$L\psi = \frac{1}{2}(\operatorname{div}(F) + h).$$

If $\psi \in L^\infty_{\mathrm{loc}}(\Omega)$ then: $\psi \in W^{1,p}_{\mathrm{loc}}(\Omega)$ for every $p < \infty$ and the above equation holds in all of Ω.

The above Lemma is a rather standard result on removable singularities. A proof is given, for the sake of completeness, at the end of this Section.

In the sequel we will make use of Green's function for the elliptic operator L in Ω, namely the function $G = G(x, y)$ which solves the Dirichlet problem

$$\begin{cases} LG(\cdot, y) = -\delta(\cdot - y), y \in \Omega, \\ G(x, y) = 0, x \in \partial\Omega, y \in \Omega. \end{cases} \tag{2.17}$$

The following well-known properties of G will be used. See for instance: Hartman and Wintner [8], Agmon, Douglis and Nirenberg [1].

$$0 < G(x, y) \leq -\frac{1}{2\pi} \log|x - y| + c(\lambda, E, \text{ diam } \Omega), x, y \in \Omega, \tag{2.18}$$

$$G(x, y) = -\frac{1}{4\pi} \log(A^{-1}(y)(x - y) \cdot (x - y)) + 0(1), \text{ as } x \to y \in \Omega, \tag{2.19}$$

$$G(\cdot, y) \in W^{2,p}_{\text{loc}}(\overline{\Omega} \setminus \{y\}), \text{ for every } p < \infty, y \in \Omega. \tag{2.20}$$

LEMMA 2.3 *Let the hypotheses of Theorem 2.1 be satisfied. The following representation holds in* Ω

$$v^{1/2} = \exp\{\psi - 2\pi \sum_{i=1}^{K} m_i G(\cdot, x_i)\}, \tag{2.21}$$

where ψ *is the* $W^{1,p}_{\text{loc}}(\Omega)$ *solution of the Dirichlet problem*

$$\begin{cases} L\psi = \frac{1}{2}(\text{div } (F) + h) \text{ in } \Omega, \\ \psi = \frac{1}{2}\phi (= \log v^{1/2}) \text{ on } \partial\Omega. \end{cases} \tag{2.22}$$

PROOF. Let ψ be defined by (2.21). By (2.20) and since $u \in W^{2,p}(\Omega)$ for every $p < \infty$ (see [1]) we get $\psi \in W^{1,p}_{\text{loc}}(\Omega \setminus \{x_1, \ldots, x_K\})$.

By Lemma 2.1 and by (2.17) we get, on $\Omega \setminus \{x_1, \ldots, x_K\}$,

$$L\psi = \frac{1}{2}(\text{ div } (F) + h).$$

Recall (1.1) and note that, as $x \to x_j$, $j = 1, \ldots, K$, we have

$$\begin{aligned} v(x) &= (A(x)Du(x)) \cdot Du(x) = \\ &= c_j \left[\left(A^{-1}(x_j)(x - x_j) \right) \cdot (x - x_j) \right]^{m_j} + o\left(|x - x_j|^{2m_j} \right) \end{aligned} \tag{2.23}$$

for some positive constant c_j. Thus combining (2.19), (2.21), (2.23) we obtain

$$\psi(x) = 0(1), \text{ as } x \to x_j, j = 1, \ldots, K,$$

that is $\psi \in L^{\infty}_{\text{loc}}(\Omega)$. Hence we may apply Lemma 2.2 to ψ and (2.22) follows.

PROOF of THEOREM 2.1. By (2.3), (2.18) and (2.21) we have in Ω

$$|Du| \geq [\lambda^{-1}v]^{1/2} \geq \exp\{\psi\} \prod_{i=1}^{K} \left(c(\lambda, E, \text{ diam } \Omega)|x - x_i|\right)^{m_i}. \tag{2.24}$$

Thus we need to find an interior lower bound on ψ. We will make use of L^∞-estimates and of Harnack's inequality for the linear divergence structure elliptic equation (2.22), see Gilbarg-Trudinger [6].

Let us denote

$$M = \sup_{\Omega} \psi. \tag{2.25}$$

The following bounds on the coefficients and on the inhomogeneous terms in equation (2.22), are straightforward consequences of (6), (2.1) and (2.5)-(2.7):

$$\begin{cases} \|AX\|_\infty \leq \lambda^2 E, \\ \|F\|_\infty \leq 3\lambda^2 E, \\ \|h\|_\infty \leq \lambda^3 E^2, \end{cases} \tag{2.26}$$

thus by the use of maximum principle in (2.22) we get:

$$M \leq \frac{1}{2} \max_{\partial\Omega} \ \log v + c(\lambda, E, |\Omega|) < \infty. \tag{2.27}$$

Now note that $w = M - \psi \in W^{1,p}_{\text{loc}}(\Omega)$ is a non-negative solution of

$$Lw = \frac{1}{2}(\text{div } (F) + h) \text{ in } \Omega.$$

Therefore, by (2.26), Harnack's inequality is applicable to w, and we obtain, for every $d \leq \mathrm{d}_0/2$,

$$\sup_{\Omega_d}(M - \psi) \leq c_1(\lambda, E, \mathrm{d}, |\Omega|) \left\{ \inf_{\Omega_d}(M - \psi) + c_2(\lambda, E, |\Omega|) \right\},$$

or, as is the same,

$$\inf_{\Omega_d} \psi \geq M - c_1(\lambda, E, \mathrm{d}, |\Omega|) \left\{ M - \sup_{\Omega_d} \psi + c_2(\lambda, E, |\Omega|) \right\}. \tag{2.28}$$

Let us evaluate M from below. By (2.21) and (2.18) we have:

$$M \geq \frac{1}{2} \sup_{\Omega} \ \log v \geq \log \left(\lambda^{-1/2} \sup_{\Omega} |Du| \right).$$

On the other hand, note that

$$\operatorname*{osc}_{\partial\Omega} g = \operatorname*{osc}_{\partial\Omega} u \leq \frac{1}{2}|\partial\Omega| \max_{\partial\Omega} |\frac{\partial u}{\partial \underline{t}}|,$$

where $\underline{t}$ denotes the tangent vector to $\partial\Omega$ and thus, since $|\partial\Omega| \leq c|\Omega|\mathrm{d}_0^{-1}$,

$$\operatorname*{osc}_{\partial\Omega} g \leq c|\Omega|\mathrm{d}_0^{-1} \sup_{\Omega} |Du|, \tag{2.29}$$

and hence:

$$M \geq \log \left(c\lambda^{-1/2}\mathrm{d}_0|\Omega|^{-1} \operatorname*{osc}_{\partial\Omega} g \right). \tag{2.30}$$

Now, by (2.28), (2.27),

$$M - \sup_{\Omega_d} \psi \leq \max_{\partial\Omega} \log \left(\lambda^{1/2}|Du| \right) + c(\lambda, E, |\Omega|) - \max_{\overline{\Omega}_d} \log \left(\lambda^{1/2}|Du| \right). \tag{2.31}$$

Let $\overline{x} \in \partial\Omega$ be such that:

$$\log |Du(\overline{x})| = \max_{\partial\Omega} \log |Du|,$$

let $z \in \partial\Omega_d$, be such that: $|\overline{x} - z| = \mathrm{d}$, hence we get, for every $p > 2$

$$\begin{aligned} &\max_{\partial\Omega} \log |Du| - \max_{\overline{\Omega}_d} \log |Du| \leq \\ &\leq \log |Du(\overline{x})| - \log |Du(z)| \leq \\ &\leq c_p \mathrm{d}^{1-2/p} \|D \log |Du|\|_{L^p(B_d(z))} \leq \\ &\leq c_p \mathrm{d}^{1-2/p} \|D^2 u\|_p \left(\underline{\min}_{B_d(z)} |Du| \right)^{-1} \end{aligned} \tag{2.32}$$

here use has been made of Morrey's inequality (see e.g.: Gilbarg-Trudinger [6]). Again by Morrey's inequality

$$\underline{\min}_{B_d(z)} |Du| \geq |Du(\overline{x})| - c_p \mathrm{d}^{1-2/p} \|D^2 u\|_p = \|Du\|_\infty - c_p \mathrm{d}^{1-2/p} \|D^2 u\|_p.$$

Now recall (2.29), and the following regularity estimate (see Agmon-Douglis-Nirenberg [1])

$$\|D^2 u\|_p \leq c(\lambda, E, |\Omega|, \mathrm{d}_0) \|g\|_{C^2(\overline{\Omega})},$$

consequently there exists $\mathrm{d}_1 > 0$ depending only on $\lambda, E, |\Omega|, \mathrm{d}_0$ and on the ratio

$$\|g\|_{C^2(\overline{\Omega})} / \operatorname*{osc}_{\partial\Omega} |g|,$$

such that, for every $\mathrm{d} < \mathrm{d}_1$,

$$2c_p \mathrm{d}^{1-2/p} \|D^2 u\|_p \leq \|Du\|_\infty.$$

Therefore, if $\mathrm{d} < \mathrm{d}_1$, then we get

$$\max_{\partial\Omega} \log|Du| - \max_{\overline{\Omega_d}} \log|Du| \leq 1,$$

and combining (2.31), (2.33) we obtain

$$M - \sup_{\Omega_d} \psi \leq 1 + c(\lambda, E, |\Omega|),$$

now, collecting (2.28), (2.30), (2.34), we get

$$\inf_{\Omega_d} \psi \geq \log(c\mathrm{d}_0 |\Omega|^{-1} \operatorname*{osc}_{\partial\Omega} g) - c_1(\lambda, \mathrm{d}, |\Omega|) c(\lambda, E, |\Omega|)$$

for every $\mathrm{d} < \mathrm{d}_1$, and clearly, if $\mathrm{d} > \mathrm{d}_1$:

$$\inf_{\Omega_d} \psi \geq \inf_{\Omega_{d_1}} \psi \geq \log(c\mathrm{d}_0 |\Omega|^{-1} \operatorname*{osc}_{\partial\Omega} g) \ - c_1(\lambda, \mathrm{d}_1, |\Omega|) c(\lambda, E, |\Omega|).$$

and finally, by (2.24), (2.2) is proven.

PROOF of LEMMA 2.2. This proof is inspired to arguments in Serrin [13]. Without loss of generality we may assume: $\Omega = B_R$, the disk of radius R centered at the origin, $K = 1, x_1 = 0$.

Let $\psi_1 \in W^{1,2}(B_R)$ be such that:

$$\begin{aligned} L\psi_1 &= \operatorname{div}\ (F) + h, && \text{in } B_{R/2}, \\ \psi_1 &= \psi, && \text{in } B_R \setminus B_{R/2}. \end{aligned}$$

Note that, by (2.26), the maximum principle yields

$$\sup_{B_{R/2}} |\psi_1| \leq \sup_{\partial B_{R/2}} |\psi| + c(\lambda, E, R) < \infty,$$

thus, denoting $\psi_0 = \psi - \psi_1$, we get $\psi_0 \in W^{1,2}_{\mathrm{loc}}(B_R \setminus \{0\})$, $\psi_0 \in L^\infty(B_R)$, $\psi_0 \equiv 0$ in $B_R \setminus B_{R/2}$, and, moreover,

$$L\psi_0 = 0 \text{ in } B_{R/2} \setminus \{0\}.$$

Let $h \in (0, R/2)$ and $r = |x|$, let us define

$$\eta(x) = \begin{cases} 0 & , \text{ if } r < h, \\ (\log \dfrac{r}{h})(\log \dfrac{R}{2h})^{-1}, & \text{if } h \leq r \leq R/2. \end{cases}$$

Note that: $\psi_0\eta^2 \in W_0^{1,2}(B_{R/2})$. Thus we have

$$\int_{B_{R/2}} \left[(AD\psi_0)\cdot D(\psi_0\eta^2) - (AX\cdot D\psi_0)\psi_0\eta^2\right] \mathrm{d}x = 0.$$

We readily infer

$$\int_{B_{R/2}} |D\psi_0|^2\eta^2 \mathrm{d}x \le c(\lambda, E) \int_{B_{R/2}} (|D\eta|^2 + \eta^2)\psi_0^2 \mathrm{d}x \le$$
$$\le c(\lambda, E)||\psi_0||_\infty^2((\log R/2h)^{-1} + R^2).$$

Thus, fixing r_0, $h < r_0 < R/2$, we have, as $h \to 0+$:

$$\int_{B_{R/2}\setminus B_{r_0}} |D\psi_0|^2 \le \text{ Const. } (\log R/2h)^2(\log r_0/h)^{-2} \to \text{ Const. } < \infty,$$

that is: $\psi_0 \in W_0^{1,2}(B_{R/2})$. now for every $\varsigma \in C_0^\infty(B_{R/2})$ we have

$$\int_{B_{R/2}} [AD\psi_0 \cdot D(\varsigma\eta) - (AX)\cdot D\psi_0\varsigma\eta] \,\mathrm{d}x = 0,$$

which readily yields

$$|\int_{B_{R/2}} (AD\psi_0 \cdot D\varsigma - (AX)\cdot D\psi_0\varsigma)\mathrm{d}x| \le$$
$$\le c(\lambda, E)\{||D\psi_0||_2||1-\eta||_2(||D\varsigma||_\infty + ||\varsigma||_\infty) + ||D\psi_0||_2||D\eta||_2||\varsigma||_\infty\}$$

Now a direct computation shows

$$||1-\eta||_2, ||D\eta||_2 \to 0 \text{ as } h \to 0+,$$

and therefore: $L\psi_0 = 0$ in $B_{R/2}$. Hence $\psi - \psi_1 = \psi_0 \equiv 0$ in B_R and $\psi \in W_{\mathrm{loc}}^{1,p}(B_R)$ for every $p < \infty$, since F, h and the coefficients of L are bounded.

3. - The case of maxima and minima at two levels only.

THEOREM 3.1 *Let $\partial\Omega$ be C^2-smooth. Let $a_{ij} \in C^1(\Omega)$, $b_i \in C(\Omega)$, $i.j = 1,2$ and let $g \in C^1(\overline{\Omega})$ be such that $g_{|\partial\Omega}$ has N maxima.*

Let us assume that, on the points of relative minimum and maximum, g takes two values only: $g_{\min}$ and $g_{\max}$, and let the tangential derivative of g on $\partial\Omega$ be non-zero out of such extremal points.

Let $u \in W^{2,2}_{\text{loc}}(\Omega) \cap C^1(\overline{\Omega})$ be the solution of (1), (2). If $N > 1$ then interior critical points of u exist and the respective multiplicities $m_1, \ldots, m_K$ satisfy

$$\sum_{i=1}^{K} m_i = N - 1. \tag{3.1}$$

REMARK 3.1 We wish to mention a certain analogy between the above Theorem and results in Walsh [16], concerning harmonic functions with boundary values at two levels only.

We start stating the following Rolle-type result. A sketch of a proof is given at the end of this section.

LEMMA 3.1 *Let $f \in C^1(\overline{\Omega})$. If $f_{|\partial\Omega}$ has relative maxima and minima at two levels only and if the connected components of the set of points of maximum on $\partial\Omega$ are at least two, then f has at least one critical point in $\overline{\Omega}$.*

PROOF of THEOREM 3.1. We will proceed by induction on N. As we already did is Section 1, we will make use of cuts on Ω. It is convenient, to this purpose, to drop the C^2- smoothness assumption on $\partial\Omega$. We will prove the following intermediate result:

PROPOSITION *Let all the hypotheses of Theorem 3.1, except $\partial\Omega \in C^2$, be satisfied. If, in addition, we assume that $|Du| > 0$ on $\partial\Omega$, then* (3.1) *holds.*

It is readily seen that Theorem 3.1 follows from the above Proposition. In fact, if $\partial\Omega$ is C^2 than the Hopf lemma is applicable, and thus the hypotheses on g imply: $|Du| > 0$ on $\partial\Omega$.

Now we prove the Proposition inductively.

If $N = 2$, then, by Lemma 3.1 and since $|Du| > 0$ on $\partial\Omega$, one interior critical point exists. By Theorem 1.1, (3.1) follows.

Let $\overline{N} > 2$ and assume, as induction hypothesis, that the above Proposition holds whenever $N < \overline{N}$.

Let $N = \overline{N}$. By Lemma 3.1, at least one interior critical point, y_1, exists.

Let us denote by: $y_1, \ldots, y_K$ all the other critical points of u belonging to the level line $\gamma = \{x \in \Omega | u(x) = u(y_1)\}$, let $n_1, n_2, \ldots, n_L$ be the respective multiplicities. Let us assume

$$\sum_{i=1}^{L} n_i < N - 1 \tag{3.2}$$

otherwise the proof is completed. let $t \in (g_{\min}, g_{\max})$ be the critical value: $t = u(y_1)$.

Let M be the number of connected components of the set $\Omega \setminus \gamma$.

By Theorem (H.-W.) we know that γ is the union of a finite number of simple C^1 arcs having endpoints on $\partial\Omega$, let ł be the number of those of such arcs which do not contain critical points and let ν be the number of the remaining arcs.

Recall that any such arc in γ has distinct endpoints on $\partial\Omega$, and every endpoint on $\partial\Omega$ belongs to only one arc. It follows

$$2(\ell + \nu) = 2N. \tag{3.3}$$

Moreover the following equality could be proven just rephrasing the arguments of Lemma 1.3

$$\nu = \sum_{i=1}^{L} n_i + \mu, \tag{3.4}$$

where μ is the number of connected components of γ which contain at least one critical point. Similarly we obtain

$$M = 2\sum_{i=1}^{L} n_i + \mu + \ell + 1. \tag{3.5}$$

Combining (3.3)-(3.5), we obtain

$$M = \sum_{i=1}^{L} n_i + 1 + N, \tag{3.6}$$

and thus, by (3.2),

$$N + 2 \leq M < 2N. \tag{3.7}$$

Let $\varepsilon > 0$ be sufficiently small in such a way that the set $\Omega^\varepsilon = \{x \in \Omega; |u(x) - t| > \varepsilon\}$ has as many connected components as $\Omega \setminus \gamma$, that is: M, and it contains all the critical points of u belonging to $\Omega \setminus \gamma$. Let $\Omega_1, \ldots, \Omega_M$ be the connected components of Ω^ε.

Note that, for every $j, u_{|\partial\Omega_j}$ has relative maxima and minima at two levels only: $g_{\max}$ or $t - \varepsilon$ as maxima, and $t + \varepsilon$ or $g_{\min}$ as minima.

Moreover u has no critical points on $\partial\Omega_j$ and $\partial\Omega_j$ is piecewise C^1 for every j.

Let N_j be the number of connected components of $\partial\Omega_j$ on which u takes its relative maximum value ($g_{\max}$ or $t - \varepsilon$).

It is readily seen that, since the set $\{x \in \partial\Omega; |g(x) - t| > \varepsilon\}$ has $2N$ connected components, then

$$\sum_{j=1}^{M} N_j = 2N. \tag{3.8}$$

Hence by (3.7), $N_j > 1$ for at least one $j = 1, 2. \ldots, M$, and also: $N_j < N$ for every j.

We may apply the induction hypothesis on those Ω_j 's for which $N_j > 1$.

Let m_{ji}, $i = 1, \dots K_j$ be the multiplicities of the critical points in Ω_j. We get

$$\sum_{i=1}^{M} \sum_{i=1}^{K_j} m_{ji} = \sum_{j=1}^{M} (N_j - 1) = 2N - M.$$

Finally, by (3.6), we get

$$\sum_{i=1}^{L} n_i + \sum_{j=1}^{M} \sum_{i=1}^{K_j} m_{ji} = N - 1,$$

which, by Theorem 1.1, yields (3.1).

It is worthwile to stress that the hypotheses of Theorem 3.1 on g imply also that the lower bound on the gradient of u holds up to the boundary of Ω.

THEOREM 3.2 *Let the hypotheses of Theorem 3.1 be satisfied. Let* (2.1) *hold. The following estimate holds for every* $x \in \overline{\Omega}$

$$|Du(x)| \geq C_3 \prod_{i=1}^{K} (C_2 |x - x_1|)^{m_1}. \tag{3.9}$$

Here $x_1, \dots, x_K \in \Omega$, *are the critical points of* u, $m_1, \dots, m_K$ *are the respective multiplicities and* C_2, C_3 *are positive constants:* C_2 *depends only on* diam Ω, λ *and* E, C_3 *depends only on* $|\Omega|, \mathrm{d}_0, \lambda, E$ *and* g.

PROOF. Let $z \in \partial\Omega$ be an extremal point of g, and thus of u. For instance, let us assume that it is a point of minimum, by the Harnack inequality (see [6]) we have:

$$\inf_{x \in \Omega_d} (u(x) - u(z)) \geq c(\lambda, E, |\Omega|, \mathrm{d}) \sup_{x \in \Omega_d} (u(x) - u(z)).$$

By continuity we get

$$\sup_{x \in \Omega_d} u(x) \geq \max_{\partial\Omega} g - c(\lambda, E, |\Omega|, \mathrm{d}_0) ||g||_{C^2} \mathrm{d},$$

and thus, if $d \leq \mathrm{d}_1 = \mathrm{d}_1(\lambda, E, |\Omega|, \mathrm{d}_0, ||g||_{C^2})$, then we obtain

$$\inf_{x \in \Omega_d} (u(x) - u(z)) \geq c(\lambda, E, |\Omega|, \mathrm{d}) \operatorname*{osc}_{\partial\Omega} g > 0.$$

Now by the Hopf lemma (see [6])

$$|Du(z)| \geq c(\lambda, E) \mathrm{d}_1^{-1} \inf_{x \in \Omega_{d_1}} (u(x) - u(z)),$$

and thus, on every extremal point $z \in \partial\Omega$,

$$|Du(z)| \geq c(\lambda, E, |\Omega|, \mathrm{d}_0, \|g\|_{C^2}) \operatorname*{osc}_{\partial\Omega} g > 0. \tag{3.10}$$

Since the tangential derivative of g is nonzero outside of the set E of extremal points on $\partial\Omega$, if $z \in \partial\Omega$ and its distance from E is $\delta > 0$, then

$$|Du(z)| \geq |\frac{\mathrm{d}}{\mathrm{d}\mathbf{t}} g(z)| \geq c(\delta) > 0, \tag{3.11}$$

here $\mathbf{t}$ is the tangent vector to $\partial\Omega$ at z.

Now if we combine (3.10), (3.11) and the estimate (2.2) with the aid of the Hölder continuity of Du in $\overline{\Omega}$, we obtain the result.

In Theorem 3.1 we have seen that, under certain hypotheses on g, the sum of the multiplicities is independent of the coefficients in equation (1).

The following example shows that this is not necessarily the case when the maxima and minima at the boundary are not at two levels only.

EXAMPLE 3.1 Consider $B = B_1((1,0))$ and let $u \in W^{1,2}(B)$ be the solution of the Dirichlet problem

$$\begin{cases} \operatorname{div}(a \operatorname{grad} u) = 0, & \text{in } B, \\ u = x_1^2 - x_2^2, & \text{on } \partial B, \end{cases}$$

where the coefficient a is defined as follows

$$a(x) = \begin{cases} 1, & \text{if } x_1 < 1, \\ 1 + \mu, & \text{if } x_1 > 1, \end{cases}$$

μ being any number larger than -1.

It is readily checked that u is given by

$$u(x) = \begin{cases} x_1^2 - x_2^2 + 2\mu(2+\mu)^{-1}\phi(x), & \text{if } x_1 \leq 1, \\ x_1^2 - x_2^2 + 2\mu(2+\mu)^{-1}\phi(2 - x_1, x_2), & \text{if } x_1 \geq 1, \end{cases}$$

here ϕ is the $W^{1,2}$ solution of the mixed problem

$$\begin{cases} \Delta\phi(x) = 0, x \in B, x_1 < 1, \\ \phi(x) = 0, x \in \partial B, x_1 < 1, \\ \phi_{x_1}(x) = 1, x_1 = 1, |x_2| < 1. \end{cases}$$

The maximum principle yields:

$$\phi(x) > 0, \text{ for every } x \in B, x_1 \leq 1,$$

and also:

$$\phi_{x_1}(x) > 0 \text{ for every } x \in \overline{B}, \text{ such that, } x_1 \leq 1, |x_2| < 1.$$

Consequently we may find $c > 0$ such that if $\mu \in (0, c)$ then $|Du|$ is always nonzero in B, and, if $\mu \in (-c, 0)$ then there exists $z_\mu \in (0, 1)$ such that $z_\mu \to 0$ as $\mu \to 0-$ and

$$|Du(z_\mu, 0)| = 0.$$

The above example might seem unsatisfactory since the coefficient a is discontinuous. However the reader can convince himself that a similar behaviour can be obtained if a is replaced by a smooth approximation.

PROOF of LEMMA 3.1. Let us denote $a = \min_{\partial\Omega} f$, $b = \max_{\partial\Omega} f$.

We argue by contradiction, that is we assume: $|Df| \geq C > 0$ in $\overline{\Omega}$. As a first consequence we get that f has no interior maxima and minima, and hence: $a < f(x) < b$, for every $x \in \Omega$. Note also that, by the continuity of f, the number of connected sets of points of maximum on $\partial\Omega$ is finite, let it be: $N \geq 2$. Let $A_1, \ldots, A_N$ be such connected sets of points of maximum on $\partial\Omega$.

Let $t_0 \in (a, b)$ and consider the level line $\{x \in \overline{\Omega} | f(x) = t_0\}$, this is made of a finite number of non-intersecting simple arcs.

Let γ_0 be any one of such arcs. Let P_1, P_2 be the endpoints of γ_0 on $\partial\Omega$. up to a change of sign on f, and up to a renumbering, we may assume that A_1, A_2 lie on the same side of γ_0 and that they can be reached, starting from P_1 resp. P_2, moving along arcs β_1, resp. β_2, in $\partial\Omega$ on which f is non-decreasing (see Fig. 3). Now let T be the set of those $t \in [t_0, b]$ such that the level line $\{x \in \overline{\Omega} | f(x) = t\}$ contains one arc $\gamma(t)$ having endpoints $P_1(t)$, resp. $P_2(t)$, on β_1, resp. β_2.

Note that: $(a) t_0 \in T$; $(b) T$ is closed, by the compactness of $\overline{\Omega}$; $(c) T$ is open, because, by the hypothesis $|Df| \geq C > 0$, the level lines of f depend continuosly on the level parameter.

Thus: $T = [t_0, b]$, which is impossible, because this would imply that either: $f = b$ on interior points, or: A_1 and A_2 are not disjoint.

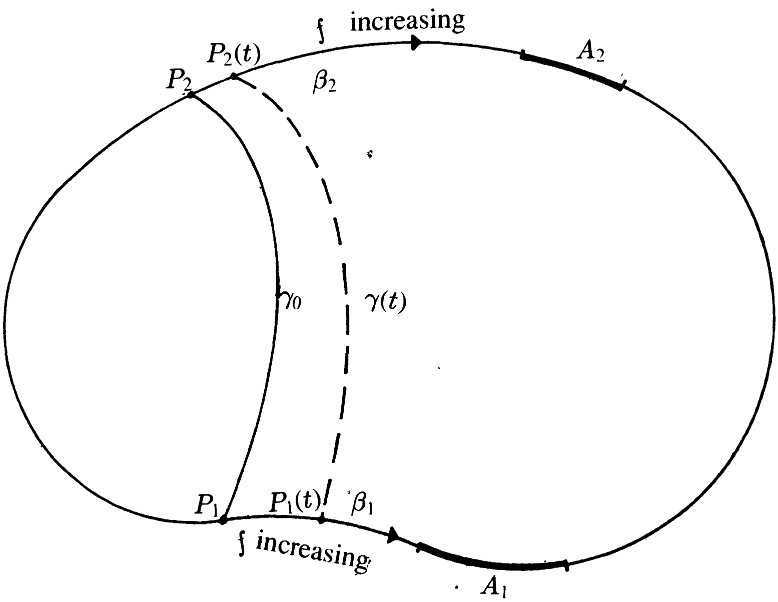

Fig. 3

4. - Equations with non-smooth coefficients.

Here we will show by means of some examples how Theorems (H.-W.) and 1.1 may fail when the regularity assumptions on the coefficients are dropped.

Let $\Omega = B_R(0)$ for some $R > 0$. We will consider equation (1) with coefficients $b_1 = b_2 = 0$ and

$$a_{ij}(x) = (1 + \gamma(r))^{-1/2} \left[\delta_{ij} + \gamma(r) r^{-2} x_i x_j\right], i, j = 1, 2, \tag{4.1}$$

Here δ_{ij} is Kronecker's symbol, $r^2 = x_1^2 + x_2^2$ and γ is a function which will be specified later.

Equations having this form have been used for various purposes by several authors, see, for instance, Gilbarg-Serrin [5].

Note that the matrix $A = (a_{ij})$ has eigenvalues $(1 + \gamma(r))^{1/2}$, $(1 + \gamma(r))^{-1/2}$. Thus condition (6) is satisfied provided

$$0 < \lambda^{-2} \leq 1 + \gamma(r) \leq \lambda^2, 0 \leq r \leq R. \tag{4.2}$$

We will consider solutions of equation (1) of the form

$$u(x) = v(r)\sin N\theta, \tag{4.3}$$

θ being the angle in polar coordinates: $x_1 = r\cos\theta$, $x_2 = r\sin\theta$, and N being a positive integer.

Note that the boundary value of u is smooth and has N maxima. Note also that if u in (4.3) satisfies (1) then, separating the variables, we are led to

$$\gamma(r)v''(r) + (v''(r) + r^{-1}v'(r) - N^2r^{-2}v(r)) = 0, \quad 0 \le r < R. \tag{4.4}$$

EXAMPLE 4.1. Pick: $R = 1, N > 1, \alpha \in (N, N^2)$ and

$$v(r) = r^{\alpha}.$$

Equation (4.4) yields

$$\gamma = \alpha^{-1}(\alpha - 1)^{-1}(N^2 - \alpha^2).$$

Note that the ellipticity condition (4.2) is satisfied.
It turns out that the coefficients a_{ij} are discontinuous at the origin and

$$u(x) = r^{\alpha}\sin N\theta$$

does not satisfy (1.1) at $x^0 = 0$ for any homogeneous harmonic polynominal H_n. Moreover as $x \to 0$

$$|Du(x)| = 0(r^{\alpha-1}) = o(r^{N-1})$$

which means that (1.2) is not true.

EXAMPLE 4.2 Pick: $N > 1$ and

$$v(r) = r^N(\log r)^{\alpha}$$

for some $\alpha \neq 0$. from (4.4) we get that

$$\gamma(r) = (\log r)^{-1}\left[(2N+1)\alpha + \alpha(\alpha-1)(\log r)^{-1}\right]\cdot$$
$$\cdot\left[N(N-1) + 2N\alpha(\log r)^{-1} + \alpha(\alpha-1)(\log r)^{-2}\right]^{-1}.$$

Note that

$$\gamma(r) = 0((\log r)^{-1}) \text{ as } r \to 0+,$$

and thus (4.2) is satisfied provided R is sufficiently small. note also that the $a'_{ij}s$ are continuous in $\overline{B}_R$. Now observe that $|Du(x)|$ is of the same order of $r^{N-1}(\log r)^{\alpha}$ as $x \to 0$. Hence Theorem (H.-W.) cannot hold.

Moreover the rate of decay of $|Du|$ as $x \to 0$ is not a power, thus a multiplicity is not defined.

However we may note that, if $\alpha < 0$, then such a rate is faster than the maximal power rate allowed by Theorem 1.2.

The following Theorem gives a result which is weaker than those of Theorems 1.1 and 1.2, but holds also for discontinuous coefficients.

THEOREM 4.1 *Let a_{ij}, $b_i \in L^\infty(\Omega)$, $ij = 1,2$. Let $g \in C(\overline{\Omega})$ have N maxima on $\partial\Omega$. Let $u \in W^{2,2}_{\text{loc}}(\Omega) \cap C(\overline{\Omega})$ satisfy* (1), (2).

The interior critical points of u are finite in number: $x_1, \dots, x_K$. Furthermore there exists $\delta \in (0,1)$ depending only on the ellipticity constant λ and on $\|B\|_\infty = \|b_1^2 + b_2^2\|_\infty^{1/2}$, such that, for every compact subset Q of Ω, we have

$$(4.5) \qquad |Du(x)| \geq C \prod_{i=1}^{K} |x - x_i|^{m_i/\delta}, x \in Q,$$

where C is a positive constant, and $m_1, \dots, m_K$ are positive integers satisfying

$$(4.6) \qquad \sum_{i=1}^{K} m_i \leq N - 1.$$

REMARK 4.1 Note that (4.6) obviously implies

$$K \leq N - 1.$$

REMARK 4.2 A sufficient condition for the existence of the solution u to (1), (2) in the function space $W^{2,2}_{\text{loc}}(\Omega) \cap C(\overline{\Omega})$ is that $\partial\Omega$ is Lipschitz. See for instance, Talenti [15], Miller [10].

The proof of Theorem 4.1 will be based on the following result due to Bers and Nirenberg [4]. We state it in a form which suits our purposes. The usual identification of the complex plane with $\mathbb{R}^2$ is understood.

THEOREM (B.-N.) *Let the hypotheses of Theorem 4.1 be satisfied. There exist:*

(a) $\delta \in (0,1]$, *depending only on* $\lambda, \|B\|_\infty$,
(b) *a complex valued function* $s \in C^\delta(\Omega)$,
(c) *a homeomorphism* $\chi : \Omega \to \Omega' \subset \mathbb{R}^2$, $\chi \in C^\delta(\Omega)$, $\chi^{-1} \in C^\delta(\Omega')$,
(d) *a harmonic function* $h = h(\xi)$ *on* Ω',

such that the following representation holds for every $x \in \Omega$,

$$(4.8) \qquad D_x u(x) = u_{x_1}(x) + i u_{x_2}(x) = e^{s(x)} \left(h_{\xi_1}(\chi(x)) + i h_{\xi_2}(\chi(x)) \right).$$

REMARK 4.3 Let us stress some consequences of Theorem (B.-N.) (see [4]).

(I) The interior critical points of u are isolated.

(II) Let $x^0 \in \Omega$, and let us denote by m the non-negative integer such that

$$h(\xi) - h(\chi(x^0)) = H_{m+1}(\xi - \chi(x^0)) + 0(|\xi - \chi(x^0)|^{m+2})$$

as $\xi \to \chi(x_0)$ where H_{m+1} is a homogeneous harmonic polynomial of degree $m+1$. We see that there exist positive constants c_1, c_2 such that, in a neighbourhood of x^0,

$$c_1|x - x^0|^{m/\delta} \leq |Du(x)| \leq c_2|x - x^0|^{m\delta} \tag{4.9}$$

(III) Using the above notation, the complex valued function

$$x \to \overline{Du(x)} = u_{x_1}(x) - iu_{x_2}(x),$$

has index m at x^0, that is

$$\lim_{r\to 0+}(2\pi)^{-1} \int\limits_{|x-x^0|=r} d(\arg \overline{Du(x)}) = m.$$

LEMMA 4.1 *Let the hypotheses of Theorem 4.1 be satisfied. Let $x^0 \in \Omega$ be a critical point, let ℓ be the number of arcs of the level line $\{x \in \Omega | u(x) = u(x^0)\}$ intersecting at x^0.*

There exists a simply connected open set $G \subset \Omega$ such that: $x^0 \in G$, ∂G is Lipschitz, and $u_{|\partial G}$ has ℓ maxima.

PROOF. (See Fig. 4). Let us first note that ℓ is finite, in fact, arguing as in Lemma 1.1, we get that if ℓ were infinite then $u_{|\partial\Omega}$ would have an infinite number of maxima.

Whithout loss of generality we may set $x^0 = 0$, $u(0) = 0$. Let $R > 0$ be such that $B_{2R}(0) \subset \Omega$ and $|Du| > 0$ in $B_{2R}(0) \setminus \{0\}$.

Let $\gamma_1, \dots, \gamma_\ell$ be the simple arcs such that $\bigcup_{i=1}^{\ell} \gamma_i$ is the connected component of the level line $\{x \in B_{2R}(0) | u(x) = 0\}$ which contains the origin.

Let $P_1, \dots, P_{2\ell}$ be ordered points on $\partial B_R(0)$ such that, for every $i = 1, \dots, \ell$, P_i and $P_{\ell+i}$ are the first two points which, starting from the origin and moving on opposite directions on γ_i, meet $\partial B_R(0)$.

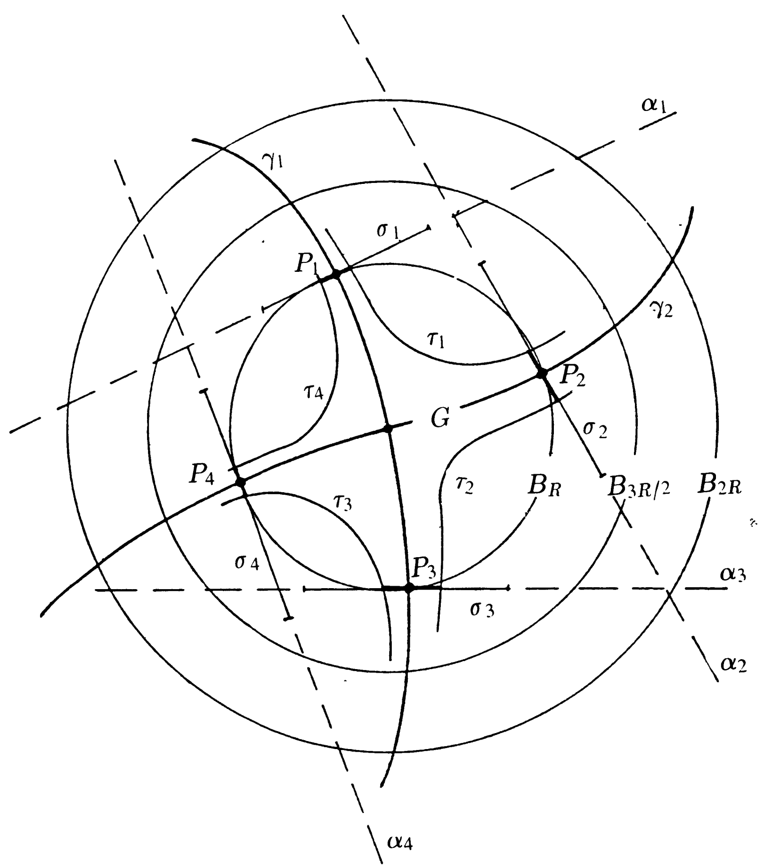

Fig. 4 ($\ell = 2$)

Let $\alpha_i, \alpha_{\ell+i}$, $i = 1, \dots, \ell$ be the straight lines normal to γ_i passing through $P_i, P_{\ell+i}$ respectively. Such lines exist and are uniquely determined since, on $\partial B_R(0)$, $|Du| \geq$ const. > 0.

We may find $t_0 > 0$ and segments $\sigma_i \subset \alpha_i$, $i = 1, \dots, 2\ell$, such that: $P_i \in \sigma_i$, $u_{|\sigma_i}$ takes all the values between $-t_0$ and t_0 and has non-zero directional derivative, $\sigma_i \subset B_{3R/2}$ for every i, $\sigma_i \cap \sigma_j = \emptyset$ for every i, j, $i \neq j$.

We may also find $t \in (0, t_0)$ such that there are 2ℓ simple arcs τ_i, $i = 1, \dots, 2\ell$ in $B_{3R/2}(0)$, having end-points on σ_i, σ_{i+1} (here we identify: $\sigma_{2\ell+1} = \sigma_1$) on which u takes alternatively the values $\pm t$.

Let σ'_i be the portion of σ_i containing P_i, on which $|u| < t$. the arcs $\tau_1, .., \tau_{2\ell}$ and the segments $\sigma'_1, .., \sigma'_{2\ell}$ form a simple closed curve surrounding 0.

Let G be the open set bounded by such curve. Note that $u_{|\partial G}$ has ℓ maxima and ℓ minima at levels t and $-t$ respectively. Moreover ∂G is Lipschitz, in fact $\tau_1, \ldots, \tau_{2\ell}$ are $C^{1,\delta}$ - smooth and at the connection points of τ_i with $\sigma'_j (j = i, i+1)$ the tangent line to τ_i is always different from α_j, since, along $\sigma_j \subset \alpha_j, u$ has non-zero directional derivative.

LEMMA 4.2 *Let the hypotheses of Theorem 4.1 be satisfied. Let $x^0 \in \Omega$ be a critical point, let ℓ be the number of arcs of the level line $\{x \in \Omega | u(x) = u(x^0)\}$ intersecting at x^0. let m be the index of $\overline{Du}$ at x^0. The following inequality holds*

$$m \leq \ell - 1. \tag{4.10}$$

PROOF. Let G be the open set described in Lemma 4.1. Let us set once more: $x^0 = 0$, $u(0) = 0$.

For every $n = 1, 2, \ldots$, and $i, j = 1, 2$, let $a^n_{ij} \in C^\infty(\Omega)$, $b^n_i \in C^\infty(\Omega)$ be such that a^n_{ij} satisfy condition (6), $(b^n_1)^2 + (b^n_2)^2 \leq E^2$, and for some $p > 2$,

$$\begin{array}{ll} a^n_{ij} \to a_{ij} & , \text{ in } L^p(\Omega), \\ b^n_i \to b_i & , \text{ in } L^p(\Omega). \end{array}$$

For every $n = 1, 2, \ldots$ let $u^n \in W^{2,2}_{\text{loc}}(G) \cap C(\overline{G})$ be the solution of

$$\begin{cases} \displaystyle\sum_{ij=1}^{2} a^n_{ij} u^n_{x_i x_j} + \sum_{i=1}^{2} b^n_i u^n_{x_i} = 0, & \text{in } G, \\ u^n = u, & \text{on } \partial G. \end{cases}$$

It is a rather standard fact that such solutions u^n are equicontinuous on $\overline{G}$ (see Miller [10]) and that for every $Q \subset\subset G$ the following uniform bounds hold (see Talenti [15])

$$\begin{aligned} \|u^n\|_{W^{2,2}(Q)} &\leq \text{ Const. }, \\ \|u^n\|_{C^{1,\delta}(Q)} &\leq \text{ Const.}. \end{aligned}$$

Consequentely we may find a subsequence, which we will continue to call $\{u^n\}$, such that, as $n \to \infty$,

$$u^n \to u \text{ in } C(\overline{G}),$$

and for every $Q \subset\subset G$:

$$u^n \to u \text{ in } C^{1,\delta'}(Q), \text{ for every } \delta' < \delta.$$

Let $r_0 > 0$ be such that $B_{r_0}(0) \subset G$, let $r \in (0, r_0)$, by the above convergence properties we have

$$\max_{|x|=r} |Du^n(x) - Du(x)| \to 0 \text{ as } n \to \infty.$$

Now, since, for $|x| = r : |Du(x)| \geq c(r) > 0$, we may find $n(r) > 0$ such that: $|Du^n(x)| \geq c(r)/2$ for every $n > n(r)$, $|x| = r$.

Hence we get

$$\max_{|x|=r} \left| \frac{Du^n(x)}{|Du^n(x)|} - \frac{Dy(x)}{|Du(x)|} \right| \leq 2 \max_{|x|=r} \frac{|Du^n(x) - Du(x)|}{|Du(x)|} \leq$$
$$\leq 2c^{-1}(r) \max_{|x|=r} |Du(x)| \to 0, \text{ as } n \to \infty.$$

Therefore, for sufficiently small r and for sufficiently large n:

$$\frac{1}{2\pi} \int_{|x-x^0|=r} d\left(\arg \overline{Du^n(x)}\right) = m.$$

Thus, by Theorem (H.-W.), m coincides with the sum of the multiplicities of the critical points of u^n in $B_r(x^0)$. And then, finally, by Theorem 1.1, we get (4.10).

PROOF of THEOREM 4.1. By the above Lemma we may rephrase the arguments leading to the proof of Theorem 1.1 just replacing the term: "multiplicity of Du" with: "index of $\overline{Du}$".

Thus we obtain (4.6) where $m_1, \ldots, m_K$ are the indices of $\overline{Du}$ at the critical points $x_1, \ldots, x_K$ respectively.

The use of the inequality on the left in (4.9) yields (4.5).

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