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Threefolds of Non Negative Kodaira Dimension with Sectional Genus Less than or Equal to 15.

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Let X be a smooth connected n dimensional subvariety of complex projective space, $\boldsymbol{P}^N$. Assume that X has non negative Koidara dimension, i.e. that some positive power of the canonical bundle K_X has a non-trivial holomorphic section. In this paper we use the results of [So3] to investigate what the numerical invariants of such X are under the assumption that the sectional genus g of X (i.e. the genus of $X \cap \boldsymbol{P}^{N-n+1}$ for a generic linear $\boldsymbol{P}^{N-n+1} \subseteq \boldsymbol{P}^N$) is small. This problem is studied most throughly under the assumptions that $n = 3$ and $g \leqslant 15$, but a number of partial results for arbitrary g and n are shown.

In § 0 we recall background material and especially the results of [So3]. The latter results (see (0.8)) relate the surface $S = X \cap \boldsymbol{P}^{N-n+2}$ for a general linear $\boldsymbol{P}^{N-n+2}$ to its minimal model S'. This lets us use the arguments from the theory of minimal surface to prove a number of results about the invariants of X. We also generalize a result of Griffiths and Harris [Gr-H] by relaxing a hypothesis about a projective n fold X from « $h^{n,0}(X) \neq 0$ » to « X is of non negative Kodaira dimension ».

In § 1 we prove a number of general results. One example is the following.

(1.1) THEOREM. *Let X be an n dimensional connected submanifold of $\boldsymbol{P}^N$ not contained in any hyperplane. Let d denote the degree of X in $\boldsymbol{P}^N$ and assume that $K_X^t \approx \mathcal{O}_X$ for some $t \neq 0$. If $d < n(N+1)$ then the order of the*

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fundamental group of X is finite and bounded by:

$$\frac{n^2+n-2}{n(N+1)-d}.$$

In particular X is simply connected if

$$d \leqslant n\left(N-\frac{n-1}{2}\right).$$

In section 2 and 3 the threefolds in $\boldsymbol{P}^5$ and $\boldsymbol{P}^6$ are dealt with the following tables.

Let X be a smooth connected threefold of non negative Kodaira dimension and let L be a very ample line bundle on X. The following table summarizes our knowledge for such X with $K_X^t \approx \mathcal{O}_X$ for some $t \neq 0$ and with sectional genus $g \leqslant 15$. Here S denotes a smooth $S \in |L|$.

TABLE I

$h^0(L) = \chi(\mathcal{O}_S)$	$d = K_S \cdot K_S = g-1$	$e(X)$	Structure
5	5	-200	
6	8	-176	Cor. (0.7.5)
6	9	-144	
7	11	-170	?
7	12	-144	Rem. (0.7.5.1)
7	13	-120	
7	14	-98	?
8	14	$-128 \geqslant e(X) \geqslant -416$	

All are simply connected with K_X trival. For all $2(h^{1,1}(X) - h^{1,2}(X)) = e(X)$, the Euler characteristic of X. For all $h^{i,0}(X) = 0$ for $i = 1$ and 2. ? means that we know no examples with these invariants.

The next table summarizes all the examples with $g \leqslant 15$ not already listed and with $h^0(L) \leqslant 6$.

TABLE II

$h^0(L)$	d	$K_S \cdot L_S$	$K_S \cdot K_S$	g	$\chi(\mathcal{O}_S)$	$\chi(\mathcal{O}_X)$
5	6	12	24	10	10	5
5	7	21	63	15	20	15
6	9	13	17	12	9	$-1 \leqslant \chi \leqslant 1$
6	10	12	12	12	7	0
6	10	14	19	13	9	$\chi \leqslant 1$
6	10	16	20	14	10	$\chi \leqslant 1$
6	10	18	27	15	12	$\chi \leqslant 2$
6	11	15	16	14	8	0
6	11	17	17	15	9	0
6	11	17	23	15	10	$\chi \leqslant 1$
6	12	16	14	15	7	0
6	12	16	20	15	8	$\chi \leqslant 1$

All the above are simply connected and thus have $h^{1,0}(X) = h^{1,0}(S) = 0$. For more information about the minimality of S see Table VI.
$K_X \cdot K_X \cdot K_X$ and $e(X)$ are linked by relation (0.7.3a) all of whose terms but these 2 can be calculated from the information above. No examples are known with any of the invariants above when $h^0(L) = 6$.

We would like to call here attention to [Ho-Sc] which studies a complementary question.

Finally we list those X with $h^0(L) \geqslant 7$ that are not in table I. For all of them $h^0(L) = 7$.

TABLE III

d	g	$K_S \cdot L$	$K_S \cdot K_S$	$\chi(\mathcal{O}_S)$	Comments
12	14	14	12	7	S is minimal model with 1 point blown up
12	14	14	14	$\leqslant 8$	minimal
12	14	14	15, 16	$\leqslant 7$	minimal $h^{3,0} = 0,\ h^{1,0} \neq 0$
12	15	16	12	7 or 8	S is minimal model with 2 points blown up
12	15	16	$14 \leqslant K_S \cdot K_S \leqslant 21$	?	minimal if $K_S \cdot K_S \geqslant 17$
13	15	15	13	7 or 8	S is minimal with 1 point blown up
13	15	15	$15 \leqslant K_S \cdot K_S \leqslant 17$	?	minimal if $K_S \cdot K_S \leqslant 16$

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0. – Notation and background material.

We review our notation which agrees with that of [So2] and [So3].

(0.1) Given a sheaf $\mathcal{S}$ of abelian groups on a topological space X, we denote the global sections of $\mathcal{S}$ over X by $\Gamma(\mathcal{S})$, or when some confusion can result, by $\Gamma(X, \mathcal{S})$. We let $h^i(X, \mathcal{S})$ or $h^i(\mathcal{S})$ for short denote $\dim h^i(X, \mathcal{S})$.

(0.2) *All* spaces, manifolds, vector bundles and maps are *complex analytic*, and all dimensions are over $\boldsymbol{C}$. We often abbreviate complex analytic to complex. Given an analytic space X, we denote its structure sheaf by $\mathcal{O}_X$, its smooth points by X_{reg}, and its singular points by Sing (X). We do not distinguish between holomorphic vector bundle E on a complex space, X, and its sheaf of germs of holomorphic sections. Thus when a coherent analytic sheaf and a holomorphic vector bundle are tensored together, the meaning is clear; the appropriate sheaves are being tensored over $\mathcal{O}_X$.

(0.3) We often denote complex projective space by $\boldsymbol{P}_{\boldsymbol{C}}$ when its exact dimension is irrelevant. A line bundle L on a complex variety, X, is *very ample* if $\Gamma(L)$ spans L and the map $\varphi\colon X \to \boldsymbol{P}_{\boldsymbol{C}}$ associated to $\Gamma(L)$ is an embedding. A line bundle L on a variety, X, is *ample* if L^n is very ample for some $n > 0$ ([Ha1] is a good reference for ampleness).

(0.4) For X a complex manifold of pure dimension, K_X denotes the canonical bundle of X, $\det(T_X^*)$ where T_X^* is the holomorphic cotangent bundle of X. For almost everything we do this is sufficient, but in order to state theorem (0.6) in its proper generality we need the notion of the canonical sheaf ω_X for X a normal complex variety ([R] is a good reference). Over X_{reg}, $\omega_{X_{\text{reg}}}$, is the sheaf of germs of local sections of $K_{X_{\text{reg}}}$. By definition ω_X equals the direct image, $i_*\omega_{X_{\text{reg}}}$ where $i\colon X_{\text{reg}} \to X$ is the inclusion map. We let $\omega_X^{(t)} = i_*\omega_X^t$ for $t > 0$. The sheaves $\omega_X^{(t)}$ are torsion free coherent sheaves. If $\omega_X^{(t)}$ is invertible for some $t > 0$, then X is said to be $\boldsymbol{Q}$-Gorenstein. If $h^0(\omega_X^{(t)}) \neq 0$ for some $t > 0$, then X is said to have non-negative Kodaira dimension.

(0.5) LEMMA. *Let X be an n dimensional, irreducible, normal projective variety of non negative Kodaira dimension. Let $f\colon X \to \boldsymbol{P}^N$ be a finite to one,* *generically one to one, holomorphic map such that $f(X)$ is not contained in any hyperplane. Let d equal the degree of $f(X)$ in $\boldsymbol{P}^N$ and let g equal the genus of C, a smooth curve obtained by pulling back a generic linear $\boldsymbol{P}^{N-n+1} \subseteq \boldsymbol{P}^N$ under f. Then:*

$$d \leqslant \frac{2(g-1)}{n-1} \tag{0.5.1}$$

with equality if and only if $\omega_X^{(t)} \approx \mathcal{O}_X$ for some $t > 0$.

PROOF. Let $L = f^*(\mathcal{O}_{\boldsymbol{P}^N}(1))$. Note that $d = \text{degree}\,(L_C)$. By the adjunction formula $K_C \approx (\omega_X \otimes L^{n-1})_C$. Thus:

$$2g-2 = \text{degree}\,(\omega_{X,C}) + (n-1)d \geqslant (n-1)d\,.$$

To see this note that $\omega_X^{(t)}$ has a section for some $t > 0$. Since $\operatorname{Sing}(X)$ is of codimension at least 2 by normality it follows that for a general $\boldsymbol{P}^{N-n+1}$, C doesn't meet $\operatorname{Sing}(X)$ and therefore:

$$\text{degree}\,(\omega_{X,C}) = \frac{1}{t}\,\text{degree}\,(\omega_{X,C}^{(t)}) \geqslant 0\,.$$

If equality happens in (0.5.1) then the section of $\omega_X^{(t)}$ cannot vanish anywhere in $X_{\text{reg}(X)}$. If it did then by ampleness of L, C would have to meet the divisor on X_{reg} where the section vanished. Therefore $\omega_{\text{reg}}^{(t)} \approx \mathcal{O}_{X_{\text{reg}}}$ and by definition of $\omega_X^{(t)}$ we conclude that $\omega_X^{(t)} \approx \mathcal{O}_X$. The converse is clear. □

(0.6) THEOREM. *Let $f\colon X \to \boldsymbol{P}^N$ be finite to one, generically one to one, holomorphic map where X is an n dimensional irreducible, normal, projective variety of non negative Kodaira dimension. Assume that $f(X)$ belongs to no hyperplane and that d equals the degree of $f(X)$ in $\boldsymbol{P}^N$. Then $d \geqslant n(N-n)+2$ with equality only if $\omega_X^{(t)} \approx \mathcal{O}_X$ for some $t > 0$.*

PROOF. Choose C as in the last lemma.

By the last lemma

$$d \leqslant \frac{2g-2}{n-1}\,. \tag{$*$}$$

By Castelnuovo's lemma (see [Ba], [Gr-H]):

$$g \leqslant \left[\frac{d-2}{N-n}\right]\left(d-N+n-1-\left(\left[\frac{d-2}{N-n}\right]-1\right)\frac{N-n}{2}\right) \tag{$**$}$$

where $[x]$ denotes the greatest integer $\leqslant x$.

Assume that

$$(***) \qquad d < n(N-n) + 2\,.$$

Then $[(d-2)/(N-n)] \leqslant n-1$. Thus by (**) and by

$$d - N + n - 1 = 2\left(\frac{d-2}{N-n} - 1\right)\left(\frac{N-n}{2}\right) + 1$$

it follows that

$$g \leqslant (n-1)\left(1 + \frac{N-n}{2}\left(\frac{d-2}{N-n} - 1 + \frac{d-2}{N-n} - \left[\frac{d-2}{N-n}\right]\right)\right)$$

but

$$0 \leqslant \frac{d-2}{N-n} - \left[\frac{d-2}{N-n}\right] \leqslant 1$$

hence

$$g \leqslant (n-1)\left(1 + \frac{d-2}{2}\right)$$

which implies the absurdity

$$2g \leqslant (n-1)d \leqslant 2(g-1)\,.$$

Therefore $d \geqslant n(N-n) + 2$.

If we have equality then (**) becomes

$$g \leqslant n\left(d - N + n - 1 - (n-1)\frac{N-n}{2}\right) \quad \text{or} \quad 2g \leqslant n(d - N + n)\,.$$

If $\omega_X^{(t)} \neq \mathcal{O}_X$ for all $t > 0$ then by the last lemma we have $d < (2g-2)/(n-1)$. Combining this with the last inequality we get:

$$2g < d + 2g - 2 + n(n-N) \leqslant 2g\,.$$

This contradiction $2g < 2g$ implies that $d = (2g-2)/(n-1)$ and hence $\omega_X^{(t)} = \mathcal{O}_X$. □

(0.6.1) REMARK. The above result is a slight generalization of [Gr-H; Cor. 2.23]. They require that $h^{n,0}(X) > 0$ rather than just that X has non negative Kodaira dimension. The rest of their result where they relax their hypothesis to $h^0(K_X \otimes L^r) > 0$ similarly generalizes with the weaker hypothesis that $h^0((K_X \otimes L^r)^M) > 0$ for some $M > 0$.

(0.7) Let X be 3 dimensional connected projective submanifold of $\boldsymbol{P}^N$. Let d denote the degree of X, g the sectional genus of X, and let S denote a smooth hyperplane section of X. From [Ha2, Appendix] we have:

$$(0.7.1) \qquad \begin{cases} \text{if } N\leqslant 5 \text{ then:} \\ d^2-5d-10(g-1)+12\chi(\mathcal{O}_S)=2K_S\cdot K_S. \end{cases}$$

The same reasoning as that yielding (0.7.1) applied to $\boldsymbol{P}^6$ yields:

$$(0.7.2) \qquad \begin{cases} \text{if } N\leqslant 6 \text{ then } -e(X)=7c_2(X)\cdot L-48\chi(\mathcal{O}_X) \\ d^2-35d-21K_X\cdot L\cdot L-7K_X\cdot K_X\cdot L-K_X\cdot K_X\cdot K_X \end{cases}$$

where $e(X)$ denotes the Euler characteristic of X and $c_2(X)$ denotes the second Chern class of X.

The next few inequalities are modeled in [So3; Lemma (3.1)].

(0.7.3) THEOREM. *Let X be as above. Then:*

a) $K_X\cdot K_X\cdot K_X+6K_X\cdot K_X\cdot L+15K_X\cdot L\cdot L+20d-e(X)$
$+48\chi(\mathcal{O}_X)-6c_2(X)\cdot L\geqslant 0$ *with equality if* $N\leqslant 5$,

b) $K_S\cdot K_S+4K_S\cdot L_S+6d\geqslant e(S)$,

c) $2e(S)\geqslant e(X)-2g+2$,

d) $-24\chi(\mathcal{O}_X)+3K_X\cdot K_X\cdot L+15K_X\cdot L\cdot L+2L\cdot c_2(X)+20d+e(X)\geqslant 0$,

e) [So3, Lemma (3.1)] *if X has nonnegative Kodaira dimension then*

$$8\chi(\mathcal{O}_X)\leqslant K_S\cdot K_S-d.$$

PROOF. Since $J_1(X, L)$ the first holomorphic jet bundle of L in X, is spanned by $N+1$ sections we have a holomorphic map:

$$\Phi\colon \boldsymbol{P}(J_1(X, L))\to \boldsymbol{P}^N,$$

where $\boldsymbol{P}(J_1(X, L))=(J_1(X, L)^*-X)/\boldsymbol{C}^*$.

Letting ξ denote the tautological line bundle on $\boldsymbol{P}(J_1(X, L))$ such that $\xi\approx\Phi^*\mathcal{O}_{\boldsymbol{P}^N}(1)$ we conclude that the degree $\xi\cdot\xi\cdot\xi\cdot\xi\cdot\xi\cdot\xi$ is $\geqslant 0$ and equal to 0 if $N\leqslant 5$ $\bigl($since then $\dim\Phi\bigl(\boldsymbol{P}(J_1(X, L))\bigr)<\dim\boldsymbol{P}(J_1(X, L))\bigr)$. Letting

$c_i = c_i(J_1(X, L))$ and using the tautological relation:

$$\xi\cdot\xi\cdot\xi - c_1\cdot\xi\cdot\xi + c_2\cdot\xi - c_3 = 0$$

we conclude that $\xi\cdot\xi\cdot\xi\cdot\xi\cdot\xi\cdot\xi = c_1^3 - 2c_1\cdot c_2 + c_3$ which computed out gives *a*) above.

Since $J_1(X, L)$ is spanned by global sections we conclude (e.g. from [F], page 216) that

$$L\cdot(c_1^2 - c_2)\geqslant 0\,, \quad c_3\geqslant 0\,, \quad c_1\cdot c_2\geqslant c_3$$

which give *b*), *c*), and *d*) above. □

(0.7.4) COROLLARY. *If in* (0.7) *we assume that* $K_X^t \approx \mathcal{O}_X$ *for some* $t\neq 0$ *then*:

a) $26d - e(X) - 6e(S)\geqslant 0$ *with equality in* $N\leqslant 5$,

b) $2e(S)\geqslant e(X) - 2d$,

c) $2e(S) + 18d + e(X)\geqslant 0$,

d) *if* $N = 5$ *then* $d^2 - 17d + 12\chi(\mathcal{O}_X) = 0$.

PROOF. (0.7.3*a*) yields *a*) above, and (0.7.3*c*) and (0.7.3*d*) yield respectively *b*), and *c*) above. Finally (0.7.1) yields *d*). □

(0.7.5) COROLLARY. *Assume that* X *is a smooth projective threefold such that* $K_X^t \approx \mathcal{O}_X$ *for some* $t > 0$. *If* L *is a very ample line bundle on* X *and* $h^0(L)\leqslant 6$, *then either*:

a) $h^0(L) = 5$ *and* X *is a degree* 5 *hypersurface in* $\boldsymbol{P}^4$, *or*

b) $h^0(L) = 6$, $d = 8$, *and* X *is the transverse intersection of a degree* 2 *and a degree* 4 *hypersurface in* $\boldsymbol{P}^5$, *or*

c) $h^0(L) = 6$, $d = 9$, *and* X *is the transverse intersection of* 2 *degree* 3 *hypersurfaces of* $\boldsymbol{P}^5$.

PROOF. *a*) is obvious. Therefore we can assume that $h^0(L) = 6$, with X imbedded in $\boldsymbol{P}^5$. By the Barth-Larsen theorem [Ba], [Sol], it can be assumed that X is simply connected. Thus $K_X \approx \mathcal{O}_X$. From this and the Kodaira vanishing theorem it follows that

$$\chi(\mathcal{O}_X) = \chi(K_S) \quad \text{and} \quad \chi(L) = h^0(L) = 6\,.$$

Using this and:

$$0 \to \mathcal{O}_X \to L \to L_S \to 0$$

we conclude that $\chi(\mathcal{O}_S) = 6$ and by (0.7.4 d) that $d^2 - 17d + 84 = 0$ or that $d = 8$ or 9.

We will proceed to finish the proof only for $d = 8$ since the proof for $d = 9$ is identical.

Choose two general sections $\langle s, t\rangle$ of L and denote the generic smooth curve of common zeros by C. We have an exact sequence:

$$0 \to L^{\ell} \to L^{\ell+1} \oplus L^{\ell+1} \to L^{\ell+2} \to K^C \otimes L_C^{\ell} \to 0$$

gotten from the Koszul complex associated to the section $s \oplus t$ of $L \oplus L$ by tensoring with L^{ℓ} and by noting that $L_C^2 \approx K_C$. Using the Kodaira vanishing theorem, $K_X \approx \mathcal{O}_X$, $h^0(L) = 6$ and the hypercohomology spectral sequence associated to the above exact sequence for $\ell = 0, 1$, and 2 we conclude that:

(*) $$h^0(L^2) = 20 \quad \text{and} \quad h^0(L^4) = 104\,.$$

Note that:

$$h^0(\mathcal{O}_{P^5}(2)) = 21 \quad \text{and} \quad h^0(\mathcal{O}_{P^5}(4)) = 126\,.$$

We see from considering the restriction map

$$\Gamma(\mathcal{O}_{P^5}(2)) \to \Gamma(X, L^2)$$

and using (*) and (**) that there is at least one quadric $\Delta \supseteq X$. From the fact that restriction gives $\Gamma(\mathcal{O}_{P^5}(1)) \approx \Gamma(X, L)$ we conclude that Δ is irreducible.

From (*) and (**) we similarly conclude that there is a vector space $V \subseteq \Gamma(\mathcal{O}_{P^5}(4))$ of sections of $\mathcal{O}_{P^5}(4)$ vanishing on X with $\dim V \geqslant 22$. Since $h^0(\mathcal{O}_{P^5}(4) \otimes [\Delta]^{-1}) = h^0(\mathcal{O}_{P^5}(2)) = 21$ we conclude that there is at least one quadric $H \supseteq X$ which is not of the form $\Delta + \Delta'$. Thus $H \cap \Delta$ is of codimension 2 and since it has degree 8 and contains X which is also of degree 8 and codimension 2 it follows that $H \cap \Delta = X$ ideal theoretically. In particular H and Δ must be smooth in a neighborhood of X and the intersection is transverse. □

(0.7.5.1) REMARK. If $h^0(L) = 7$, $d = 12$, $K_X^t = \mathcal{O}_X$ then it can be shown that X is the transversal intersection of two quadratic and a cubic hypersurface of P^5.

We will need the following result of Castelnuovo-Beauville [Ba-P-V]

(0.7.6) LEMMA. *Let S be a minimal model of general type. If $\Gamma(K_S)$ gives a birational map of S then*

$$K_S \cdot K_S \geqslant 3h^{2,0}(S) - 7 .$$

The following is the main theorem of [So3].

(0.8) THEOREM. *Let X be 3 dimensional projective manifold of non negative Kodaira dimension. Let L be an ample line bundle on X. Every smooth $S \in |L|$ is of general type and satisfies $K_S \cdot K_S \geqslant d = S \cdot S \cdot S$. Further there exists an ample line bundle L' on an algebraic manifold X' such that:*

a) X is the blow-up $\pi \colon X \to X'$ of X' at a finite set F of points,

b) every smooth $S \in |L|$ is the proper transform of a smooth $S' \in |L'|$ with $S' \supseteq F$ and $\pi_S \colon S \to S'$ the map of S onto its minimal model,

c) for S' and S as in b):

$$K_{S'} \cdot K_{S'} \geqslant K_{S'} \cdot L'_{S'} \geqslant L'_{S'} \cdot L'_{S'} .$$

The equality $K_{S'} \cdot L'_{S'} = L'_{S'} \cdot L'_{S'}$ is equivalent to $K_S \cdot K_S = d$ and this is equivalent to $K^t_{X'}$ being the trivial bundle for some $t \neq 0$. If X is the transversal intersection of $n-3$ divisors $\langle D_i \rangle \leqslant |\mathfrak{L}|$ where $\mathfrak{L}$ is an ample line bundle on an n fold of non negative Kodaira dimension and $\mathfrak{L}_X = L$ then it is also shown [So3; (1.9.4)] *that:*

$$(n-2) L'_{S'} \cdot K_{S'} \leqslant K_{S'} \cdot K_{S'} ,$$

$$(n-2) L'_{S'} \cdot L'_{S'} \leqslant K_{S'} \cdot L_{S'} ,$$

$$\frac{(n-2)^2}{n-1} (2g-2) \leqslant K_{S'} \cdot K_{S'} .$$

The following lemma is used throughout [So3].

(0.8.1) LEMMA. *If S, L and X are as in* (0.8) *and if*

$$(K_S \cdot K_S + 1)(d+1) > (K_S \cdot L - 1)^2$$

then smooth $S \in |L|$ are minimal models.

PROOF. If S is not minimal by the above theorem $L'_{S'}\cdot L'_{S'} > L_S\cdot L_S = d$, $K_{S'}\cdot K_{S'} > K_S\cdot K_S$ and $K_S\cdot L_S > K_{S'}\cdot L'_{S'}$. Thus by the Hodge index theorem we get

$$(d+1)(K_S\cdot K_S+1)\leqslant(L'_{S'}\cdot L'_{S'})(K_{S'}\cdot K_{S'})\leqslant(K_{S'}\cdot L'_{S'})^2<(K_S\cdot L-1)^2$$

contradicting the hypothesis of the lemma. □

We adopt the notation $d'=L'_{S'}\cdot L'_{S}$, from here on.

(0.8.2) LEMMA. *Let X, L and S be as in Theorem* (0.8). *Then*

$$K_S\cdot K_S\neq d+1\ .$$

PROOF. If $K_S\cdot K_S=d+1$ then $K_{S'}\cdot K_{S'}=d'+1$. By the inequality

$$d'\leqslant K_{S'}\cdot L'_{S'}\leqslant K_{S'}\cdot K_{S'}$$

and the fact that the parities of d' and $K_{S'}\cdot L'_{S'}$ are the same one so that $d'=K_{S'}\cdot L'_{S'}$. But this implies by (0.8) that $d=K_S\cdot K_S$. □

(0.8.3) THEOREM. *Let X, L, and S be as above. Assume further that L is very ample. Then $K_S\cdot K_S=d+3$ implies that $h^{3,0}(X)=0$, $h^2(L'_{S'})=0$, $h^0(L)\geqslant 7$, $h^{1,0}(X)\neq 0$, and $K_{S'}\cdot L'_{S'}=d'+2$. If $K_S\cdot K_S=d+5$ then $K_{S'}\cdot L'_{S'}=d'+4$ and $h^{3,0}(X)\leqslant 1$.*

PROOF. If $K_S\cdot K_S=d+3$ then $K_{S'}\cdot K_{S'}=d'+3$ and $K_{S'}\cdot L'_{S'}=d'+2$ by the same reasoning as (0.8.2). If $h^{3,0}(X)\neq 0$ we can choose a $D\in|K_{X'}|$ and a smooth $S'\in|L'|$ such that $S'\cap D$ is a curve C. Note that

$$C\cdot L'_{S'}=K_{X'}\cdot L'\cdot L'=K_{S'}\cdot L'-d'=2\ .$$

Since $\Gamma(L'_C)$ gives an embedding on a dense open set of C one conclude that either C is a smooth rational curve or a union of 2 smooth rational curves $E+F$. Since the intersection of C with itself on S' is:

$$C\cdot C=K_{X'}\cdot K_{X'}\cdot L'_{S'}=K_{S'}\cdot K_{S'}-2K_{S'}\cdot L'_{S'}+d'=-1$$

we see that either C is an exceptional curve of S' contradicting the minimality asserted in (0.8) or

$$-1=(E+F)\cdot(E+F)=E\cdot E+F\cdot F+2E\cdot F\,.$$

This would imply that either $E\cdot E\geqslant 0$ or $F\cdot F\geqslant 0$. If say $E\cdot E\geqslant 0$ then since degree $(N_E)=E\cdot E$ we conclude that $h^0(N_E)\neq 0$ and $h^1(N_E)=0$. By deformation theory this implies that S' contains a pencil of rational curves. This contradicts the fact that S' is of general type. Note that if $h^2(L_{S'})\neq 0$ then $h^0(K_{S'}\otimes L_{S'}^{\prime -1})\neq 0$. Thus $h^0(K_{X',S'})\neq 0$ and the same argument as above gives a contradiction.

By (0.7.3 *e*) $\chi(\mathcal{O}_X)\leqslant 0$. Since $h^{3,0}(X)=0$ we see that $h^{1,0}(X)>0$. Further since threefolds in $\boldsymbol{P}^5$ are simply connected by the Barth-Larsen theorem [Ba], [S01] we see that $h^0(L)\geqslant 7$.

If $K_S\cdot K_S=d+5$ then $K_{S'}\cdot K_{S'}=d'+5$. By the same argument as above $K_{S'}\cdot L'_{S'}=d'+2$ or $d'+4$. The former is impossible by the Hodge index theorem:

$$d'(d'+5)\leqslant(K_{S'}\cdot L_{S'})^2=d'^2+4d'+4\,.$$

Thus $d'+4=K_{S'}\cdot L'_{S'}$. By (0.7.3 *e*) $\chi(\mathcal{O}_X)\leqslant 0$. If $h^{3,0}(X)\geqslant 2$ then for a generic $S\in|L|$, $h^0(K_{X',S'})\geqslant 2$. Since $K_{X',S'}\cdot K_{X',S'}=-1$ this is impossible unless the generic member of $|K_{X',S'}|$ is reducible. Thus the moving part of $|K_{X',S'}|$ has degree $<K_{X',S'}\cdot L'_{S'}=K_{S'}\cdot L'_{S'}-d'=4$. This implies that the moving part is elliptic or rational. This is impossible since we can't have a pencil of elliptic or rational curves on a general type surface. □

(0.8.4) LEMMA. *Set L, X and S be as in* (0.8). *Assume further that L is very ample. If $4+d=K_S\cdot K_S$ and $2+d'=K_{S'}\cdot L'_{S'}$ then $h^{3,0}(X)=0$, $h^{1,0}(X)>0$ and $h^0(L)\geqslant 7$.*

PROOF. By (0.7.3 *e*) we conclude that $\chi(\mathcal{O}_X)\leqslant 0$. If we show that $h^{3,0}(X)=0$ then we will be done by the reasoning of (0.8.3). If $h^{3,0}(X)>0$ then we can choose a curve $C\in|K_{X',S'}|$ for a generic smooth $S'\in|L'|$ such that:

$$C\cdot L'=2 \quad \text{and} \quad C\cdot C=0 \quad \text{on } S'\,.$$

The former as before implies that C is either a smooth rational curve or a union of two smooth rational curves $E+F$ with $E\cdot F\leqslant 1$. In either case we get a smooth rational curve on S' with non negative intersection on S'. As in (0.8.3) this implies by deformation theory that there is a pencil of rational curves in S'. □

(0.8.5) LEMMA. *Let X, L, and S be as in* (0.8). *Assume further that L is very ample. Then $h^{3,0}(X) \neq 0$ and $h^{1,0}(X) = 0$ implies that $\chi(\mathcal{O}_S) \geqslant (h^0(L_S) + 1)$ In particular if $K_S \cdot K_S \leqslant d + 7$ and $h^0(L) = 6$ then $\chi(\mathcal{O}_S) \geqslant 6$.*

PROOF. If $h^{3,0}(X) \neq 0$ then choose a smooth $S \in |L|$ such that

$$\Gamma(K_{X,S}) \neq 0 .$$

Then by the first Lefschetz theorem $h^{1,0}(X) = h^{1,0}(S)$ and therefore

$$\chi(\mathcal{O}_S) = h^0(K_S) + 1 \leqslant h^0(L_S) + 1$$

since

$$K_{X,S} \otimes L_S \approx K_S . \qquad \square$$

We now use (0.8) to classify the possible numerical invariants of threefolds X of nonnegative Kodaira dimension with a very ample line bundle L such that $h^0(L) = 6$ and $d = g - 2$ or $d = g - 3$. These results which extend are summarized in table V below and will be used in section 2.

Since $d + K_S \cdot L_S = 2g - 2$ we conclude that $K_S \cdot L_S = g = d + 2$ and therefore by the Hodge index theorem

$$K_S \cdot K_S \leqslant \frac{(d+2)^2}{d} = d + 4 + \frac{4}{d} .$$

Since $d \geqslant 8$ we see that $K_S \cdot K_S \leqslant d + 4$. Note by (0.8.1) that $K_S \cdot L_S = d + 2$ implies that smooth $S \in |L|$ are minimal models. By (0.8.2), (0.8.3), (0.8.4) and (0.8.5) we conclude:

$$K_S \cdot K_S \in \{d, d+2\} \quad \text{and} \quad \chi(\mathcal{O}_S) \geqslant 6 .$$

The relation (0.7.1) becomes

$$d^2 - 15d - 10 + 12\chi(\mathcal{O}_S) = 2K_S \cdot K_S .$$

By Castelnuovo inequality we can assume $d \geqslant 9$.

Checking this relation systematically for $K_S \cdot K_S = d$ and $d + 2$ and using $\chi(\mathcal{O}_X) \geqslant 6$ we see that $d = 10$, $K_S \cdot L_S = 12$, $g = 12$, $K_S \cdot K_S = 12$, $\chi(\mathcal{O}_S) = 7$ and S is minimal. Also $h^{1,0}(X) = h^{2,0}(X) = 0$ and $h^{3,0}(X) = 1$.

Now assume that $d = g - 3$. By the Hodge index theorem as above we see that

$$d \leqslant K_S \cdot K_S \leqslant d + 9 .$$

We know that $K_S \cdot K_S \neq d + 1$ or $d + 3$ by (0.8.1) and (0.8.3). The relation (0.7.1) becomes:

$$d^2 - 15d - 20 + 12\chi(\mathcal{O}_S) = 2K_S \cdot K_S .$$

We now run systematically using this relation from $K_S \cdot K_S = d$ to $d + 9$. Note also that $\chi(\mathcal{O}_S) \geqslant 6$ if $K_S \cdot K_S \leqslant d + 7$ by (0.8.5). We find that this cuts the possible invariants down to the list:

TABLE IV

d	$K_S \cdot L_S$	$K_S \cdot K_S$	g	$\chi(\mathcal{O}_S)$
13	17	13	16	6
9	13	11	12	8
12	16	14	15	7
10	14	13	13	8
11	15	16	14	8
14	18	19	17	6
13	17	19	16	7
9	13	17	12	9
12	16	20	15	8
17	21	25	20	3
10	14	19	13	9
10	12	12	12	7

Using (0.8) and the reasoning of (0.8.1) we see that for the second line a smooth $S \in |L|$ will be a minimal model S' with one point blown up. Thus $K_{S'} \cdot K_{S'} = 12$ which contradicts (0.7.6).

Line 4 doesn't exist. Using (0.8) and the reasoning of (0.8.1) we see that on the minimal model S' of a smooth $S \in |L|$,

$$d' = 11 , \qquad K_{S'} \cdot L'_{S'} = 13 , \qquad K_{S'} \cdot K_{S'} = 14$$

contradicting (0.8.3).

The final table is as follows.

TABLE V

d	$K_S \cdot L_S$	$K_S \cdot K_S$	g	$\chi(\mathcal{O}_S)$	$\chi(\mathcal{O}_X)$	Comments
13	17	13	16	6	0	$K_{X'} \simeq \mathcal{O}_{X'}$
12	16	14	15	7	0	S is minimal model with 1 point blown up
11	15	16	14	8	0	S minimal
14	18	19	17	6	0	S minimal
13	17	19	16	7	0	S minimal
9	13	17	12	9	$\leqslant 1$	S minimal
12	16	20	15	8	$\leqslant 1$	S minimal
10	14	19	13	9	$\leqslant 1$	S minimal
10	12	12	12	7	0	S minimal
17	21	25	20	3	0 or 1	S minimal

Possible invariants of smooth threefolds X of non negative Kodaira dimension with a very ample line bundle L such that $h^0(L) = 6$ and $d = g - 3$ or $d = g - 2$. No examples with the above invariants are known.

1. – Some general results.

We start with some general results about n folds with trivial canonical class.

(1.1) THEOREM. *Let X be an n dimensional connected projective submanifold of $\boldsymbol{P}^N$ not contained in any hyperplane. Let d denote the degree of X in $\boldsymbol{P}^N$ and assume that $K_X^t \approx \mathcal{O}_X$ for some $t \neq 0$. If $d < n(N+1)$, then the order of the fundamental group of X is finite and bounded by:*

$$\frac{n^2 + n - 2}{n(N+1) - d}.$$

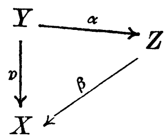
In particular X is simply connected if,

$$d \leqslant n\left(N - \frac{n-1}{2}\right).$$

PROOF. Let $p\colon Y \to X$ be a δ sheeted unramified cover where Y is smooth and connected. Let $L_Y = p^*L$ where $L = \mathcal{O}_{P^N}(1)|_X$

$$h^0(L_Y) = \chi(L_Y) = \delta\chi(L) = \delta h^0(L).$$

From this we see that the number of sections of the pullback of L is greater than the number of sections of L if $\delta > 0$. We get a diagram:



where Z is the normalization of the image of the map associated to $\Gamma(L_Y)$ and α is the lift of the given map. From this diagram and $dp = d\beta \circ d\alpha$ we easily conclude that Z is smooth and α, β are both unramified covers. Assume that α is a sheeted and β is b sheeted. If α is not a biholomorphism then $\alpha^*\Gamma(\beta^*L) = \Gamma(L_Y)$ which implies $bh^0(L) = \delta h^0(L)$ which contradicts $\delta h^0(L) = a \cdot bh^0(L)$ with $a > 1$.

Thus the map associated to $\Gamma(L_Y)$ is finite and generically one to one. Thus by (0.6)

$$\delta d \geqslant n\big(\delta h^0(L) - 1 - n\big) + 2 \qquad \text{or} \qquad n^2 + n - 2 \geqslant \delta\big(h^0(L)n - d\big)$$

which if $d < n(N+1) \leqslant n\big(h^0(L) + 1\big)$ implies that

$$\delta < \frac{n^2 + n - 2}{n(N+1) - d}.$$

This of course deals with the algebraic fundamental group but we get our conclusion about the fundamental group by using the basic structure theorem for Kahler manifolds with $K_X^t = \mathcal{O}_X$ for some $t > 0$. This theorem ([Be] is a good reference) asserts that a finite cover of such an X is a product of a torus and a simply connected manifold. $\square$

(1.1.1) REMARK. The above result is sharp. Let S be a smooth quartic surface in P^3 and let E be a smooth cubic curve in P^2. Embed $X = S \times E$ in P^{11} by the Veronese embedding. The degree, d, of X is 36. Thus

$$d = (\dim X)\cdot(11 + 1) = 36 .$$

(1.1.2) Corollary. *If in the above theorem* $\dim X = 3$ *then* $K_X \approx \mathcal{O}_X$ *if* $d \leqslant 3N + 2$ *and* X *is simply connected if* $d \leqslant 3N - 3$.

Proof. If $K_X^t \approx \mathcal{O}_X$ for $t \neq 0$ and $\dim X = 3$ then by the Hirzebruch-Riemann-Roch formula, $\chi(\mathcal{O}_X) = 0$. Since $d \leqslant 3N + 2$ implies that X has finite fundamental group it follows that $h^1(\mathcal{O}_X) = 0$ and therefore $h^3(\mathcal{O}_X) \neq 0$. Thus K_X has a non-trivial section and we can conclude that $K_X \approx \mathcal{O}_X$. The rest of the corollary is an immediate consequence of Theorem (1.1).

(1.1.3) Corollary. *Let* $X \subseteq P^N$ *be as in Theorem* (1.1). *If* X *is not simply connected then*

$$d \geqslant \frac{3n^2 + n + 2}{2} .$$

In particular if $\dim X = 3$ *then* $d \geqslant 16$.

Proof. If $N < 2n$ then X is simply connected by the Barth-Larsen theorem (see [Ba], [So1]). The above inequality follows from $N \geqslant 2n$ and the inequality of Theorem (1.1).

(1.1.4) Remark. Theorem (1.1) can be generalized in a number of directions.

First X doesn't have to be embedded. We can assume instead that there is a finite to one, generically one to one holomorphic map $f\colon X \to P^N$ with $f(X)$ contained in no hyperplane.

Second if we use the algebraic fundamental group and the notion of « algebraically simply connected » then the same theorem holds with the condition that $K_X^t \approx \mathcal{O}_X$ for some $t > 0$ replaced by the condition that

$$L = K_X^{-1} \otimes f^* \mathcal{O}_{P^N}(1)$$

is numerically effective and satisfies

$$\underbrace{c_1(L) \wedge \dots \wedge c_1(L)}_{n\text{-th power}} > 0 .$$

By the Kawamata-Viehweg vanishing theorem ([K], [V]) this condition implies that for any unramified cover

$$g\colon\ Y\to X\,,\qquad h^0\big(g^*f^*\mathcal{O}_{\boldsymbol{P}^N}(1)\big)=h^0\big((g^*L)\otimes K_Y\big)\approx\chi\big(g^*L)\otimes K_Y\big)$$
$$=\deg(g)\cdot\chi(K_X\otimes L)=\deg(g)\,h^0\big(f^*\mathcal{O}_{\boldsymbol{P}^N}(1)\big)\,,$$

which is the key fact used in the proof.

Third we can allow singularities that are mild enough not to falsify the Kodaira vanishing theorem in the form discussed in the preceding paragraph. For example we could assume that X is normal and Gorenstein, i.e. ω_X is invertible and X is Cohen Macaulay, and that except at a finite set of points X has rational singularities. The derivation of the needed vanishing theorem from the Kawamata-Viehweg vanishing theorem is easy (see [Sh-So]).

(1.2) THEOREM. *Let X be an n dimensional connected submanifold of $\boldsymbol{P}^N$. Assume that X is of non negative Kodaira dimension. Then let S denote a smooth surface (necessarily of general type) gotten by intersecting X with a general linear $\boldsymbol{P}^{N-n+2}\subseteq\boldsymbol{P}^N$. Let S' denote the minimal model of S. Then:*

$$\frac{c_1^2[S']}{e[S']}\geqslant\frac{(n-2)^2}{n^2+2}\,.$$

In particular

$$\liminf_{n\to\infty}\frac{c_1^2[S']}{e[S']}\geqslant 1\,.$$

PROOF. By (0.7.3 e),

$$K_S\cdot K_S+4K_S\cdot L_S+6d\geqslant e(S)\,.$$

Using the notation of (0.8) and (0.7.3 e)

$$K_{S'}\cdot K_{S'}+4K_{S'}\cdot L'_{S'}+6L'_{S'}\cdot L'_{S'}\geqslant K_S\cdot K_S+4K_S\cdot L_S+6L_S\cdot L_S\geqslant e(S)\geqslant e(S')$$

and

$$(n-2)K_{S'}\cdot L'_{S'}\leqslant K_{S'}\cdot K_{S'}\,,\qquad (n-2)^2L'_{S'}\cdot L'_{S'}\leqslant K_{S'}\cdot K_{S'}$$

we get the above theorem. □

We now restrict to dimension 3. Table I summarizes our knowledge of the smooth 3 folds X with $K_X^t\approx\mathcal{O}_X$ for some $t\neq 0$ and with sectional genus $g\leqslant 15$. The notation is as in (0.7)

2. - $X \subseteq \boldsymbol{P}^5$.

Throughout this section it is assumed that X is a 3 dimensional connected projective manifold of non negative Kodaira dimension and that L is a very ample line bundle with $h^0(L) = 6$. We further assume that $g \leqslant 15$ where g is the genus of the transversal intersection of two smooth $S \in |L|$.

A priori

$$8 \leqslant d \leqslant g - 1 \leqslant 14 .$$

Since we have already classified all (X, L) with $d \geqslant g - 3$ and $h^0(L) = 6$ in tables I and V, we can assume that $d \leqslant g - 4$. Using Castelnuovo's inequality it is easily checked that the only possible g and d left are:

$$g = 14 , \qquad d = 10 ,$$

$$g = 15 , \qquad d = 10, 11 .$$

If $d = 10$ and $g = 14$ then we have $K_S \cdot L = 16$ and by the Hodge index theorem $K_S \cdot K_S \leqslant 25$. By the relation (0.7.1) we have:

$$40 + K_S \cdot K_S = 6\chi(\mathcal{O}_S) .$$

Thus $K_S \cdot K_S = 14$ or 20 with $\chi(\mathcal{O}_X) = 9$ and 10 respectively. In the former case we conclude by the reasoning of (0.8.1) that on the minimal model S' of S, $L_{S'} \cdot L_{S'} = 15$ or 16. This contradicts (0.7.6). If $K_S \cdot K_S = 20$ then by (0.8.1) S is minimal.

If $d = 10$ and $g = 15$ then by (0.7.1)

$$45 + K_S \cdot K_S = 6\chi(\mathcal{O}_S) .$$

Using this and the Hodge index theorem as above we see that $K_S \cdot K_S = 15$, 21, 27, and $\chi(\mathcal{O}_S) = 10$, 11, 12 respectively. Let S' be the minimal model of S. If $K_S \cdot K_S = 15$ then $K_{S'} \cdot K_{S'} \leqslant 19$ by (0.8) and by (0.7.6) we get a contradiction. If $K_S \cdot K_S = 21$ then by the reasoning of (0.8.1) $K_{S'} \cdot K_{S'} \leqslant 22$ contradicting (0.7.6).

If $d = 1$ and $g = 15$ then reasoning as above we get that $K_S \cdot K_S = 11$, 17, 23, with $\chi(\mathcal{O}_X) = 8$, 9, 10, respectively. Note that for $K_S \cdot K_S = 11 = d$ it follows that $K_{S'} \cdot K_{S'} = 14 = d'$ and $h^0(L') = \chi(L') = \chi(\mathcal{O}_{S'}) = 8$. Thus $h^0(L'_{C'}) \geqslant 6$ for the intersection of two generic $S' \in |L'|$. The genus of C' is 15 but Castelnuovo's inequality gives the absurdity $g \leqslant 12$.

Table VI summarizes the above.

TABLE VI

g	d	$K_S \cdot L_S$	$K_S \cdot K_S$	$\chi(\mathcal{O}_S)$	$\chi(\mathcal{O}_X)$	Comments
12	9	13	17	9	$-1 \leqslant \chi \leqslant 1$	S minimal
12	10	12	12	7	0	$h^{2,0}(X) = 0$ S minimal
13	10	14	19	9	$\chi \leqslant 1$	S minimal
14	10	16	20	10	$\chi \leqslant 1$	S is minimal
14	11	15	16	8	0	$h^{2,0}(X) = 0$ S is minimal
15	10	18	27	12	$\chi \leqslant 2$	S is minimal
15	11	17	17	9	0	S is possibly the blown-up at one point of the minimal model
15	11	17	23	10	$\chi \leqslant 1$	S is minimal
15	12	16	14	7	0	$h^{2,0}(X) = 0$ S is the blown-up of the minimal model at one point
15	12	16	20	8	$\chi \leqslant 1$	S is minimal

Possible invariants of smooth threefolds X of non negative Kodaira dimension with a very ample line bundle such L that $h^0(L) \leqslant 6$, $g \leqslant 15$ and which are not in Table II. All of the above are simply connected and thus have $h^{1,0}(X) = h^{1,0}(S) = 0$. No examples with the above invariants are known.

3. – The remaining cases.

Throughout this section X is a threefold of non negative Kodaira dimension with a very ample line bundle L such that $h^0(L) \geqslant 7$ and $g \leqslant 15$. By Castelnuovo's lemma we quickly check that $h^0(L) = 7$. By (0.6) and table I we only must consider:

$$d = 12\,, \qquad g = 14, 15\,,$$

$$d = 13\,, \qquad g = 15\,.$$

We can only say a little about these. We will carry out the calculation for $d = 12$, $g = 14$; all proceeds in the same way.

If $d = 12$, $g = 14$ then $K_S \cdot L_S = 14$. Thus by the Hodge index theorem

$$K_S \cdot K_S \leqslant 16 .$$

By the lemma's in (0.8) only $K_S \cdot K_S = 12, 14, 15$ and 16 are possible. If $K_S \cdot K_S = 12 = d$ then by (0.8) $K_{X'}^t = \mathcal{O}_{X'}$ and S is the minimal model S' with one point blown-up. Thus $K_{S'} \cdot K_{S'} = 13$ and therefore by (0.7.6) $3\chi(\mathcal{O}_S) - 10 \leqslant 13$ or $\chi(\mathcal{O}_S) \leqslant 7$. If $K_S \cdot K_S = 14$ then by (0.8.4) $h^{3,0}(X) = 0$ and $h^{1,0}(S) = h^{1,0}(X) > 0$ and S is minimal. Thus $14 \geqslant 3h^{2,0}(S) - 7 \geqslant 3\chi(\mathcal{O}_S) - 7$ or $\chi(\mathcal{O}_S) \leqslant 7$. If $K_S \cdot K_S = 15$ then use (0.8.3). The results are summarized in table III.

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