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Existence results for embedded minimal surfaces of controlled topological type, II

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Existence Results for Embedded Minimal Surfaces of Controlled Topological Type, II.

JÜRGEN JOST (*)

0. – Introduction.

In the first part of these investigations, hereafter referred to as [I], we developed boundary regularity results (at free and fixed boundaries) for varifolds arising from minimizing the area among embedded surfaces, and we then proceeded to produce embedded minimal surfaces under certain boundary conditions via a minimizing procedure, and we imposed as much control on the topological type of these surfaces as the topological and geometric data of the respective problem allowed.

In the present second part, we want to combine these techniques and results with saddle point arguments in order to produce embedded minimal surfaces of controlled topological type solving free boundary problems where the minimum of area would degenerate into a point as there is no topological or geometric constraint preventing this.

These saddle point or, as they are often called, minimaxing methods originated from the work of Pitts [P] and were further developed by Simon-Smith [SS] to prove the existence of an embedded minimal two-sphere in a Riemannian manifold diffeomorphic to the threedimensional sphere.

The present article will start in a rather technical vein by constructing a certain deformation of a minimaxing path of surfaces. The setting will always be a subset A of some threedimensional manifold (occasionally restricted in later applications to be Euclidean space) where we require that the closure of A is diffeomorphic to the threedimensional ball, and we look for embedded minimal surfaces inside A that meet the boundary of A orthogonally. We then combine one consequence of this deformation, the existence of a varifold satisfying a certain « almost minimizing » property

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(see § 1 for the precise definition), with the regularity results of [I] and show that if ∂A has positive mean curvature w.r.t. the interior normal vector, then A contains an embedded minimal disk meeting ∂A orthogonally (provided there exist no embedded minimal two-spheres inside A). This result generalizes the main result of [GJ2].

This result of course raises the questions what happens if we drop this requirement on the mean curvature of ∂A. Our point of view can be clarified by discussing a recent paper of Struwe [St]. He treated the corresponding parametric free boundary value problem (assuming that A is a subset of Euclidean space) and showed that there exists a parametric minimal disk meeting ∂A orthogonally at its boundary which, however, need neither be embedded (actually not even necessarily immersed) nor contained in A, i.e. is allowed to penetrate ∂A in an unphysical way. It seems that from the point of view of [St], this result cannot be much improved (apart from showing that the solution is immersed), because in general one cannot expect that A contains a disktype minimal surface without imposing curvature restrictions on ∂A. For this reason, we allow a somewhat more general topological type, namely look for a surface of arbitrary (finite) connectivity, but still of genus zero, i.e. topologically a disk with holes. Working in this more general class we are able to show that there exists a minimal surface meeting ∂A orthogonally which is embedded and, what is even more important, confined to lie inside A. (We note that the control of the genus of the limit surface depends on the argument of [SS]).

Actually, from the famous results of Lusternik and Schnirelmann on the existence of three closed geodesic without selfintersections on any compact surface of genus zero, one should expect that there exist not only one but three embedded minimal surfaces solving the free boundary value problems considered in the present paper. The basic technical problem one has to face, however, when attempting to prove this is that the solution of a higher order saddle point construction might just be a multiple covering of the solution obtained in the first step, thus not being geometrically different. While we are not able to resolve this difficulty in its full generality, we nevertheless do show the existence of three embedded minimal disks meeting ∂A orthogonally under the assumption that ∂A is a strictly convex surface in Euclidean space and that if R_1 is the radius of the largest ball contained in $\bar{A}$ and R_2 the radius of the smallest ball containing A the ratio R_2/R_1 does not exceed $\sqrt{2}$. An important role in the demonstration of this result is played by the following estimate: *The area of any embedded minimal disk in A meeting ∂A orthogonally is at least πR_1^2 if ∂A is strictly convex* (1).

(1) This estimate already appeared in [Sm]. We shall give a different proof.

We shall also use the topological arguments of Lusternik and Schnirelmann, and we shall avail ourselves of the coincise presentation of this theory in the appendix of Klingenberg's book [Kl].

Finally, we want to mention that if A instead is a simplex in Euclidean space, i.e. having a boundary consisting of four planar pieces, then the existence of three embedded minimal disks was recently shown by Smyth [Sm], thus taking up an old investigation of H. A. Schwarz. Because of the more special nature of this boundaιy configuration, this case can be dealt with by rather elementary methods, quite in contrast to ours which seems to require an elaborate machinery.

1. – Notations, definitions, and preliminaries.

$$I := [0, 1] ,$$

$$D := \{x \in \mathbb{R}^2 : x = (x_1, x_2) : x_1^2 + x_2^2 \leqslant 1\} ,$$

$$B := \{x \in \mathbb{R}^3 : x = (x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 \leqslant 1\} .$$

In I^q, the q-fold Cartesian product of I with itself, we define for $t = (t_1, \dots, t_q)$, $\tau = (\tau_1, \dots, \tau_q)$

$$|t - \tau| := \sup_{1 \leqslant j \leqslant q} |t_j - \tau_j|$$

and for $\sigma > 0$

$$R(t, \sigma) := \{\tau \in I^q : |t - \tau| \leqslant \sigma\} .$$

We let A be a bounded open subset of a threedimensional Riemannian manifold X. We assume that $\bar{A}$ is diffeomorphic to the threedimensional ball B and that ∂A is of class C^4.

Let U be an open subset of X. We consider two situations:

1) ∂A has positive mean curvature w.r.t. the interior normal. In this case, we define

$$I_1(U) := \{\psi = \{\psi_t\},\ \psi_t : X \to X\ \ (t \in [0, 1]),\ \text{isotopy of class } C^1$$
$$\text{(in particular } \psi_0 = \mathrm{id},\ \psi_t L X \setminus K = \mathrm{id} \text{ for some } K \subset\subset U$$
$$\text{and all } t \in [0, 1],\ \psi_t(\partial A) \subset \partial A\},$$

$$\mathcal{M}_1 := \{\varphi(D) : \varphi : D \to X \text{ injective of class } C^2,\ \varphi(D) \subset \bar{A},$$
$$\varphi(\partial D) = \varphi(D) \cap \partial A,\ \varphi(D) \text{ meets } \partial A \text{ transversally}\} .$$

2) Here, we assume no curvature condition on ∂A and define

$$I_2(U) := \{\psi = \{\psi_t\},\ \psi_t : X \to X\ (t \in [0,1]),\ \text{isotopy of class } C^1,$$
$$\psi_t \llcorner X \setminus K = \text{id for some } K \subset\subset U \text{ and all } t \in [0,1[\},$$

$$\mathcal{M}_2 := \{\varphi(D) : \varphi : D \to X \text{ injective of class } C^2,\ \varphi(D) \cap A \neq \emptyset,\ \varphi(\partial D) \subset \partial A\}\,.$$

Note that here we do not require that $\varphi(D) \subset A$ or that $\varphi(\partial D) = \varphi(D) \cap \partial A$.

Also, in the second case, we do not require that the surfaces intersect ∂A transversally. Note that $I_1(U)$ also depends on A, but we have suppressed this dependence in our notation, in order to gain a uniform notation for both cases. When we shall use a subscript l in the sequel, it can take either the value 1 or 2 and will refer to the corresponding situation.

If Σ is a rectifiable surface in X, we put

$$|\Sigma| := \text{area}\,(\Sigma \cap \bar{A})\,.$$

For $\Sigma \in \mathcal{M}_l$, we define $(\alpha > 0)$

$$I_l(\Sigma, U, \alpha) := \{\psi = \{\psi_t\}_{0 \leqslant s \leqslant 1} \in I_l(U) : |\psi_t(\Sigma)| \leqslant |\Sigma| + \alpha \text{ for all } t \in [0,1]\}.$$

Furthermore, for $\varepsilon > 0,\ \alpha = 0$

$$S_l(U, \varepsilon, \alpha) := \{\Sigma \in \mathcal{M}_l : \text{if } \psi \in I_l(\Sigma, U, \alpha), \text{ then } |\psi_1(\Sigma)| \geqslant |\Sigma| - \varepsilon\}.$$

If $\chi : X \to X$ is a diffeomorphism, we also define supp χ as the closure of the set of x with $\chi(x) \neq x$.

Let

$$\mathfrak{U}_m := \{U_1, \ldots, U_m) : U_i \subset X \text{ open},\ U_i \cap A \neq \emptyset$$
$$\text{dist}\,(U_i, U_j) \geqslant \min\{\text{diam}\,(U_i), \text{diam}\,(U_j)) \text{ for all } i, j \in \{1, \ldots, m\},\ i \neq j\}.$$

Moreover,

$$\mathfrak{U}_m^\varrho := \{(U_1, \ldots, U_m) \in \mathfrak{U}_m : \text{diam}\, U_i \leqslant \varrho,\ i = 1, \ldots, m\}\,.$$

By the Nash embedding theorem, X can be isometrically embedded into some $\mathbf{R}^n$, and as usual, we have the corresponding inclusion for the space of k-dimensional varifolds on X.

$$V_k(X) = \{V \in V_k(\mathbf{R}^n) : \text{spt}\, \|V\| \subset X\}.$$

On $V_k(\mathbf{R}^n)$, we have the flat metric

$$\boldsymbol{F}(V, W) := \sup\left\{|V(f) - W(f)| : f \in C^{0,1}(G_k(\mathbf{R}^n)),\ |f| \leqslant 1,\ \operatorname{Lip}(f) \leqslant 1\right\}.$$

If M is a submanifold of X, we denote by

$$\boldsymbol{v}(M)$$

the associated varifold. Vice versa, a varifold V gives rise to a measure $\|V\|$ on X with support $\operatorname{spt}\|V\|$.

DEF. *In case l), a varifold* $V \in V_2(A) := \{W \in V_2(X) : \operatorname{spt}\|W\| \subset \bar{A}\}$, $V \neq 0$, *ist called* ϱ, m *almost minimizing if for each* $\varepsilon > 0$ *there exists* $\alpha > 0$ *and* $\Sigma \in \mathcal{M}_l$ *with* $\boldsymbol{F}(V, \boldsymbol{v}(\Sigma \cap A)) < \varepsilon$ *and, for any* $(U_1, \ldots, U_m) \in \mathcal{U}_m^\varrho$,

$$\Sigma \in S_l(U_i, \varepsilon, \alpha)$$

for at least one $i \in \{1, \ldots, m\}$. *We also say that (for this* i*)* V *is almost minimizing on* U_i.

NOTE. This notion is different from the original notion of almost minimizing of Pitts [P]. It is a modification along the lines of [SS] and [GJ2].

We shall usually suppress the dependence on ϱ and m in the notation.

Let $u(0, 0)$ assign to $t \in I$ the set

$$u(0, 0)(t) = \{x_3 = 2t - 1\} \cap B.$$

Let $\varrho^i(\tau)$ be the positive rotation by $\pi \cdot \tau$ around the x_i-axis. We let $u(0, 1)$ assign to a pair $(t_1, t_2) \in I^2$ the set

$$u(0, 1)(t_1, t_2) = \varrho^1(t_2)u(0, 0)(t_1)$$

and $u(1, 1)$ assign to a triple $(t_1, t_2, t_3) \in I^3$ the set

$$u(1, 1)(t_1, t_2, t_3) := \varrho^2(t_3)u(0, 1)(t_1, t_2).$$

Finally, we assign to each $(t_1, t_2, t_3) \in I^3$ an orientation of $u(1, 1)(t_1, t_2, t_3)$ in a continuous way. A change of orientation will be denoted by a minus (—) sign. This also induces an orientation of $u(0, 1)(t_1, t_2)$. In particular

$$u(0, 1)(t_1, 1) = -u(0, 1)(1 - t_1, 0) \tag{1.1}$$

and also

$$u(1,1)(t_1, 1, t_3) = -u(1,1)(1-t_1, 0, t_3) \tag{1.2}$$

and

$$u(1,1)(t_1, t_2, 1) = -u(1,1)(1-t_1, 1-t_2, 0)\,. \tag{1.3}$$

Let $\lambda : B \to \bar{A}$ be a diffeomorphism, existing by assumption, and let $v(,i\,j)$ be the image of $u(i,j)$ under λ, e.g.

$$v(0,0)(t) = \lambda \circ u(0,0)(t)\,.$$

We consider $V(0,0)$, the class of v that can be obtained from $v(0,0)$ by a homotopy h with the following properties:

$$h : [0,1] \times (\mathring{I}, \partial I) \to (\mathcal{M}_l, \partial A)$$

with

$$h(\{0\} \times \{t\}) = v(0,0)(t)\,,$$

$$h(\{1\} \times \{t\}) = v(t)\,.$$

Furthermore, we require that there exists a C^1-mapping

$$k : I^2 \times D \to X$$

and that

$k(\tau, t)\ \cdot)$ is a diffeomorphism from D onto $h(\tau, t)$ for each $\tau \in I$ and $t \in \mathring{I}$.

Thus $h(\tau, t)$ assigns to $t \in (0,1)$ an embedded disk meeting A and to $t \in \{0,1\}$ a point in ∂A. We let $V(0,1)$ be the class of v that can be obtained from $v(0,1)$ via a homotopy h satisfying

$$h : [0,1[\times \big(\mathring{I} \times I, \{(t_1, t_2) : t_1 \in \{0,1\}\}\big) \to (\mathcal{M}_l, \partial A)$$

with

$$\begin{cases} h(\tau, t_1, 1) = -h(\tau, 1-t_1, 0) & \text{for all } (\tau, t_1) \in I \times I\,, \\ h(0, t_1, t_2) = v(0,1)(t_1, t_2)\,, \\ h(1, t_1, t_2) = v(t_1, t_2)\,. \end{cases} \tag{1.4}$$

We again require the existence of a C^1-map k

$$k : I^3 \times D \to X$$

so that $k(\tau, t_1, t_2, \cdot)$ is a diffeomorphism from D onto $h(\tau, t_1, t_2)$ for $\tau, t_2 \in I$, $t_1 \in \mathring{I}$.

Finally, $V(1, 1)$ is the class of v that can be obtained from $v(1, 1)$ via a homotopy h satisfying

$$h:[0, 1] \times (\mathring{I} \times I^2, \partial I \times I^2) \to (\mathcal{M}_l, \partial A)$$

with

$$h(\tau, t_1, 1, t_3) = -h(\tau, 1 - t_1, 0, t_3) \tag{1.5}$$

$$h(\tau, t_1, t_2, 1) = -h(\tau, 1 - t_1, 1 - t_2, 0) \tag{1.6}$$

for all $(\tau, t_1, t_2, t_3) \in I^4$, and of course

$$h(0, t_1, t_2, t_3) = v(1, 1)(t_1, t_2, t_3)\,,$$

$$h(1, t_1, t_2, t_3) = v(t_1, t_2, t_3)\,.$$

Again, we require the existence of a covering of h by a C^1-map

$$k: I^4 \times D \to X$$

for which $k(\tau, t_1, t_2, t_3, \cdot)$ is a diffeomorphism of D onto $h(\tau, t_1, t_2, t_3)$ for $\tau, t_2, t_3 \in I$, $t_1 \in I$.

We note that each $v \in V(i, j)$ is a $\mathbb{Z}_2$-cycle of $\mathcal{M}_l$ mod ∂A ($x \in \partial A$ here is considered as a surface degenerated into a point), for, if e.g. $v \in V(1, 1)$,

$$v(\partial I \times I^2) \subset \partial A$$

and

$$v(I \times \partial I^2) = -2v(I \times \{0\} \times I) + 2v(I \times I \times \{0\})$$

because of (1.5) and (1.6).

We put

$$\varkappa(0, 0) := \inf_{v \in V(0,0)} \sup_{t \in I} |v(t)|\,,$$

$$\varkappa(0, 1) := \inf_{v \in V(0,1)} \sup_{(t_1, t_2) \in I^2} |v(t_1, t_2)|\,,$$

$$\varkappa(1, 1) := \inf_{v \in V(1,1)} \sup_{(t_1, t_2, t_3) \in I^3} |v(t_1, t_2, t_3)|\,.$$

We define the corresponding sets $C(i, j)$ of critical varifolds as

$$C(i, j) := \Big\{V \in V_2(A) : V = \lim_{k\to\infty} \boldsymbol{v}\big(v^k(t^k) \cap \bar{A}\big)$$

where $t^k \in I^q$, $q = i + j + 1$, $v^k \in V(i, j)$,

$$\lim_{k\to\infty} |v^k(t^k)| = \varkappa(i, j), \qquad \lim_{k\to\infty} \big(\sup_{t\in I^q} |v^k(t)| - |v^k(t^k)|\big) = 0\Big\}.$$

A corresponding sequence $v^k(t^k)$ is called a minimaxing sequence. We let $C'_\varrho(i, j)$ be the set of (ϱ, m)-almost minimizing varifolds (for $m = (3^q)^{3^q}$, $q = i + j + 1$; ϱ will be specified later on) contained in $C(i, j)$ and

$$C''_\varrho(i, j) := C(i, j) \setminus C'_\varrho(i, j).$$

If $V \in V_2(A)$ and $\delta > 0$, we put

$$N(V, \delta) := \{V' \in V_2(A) : F(V, V') < \delta\}.$$

We shall need the following selection lemma.

LEMMA 1.1. *Let* $Q^q := \big\{\sigma : \sigma = \{(t_1, \ldots, t_q) \in I^q : (n_i - 1)/N \leq t_i \leq n_i/N,\ n_i \in \mathbf{N},\ 1 \leq n_i \leq N,\ i = 1, \ldots, q\}\big\}$ *(for a given* $N \in \mathbf{N}$*) be a standard partition of* I^q *into cubes.*

For each $\sigma \in Q^q$, *let* $A(\sigma)$ *be a collection of* $m := (3^q)^{3^q}$ *open subsets* $U_1, \ldots, U_m$ *of* X *and* $\operatorname{dist}(U_i, U_j) \geq \min(\operatorname{diam} U_i, \operatorname{diam} U_j)$ *for* $i, j \in \{1, \ldots, m\}$, $i \neq j$.

Then there exists a function $\alpha(\sigma)$ *assigning to each* σ *an element of* $A(\sigma)$ *with* $\alpha(\sigma) \cap \alpha(\tau) = \emptyset$ *whenever* $\sigma \cap \tau \neq \emptyset$.

PROOF (following Pitts [P, 4.8 f.]). We can divide Q^q into 3^q classes $E_1, \ldots, E_{3^q}$ so that each E_j $(j = 1, \ldots, 3^q)$ satisfies:

> if $\sigma, \tau \in E_j$, $\sigma \neq \tau$, and $\sigma \cap \lambda \neq \emptyset$ for some $\lambda \in Q^q$, then $\tau \cap \lambda = \emptyset$ (i.e. no two elements of the same class have a common neighbour).

For $\sigma \in Q^q$, we put

$$u(\sigma) := \{\tau \in Q^q : \tau \cap \sigma \neq \emptyset\}.$$

First for each $\tau \in E_1$ and $\tau \in u(\sigma)$ we want to choose $A_1(\tau) \subset A(\tau)$ with $(3^q)^{3^q - 1}$ elements so that

$$\cup \{A_1(\tau) : \tau \in u(\sigma)\}$$

is a disjointed family.

Given σ, we put

$$A'(\sigma) := \bigcup_{\tau \in u(\sigma)} A(\tau) .$$

We choose u_1 as an element of $A'(\sigma)$ of smallest diameter. Then, for each $\tau \in u(\sigma)$, there is at most one $U \in A(\tau)$ with

$$U_1 \cap U \neq \emptyset .$$

Let

$$A^{(1)}(\tau) := \{U \in A(\tau) : U \cap U_1 = \emptyset\} .$$

We then choose an element U_2 of smallest diameter in $\bigcup_{\tau \in u(\sigma)} A^{(1)}(\tau)$. Again, for each $\tau \in u(\sigma)$, there is at most one $U \in A^{(1)}(\tau)$ with

$$U_2 \cap U \neq \emptyset .$$

Let

$$A^{(2)}(\tau) := \{U \in A^{(1)}(\tau) : U \cap U_2 = \emptyset\} .$$

We choose $U_3 \in \bigcup_{\tau \in u(\sigma)} A^{(2)}(\tau)$ of smallest diameter and repeat this procedure, until, for some τ, we have chosen $(3^q)^{3^q-1}$ elements from $A(\tau)$. These elements then form $A_1(\tau)$, and we can discard the remaining elements of $A(\tau)$. We then proceed until we have chosen $(3^q)^{3^q-1}$ elements for every $\tau \in u(\sigma)$. In the same way, for each $\sigma \in E_2$ and $\tau \in u(\sigma)$, we choose $A_2(\tau) \subset A_1(\tau)$ with $(3^q)^{3^q-2}$ elements so that

$$\cup \{A_2(\tau) : \tau \in u(\sigma)\}$$

is a disjointed family.

If we repeat this construction, then $A_{3^q}(\tau)$ contains precisely one element, and letting $\alpha(\tau)$ be this element, we have proved the lemma.

2. – The deformation.

From the definition of $C''_\varrho(i, j)$ we infer

$$\forall V \in C''_\varrho(i, j) , \quad \exists \varepsilon_V > 0 , \quad \forall N \in \mathcal{M}_l \quad \text{with } F(V, v(M \cap \bar{A})) < \varepsilon_V ,$$

$\forall \delta > 0$, $\exists U_i = U_i(M, \delta, V)$ for $i = 1, \ldots, m$ and

$$(U_1, \ldots, U_m) \in \mathfrak{U}^\varrho_m \quad \text{and} \quad \psi^i = \{\psi^i_s\}_{0 \leqslant s \leqslant 1} = \psi^i(M, \delta, V)$$

with

$$\operatorname{supp} \psi_s^i \subset U_i \qquad (0 \leqslant s \leqslant 1)\,, \tag{2.1}$$

$$|\psi_s^i(M)| \leqslant |M| + \delta \qquad (0 \leqslant s \leqslant 1)\,. \tag{2.2}$$

$$|\psi_1^i(M)| < |M| - \varepsilon_V \qquad \text{for } i = 1, \ldots, m\,. \tag{2.3}$$

Furthermore, by a redefinition of the parameter s, we can also achieve

$$\frac{d}{ds}\psi_s^i|_{(s=0)} = 0\,. \tag{2.4}$$

Since massbounded sets of varifolds are compact in the F-topology, for given $\eta > 0$, we can find $V_1, \ldots, V_k \in C''_\varrho(i, j)$ with

$$C(i, j) \subset N\big(C'_\varrho(i, j), \eta/2\big) \cup \bigcup_{k=1}^{K} N(V_k, \varepsilon_{V_k})\,. \tag{2.5}$$

If $C'_\varrho(i, j) = \emptyset$, we can take $\eta = 0$.

Let

$$\varepsilon_1 := \min_{1 \leqslant k \leqslant K} \{\varepsilon_{V_k}\}\,.$$

Furthermore, by a standard compactness argument, there exists $\varepsilon_2 > 0$ with the property that

$$\text{if } \sup_{t \in I^q} |v(t)| < k(i, j) + \varepsilon_2 \quad \text{and} \quad |v(t^*)| > k(i, j) - \varepsilon_2$$

for $v \in V(i, j)$ and $t^* \in I^q$ then

$$\boldsymbol{v}\big(v(t^*) \cap \bar{A}\big) \subset N\big(C'_\varrho(i, j), \eta/2\big) \cap \bigcup_{k=1}^{K} N(V_k, \varepsilon_{V_k})\,. \tag{2.6}$$

Let

$$\varepsilon := \tfrac{1}{2} \min\big(\varepsilon_0, \varepsilon_1, \varepsilon_2, \eta, k(i, j), 3^{-q-1}\big)$$

ε_0 will be specified in § 5. It can be assigned an arbitrary positive value in § 4.

Let $v \in V(i, j)$ satisfy

$$\sup_{t \in I^q} |v(t)| \leqslant k(i, j) + \varepsilon/4 \tag{2.7}$$

v will be fixed for the rest of this section. Then, if

$$|v(t)| \geq k(i, j) - 2\varepsilon \tag{2.8}$$

and

$$F\big(C'_\varrho(i, j), \boldsymbol{v}(v(t) \cap \bar{A})\big) \geq \eta/2 , \tag{2.9}$$

then

$$\boldsymbol{v}(v(t) \cap \bar{A}) \subset N(V_k, \varepsilon_{V_k}) \tag{2.10}$$

for some $k \in \{1, \dots, K\}$ by (2.6), and there exist sets $U_i(v(t), \varepsilon^2, V_k)$ and isotopies $\psi^i \in I_l(v(t), U_i, \varepsilon^2)$ satisfying (2.1)-(2.4).

Furthermore, we can find $\lambda = \lambda(\varepsilon, t) > 0$ so that for all $\tau \in I^q$ with $|t - \tau| < \lambda$, all $s \in [0, 1[$ and all $i \in \{1, \dots, m\}$

$$\big||\psi^i_s(v(\tau))| - |\psi^i_s(v(t))|\big| < \varepsilon^2 . \tag{2.11}$$

(Here, we use the smoothness of $v \in V(i, j)$ following from the properties of the homotopy h in § 1), and also

$$F\Big(\boldsymbol{v}\big(\psi^i_s(v(\tau)) \cap \bar{A}\big), \boldsymbol{v}\big(\psi^i_s(v(t)) \cap \bar{A}\big)\Big) < \varepsilon^2 . \tag{2.12}$$

We choose $t_1, \dots, t_Q$, using compactness, satisfying

$$I^q \subset \bigcup_{j=1}^{Q} R(t_j, \tfrac{1}{4}\lambda_j) \qquad \text{with } \lambda_j := \lambda(\varepsilon, t_j) .$$

We choose N as the smallest positive integer with the property that for any $t \in I^q$ there exists $j \in \{1, \dots, Q\}$ so that

$$R(t, 1/N) \subset R(t_j, \lambda_j) .$$

As in Lemma 1.1, we choose a partition of I^q into cubes of side length $1/N$. We denote these cubes by R_l, $l = 1, \dots, N^q$, and their centers by t^l, i.e. $R_l = \{\tau \in I^q : |\tau - t^l| \leq 1/N\}$. For each l, we choose $j = j(l) \in \{1, \dots, Q\}$ so that $R_l \subset R(t_j, \lambda_j)$. If (2.8) and (2.10) hold for $t = t_j$ $\big(j \in \{1, \dots, Q\}\big)$, we have corresponding sets $U_i\big(v(t_j), \varepsilon^2, V_k\big) =: U_i(t_j)$, $i = 1, \dots, m$. If (2.8) or (2.10) is not valid for $t = t_j$, we choose any sets $\big(U_1(t_j), \dots, U_m(t_j)\big) \in \mathfrak{U}_m$. Then if,

$$R_l \subset R(t_j, \lambda_j) \qquad \big(l \in \{1, \dots, N^q\}, j \in \{1, \dots, Q\}\big)$$

we put

$$A(R_l) := \{U_1(t_j), \ldots, U_m(t_j)\} .$$

The function α of Lemma 1.1 assigns to each R_l an open set $\alpha(R_l) \in A(R_l)$ so that

$$\alpha(R_l) \cap \alpha(R_k) = \emptyset$$

whenever $R_l \cap R_k \neq \emptyset$ i.e. the sets assigned to neighbouring cubes are disjoint.

Furthermore, it is clear from the above construction and the proof of Lemma 1.1 that this assignment can also be required to satisfy the boundary conditions (1.4)-(1.6), i.e. that for example for $q = 2$ the same open set is assigned to the cubes containing the point $(t_1, 1)$ and $(1 - t_1, 0)$, resp.

Let $s : \mathbf{R}^q \to [0, 1]$ be a smooth function with

$$s(t) = 1 \qquad \text{for } |t| \leqslant 1/N ,$$

$$s(t) = 0 \qquad \text{for } |t| \geqslant 3/2N .$$

(Here, we use again the sup-norm

$$|t| := \sup_{1 \leqslant i \leqslant q} |t_i| \quad \text{ if } t = (t_1, \ldots, t_q)) .$$

If $R_l \subset R(t_j, \lambda_j)$ and if for some $t \in R_l$

$$|v(t)| \geqslant \varkappa(i, j) - \varepsilon \tag{2.13}$$

then by (2.11) for $s = 0$,

$$|v(t_j)| \geqslant \varkappa(i, j) - 2\varepsilon . \tag{2.14}$$

If on the other hand

$$|v(t_j)| \leqslant \varkappa(i, j) - 2\varepsilon \tag{2.15}$$

then for $t \in R_l$

$$|v(t)| \leqslant \varkappa(i, j) - \varepsilon . \tag{2.16}$$

Moreover, by (2.12), if for some $t \in R_l$

$$F\big(v(v(t) \cap \bar{A}), C'_\varrho(i, j)\big) \geqslant \eta \tag{2.17}$$

then

$$F\big(\boldsymbol{v}(v(t_j)\cap \bar{A}),\ C'_\varrho(i,j)\big)\geqslant \eta/2\ . \tag{2.18}$$

Now, if for some $t\in R_l$ (2.13) and (2.17) hold then we choose $\psi^i=\psi^i\cdot\big(v(t_j),\varepsilon^2,V_k\big)$, where $j=j(l)$, i is the index with $\alpha(R_l)=U_i\big(v(t_j),\varepsilon^2,V_k\big)$ and k comes from (2.10).

On $\alpha(R_l)$, we then replace $v(t)$ by

$$\bar{v}(t):=\psi_{s(t_l-t)}\big(v(t)\big)\ .$$

If (2.13) or (2.17) does not hold for any $t\in R_l$, we put

$$\bar{v}(t)=v(t)\ \text{on this cube}\ R_l\,.$$

Outside $\bigcup_{l=1}^{N^q}\alpha(R_l)$, we have of course also

$$\bar{v}(t)=v(t)\ .$$

Since $s(t_l-t)=1$ on R_l,

$$|v(t)\cap\alpha(R_l)|-|\bar{v}(t)\cap\alpha(R_l)|\geqslant\varepsilon \tag{2.19}$$

by (2.11), for $t\in R_l$, if (2.13) and (2.17) hold for some $t\in R_l$. On the other hand, on each neighbouring cube R_k (i.e. $R_k\cap R_l\neq\emptyset$, $k\neq l$)

$$|\bar{v}(t)\cap\alpha(R_k)|-|v(t)\cap\alpha(R_k)|\leqslant 3\varepsilon^2\ . \tag{2.20}$$

Therefore, since each cube has 3^q-1 neighbours, we have for each cube R_l containing some t for which (2.13) and (2.17) hold

$$|v(t)|-|\bar{v}(t)|\geqslant\varepsilon-3(3^q-1)\,\varepsilon^2\geqslant\varepsilon/2\ . \tag{2.21}$$

Since (2.7) was assumed, we therefore have on each cube R_l containing a t satisfying (2.17)

$$|\bar{v}(t)|\leqslant\varkappa(i,j)-\varepsilon/4 \tag{2.22}$$

for all $t\in R^l$.

The construction guarantees that $\bar{v}\in V(i,j)$.

As a first consequence, we note

LEMMA 2.1. *$C'_\varrho(i,j)\neq\emptyset$ for any $\varrho>0$.*

PROOF. Otherwise, the above deformation would yield $\bar{v} \in V(i, j)$ satisfying (2.22), thus contradicting the definition of $\varkappa(i, j)$. (Note again that we were allowed to take $\eta = 0$ in the above construction if $C'_\varrho(i, j) = \emptyset$).

Other consequences will be derived in § 5.

3. – Regularity and control of the topological type of almost minimizing varifolds.

LEMMA 3.1. *Any almost minimizing varifold $V \in C'_\varrho(i, j)$ ($\varrho > 0$) corresponds to a disjoint collection of embedded minimal surfaces,*

$$\operatorname{spt} \|V\| = \sum_{j=1}^{J} n_j M_j \tag{3.1}$$

where $n_j \in \mathbb{N}$ and M_j is an embedded minimal surface in A meeting ∂A orthogonally, or M_j is a closed embedded minimal surface inside A $(j \in \{1, \ldots, J\})$.

PROOF. This follows from §§ 3f of [GJ2] if we replace § 2 of [GJ2] by Thm. 5.2 of [I] in case ∂A has positive mean curvature w.r.t. the interior normal and by Thm. 5.1 of [I] in the general case.

LEMMA 3.2. *Let $A \subset X$, $\bar{A}$ being diffeomorphic to the unit ball. If ∂A has positive mean curvature w.r.t. the interior normal then each M_j occuring in* (3.1) *is diffeomorphic to a disk or a two-sphere.*

PROOF. We can apply the arguments of § 5 of [GJ2] (which were taken from [SS]) verbatim to demonstrate the claim. Note that we have to use the assumption that $\bar{A}$ is diffeomorphic to the unit ball in order to conclude that M_j, since embedded, has to be orientable.

LEMMA 3.3. *Let again $\bar{A}$ be diffeomorphic to the unit ball. If we do not require a curvature condition for ∂A, then we can still conclude that each M_j in* (3.1) *is an oriented surface of genus zero, i.e. topologically a disk with holes or again a two-sphere.*

PROOF. Again, M_j is orientable. The argument of § 5 of [GJ2] now shows that for each simple closed curve γ in M_j there exists $l \in \mathbb{N}$ so that $l\gamma$ can be homotoped into ∂A or to a point. Since M_j is orientable, the same holds for γ, and the claim follows.

4. – Existence results.

THM. 4.1. *Suppose A is a bounded open subset of a threedimensional Riemannian manifold X, $\bar{A}$ being diffeomorphic to the unit ball, and let ∂A have positive mean curvature w.r.t. the interior normal and be of class C^4. Suppose that A contains no minimal embedded two-sphere.*

Then there exists an embedded minimal disk M in A which meets ∂A orthogonally.

PROOF. Lemma 2.1 implies that $C'_\varrho(0, 0) \neq \emptyset$ (for any $\varrho > 0$). The result then follows from Lemmata 3.1 and 3.2.

THM. 4.2. *Suppose A is a bounded open subset of a threedimensional Riemannian manifold, $\bar{A}$ again being diffeomorphic to the unit ball and $\partial A \in C^4$, and suppose that A does not contain an embedded minimal twosphere.*

Then there exists an embedded minimal surface in A meeting ∂A orthogonally which is of genus zero, i.e. topologically a disk (possibly) with holes.

PROOF. The result this time follows from Lemmata 2.1, 3.1 and 3.3.

REMARK. The existence of embedded minimal two-spheres in A is excluded if A has nonpositive sectional curvature, or, more generally, if it carries a strictly convex function.

5. – Embedded minimal disks in convex bodies.

The purpose of this section is the demonstration of

THM. 5.1. *Let A be a bounded open subset of $\mathbb{R}^3$ with strictly convex boundary $\partial A \in C^4$.*

Let R_1 be the largest positive number for which there exists a ball $B(x_1, R_1)$ with

$$B(x_1, R_1) \subset \bar{A} ,$$

and let R_2 be the smallest positive number for which there exists a ball $B(x_2, R_2)$ with

$$A \subset B(x_2, R_2)$$

and suppose

$$R_1/R_2 \geqslant 1/\sqrt{2} .$$

Then there exist three embedded minimal disks in A meeting ∂A orthogonally.

We shall first derive some auxiliarly results.

Let $w\colon I\to\mathcal{M}_1$ be a continuous path. Since $w(t)$, $t\in I$, is an oriented surface, it divides A into two parts $A_1(t)$, $A_2(t)$, where say, $A_1(t)$ lies on the side of $w(t)$ determined by the positive normal vector. More generally, we also consider paths where we allow that $w(0)$ and $w(1)$ are points in ∂A.

LEMMA 5.1. *Let $w\colon I\to\mathcal{M}_1$ or $w\colon(I,\partial I)\to(\mathcal{M}_1,\partial A)$ be a path with $A_1(1)=A_2(0)$. Then, if $B(x_0,r)$ is any ball contained in $\bar{A}$, we have*

$$\sup_{t\in I}|w(t)\cap B(x_0,r)|\geqslant\pi r^2\,. \tag{5.1}$$

PROOF. If $B\subset\bar{A}$, then we denote by

$$Z_2(B,\partial B,G)$$

the 2-dimensional cycles in $B \bmod \partial B$ with coefficients in a group G. It follows from [A] that

$$H_m(B,\partial B;G)=\pi_1\big(Z_2(B,\partial B;G),\{0\}\big)\,,$$

and furthermore that the nontrivial elements in $\pi_1\big(Z_2(A,\partial A;G),\{0\}\big)$ are precisely those represented by paths w satisfying

$$A_1(1)=A_2(0)\,.$$

This also implies that a nontrivial element of $\pi_1\big(Z_2(A,\partial A;G),\{0\}\big)$ induces a nontrivial element of $\pi_1\big(Z_2(B,\partial B;G),\{0\}\big)$ by restriction onto $B\subset\bar{A}$.

Thus, in particular, w induces a nontrivial element in $\pi_1\Big(Z_2\big(B(x_0,r),\partial B(x_0,r);G\big),\{0\}\Big)$, and (5.1) then is well-known (A nonelementary proof can also be given along the lines of this paper: a minimaxing procedure over nontrivial paths yields an embedded minimal disk in $B(x_0,r)$, cf. [GJ2], and each such disk is planar and has area πr^2 (cf. [N]); therefore

$$\inf\sup_{t\in I}|v(t)\cap B(x_0,r)|=\pi r^2$$

where the infimum is taken over all nontrivial paths v).

The following lemma si essentially due to J. Steiner.

LEMMA 5.2. *Let $A\subset\mathbf{R}^3$ be a bounded open set with a strictly convex boundary ∂A of class C^2. Let M be an embedded oriented minimal surface* *in A meeting ∂A orthogonally (in particular $\partial M = M \cap \partial A$). Let $n(x)$ be a unit normal vector at $x \in M$, consistently chosen with the help of the orientation. For $\varepsilon \in \mathbf{R}$ let*

$$M_\varepsilon := \{x + \varepsilon n(x) : x \in M\}$$

be a parallel surface. Then, if $|\varepsilon|$ is sufficiently small, $\varepsilon \neq 0$, M_ε is an surface with embedded

$$\partial M_\varepsilon \cap \bar{A} = \emptyset \tag{5.2}$$

and

$$|M_\varepsilon \cap \bar{A}| < |M| . \tag{5.3}$$

PROOF. (5.2) follows since ∂A is strictly convex and M meets ∂A orthogonally. The second variation formula gives

$$\frac{d^2}{d\varepsilon^2} \operatorname{Area}(M_\varepsilon)|_{\varepsilon=0} = 2\int K \, dM$$

where K is the Gauss curvature of M, and (5.3) follows from $K \leqslant 0$ and (5.2).

LEMMA 5.3. *Let $A \subset \mathbf{R}^3$ again have a strictly convex boundary ∂A. Let M be an embedded minimal disk in A meeting ∂A orthogonally. Then thereex ists a path $v \in V(0, 0)$ with $v(\frac{1}{2}) = M$ and*

$$\sup_{t \in I} |v(t)| = |M| .$$

PROOF. M divides A into two components $A_1(M)$ and $A_2(M)$. For $i \in \{1, 2\}$, we define

$$J_i := \{\psi = \{\psi_s\}_{0 \leqslant s \leqslant 1}, \psi_s : \mathbf{R}^3 \to \mathbf{R}^3, \text{ isotopy of class } C^1 \text{ (in particular } \psi_0 = \mathrm{id}),$$
$$\psi_s(A_i) \subset A_i,\ \psi_s(\partial A_i) \subset \partial A_i,\ |\psi_s(M)| \leqslant |M| \text{ for } 0 \leqslant s \leqslant 1\}$$

and

$$\mu_i := \inf_{\varphi \in J_i} |\psi_1(M)| .$$

The claim easily follows if we can show $\mu_i = 0$ for $i = 1, 2$. Thus, let us assume

$$\mu_i > 0 \qquad \text{for } i = 1 \text{ or } i = 2 . \tag{5.4}$$

Lemma 5.2 implies

$$\mu_i < |M| ,$$

and therefore Lemma 1 in § 4 of [GJ2] shows that the constraint $|\psi_s(M)| \leq |M|$ for all $s \in (0, 1)$ in the definition of J_i plays no role for minimizing.

Thus suppose that we have a sequence $\psi^j \in J_i$ with

$$|\psi_1^j(M)| \to \mu_i .$$

After selection of a subsequence, we get varifold convergence

$$\boldsymbol{v}(\psi_1^j(M)) \to W , \quad \text{say} , \qquad (W \neq 0 \text{ by } (5.4))$$

and the arguments of [I], in particular § 1, Thm. 5.2, Thms. 6.1 and 6.2, imply that

$$\operatorname{supp}(W) = \sum_{k=1}^{K} n_k M_k$$

where $n_k \in \mathbb{N}$ and M_k are mutually disjoint embedded minimal disks in A_i (note that in order to guarantee that the M_k are oriented we have to use that there are no unorientable surfaces in the unit ball meeting its boundary transversally, and in order to guarantee that there are no closed components we have to use the Euclidean structure (or at least the non-positivity of the curvature of) the ambient space). Also, each M_k meets ∂A orthogonally.

The minimizing property of M_k contradicts Lemma 5.2, however, and thus (5.4) is not possible, and the claim is proved.

LEMMA 5.4 (²). *Let $A \subset \mathbb{R}^3$ again have a strictly convex boundary ∂A. Let $B(x_0, R)$ be a largest possible ball contained in $\bar{A}$.*

Let M be an embedded minimal disk in A meeting ∂A orthogonally. Then

$$|M| \geq \pi R^2 . \tag{5.5}$$

PROOF. By Lemma 5.1, for any $v \in V(0, 0)$

$$\sup_{t \in I} |v(t)| \geq \pi R^2 ,$$

and hence the claim follows from Lemma 5.3.

(²) Cfr. [Sm].

REMARK. It is more elementary to prove

$$|M| \geqslant \pi \bar{R}^2 . \tag{5.6}$$

where $\bar{R}$ is the largest number with the property that for each $y \in \partial A$ there exists a ball $B(x, \bar{R}) \subset \bar{A}$ with $y \in \partial B(x, \bar{R})$. We can use Küster's estimate ([Kü], improving the estimate of Hildebrandt-Nitsche [HN])

$$l \leqslant \frac{2}{\bar{R}} |M|$$

with $l := \text{length}(\partial M)$ together with the isoperimetric inequality

$$|M| \leqslant \frac{1}{4\pi} l^2$$

to derive (5.6).

We can now carry out the *proof of Thm.* 5.1.

Lemma 2.1 implies that corresponding to $\varkappa(0,0)$, $\varkappa(0,1)$, and $\varkappa(1,1)$, we find almost minimizing varifolds $V_{0,0}$, $V_{0,1}$ and $V_{1,1}$. By the regularity results of § 3,

$$\sup(V_{i,j}) = \sum_{k=1}^{K_{i,j}} n_k M_{i,j}^k$$

where $n_k \in \mathbf{N}$ and $M_{i,j}^k$ are embedded minimal disks meeting ∂A orthogonally.

Obviously,

$$\sum_{k=1}^{K_{i,j}} n_k |M_{i,j}^k| = \varkappa(i,j) \leqslant \varkappa(1,1) \leqslant \pi R_2^2 .$$

On the other hand, by Lemma 5.4

$$|M_{i,j}^k| \geqslant \pi R_1^2 \qquad \text{for } k \in \{1, \ldots, K_{i,j}\}.$$

Therefore, if $R_1/R_2 > 1/\sqrt{2}$, in each sum, there is only one nontrivial surface, and this surface is of multiplicity 1.

If $R_1/R_2 = 1/\sqrt{2}$, we can find a path $v \in V(1,1)$ so that each $v(t)$ is the intersection of A with a plane and

$$\sup_{t \in I^s} |v(t)| \leqslant \pi R_2^2 .$$

If $\varkappa(1,1)=\pi R_2^2$, it follows that there exists $t\in I^3$ for which $\boldsymbol{v}(v(t))$ is almost minimizing, *i.e.* it meets ∂A orthogonally, and we see again that in each sum there is only one nontrivial surface, of multiplicity 1 as before.

If on the other hand $\varkappa(1,1)<\pi R_2^2$, then this property is deduced as before.

Therefore, the result follows if we have the strict inequalities

$$\varkappa(0,0)<\varkappa(0,1)<\varkappa(1,1)\,. \tag{5.7}$$

Thus, we only have to discuss the cases where instead of (5.7), we have equality somewhere. For this discussion, we need the constructions of Lusternik-Schnirelmann theory. We shall use the presentation in the appendix of [Kl].

We suppose that $v\in V(0,1)$ satisfies (2.7), i.e.

$$\sup_{t\in I^2}|v(t)|\leqslant\varkappa(0,1)+\varepsilon/4\,.$$

We then apply the deformation of § 2 to construct a path $\bar v\in V(0,1)$ satisfying in particular (2.22).

Let

$$l\colon I\to I^2$$

be continuous with $l(\varphi)=\big(t_1(\varphi),t_2(\varphi)\big)$ and

$$t_1(1)=1-t_0(0)\neq 0\,,$$

$$t_2(0)=0\,,\ t_2(1)=1\,.$$

Then $\bar v\circ l$ is a 1-cycle mod 2 of $\mathcal{M}_1$, homologous mod 2 to $\bar v|I\times\{0\}$ modulo ∂A, which is seen as follows.

Let

$$L\colon I^2\to I^2$$

with

$$L(\lambda,\varphi)=\big(\lambda t_1(\varphi),t_2(\varphi)\big)\,.$$

Thus

$$\begin{aligned}\bar v\circ\partial L&=\bar v\circ L|\{1\}\times I-\bar v\circ L|I\times\{1\}\\&\quad+\bar v\circ L|I\times\{0\}\ \text{modulo}\ \partial A,\ \text{since}\ \bar v\circ L|\{0\}\times I\ \text{is a curve in}\ \partial A\\&=\bar v\circ l-\big\{\bar v\big(\lambda(1-t_1(0)),1\big);\,0\leqslant\lambda\leqslant1\big\}+\big\{\bar v\big(\lambda t_1(0),0\big);\,0\leqslant\lambda\leqslant1\big\}\\&=\bar v\circ l+\bar v|I\times\{0\}\ \text{mod}\ 2\end{aligned}$$

because $\bar v$ satisfies the boundary condition (1.1) (via (1.4)).

We now assume $\varkappa(0,1) = \varkappa(0,0) =: \varkappa$.

We discuss two cases:

1)

$$\sup_{\varphi \in I} |\bar{v} \circ l(\varphi)| < \varkappa \tag{5.8}$$

for some positive values of the parameters ϱ, η, ε_0 (note that the deformation leading to $\bar{v}$ did depend on these parameters). We then look at all w that can be obtained from $\bar{v} \circ l$ via a homotopy h of the following type

$$\begin{cases} h: I \times I \to \mathcal{M}_1 , \\ h(0, \varphi) = \bar{v} \circ l(\varphi) , \\ h(1, \varphi) = w(\varphi) , \\ h(\tau, 1) = -h(\tau, 0) , \end{cases} \tag{5.9}$$

and we require that there exists a C^1-map

$$k: I^2 \times D \to X$$

so that $k(\tau, \varphi, \cdot)$ is a diffeomorphism from D onto $h(\tau, \varphi)$ for $(\tau, \varphi) \in I^2$.

We denote the class of such w by W.

Let

$$\varkappa' := \inf_{w \in W} \sup_{\varphi \in I} |w(\varphi)| .$$

Since any $w \in W$ satisfies $A_1(1) = A_2(0)$ because of (5.9), Lemma 5.1 implies

$$\varkappa' \geqslant \pi R_1^2 > 0 . \tag{5.10}$$

The deformation of § 2 can be used with $\varkappa'$ instead of $\varkappa(i, j)$ and W instead of $V(i, j)$ to yield the existence of a ϱ, 27-almost minimizing varifold V (for any $\varrho > 0$). By the regularity results of § 3, V yields an embedded minimal surface M_4 of genus zero meeting ∂A orthogonally, with multiplicity $m \geqslant 1$.

Lemma 5.2 implies

$$|M_4| \geqslant \pi R_1^2 .$$

Since on the other hand

$$m|M_4| \leqslant \varkappa' < \varkappa < \pi R_2^2 ,$$

we infer $m = 1$. The same argument shows that the support of V is connected, i.e. V yields precisely one embedded minimal disk. Also, since

$$|M_4| = \varkappa' < \varkappa = |M_1|,$$

we infer

$$M_4 \neq M_1 .$$

2)

$$\sup_{\varphi\in I} |vol(\varphi)| \geqslant \varkappa \tag{5.11}$$

for all positive ϱ, η, ε_0.

We let $N'(C'_\varrho(0, 1), \eta)$ be the image of $\boldsymbol{v}(v(I^2)) \cap N(C'_\varrho(0, 1), \eta)$ under the deformation of § 2.

We want to show that $C'_\varrho(0, 1)$ contains arbitrarily many elements if ϱ is chosen small enough. Thus, in particular, by the regularity results of § 3, we can obtain infinitely many embedded disks.

Let us assume on the contrary that card $(C'_\varrho(0, 1))$ is bounded from above independently of ϱ. Since each element of $C'_\varrho(0, 1)$ corresponds to an embedded minimal disk (using § 3, as already mentioned) of area (at most) $\varkappa$, we can find a ball $B(x_0, 4r) \subset \bar{A}$ with

$$B(x_0, 4r) \cap \operatorname{supp} C'_\varrho(0, 1) = \emptyset$$

for some $r > 0$ and all $\varrho > 0$.

(Note that $C'_{\varrho_1}(0, 1) \subset C'_{\varrho_2}(0, 1)$ if $\varrho_1 \leqslant \varrho_2$).

We now fix

$$\varrho = r .$$

We then choose η so small that for all $\Sigma \in \mathcal{M}_1$, with $\boldsymbol{v}(\Sigma) \in N(C'_r(0, 1), \eta)$

$$|\Sigma \cap B(x_0, 2r)| < \mu$$

for some prescribed $\mu > 0$.

By (2.11), if $\boldsymbol{v}(v(R_l)) \cap N(C'_r(0, 1), \eta) \neq \emptyset$, then for all $t \in R_l$

$$|v(t) \cap B(x_0, 2r)| < \mu + \varepsilon^2 .$$

We now let $(U_1, \ldots, U_m) \in \mathcal{U}^r_m$.

If $U_i \cap B(x_0, r) \neq \emptyset$, thus $U_i \subset B(x_0, 2r)$ and hence for $t \in R_l$

$$|\psi^i_s(v(t)) \cap U_i| \leqslant \mu + 2\varepsilon^2 \qquad \text{for all } s \in [0, 1]$$

using (2.2) (note that in the deformation of § 2, we had chosen $\delta = \varepsilon^2$).

Hence for t with $\boldsymbol{v}(v(t)) \subset N(C'_r(0,1), \eta)$

$$|\boldsymbol{v}(t) \cap B(x_0, r)| \leqslant \mu + 2m\varepsilon^2 . \tag{5.12}$$

We then choose μ and ε_0 (and consequently η and ε; note that $\varepsilon \leqslant \varepsilon_0$) small enough to guarantee

$$\mu + 2m\varepsilon^2 < \pi r^2 . \tag{5.13}$$

Therefore, if $\boldsymbol{v}(\bar{v}(t)) \in N'(C'_r(0,1), \eta) \cap \{\boldsymbol{v}(\Sigma) : \Sigma \in \mathcal{M}_1, |\Sigma| \geqslant \varkappa\}$, since then $\boldsymbol{v}(v(t)) \in N(C'_r(0,1), \eta)$, we have

$$|\bar{v}(t) \cap B(x_0, r)| < \pi r^2 . \tag{5.14}$$

Let

$$\Omega := \bar{v}^{-1}\Big(\{\Sigma : \boldsymbol{v}(\Sigma) \in N'(C'_r(0,1), \eta)\} \cap \{\Sigma \in \mathcal{M}_1, |\Sigma| \geqslant \varkappa\}\Big) \subset I^2 .$$

Because of (5.11),

$$l(I) \cap \Omega \neq \emptyset . \tag{5.15}$$

Since the boundary conditions (1.4) lead to an identification of $I \times \{0\}$ and $I \times \{1\}$ making I^2 into a Möbius strip, and therefore any two nontrivial 1-cycles in general position intersect in an odd number of points, (5.15) implies that Ω carries a nontrivial 1-cycle. (Clearly, we can make the construction smooth enough that every connected component of Ω is path-connected.) This means that there exists a map

$$\lambda : I \to I^2 , \qquad \lambda(\sigma) = (t_1(\sigma), t_2(\sigma))$$

so that

$$\boldsymbol{v}\big(\bar{v}(t_1(\sigma), t_2(\sigma))\big) \subset N'(C'_r(0,1), \eta) \cap \{\boldsymbol{v}(\Sigma) : \Sigma \in \mathcal{M}_1, |\Sigma| \geqslant \varkappa\} ,$$

$$\bar{v}(t_1(1), 1) = -\bar{v}(t_1(0), 0) .$$

Since $\bar{v} \circ \lambda$ represents a nontrivial path, however (5.14) contradicts Lemma 5.1.

This proves that in case 2), we get infinitely many embedded minimal disks.

Similarly, let $v \in V(1,1)$ satisfy

$$\sup_{t \in I^2} |v(t)| \leqslant \varkappa(1,1) + \varepsilon/4 .$$

The deformation of § 2 yields $\bar{v} \in V(1,1)$ satisfying (2.22).

Let

$$l_1 : I^2 \to I^3$$

be continuous with

$$l_1(\varrho_1, \varrho_2) := \big(t_1(\varrho_1, \varrho_2), t_2(\varrho_1, \varrho_2), t_3(\varrho_1, \varrho_2)\big)$$

and

$$\begin{aligned} t_1(1, \varrho_2) &= 1 - t_1(0, \varrho_2)\,, \\ t_1(\varrho_1, 1) &= 1 - t_1(\varrho_1, 0)\,, \\ t_2(1, \varrho_2) &= 1\,, \\ t_2(0, \varrho_2) &= 0\,, \\ t_2(\varrho_2, 1) &= 1 - t_2(\varrho_2, 0)\,, \\ t_3(1, \varrho_2) &= t_3(0, \varrho_2)\,, \\ t_3(\varrho_1, 1) &= 1\,, \\ t_3(\varrho_1, 0) &= 0 \end{aligned}$$

and

$$\begin{aligned} &L_1 : I^3 \to I^3\,, \\ &L_1(\lambda, \varrho_1, \varrho_2) := \big(\lambda t_1(\varrho_1, \varrho_2), t_2(\varrho_1, \varrho_2), t_3(\varrho_1, \varrho_2)\big)\,. \end{aligned}$$

Then

$$\begin{aligned} \bar{v}\circ\partial L_1 = \bar{v}\circ L_1|\{1\}\times I^2 &- \bar{v}\circ L_1|I\times\{1\}\times I \\ &+ \bar{v}\circ L_1|I\times\{0\}\times I + \bar{v}\circ L_1|I\times I\times\{1\} - \bar{v}\circ L_1|I\times I\times\{0\} \bmod \partial A \\ &= \bar{v}\cdot l_1 - \Big\{\bar{v}\Big(\lambda(1 - t_1(0, \varrho_2)), 1, t_3(0, \varrho_2)\Big) : (\lambda, \varrho_2)\in I^2\Big\} \\ &+ \big\{\bar{v}\big(\lambda t_1(0, \varrho_2), 0, t_3(0, \varrho_2)\big) : (\lambda, \varrho_2)\in I^2\big\} \\ &+ \Big\{\bar{v}\Big(\lambda(1 - t_1(\varrho_1, 0)), 1 - t_2(\varrho_1, 0), 1\Big) : (\lambda, \varrho_1)\in I^2\Big\} \\ &- \big\{\bar{v}\big(\lambda t_1(\varrho_1, 0), t_2(\varrho_1, 0), 0\big) : (\lambda, \varrho_1)\in I^2\big\} = \bar{v}\circ l_1 + \bar{v}|I^2\times\{0\} \bmod 2 \end{aligned}$$

since the second and third term together give rise to a trivial cycle.

Thus, $\bar{v}\circ l_1$ is homologous mod 2 to $\bar{v}|I^2\times\{0\}$ modulo ∂A.

Also, $\bar{v}\circ l_1$ is homotopic to $\bar{v}|\{\frac{1}{2}\{\times I^2$. We are now able to discuss the case

$$\varkappa(0, 1) = \varkappa(1, 1) =: \varkappa_1\,.$$

As before, we distinguish two cases

1) $$\sup_{(\varrho_1,\varrho_2)\in I^2} |\bar{v}\circ l_1(\varrho_1, \varrho_2)| < \varkappa_1 .$$

Then we again look at the class W_1 of those w that can be obtained from $\bar{v}\circ l_1$ through suitable homotopies (satisfying the appropriate boundary and smoothness conditions; we omit the details). Let

$$\varkappa'' := \inf_{w\in W_1} \sup_{(\varrho_1,\varrho_2)\in I^2} |w(\varrho_1, \varrho_2)| .$$

We note

$$\varkappa' \leqslant \varkappa'' < \varkappa_1 .$$

Thus, the previous construction implies that we can find two distinct embedded minimal disks of area $\leqslant \varkappa''$.

2) $$\sup_{(\varrho_1,\varrho_2)\in I^2} |\bar{v}\circ l_1(\varrho_1, \varrho_2)| \geqslant \varkappa_1$$

(for all positive ϱ, η, ε_0).

Then one finds again a nontrivial path with image in $N'(C_r'(1, 1), \eta) \cap \{v(\Sigma): \Sigma \in \mathcal{M}_1;\ |\Sigma| \geqslant \varkappa_1\}$ (this time, the path is homotopic to $\bar{v}|\{\frac{1}{2}\} \times I \times \{0\}$). Therefore, again the set of almost minimizing varifolds and hence of embedded minimal disks has to be infinite.

This completes the proof of Thm. 5.1.

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