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Intermediate spaces and the complex method of interpolation for families of Banach spaces

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Intermediate Spaces and the Complex Method of Interpolation for Families of Banach Spaces.

EUGENIO HERNÁNDEZ (*)

1. – Introduction.

Recently, R. Coifman, M. Cwikel, R. Rochberg, Y. Sagher and G. Weiss have developed a theory of complex interpolation for families of Banach spaces ([3], [4]). They start with a family of Banach spaces associated with the boundary of the unit disk Δ in $\mathbb{C}$ (the set of complex numbers) and, for each complex number in the interior of Δ, they are able to define an intermediate space with properties that are appropriate for interpolation. (For a summary of this construction and its properties see section 2 below). This method generalizes that of Calderón for pairs of Banach spaces ([2]).

In the same papers they proved that the intermediate spaces of L^p spaces are also L^p spaces. Specifically, if p is a measureable function defined on T, the boundary of Δ, whose range is contained in $[1, \infty]$, then the intermediate space at the point z, interior to Δ, of the family of Banach spaces $\{L^{p(\xi)}\}$, $\xi \in T$, is $L^{p(z)}$, where $1/p(z)$ is the harmonic function on Δ whose boundary values are $1/p(\xi)$.

In this paper we continue the identification of other spaces of measurable functions as well as spaces of vector valued sequences (this work was suggested in [4]). More precisely, we identify the intermediate spaces of weighted L^p spaces, L^p spaces of Banach space valued functions, Lorentz spaces, l_p^s spaces of vector valued sequences, Sobolev and Besov-Lipschitz spaces. This is accomplished by developing a theory of interpolation of Banach

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lattices that generalizes that of A. P. Calderón ([2]). Since only the general theory is developed in [3], this work is a natural complement to that paper.

As for notation we systematically use the letter θ instead of $e^{i\theta}$ to denote an element of $T = \{z \in C: |z| = 1\}$. Also, $P_z(\theta)$ will denote the Poisson kernel of Δ for evaluation at $z \in \Delta$, $Q_z(\theta)$ will denote the conjugate Poisson kernel and $H_z(\theta) = P_z(\theta) + iQ_z(\theta)$ will denote the Herglotz kernel.

We assume that the reader is familiar with the basic facts of the real and complex interpolation methods. Unless otherwise stated the norm on a Banach space B will be denoted by $\| \ \|_B$.

2. – The complex interpolation method.

We now describe the complex interpolation method for families of Banach spaces and summarize some of its properties. Let $\{B(\theta)\}$, $\theta \in T$, be a family of Banach spaces associated with the boundary of the unit disk in $\mathbf{C}$. We say that this family is an *interpolation family of Banach spaces* (or *interpolation family*, for short) if each $B(\theta)$ is continuously embedded in a Banach space $(U, \| \ \|_U)$, the function $\theta \to \|b\|_{B(\theta)}$ is measurable for each $b \in \bigcap_{\theta \in T} B(\theta)$, and if

$$\beta = \left\{ b \in \bigcap_{\theta \in T} B(\theta) / \int_T \log^+ \|b\|_{B(\theta)} \, d\theta < \infty \right\}$$

we have $\|b\|_U \leq k(\theta) \, \|b\|_{B(\theta)}$, for all $b \in \beta$, where $\log^+ k(\theta) \in L^1$ (the space β is called the *log-intersection* space of the given family and U is called a containing space).

We let $N^+(B(\cdot))$ be the space of all β-valued analytic functions of the form

$$g(z) = \sum_{j=1}^{\infty} \psi_j(z) \, b_j$$

for which $\|g\|_\infty = \sup \|g(\theta)\|_{B(\theta)} < \infty$, where $\psi_j \in N^+$ and $b_j \in \beta$, $j = 1, 2, \ldots, m$. (N^+ denotes the positive Nevalinna class for Δ (see [5], Chapter 2)). The completion of the space $N^+(B(\cdot))$ with respect to $\| \ \|_\infty$ is denoted by $\mathcal{F}(B(\cdot))$. (It is not difficult to show that $\mathcal{F}(B(\cdot))$ is a closed subspace of a Banach space of analytic functions). The space $[B(\theta)]_z$, which will also be denoted by $B(z)$, consists of all elements of the form $f(z)$ for $f \in \mathcal{F}(B(\cdot))$. A Banach space norm is defined on $B(z)$ by

$$\|v\|_z \equiv \|v\|_{B(z)} = \{\|f\|_\infty : f \in \mathcal{F}(B(\cdot)), \ f(z) = v\}$$

$v \in B(z)$. It can be proved that $(B(z), \| \|_z)$ is a Banach space and β is dense in each $B(z)$. The space $B(z)$ is called an intermediate space of the family $\{B(\theta)\}$, $\theta \in T$.

This construction has the following two fundamental properties:

THEOREM (2.1). (*Subharmonicity*). *For each $g \in \mathcal{F}(B(\cdot))$ and each $z \in \Delta$ we have*

$$\|g(z)\|_{B(z)} \leq \exp \int_T \log \|g(\theta)\|_{B(\theta)} P_z(\theta)\, d\theta .$$

THEOREM (2.2). (*Interpolation theorem*). *Let T be a linear operator which maps U continuously into V, where U and V are containing spaces for the families $\{A(\theta)\}$ and $\{B(\theta)\}$, respectively. Suppose further that T maps $\mathcal{A}$ into $\bigcap_{\theta\in T} B(\theta)$ with $\|Ta\|_{B(\theta)} \leq M(\theta)\ \|a\|_{A(\theta)}$ for all $a \in \mathcal{A}$, $\theta \in T$, where* $\log M(\theta)$ *is absolutely integrable on T and $\mathcal{A}$ is the log-intersection space of the family $\{A(\theta)\}$. Then, T maps $A(z)$ into $B(z)$ with norm not exceeding*

$$M(z) = \exp \int_T \big(\log M(\theta)\big) P_z(\theta)\, d\theta, \ \ z \in \Delta.$$

The duality and reiteration theorems hold as well as an interpolation theorem for « analytic » families of linear operators.

See [3] for details and proofs. In [3] a relation between this interpolation construction and the complex interpolation method of Calderón is also given.

3. – The fundamental inequality.

Let (M, dx) be a fixed measure space. Suppose that the function $p: \bar{\Delta} \to [1, \infty]$ is such that $1/p(z)$ is harmonic on Δ. A measurable function $F: T \times M \to \mathbb{R}$ is called *p-admissible* if $\int_T d\theta P_z(\theta) \|F(\theta, \cdot)\|_{L^{p(\theta)}} < \infty$ for some $z \in \Delta$ (and hence for all z).

PROPOSITION (3.1). *For a p-admissible function F we have*

$$\log \|u_F(z, \cdot)\|_{L^{p(z)}} \leq \int_T d\theta P_z(\theta) \log \|F(\theta, \cdot)\|_{L^{p(\theta)}}$$

where $u_F(z, x) = \exp \Big\{\int_T d\theta H_z(\theta) \log |F(\theta, x)|\Big\}$, $z \in \Delta$.

PROOF. We start by proving the inequality for $p = 1$. In other words, we assume that F is 1-admissible and we want to show

$$\log \int_M |u_F(z, x)|\, dx \leqq \int_T d\theta P_z(\theta) \log \|F(\theta, \cdot)\|_{L^1}. \tag{3.1}$$

The right hand side of the above inequality is a harmonic function on Δ. Since F is 1-admissible, an application of Jensen's inequality and Fubini's theorem imply that $u_F(z, \cdot) \in L^1$ for all $z \in \Delta$. Since u_F is analytic, a theorem of E.M. Stein and G. Weiss (see [10]) implies that the function $\log \int_M |u_F(z, x)|dx$ is subharmonic. Since both functions have the same boundary values, namely $\log \|F(\theta, \cdot)\|_{L_1}$, the values of the subharmonic function must be smaller than the values of the harmonic one. This proves inequality (3.1).

To prove proposition (3.1), fix $z_0 \in \Delta$ and let $g \geqq 0$ be a simple function on M satisfying $\|g\|_{L^{q(z_0)}} \leqq 1$, where $(1/p(z)) + (1/q(z)) = 1$.

Denote by $a(z)$ the unique analytic function in Δ whose real part has boundary values $1/q(\theta)$ and $a(z_0) = 1/q(z_0)$. Consider $g(z, x) = [g(x)]^{a(z)q(z_0)}$, $z \in \bar{\Delta}$.

Simple calculations show $\int_T d\theta P_{z_0}(\theta) \log|g(\theta, x)| = \log g(x)$. From here and (3.1) we deduce

$$\int_M g(x)|u_F(z_0, x)|dx \leqq \exp\left\{\int_T d\theta P_{z_0}(\theta) \log\left(\int_M |F(\theta, x)\, g(\theta, x)|\, dx\right)\right\}.$$

Since $\|g(\theta, \cdot)\|_{L^{q(z)}} \leqq 1$ for all $\theta \in T$, the above inequality together with Hölder's inequality implies

$$\int_M g(x)|u_F(z_0, x)|\, dx \leqq \exp\left\{\int_T d\theta P_{z_0}(\theta) \log \|F(\theta, \cdot)\|_{L^{p(\theta)}}\right\}.$$

From here, the fundamental inequality follows by observing that $\|u_F(z_0, x)\|_{L^{p(z_0)}} = \sup\left\{\int_M g(x)|u_F(z_0, x)\, dx / g \geqq 0\right\}$ simple and $\|g\|_{L^{q(z_0)}} \leqq 1$.

We remark that a particular case of inequality (3.1) is Hölder's inequality. To see this take $f, g \in L^1$ and $0 \leqq s \leqq 1$, and apply (3.1) to $F(\theta, x) = f(x)\chi_{[0, 2\pi s)}(\theta) + g(x)\chi_{[2\pi s, 2\pi)}(\theta)$ at $z = 0$, where χ_E denotes the characteristic function of the set E. The result is Hölder's inequality with $p = 1/s$ and $q = 1/1 - s$.

4. – Banach lattices and examples.

A subclass X of the class of measurable functions on a measure space (M, dx) is called a *Banach lattice* if there exists a norm $\|\ \|_X$ on X such that

$(X, \| \|_X)$ is a Banach space and if $f \in X$ and g is a measurable function such that $|g(x)| \leqq |f(x)|$ *a.e.* on M, then $g \in X$ and $\|g\|_X \leqq \|f\|_X$.

Given a Banach lattice X on (M, dx) we present below a way to construct others. Let $\varphi(x, t)$ be a real valued function defined on $M \times [0, \infty]$ such that $\varphi(\cdot, 0) \equiv 0$ on M and for each $x \in M$, $\varphi(x, t)$ is a concave increasing function on t. Denote by $\varphi(X)$ the class of measurable functions g on M for which there exist $\lambda > 0$ and $f \in X$ with $\|f\|_X \leqq 1$ such that

$$|g(x)| \leqq \lambda \varphi(x, |f(x)|) .$$

The norm of an element $g \in \varphi(X)$, denoted by $\|g\|_{\varphi(X)}$, is defined as the infimum of the values of λ for which the above inequality holds. It is well known ([2], § 13.3 and 33.3) that $(\varphi(X), \| \|_{\varphi(X)})$ is a Banach lattice.

We now give some examples of Banach lattices, which will be needed in the sequel.

EXAMPLE 1. If $X = L^1 \equiv L^1(M)$, w is a positive measurable function on M and $\varphi_{p,w}(x, t) = [w(x)]^{-1/p} t^{1/p}$, $1 \leqq p \leqq \infty$, then $\varphi_{p,w}(L^1)$ coincides with L^p_w, the L^p space with respect to the weight w.

EXAMPLE 2. Let $s \in \mathbb{R}$, $1 \leqq p \leqq \infty$ and $\varphi_{p,s}(n, t) = 2^{-snt^{1/p}}$, $n \in \mathbb{N}$. Then, $\varphi_{p,s}(l_1)$ is the space l^s_p of all real valued sequences $a = (a_n)_{n=1}^{\infty}$ such that $\|a\|_{l^l_p} = \left\{ \sum_{n=1}^{w} [2^{sn}|a_n|]^p \right\}^{1/p} < \infty$. (When $p = \infty$ we write $\|a\|_{l^s_p} = \sup_n 2^{ns}|a_n|$). When $s = 0$ we shall write l_p instead of l^0_p for obvious reasons.

EXAMPLE 3. For $x \in (0, \infty)$, $p \in \mathbb{R} (p \neq 0)$ and $1 \leqq q \leqq \infty$ we define $\varphi_{p,q}(x, t) = x^{1/q - 1/p} t^{1/q}$. If we consider the Lebesgue measure dx on the set $(0, \infty)$, the Banach lattice $\varphi_{p,q}(L^1) \equiv \varphi_{p,q}(L^1(0, \infty))$ is the space $X_{p,q}$ of all measurable functions g on $(0, \infty)$ such that

$$\|g\|_{X_{p,q}} = \left\{ \int_0^{\infty} [x^{1/p} |g(x)|]^q \frac{dx}{x} \right\}^{1/q} < \infty .$$

What needs to be proved in examples 1 and 3 is straightforward; example 2 is contained in example 1 by taking $M = \mathbb{N}$ with the discrete measure and $w(n) = 2^{snp}$, $n \in \mathbb{N}$.

5. – Interpolation of Banach lattices.

Let $\{X(\theta)\}$, with $\theta \in T$, be a family of Banach lattices on a fixed measure space (M, dx). For $z \in \Delta$ we denote by $[X(\theta)]^z$ the class of measurable functions f on M for which there exist $\lambda > 0$ and a measurable function $F\colon T\times M \to \mathbb{R}$ with $\|F(\theta, \cdot)\|_{X(\theta)} \leqq 1$ *a.e.* such that

$$|f(x)| \leqq \lambda \exp\left\{\int_T d\theta P_z(\theta) \log |F(\theta, x)|\right\}.$$

We let $\|f\|^z \equiv \|f\|_{[X(\theta)]^z}$ be the infimum of the values of λ for which such an inequality holds.

LEMMA (5.1). *$\left([X(\theta)]^z, \|\ \|^z\right)$ is a Banach lattice on (M, dx).*

PROOF. The homogeneity of the norm is clear. The subadditivity is not so clear. To prove it we proceed as follows. Let f_n be a sequence of functions in $[X(\theta)]^z$ such that $\sum_{n=1}^{\infty} \|f_n\|^z < \infty$. Then, given $\varepsilon > 0$, there exist λ_n and measurable functions $F_n\colon T\times M \to \mathbb{R}$ satisfying $\|F_n(\theta, \cdot)\|_{\varphi(X)} \leqq 1$, $\lambda_n \leqq \|f_n\|^z + \varepsilon/2^n$ and

$$|f_n(x)| \leqq \lambda_n \exp\left\{\int_T d\theta P_z(\theta) \log|F_n(\theta, x)|\right\},$$

$n = 1, 2, \ldots$. Use proposition (3.1) with $M = \mathrm{N}$, the discrete measure on N and $p = 1$ to obtain

$$\sum_{n=1}^{\infty} |f_n(x)| \leqq \lambda \exp\left\{\int_T d\theta P_z(\theta) \log\left(\sum_{n=1}^{\infty} \frac{\lambda_n}{\lambda} F_n(\theta, x)|\right)\right\}, \tag{5.1}$$

where $\lambda = \sum_{n=1}^{\infty} \lambda_n$. Since $\left\|\sum_{n=1}^{\infty} (\lambda_n/\lambda)|F_n(\theta, x)|\right\|_{X(\theta)} \leqq 1$, a convergence in measure argument (see [2], § 13.2 and 33.2) shows that the above series converges to an element $g(\theta, x) \in X(\theta)$ such that $\|g(\theta, \cdot)\|_{X(\theta)} \leqq 1$. The inequality (5.1) then implies that $\sum_{n=1}^{\infty} |f_n(x)| \in [X(\theta)]^z$ and $\left\|\sum_{n=1}^{\infty} |f_n(x)|\right\|^z \leqq \lambda \leqq \sum_{n=1}^{\infty} \|f_n\|^z + \varepsilon$. Since ε was arbitrary we deduce $\left\|\sum_{n=1}^{\infty} |f_n(x)|\right\|^z \leqq \sum_{n=1}^{\infty} \|f_n\|^z$. This proves the subadditivity of the norm as a particular case. The only remaining property of the norm that is not clear is that $\|f\|_z = 0 \Rightarrow f \equiv 0$ a.e. Assume $\|f\|_z = 0$. For each integer n, there exist functions $F_n\colon T\times M \to \mathbb{R}$ with $\|F_n(\theta, \cdot)\|_{X(\theta)} \leqq 1$ a.e. such that

$$|f(x)| \leqq \exp\left\{\int_T d\theta P_z(\theta) \log \frac{1}{n^2} |F_n(\theta, x)|\right\}. \tag{5.2}$$

Then $\sum_{n=1}^{\infty} \|(1/2^n)|F_n(\theta, x)|\|_{X(\theta)} \leqq \sum_{n=1}^{\infty} 1/n^2 < \infty$. As above, a convergence in measure argument shows that $(1/n^2)\, F_n(\theta, \cdot)$ tends to zero a.e. as $n \to \infty$. Inequality (5.2) now implies $f = 0$ a.e.

It remains to be proved that $([X(\theta)]^z, \|\ \|^z)$ is complete. Let f_n be a sequence of functions in $[X(\theta)]^z$ such that $\sum_{n=1}^{\infty} \|f_n\|^z < \infty$. We have proved that $\sum_{n=1}^{\infty} |f_n(x)| \in [X(\theta)]^z$ and $\left\|\sum_{n=1}^{\infty} |f_n(x)|\right\|^z \leqq \sum_{n=1}^{\infty} \|f_n\|^z$. Thus, $\sum_{n=1}^{\infty} |f_n| < \infty$, a.e. and we can consider f as the pointwise sum of the series $\sum_{n=1}^{\infty} f_n$. Since $|f(x)| \leqq \leqq \sum_{n=1}^{\infty} |f_n(x)|$ we see that $f \in [X(\theta)]^z$. Finally, it is easy to see that $\sum_{n=1}^{\infty} f_n$ converges to f in the space $([X(\theta)]^z, \|\ \|^z)$, which proves the completeness of this space. ■

We now apply this interpolation construction to particular Banach lattices. Let X be a Banach lattice on a measure space (M, dx) and let $\{\varphi_\theta\}$, $\theta \in T$, be a family of real valued functions defined on $M \times [0, \infty)$, measurable on θ, such that $\varphi_\theta(\cdot, 0) \equiv 0$ on M, a.e. θ, and for almost every θ and for each $x \in M$, $\varphi_\theta(x, t)$ is a concave increasing function of t. Suppose further, that for some $z \in \Delta$ (and hence for all)

$$\varphi_z(x, t) = \exp\left\{\int_T d\theta P_z(\theta) \log \varphi_\theta(x, t)\right\} < \infty \tag{5.3}$$

for all $x \in M$, $t \in [0, \infty)$.

Lemma (5.2). *$\varphi_z(x, t)$ is a concave increasing function of t for all $z \in \Delta$, $x \in M$. Moreover, $\varphi_z(X) \subset [\varphi_\theta(X)]^z$ and the inclusion is norm decreasing.*

Proof. Let $0 \leqq t_1 \leqq t_2$ and $0 < \lambda < 1$. Inequality (3.1) applied to a two point measure space gives us

$$(1-\lambda)\, \varphi_z(x, t_1) + \lambda \varphi_z(x, t_2) \leqq \exp\left\{\int_T d\theta P_z(\theta) \log |(1-\lambda)\, \varphi_\theta(x, t_1) + \lambda \varphi_\theta(x, t_2)|\right\}.$$

The concavity of φ_z now follows from the concavity of each φ_θ.

To prove the inclusion, take $g \in \varphi_z(X)$ and $\varepsilon > 0$. Then, there exists $f \in X$ with $\|f\|_X \leqq 1$ such that

$$|g(x)| \leqq (1+\varepsilon) \|g\|_{\varphi_z(X)} \exp\left\{\int_T d\theta P_z(\theta) \log \varphi_\theta(x, |f(x)|)\right\}. \tag{5.4}$$

Since, clearly, $\varphi_\theta(x, |f(\cdot)|) \in X(\theta)$ and $\|\varphi_\theta(x, |f(\cdot)|)\|_{X(\theta)} \leqq 1$, the definition of $[X(\theta)]^z$ and (5.4) imply $g \in [\varphi_\theta(X)]^z$ and $\|g\|^z \leqq (1+\varepsilon)\|g\|_{\varphi_z(X)}$, which allows us to obtain the desired conclusion upon letting $\varepsilon \to 0$. ∎

Let now $1 \leqq p(\theta) \leqq \infty$ be a measurable function on T and for each $\theta \in T$ let $w_\theta(x)$ be a measurable function on M. Assume that for some $z \in \Delta$ (and hence for all)

$$w_z(x) = \exp\left\{p(z)\int_T d\theta P_z(\theta)\,(1/p(z))\log w_\theta(x)\right\} < \infty \tag{5.5}$$

a.e. $x \in M$, where $1/p(z)$ is the harmonic function on Δ whose boundary values are $1/p(\theta)$ (i.e. $1/p(z) = \int_T d\theta(1/p(\theta))P_z(\theta)$. Lemma (5.2) together with example 1 of section 4 implies $L^{p(z)}_{w_z} \subset [L^{p(\theta)}_{w_\theta}]^z$, and the inclusion is norm decreasing. In this case the reverse inclusion is also true and it is a consequence of proposition (3.1). To see this, take $f \in [L^{p(\theta)}_{w_\theta}]^z$ and $\varepsilon > 0$. Choose a measurable function $F: T \times M \to \mathbb{R}$ such that $F(\theta, \cdot) \in L^{p(\theta)}_{w_\theta}$ with

$$\|F(\theta, \cdot)\|^{p(\theta)}_{w_\theta} \leqq 1$$

and

$$|f(x)| \leqq (1+\varepsilon)\|f\|^z \exp\left\{\int_T d\theta P_z(\theta)\log|F(\theta, x)|\right\}.$$

Proposition (3.1) now implies

$$\|f\|^{p(z)}_{w_z} \leqq (1+\varepsilon)\|f\|^z \exp\left\{\int_T d\theta P_z(\theta)\log\|F(\theta, \cdot)\|^{p(\theta)}_{w_\theta}\right\} \leqq (1+\varepsilon)\|f\|^z.$$

This proves the following result:

PROPOSITION (5.3). *Let $1 \leqq p(\theta) \leqq \infty$ be a measurable function of T and for each $\theta \in T$ let $w_\theta(x) \geqq 0$ be a measurable function on M. If $w_z(x) < \infty$ a.e. x, where w_z is given in (5.5), we have $[L^{p(\theta)}_{w_\theta}]^z = L^{p(z)}_{w_z}$, $z \in \Delta$, with equality of norms, where $1/p(z)$ is the harmonic function on Δ whose boundary values are $1/p(\theta)$.*

COROLLARY (5.4). *Let $1 \leqq p(\theta) \leqq \infty$ and $s(\theta)$ be two real values measurable functions on T such that $-\infty < s(z) = \int_T s(\theta)P_z(\theta)d\theta < \infty$. Then, $[l^{s(\theta)}_{p(\theta)}]^z = l^{s(z)}_{p(z)}$ where $1/p(z) = \int_T (1/p(\theta))P_z(\theta)\,d\theta$.*

This is an easy consequence of the above proposition and example 2 of section 4. An argument similar to that used to prove proposition (5.3) can be applied to the Banach lattices given in example 3, section 4, to obtain the following result:

PROPOSITION (5.5). *Let p, q be two measurable functions defined on T such that $1 \leqq q(\theta) \leqq \infty$ and $1 \leqq p(\theta) \leqq \infty$, $\theta \in T$. Assume that $1/p(z) = \int_T 1/p(\theta) P_z(\theta)\, d\theta$ and $1/q(z) = \int_T 1/q(\theta) P_z(\theta)\, d\theta$. Then, $[X_{p(\theta), q(\theta)}]^z = X_{p(z), q(z)}$ with equal norms, $z \in \Delta$.*

We remark that to obtain this result we need to use proposition (3.1) for $q(\theta)$ and the measure space $([0, \infty), dx/x)$.

6. – The relation with the complex method of interpolation.

Let B be a Banach space. A function defined on a measure space (M, dx) with values in B is said ro be *measurable* if it is the limit almost everywhere of « simple B-values functions ». A function with values in B is said to be *simple* if it takes finitely many values, each on a measurable subset of M. Given a Banach lattice X on M we denote by $X(B)$ the class of B-values measurable functions $f(x)$ such that $\|f(x)\|_B \in X$ and we define $\|f\|_{X(B)} = \big\| \|f(x)\|_B \big\|_X < \infty$. It is known that $(X(B), \| \; \|_{X(B)})$ is a Banach space (see [2], 13.6 and 33.6).

We say that a Banach lattice X has the *dominated convergence property* if, given $f \in X$ and $\{f_n\}_{n=1}^{\infty}$ such that $|f_n| \leqq |f|$, $n = 1, 2, \ldots$ and $f_n \to 0$ as $n \to \infty$, then $\|f_n\|_X \to 0$ as $n \to \infty$. Notice that all the Banach lattices given in the examples of section 4 have the dominated convergence property.

A family of Banach lattices $\{X(\theta)\}$, $\theta \in T$, is called an *interpolation family* if it is an interpolation family of Banach spaces for which the containing space is also a Banach lattice and $\|f(x, \theta)\|_{X(\theta)}$ is a measurable function of θ for all measurable $f\colon M \times T \to \mathbb{R}$ such that $f(\cdot, \theta) \in X(\theta)$ a.e. θ.

THEOREM (6.1). *Suppose that $\{B(\theta)\}$ and $\{X(\theta)\}$, $\theta \in T$, are nterpolation families of Banach spaces and that in addition each $X(\theta)$ is a Banach lattice on M and $\beta = \bigcap_{\theta \in T} B(\theta)$, where β is the log-intersection of the family $\{B(\theta)\}$. Then $\{X(\theta)\,(B(\theta))\}$, $\theta \in T$, is an interpolation family of Banach spaces and $[X(\theta)\,(B(\theta))]_z \subset [X(\theta)]^z\,(B(z))$. If, in addition, we assume that $[X(\theta)]^z$ has the dominated convergence property, the spaces $[X(\theta)\,(B(\theta))]_z$ and $[X(\theta)]^z\,(B(z))$ coincide and their norms are equal.*

PROOF. We check first that $\{X(\theta)\,(B(\theta))\}$, $\theta \in T$, is an interpolation family of Banach spaces. If U is a containing Banach space of the family

$\{B(\theta)\}$ and V is a containing Banach lattice of the family $\{X(\theta)\}$, the Banach space $V(U)$ is a containing space for $\{X(\theta)\,(B(\theta))\}$. If $f \in \bigcap_{\in T} X(\theta)\,(B(\theta))$, the function $\|f(x)\|_{B(\theta)}$ is measurable in θ for almost every $x \in M$; by hypothesis $\|f\|_{X(\theta)(B(\theta))} = \big\| \|f(x)\|_{B(\theta)} \big\|_{X(\theta)}$ is measurable in θ. Finally, if f belongs to the log-intersection space of the family $\{X(\theta)\,(B(\theta))\}$, we have $f(x) \in \bigcap_{\theta \in T} B(\theta) = \beta$ for almost every $x \in M$; thus, $\|f(x)\|_U \leqq k_U(\theta)\|f(x)\|_{B(\theta)}$ and consequently $\|f(\cdot)\|_U \in \bigcap_{\theta \in T} X(\theta)$. Moreover, $\int_T \log^+\|f\|_{X(\theta)(B(\theta))}\, d\theta < \infty$ implies

$$\int_T \log^+ \big\|\|f(\cdot)\|_U\big\|_{X(\theta)} d\theta \leqq \int_T \log^+ k_U(\theta)\, d\theta + \int_T \log^+ \|f\|_{X(\theta)} d\theta < \infty,$$

which shows that $\|f(\cdot)\|_U \in \chi$, where χ denotes the log-intersection space of the family $\{X(\theta)\}$. Since $\{X(\theta)\}$ is an interpolation family we have $\|f\|_{V(U)} \leqq \big\|k_U(\theta)\|f(x)\|_U\big\|_{X(\theta)} \leqq k_V(\theta)\, k_U(\theta) \|f\|_{X(\theta)(B(\theta))}$ where $\int_T \log^+ k_V(\theta)\, k_U(\theta)\, d\theta < \infty$. This proves the desired result.

By an obvious density argument, the inclusion $[X(\theta)\,(B(\theta))]_z \subset [X(\theta)]^z (B(z))$ will follow from the inequality

$$\|g(z, \cdot)\|_{[X(\theta)]^z(B(z))} \leqq \|g\|_\infty, \tag{6.1}$$

which is true for any g of the form $g(\xi, x) = \sum_{j=1}^N \Psi_j(\xi) f_j(x)$, where f_j belongs to the log-intersection space of the family $\{X(\theta)(B(\theta))\}$ and $\Psi_j \in N^+$. To prove (6.1) we observe that for almost every $x \in M$, $f_j(x) \in \bigcap_{\theta \in T} B(\theta) =$ and consequently $g(\xi, x) \in N^+(\beta)$ for a.e. $x \in M$. By theorem (2.1) we have

$$\|g(z, x)\|_{B(z)} \leqq \|g\|_\infty \exp\left\{\int_T d\theta P_z(\theta) \log \frac{\|g(\theta, x)\|_{B(\theta)}}{\|g\|}\right\} \tag{6.2}$$

a.e. $x \in M$, where $\|g\|_\infty = \operatorname{ess\,sup}_{\theta \in T} \|g(\theta)\|_{X(\theta)(B(\theta))}$ (notice that we can always assume $\|g\|_\infty \neq 0$). Since $\big\| \|g(\theta, x)\|_{B(\theta)}/\|g\|_\infty \big\|_{X(\theta)} \leqq 1$, the definition of $[X(\theta)]^z$ and (6.2) imply (6.1).

Before proving the reverse inclusion and the corresponding norm inequality we need the following lemma. The proof of this lemma is a straightforward modification of the proof of a lemma that can be found in [2] (33.6). Details can be found in [6].

LEMMA (6.2). *Assume that $[X(\theta)]^z$ has the dominated convergence property. Given $\varepsilon > 0$, let S_ε be the class of simple $k \in [X(\theta)]^z(B(z))$ such that there exists*

$K: T \times M \to \mathbb{R}$ with $\|K(\theta, \cdot)\|_{X(\theta)} \leqq 1$ for all $\theta \in T$, satisfying

$$\|k(x)\|_{B(z)} = (1 + \varepsilon)\|k\|_{[X(\theta)]^z(B(z))} \exp\left\{\int_T d\theta P_z(\theta) \log |k(\theta, x)|\right\}$$

and such that the non-zero values of each $k(\theta, \cdot)$ have positive upper and lower bounds. Then, S_ε^1 is dense in $[X(\theta)]^z(B(z))$.

We proceed now to prove the reverse inclusion. Let $k \in S_\varepsilon$ and write $k(x) = \sum_1^N \chi_j(x) a_j$ where $a_j \in B(z)$ and the χ_i are characteristic functions of disjoint measurable sets on M. We can find $\psi_j \in \mathcal{F}(B(\cdot))$ such that $\psi_j(z) = a_j/\|a_j\|_{B(z)}$, $j = 1, \ldots, N$, and $\|\psi_j\|_\infty \leqq 1 + \varepsilon$. Define

$$g(\xi, x) = (1 + \varepsilon)\|k\|_{[X(\theta)]^z(B(\theta))} \exp\left\{\int_T d\theta P_z(\theta) \log |k(\theta, x)|\right\} \sum_{j=1}^N \chi_j(x) \psi_j(\xi)$$

where $k(\theta, x)$ is the function corresponding to $k \in S_\varepsilon$. Since each ψ_j is a limit of functions in $N^+(B(\cdot))$ one can show that $g \in \mathcal{F}(X(\theta))(B(\cdot))$. An elementary computation shows that $g(z, x) = k(x)$; thus, $k \in [X(\theta)\ (B(\theta))]_z$. Moreover, $\|\psi_j(\theta)\|_{B(\theta)} \leqq \|\psi_j\|_\infty \leqq 1 + \varepsilon$ implies

$$\|k\|_{[X(\theta)(B(\theta))]_z} \leqq \|g\|_\infty \leqq (1 + \varepsilon)^2 \|k\|_{[X(\theta)]^z(B(z))}\,. \tag{6.3}$$

Let now $f \in [X(\theta)]^z(B(z))$. By lemma (6.2) we construct a sequence of functions $k_m \in S_\varepsilon$ such that

$$\left\| f - \sum_{m=1}^N k_m \right\|_{[X(\theta)]^z(B(z))} \leqq \frac{1}{2^N} \|f\|_{[X(\theta)]^z(B(z))} \tag{6.4}$$

and

$$\|k_m\|_{[X(\theta)]^z(B(z))} \leqq \frac{1}{2^m} (1 + \varepsilon)\|f\|_{[X(\theta)]^z(B(z))} \tag{6.5}$$

$m = 1, 2, \ldots$. By (6.4) the partial sum of the series $\sum_{m=1}^\infty k_m$ converges to f in $[X(\theta)]^z(B(z))$. On the other hand, (6.5) and (6.3) imply that $\sum_{m=1}^\infty k_m$ also converges in $[X(\theta)\ (B(\theta))]_z$ and its norm is smaller than $(1 + \varepsilon)^3 \|f\|_{[X(\theta)]^z(B(z))}$ But the two series coincide and so we have $f \in [X(\theta)\ (B(\theta))]_z$ with norm not exceeding $(1 + \varepsilon)^3 \|f\|_{[X(\theta)]^z(B(z))}$. The result follows from here since ε is arbitrary. ∎

REMARK. We notice that, by taking $B(\theta) = \mathbb{R}$ for all $\theta \in T$, theorem (6.1) ensures us that, for $z \in \Delta$, $[X(\theta)]^z = X(z)$, provided $[X(\theta)]^z$ has the dominated convergence property.

7. – Interpolation of $L^p_w(B)$ and l^s_p spaces.

Let w be a positive measurable function on a measure space (M, dx) and $1 \leqq p \leqq \infty$. We say that $f \in L^p_w$ if $\|f\|_{L^p_w} = \left\{\int_M |f(x)|^p w(x)\, dx\right\}^{1/p} < \infty$. If B is a Banach space $L^p_w(B)$ is defined as in section 6.

Suppose that $p: T \to [1, \infty]$ is a measurable function and $\{w_\theta\}$, $\theta \in T$, is a family of positive measurable functions on M such that

(7.1) $\theta \to w_\theta(x)$ is measurable for all $x \in M$

and

(7.2) there exist $k: T \to (0, \infty)$ and $w: M \to \mathbb{R}_+ (w > 0)$ measurable such that $w(x) \leqq k(\theta) w_\theta(x)$ a.e. $x \in M$, $\theta \in T$, such that $\int d\theta \log^+ k(\theta) < \infty$.

We claim that $\{L^{p(\theta)}_{w_\theta}\}$, $\theta \in T$, is an interpolation family of Banach lattices. To see this observe that if $f \in L^{p(\theta)}_{w_\theta}$ we have $f \in L^{p(\theta)}_w$ and $\|f\|_{L^{p(\theta)}_w} \leqq [k(\theta)]^{1/p(\theta)}$ $\|f\|_{L^{p(\theta)}_{w_\theta}}$. Moreover, $L^{p(\theta)}_w \subset L^1_w + L^\infty$ and $\|f\|_{L^1_w + L^\infty} \leqq \|f\|_{L^{p(\theta)}_w}$ for all $f \in L^{p(\theta)}_w$ (see [13], 1.9.3). Therefore, we can take $U = L^1_w + L^\infty$ as a containing space. The measurability of $\theta \to \|f\|_{L^{p(\theta)}_{w_\theta}}$ follows from (7.1) and the measurability of p. Finally, $\|f\|_U \leqq [k(\theta)]^{1/p(\theta)} \|f\|_{L^{p(\theta)}_{w_\theta}}$ for all $f \in \bigcap_{\theta \in T} L^{p(\theta)}_w$ and $\int_T \log^+ [k(\theta)]^{1/p(\theta)} d\theta < \infty$.

By applying theorem (6.1) and proposition (5.3) we have the following result

PROPOSITION (7.1). *Let $p: T \to [1, \infty]$ be a measurable function and $\{w_\theta\}$, $\theta \in T$, be a family of positive measurable functions on M satisfying* (7.1) *and* (7.2) *and such that*

$$w_z(x) = \exp\left\{p(z) \int d\theta P_z(\theta) \frac{1}{p(z)} \log w_\theta(x)\right\} < \infty .$$

Assume also that $\{B(\theta)\}$, $\theta \in T$, is an interpolation family of Banach spaces such that $\bigcap_{\theta \in T} B(\theta) = \beta$. Then $[L^{p(\theta)}_{w_\theta}(B(\theta))]_z = L(^{p(z)}_{w_z} B(z))$ and their norms coincide, where $1/p(z)$ is the harmonic function on Δ whose boundary values are $1/p(\theta)$.

REMARKS. The proposition and the interpolation theorems of [3] generalize an interpolation theorem for operators acting on L^p spaces with change of measures, due to E.M. Stein and G. Weiss (see [11]). By taking $w_\theta \equiv 1$ and $B(\theta) \equiv \mathbb{R}$ we obtain $[L^{q(\theta)}]_z = L^{q(z)}$, which has already been obtained in [3].

COROLLARY (7.2). *Let $q: T \to [1, \infty]$ and $s: T \to \mathbb{R}$ be measurable functions on T such that s is bounded below and $s(z) = \int_T s(\theta) P_z(\theta)\, d\theta < \infty$. If $\{B(\theta)\}$, $\theta \in T$, is an interpolation family of Banach spaces such that $\bigcap_{\theta \in T} B(\theta) = \beta$ we have*

1) $[l_{q(\theta)}(B(\theta))]_z = l_{q(z)}(B(z))$ and

2) $[l^{s(\theta)}_{q(\theta)}(B(\theta))]_z = l^{s(z)}_{q(z)}(B(z))$

with equality of norms, where $1/q(z) = \int_T (1/q(\theta)) P_z(\theta)\, d\theta$.

PROOF. Take $M = \mathbb{N}$ with the discrete measure and $w_\theta(n) = 2^{s(\theta)np(\theta)}$ if $q(\theta) < \infty$ and $w_\theta(n) = 2^{s(\theta)n}$ if $q(\theta) = \infty$ and apply proposition (7.1). ∎

8. – Interpolation of Sobolev and Besov-Lipschitz spaces.

The definitions of Sobolev and Besov-Lipsichtz spaces that we shall use are taken from [1] (chapter 6). Let $\mathcal{S}$ be the class of Schwartz functions on $\mathbb{R}^n$ and let $\mathcal{S}'$, the dual of $\mathcal{S}$, be the space of tempered distributions. For $s \in \mathbb{R}$ and $f \in \mathcal{S}'$ we define $J^s f = \mathcal{F}^{-1}\{(1 + |\cdot|^2)^{s/2} \mathcal{F} f\}$, where $\mathcal{F}$ denotes the Fourier transform of f and $\mathcal{F}^{-1}$ its inverse. For $s \in \mathbb{R}$ and $1 \leqq p \leqq \infty$ we define the Sobolev space, $H^s_p \equiv H^s_p(\mathbb{R}^n)$ as the space of all $f \in \mathcal{S}'$ for which $\|f\|^s_p = \|J^s f\|_{L^p} < \infty$. It is known that H^s_p is a Banach space.

PROPOSITION (8.1). *Let $p: T \to (1, \infty)$ and $s: T \to \mathbb{R}$ be measurable functions on T such that s is bounded. Then, $\{H^{s(\theta)}_{p(\theta)}\}$, $\theta \in T$, is an interpoation family of Banach spaces and if*

$$(A)\int_T d\theta \log p(\theta) < \infty \, and \, (B)\int_T d\theta \log(1/p(\theta) - 1) < \infty$$

we have

$$[H^{s(\theta)}_{p(\theta)}]_z = H^{s(z)}_{p(z)}$$

with equivalent norms, where

$$1/p(z) = \int_T (1/p(\theta))\, P_z(\theta)\, d\theta \quad \textit{and} \quad s(z) = \int_T s(\theta) P_z(\theta)\, d\theta\,.$$

Before proving this proposition we state the corresponding result for Besov-Lipschitz spaces. Take a function $\varphi \in \mathcal{S}$ such that $\operatorname{supp} \varphi = \{x \in \mathbb{R}^n \colon 2^{-1} \leqq |x| \leqq 2\}$, $\varphi(x) > 0$ for $2^{-1} < |x| < 2$ and $\sum_{k=-\infty}^{\infty} \varphi(2^{-k}x) = 1 (x \neq 0)$ (the existence of such a function is not difficult to prove). Define φ_k, $k = 0, \pm 1, \pm 2, \ldots$ and ψ by

$$\varphi_k(x) = \varphi(2^{-k}x) \quad \text{and} \quad \psi(x) = 1 - \sum_{k=1}^{\infty} \varphi(2^{-k}x)\,.$$

Evidently, $\varphi_k \in \mathcal{S}$ and $\psi \in \mathcal{S}$. Let $s \in \mathbb{R}$, $1 \leqq p$, $q \leqq \infty$. We define the Besov-Lipschitz space $B^s_{p,q} \equiv B^s_{p,q}(\mathbb{R}^n)$ as the set of all $f \in \mathcal{S}$ for which

$$\|f\|^s_{p,q} = \|\psi * f\|_p + \left\{ \sum_{k=1}^{\infty} |2^{ks} \|\varphi_k * f\|_p|^q \right\}^{1/q} < \infty\,.$$

In [12], M. Taibleson has given equivalent definitions of these spaces for $s > 0$. In particular, he was able to prove that if $0 < s < 1$, $B^s_{\infty,\infty} = \mathrm{lip}(s)$ and $B^s_{p,\infty} = \mathrm{lip}(s, p)$ (see [12], theorem 4).

PROPOSITION (8.2). *Let $q \colon T \to [1, \infty]$ and $s \colon T \to \mathbb{R}$ be measurable functions on T such that s is bounded below and $s(z) = \int_T s(\theta) P_z(\theta)\, d\theta < \infty$. Then, if $1 \leqq p \leqq \infty$, $\{B^{s(\theta)}_{p,q(\theta)}\}$, $\theta \in T$, is an interpolation family of Banach spaces and*

$$[B^{s(\theta)}_{p,q(\theta)}]_z = B^{s(z)}_{p,q(z)}.$$

with equivalent norms, where $1/q(z) = \int_T (1/q(\theta))\, P_z\, d\theta$.

Before proving these two propositions we need three lemmas; these three results are well known and can be found in interpolation monographs such as [1] and [13].

LEMMA 8.1. (1) *If $s_1 < s_2$ we have $H^{s_2}_p \subset H^{s_1}_p (1 \leqq p \leqq \infty)$ and if $f \in H^{s_2}_p$, $\|f\|^{s_1}_p \leqq C[1 + (2^{s_2 - s_1})^{-1}]\, \|f\|^{s_2}_p$ where C is independent of s_1, s_2 and p.*

(2) *If $s_1 < s_2$ we have $B^{s_2}_{pq} \subset B^{s_1}_{pq} (1 \leqq p,\ q \leqq \infty)$ and if $f \in B^{s_2}_{pq}$, $\|f\|^{s_1}_{pq} \leqq \|f\|^{s_2}_{pq}$.*

LEMMA (8.2). *Let A_0, A_1 be an interpolation couple of Banach spaces and $\alpha: T \mapsto (0,1)$, $q: T \mapsto [1, \infty]$ be two measurable functions. Let $A(\theta) = (A_0, A_1)_{\alpha(\theta), q(\theta)}$ be the intermediate space obtained by the K-method of interpolation. Then,*

$$\|a\|_{A_0+A_1} \leqq [\alpha(\theta)\, q(\theta)]^{1/q(\theta)}\, \|a\|_{A(\theta)}$$

for all $a \in \bigcap_{\theta\in T} A(\theta)$.

LEMMA (8.3). (1) *Let $1 < p < \infty$, $s \in \mathbb{R}$. Then, there exist $P: H^s_p \to L^p(l^s_2)$ and $R: L^p(l^2_s) \to H^s_p$ linear and continuous such that $R \circ P$ is the identity on H^s_p. Moreover, $\|P\|$, $\|R\| \sim 1/(p-1)$ as $p \to 1$ and $\|P\|$, $\|R\| \sim p$ as $p \to \infty$.*

(2) *Let $1 \leqq p$, $q \leqq \infty$, $s \in \mathbb{R}$. Then, there exist $P: B^s_{p,q} \to l^s_q(L^p)$ and $R: l^s_q(L^p) \to B^s_{p,q}$ linear and continuous such that $R \circ P$ is the identity on $B^s_{p,q}$. Moreover $\|P\| \leqq 1$ and $\|R\| \sim 2^{(q-1)/q}$.*

Comments on the proof of the lemmas: Lemma (8.1) can be found in [1] (theorems 6.2.3 and 6.2.4), and lemma 8.3 is theorem 6.4.3 of [1] (We notice that P maps $\mathcal{S}'$ to the space of all sequences of tempered distributions and R maps this space to $\mathcal{S}'$). To prove lemma 8.2 we assume that the reader is familiar with the K-method of interpolation. If $a \in A(\theta)$, the fact that $K(t, a)$ is an increasing function of t([13], p. 24), together with the trivial equality $\int_1^\infty s^{-\alpha(\theta)q(\theta)}(ds/s) = 1/\alpha(\theta)q(\theta)$ imply

$$\|a\|_{A_0+A_1} = K(1, a) \leqq [\alpha(\theta)q(\theta)]^{1/q(\theta)} \left\{ \int_1^\infty K(t, a)\, s^{-\alpha(\theta)q(\theta)} \frac{ds}{s} \right\}^{1/q(\theta)}$$
$$= [\alpha(\theta)q(\theta)]^{1/q(\theta)} \|a\|_{A(\theta)},$$

which is the desired result.

PROOF OF PROPOSITION (8.1). Let $s_0 < \inf_{\theta\in T} s(\theta)$; lemma 8.1 (1) shows that $H^{s(\theta)}_{p(\theta)} \subset H^{s_0}_{p(\theta)}$ and $\|f\|^{s_0}_{p(\theta)} \leqq C \|f\|^{s(\theta)}_{p(\theta)}$, for all $f \in H^{s(\theta)}_{p(\theta)}$, where C is independent of $s(\theta)$ and $p(\theta)$. By lemma 8.2 and $H^{s_0}_{p(\theta)} = (H^{s_0}_1, H^{s_0}_\infty)_{\alpha(\theta), p(\theta)}$, where $1/p(\theta) = 1 - \alpha(\theta)$, (see theorem 6.4.5(5) of [1]) we deduce that $\|f\|_{H^{s_0}_1 + H^{s_0}_\infty} \leqq [p(\theta)]^{1/p(\theta)} \|f\|_{H^{s_0}_{p(\theta)}}$ for all $f \in H^{s_0}_{p(\theta)}$. Thus, we can take $U = H^{s_0}_1 + H^{s_0}_\infty$ as the containing space and we have

$$\|f\|_U \leqq C[p(\theta)]^{1/p(\theta)} \|f\|_{H^{s(\theta)}_{p(\theta)}}$$

for all $f \in \bigcap_{\theta \in T} H^{s(\theta)}_{p(\theta)}$. Since the measurability of $\theta \to \|f\|^{s(\theta)}_{p(\theta)} = \|J^{s(\theta)}f\|_{p(\theta)}$ is clear, we obtain the first part of the proposition. We notice that the log-intersection space of the family $\{l_2^{s(\theta)}\}$, $\theta \in T$, coincides with $l_2^{s+} = \bigcap_{\theta \in T} l_2^{s(\theta)}$, where $s_+ = \sup_{\theta \in T} s(\theta)$.

We now prove the equality of the spaces. Since P maps $H^{s(\theta)}_{p(\theta)}$ continuously into $L^{p(\theta)}(l_2^{s(\theta)})$ with norm bounded by $M(\theta)$, where $\log M(\theta)$ is absolutely integrable on T (this is due to lemma 8.3(1) and conditons (A) and (B)), we use theorem (2.2) to deduce that P also maps $[H^{s(\theta)}_{p(\theta)}]_z$ continuiusly into $[L^{p(\theta)}(l_2^{s(\theta)})]_z = L^{p(z)}(l_2^{s(z)})$ where $s(z) = \int_T s(\theta) P_z(\theta)\, d\theta$ and $1/p(z) = \int_T (1/p(\theta) P_z(\theta)\, d\theta$ (see proposition 7.1). On the other hand, R maps $L^{p(z)}(l_2^{s(z)})$ continuously onto $H^{s(z)}_{p(z)}$. Consequently, $R \circ P$, which is the identity, maps $[H^{s(\theta)}_{p(\theta)}]_z$ into $H^{s(z)}_{p(z)}$. Thus, $[H^{s(\theta)}_{p(\theta)}]_z$ is continuously embedded in $H^{s(z)}_{p(z)}$.

Now, R maps $L^{p(\theta)}(l_2^{s(\theta)})$ continuously into $H^{s(\theta)}_{p(\theta)}$ and again, by theorem (2.2), it maps $L^{p(z)}(l_2^{s(z)})$ continuously into $[H^{s(\theta)}_{p(\theta)}]_z$. But the image of $L^{p(z)}(l_2^{s(z)})$ undei R is $H^{s(z)}_{p(z)}$ and so $H^{s(z)}_{p(z)} \subset [H^{s(\theta)}_{p(\theta)}]_z$. Since we have already proved the reverse inclusion and its continuity, the open mapping theorem yields the desired conclusion. ■

The proof of proposition 8.2 is very similar to the proof just given, but it is obtained by using the result $(B^s_{p,1}, B^s_{p,\infty})_{\theta,q} = B^s_{p,q}$, $1 \leq p \leq \infty$, $s \in \mathbb{R}$ (theorem 6.4.5(2) of [1]). Details are left to the reader.

9. – Interpolation of Lorentz spaces.

Let (M, μ) be a measure space and for $f \in L_{\text{loc}}(M)$ define

$$f^{**}(t) = \frac{1}{t} \sup \int_E |f|\, d\mu\,, \qquad 0 > t > \infty$$

where the supremum is taken over all measurable sets E in M such that $\mu(E) \leq t$. If X is a Banach lattice on the halfline $0 < t < \infty$, we denote by X^* the class of measurable functions f on M such that $f^{**} \in X$ and write $\|f\|_{X^*} = \|f^{**}\|_X$: That X^* is a Banach lattice on M is a well known fact (see [2], 13.4 and 33.4).

We shall now briefly introduce the definition of Lorentz spaces, which were first studied by G. Lorentz (see [1]). For a measurable function f on a measure space (M, μ) we introduce the distribution function of f as $m(\sigma, f) = \mu\{x / |f(x)| > \sigma\}$, $\sigma > 0$. The decreasing rearrangement of f is defined as $f^*(t) = \inf\{\sigma / m(\sigma, f) \leq t\}$, $t > 0$. If $1 \leq p < \infty$ and $1 \leq q < \infty$

we let $L_{p,q}$ be the space of all measurable functions f on (M, μ) for which

$$\|f\|_{p,q} = \left\{ \int_0^\infty t^{1/p} f^*(t)^q \frac{dt}{t} \right\}^{1/q} < \infty .$$

If $1 \leqq p \leqq \infty$, $q = \infty$, we let $L_{p,\infty}$ be the space of all measurable f on (M, μ) such that $\|f\|_{p,\infty} = \sup_{t>0} t^{1/p} f^*(t) < \infty$. It is well known that $L_{1,1} = L^1$ and $L_{p,q}$, $1 < p \leqq \infty$, $1 \leqq q \leqq \infty$ are Banach spaces.

As a consequence of the equality $f^{**}(t) = 1/t \int_0^t f^*(s)\, ds$ and Hardy's inequality one obtains the following result (see [9]), which shows that $L_{p,q}$ is a particular case of the spaces X^* introduced above.

LEMMA (9.1). *If (M, μ) is non-atomic, $1 < p < \infty$, and $1 \leqq q \leqq \infty$, the spaces $L_{p,q}$ and $X^*_{p,q}$, where $X_{p,q}$ is as in example 3 of section 4, coincide and their norms are equivalent.*

What we shall do now is to obtain a general interpolation theorem for Banach lattices of the type X^* and use it, together with the above lemma, to find the intermediate spaces of Lorentz spaces. For $f \in L_{\text{loc}}(0, \infty)$ we consider the operators

$$(S_1 f)(t) = \frac{1}{t} \int_0^t f(s)\, ds , \qquad (S_2 f)(t) = \int_t^\infty \frac{f(s)}{s}\, ds .$$

THEOREM (9.2). *Let $\{X(\theta)\}$, $\theta \in T$, be a family of Banach lattices on $(0, \infty)$ contained in $L_{\text{loc}}(0, \infty)$. Assumed*

$$\|S_j f\|_{X(\theta)} \leqq c_j(\theta) \|f\|_{X(\theta)} \tag{1}$$

for all $f \in X(\theta)$, where $\int_T \big(\log c_j(\theta)\big)\, d\theta < \infty$, $j = 1, 2$. Then, the spaces $[X(\theta)^*]^z$ and $\big([X(\theta)]^z\big)^*$ coincide and their norms are equivalent.

PROOF. Before starting the proof of $[X(\theta)^*]^z \subset \big([X(\theta)]^z\big)^*$ we need the following result:

LEMMA. *Let $F\colon T \times M \to \mathbb{R}_+$ be measurable and assume that*

$$\int_T d\theta P_z(\theta) \log \|F(\theta, \cdot)\|_{L^1} < \infty$$

for same $z \in \Delta$ (*and hence for all* z). *Then,*

$$\left(\exp\left\{\int_T d\theta P_z(\theta)\log F(\theta,\cdot)\right\}\right)^{**}(t) \leqq \exp\left\{\int_T d\theta P_z(\theta)\log F^{**}(\theta,t)\right\}.$$

The proof of the lemma is an easy consequence of proposition (3.1), for it follows that the left-hand side equals

$$\frac{1}{t}\sup_{\mu(E)\leqq t}\int_E d\mu(x)\left\{\exp\int_T d\theta P_z(\theta)\log F(\theta,x)\right\}$$

which is majorated by

$$\frac{1}{t}\sup_{\mu(E)\leqq t}\exp\left\{\int_T d\theta P_z(\theta)\log\left(\int_E d\mu(x)F(\theta,x)\right)\right\} \leqq \exp\left\{\int_T d\theta P_z(\theta)\log F^{**}(\theta,t)\right\}.$$

Let now $f \in [X(\theta)^*]^z$. Given $\varepsilon > 0$ we can choose $F(\theta, x)$ with $\|F(\theta,\cdot)\|_{X(\theta)} < 1$ such that

$$|f(x)| \leqq (1+\varepsilon)\|f\|^z \exp\left\{\int_T d\theta P_z(\theta)\log|F(\theta,x)|\right\}$$

where $\|f\|^z$ denotes the norm of f as an element of $[X(\theta)^*]^z$. By the above lemma

$$f^{**}(t) \leqq (1+\varepsilon)\|f\|^z \exp\left\{\int_T d\theta P_z(\theta)\log|F^{**}(\theta,t)|\right\}.$$

Moreover, $\|F^{**}(\theta,\cdot)\|_{X(\theta)} = \|F(\theta,\cdot)\|_{X(\theta)^*} \leqq 1$ so that the above inequality implies $f^{**} \in [X(\theta)]^z$ and $\|f^{**}\|_{[X(\theta)]^z} \leqq (1+\varepsilon)\|f\|^z$. The desired inclusion and the corresponding norm inequality follow immediately.

We now prove the reverse inclusion. Given $f \in ([X(\theta)]^z)^*$ and $\lambda > \|f\|_{([X(\theta)]^z)^*}$ we can choose $F(\theta, t)$ with $\|F(\theta,\cdot)\|_{X(\theta)} \leqq 1$ such that

$$f^{**}(t) \leqq \lambda \exp\left\{\int_T d\theta P_z(\theta)\log|F(\theta,t)|\right\}.$$

Proposition (3.1) implies

$$(S_2 f^{**})(t) \leqq c(z)\lambda \exp\left\{\int_T d\theta P_z(\theta)\log\left|\frac{|S_2(F(\theta,\cdot))(t)|}{c_1(\theta)c_2(\theta)}\right|\right\} \tag{9.1}$$

where $c(z) = \exp\{\int_T d\theta P_z(\theta) \log c_1(\theta) c_2(\theta)\}$. Observing that

$$\big(S_2(S_1 g)\big)(t) = (S_1 g)(t) + (S_2 g)(t) \text{ and } f^* \leqq f^{**} = S_1 f^*$$

we deduce $f^* \leqq S_1 f^* + S_2 f^* = S_2(S_1 f^*) = S_2(f^{**})$. This inequality together with (9.1) implies

$$f^*(t) \leqq c(z)\, \lambda \exp\left\{\int_T d\theta P_z(\theta) \log h(\theta, t)\right\} \tag{9.2}$$

where $h(\theta, t) = S_2\big(F(\theta, \cdot)\big)(t)/c_1(\theta)c_2(\theta)$. Define $G(\theta, x) = h\big(\theta, m(|f(x)|, f)\big)$. Using the fact that $G^*(\theta, t) \leqq h(\theta, t)$ we have $G^{**}(\theta, t) = \big(S_1 f^*(\theta, \cdot)\big)(t) \leqq \leqq \big(S_1 h(\theta, \cdot)\big)(t) = S_1 S_2 F(\theta, \cdot)(t)/c_1(\theta)c_2(\theta)$, so that condition (1) implies $\|G^{**}(\theta, \cdot)\|_{X(\theta)} \leqq \|F(\theta, \cdot)\|_{X(\theta)} \leqq 1$. Hence

$$G(\theta, \cdot) \in \big(X(\theta)\big)^* \quad \text{and} \quad \|G(\theta, \cdot)\|_{(X(\theta))^*} \leqq 1.$$

Moreover, using the inequality $|f(x)| \leqq f^*\big(m(|f(x)|, f)\big)$ and (9.2) we obtain

$$|f(x)| \leqq c(z)\, \lambda \exp\left\{\int_T d\theta P_z(\theta) \log h\big(\theta, m(|f(x)|, f)\big)\right\}$$
$$= c(z)\, \lambda \exp\{d\theta P_z(\theta) \log G(\theta, x)\}$$

which proves the desired result. ∎

To be able to apply the theorem to Lorentz spaces we need to find a bound for the norms of the operators S_j, $j = 1, 2$ acting on $X_{p,q}$ (see the definition of $X_{p,q}$ in example 3, section 4). This is contained in the following result:

LEMMA (9.3). *If $X_{p,q}$ $1 < p < \infty$, $1 \leqq q < \infty$, is the Banach lattice of all measurable functions f on $(0, \infty)$ such that*

$$\|f\|_{X_{p,q}} = \left\{\int_0^\infty [s^{1/p}|f(s)|]^q \frac{ds}{s}\right\} < \infty ,$$

we have

$$\|S_1 f\|_{X_{p,q}} \leqq \frac{p}{p-1} \|f\|_{X_{p,q}} \quad \text{and} \quad \|S_2 f\|_{X_{p,q}} \leqq p \|f\|_{X_{p,q}}$$

for all $f \in X_{p,q}$.

PROOF. As several of the properties of Lorentz spaces, this lemma depends essentially on Hardy's inequality: if $q \leqq 1$, $r \neq 0$ and $f \geqq 0$,

$$\left\{\int_0^\infty \left(\int_t^\infty f(s)\,ds\right)^q t^r \frac{dt}{t}\right\}^{1/q} \leqq \frac{q}{|r|}\left\{\int_0^\infty [sf(s)]^q s^r \frac{ds}{s}\right\}^{1/q}. \tag{9.3}$$

The original proof of (9.3) can be found in [7] (Chapter IX). An easier proof can be obtained as an application of Jensen's inequality and Fubini's theorem (see [8], page 256).

To prove the estimate for S_1 we use Hardy's inequality with $r = (q/p) - q < 0$ to obtain

$$\|S_1 f\|_{X_{p,q}} \leqq \frac{p}{p-1}\left\{\int_0^\infty [sf(s)]^q s^{(q/p)-q} \frac{ds}{s}\right\}^{1/q} = \frac{p}{p-1}\,\|f\|_{X_{p,q}}.$$

To prove the estimate for S_2 we use Hardy's inequality for $f = q/p > 0$ and $f(s)/s$ to obtain

$$\|S_2 f\|_{X_{p,q}} \leqq p\left\{\int_0^\infty [f(s)]^q s^{q/p} \frac{ds}{s}\right\}^{1/q} = \|f\|_{X_{p,q}}. \qquad \blacksquare$$

PROPOSITION (9.4). *Let $p\colon T \to (1, \infty)$ and $q\colon T \to [1, \infty)$ be two measurable functions on T such that*

$$\int_T d\theta \log p(\theta) < \infty \quad \text{and} \quad \int_T d\theta \log \frac{1}{p(\theta)-1} < \infty. \tag{1}$$

Then, $\{L_{p(\theta),q(\theta)}\}$, $\theta \in T$, is an interpolation family of Banach spaces and

$$[L_{p(\theta),q(\theta)}]_z = L_{p(z),q(z)}$$

with equivalent norms, where

$$1/p(z) = \int_T (1/p(\theta))\, P_z(\theta)\, d\theta, \qquad 1/q(\theta) = \int_T (1/q(\theta))\, P_z(\theta)\, d\theta.$$

PROOF. To prove that $\{L_{p(\theta),q(\theta)}\}$, $\theta \in T$, is an interpolation family we observe that $(L^1, L^\infty)_{\alpha(\theta),q(\theta)} = L_{p(\theta),q(\theta)}$, where $1/p(\theta) = 1 - \alpha(\theta)$ ([1], *p.* 113). Then, we can take $U = L^1 + L^\infty$ as a containing space and by lemma 8.2

we have $\|f\|_U \leqq [\alpha(\theta) q(\theta)]^{1/q(\theta)} \|f\|_{p(\theta),q(\theta)}$, for all $f \in \bigcap_{\theta\in T} L_{p(\theta),q(\theta)}$, where

$$\int_T \log^+[\alpha(\theta)\, q(\theta)]^{1/q(\theta)} d\theta \leqq \int_T \log[q(\theta)]^{1/q(\theta)} d\theta \leqq 2\pi .$$

We now prove the equality of the spaces. By lemma (9.3) and condition (9.4) (1) we can use theorem (9.2) to obtain $[X^*_{p(\theta),q(\theta)}]^z = ([X_{p(\theta),q(\theta)}]^z)^*$. By lemma (9.1) and theorem (6.1) we have

$$[X^*_{p(\theta),q(\theta)}]^z = [L_{p(\theta),q(\theta)}]^z = [L_{p(\theta),q(\theta)}]_z .$$

On the other hand proposition (5.5) and lemma (9.1) imply

$$([X_{p(\theta),q(\theta)}]^z)^* = X^*_{p(z),q(z)} = L_{p(z),q(z)} .$$

This proves the desired result. ■

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