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Isoperimetric Inequalities in Parabolic Equations.

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0. – Introduction.

Consider the parabolic equation

$$(1)\qquad \begin{cases} \dfrac{\partial u}{\partial t} + \mathfrak{A}_t(u) + cu = f & \text{in } Q = (0, T)\times\Omega\,, \\ u = 0 & \text{on } \Sigma = (0, T)\times\partial\Omega\,, \\ u(0,\cdot) = u_0 & \text{for } t = 0\,, \end{cases}$$

where Ω is a bounded regular domain in $\mathbb{R}^N$ ($N \geqq 1$),

$$\mathfrak{A}_t(u) = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_j} a_{ij}(t, x) \frac{\partial u}{\partial x_i}\,,$$

a_{ij} satisfy the uniform ellipticity condition (with constant one)

$$\sum_{i,j=1}^{N} a_{ij}(t, x)\xi_i\xi_j \geqq |\xi|^2\,, \qquad \forall\xi \in \mathbb{R}^N;$$

c, u_0 and f are non-negative functions; their regularity will be precised later on.

Consider also the equation

$$(\tilde{1})\qquad \begin{cases} \dfrac{\partial U}{\partial t} - \Delta U = \underset{\sim}{f} & \text{in } \tilde{Q} = (0, T)\times\tilde{\Omega}\,, \\ U = 0 & \text{on } \tilde{\Sigma} = (0, T)\times\partial\tilde{\Omega}\,, \\ U(0,\cdot) = \underset{\sim}{u}_0\,, \end{cases}$$

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where $\tilde{\Omega}$ is the ball of $\mathbb{R}^N$, centered at the origin, which has the same measure as Ω, and $\underset{\sim}{u}_0$ (resp. $\underset{\sim}{f}(t,\cdot)$) is the rearrangement of u_0 (resp. $f(t,\cdot)$) in $\tilde{\Omega}$, which decreases along the radii. This rearrangement is defined as follows.

If v is a real measurable [1] function defined in Ω, the decreasing rearrangement of v is defined in $\bar{\Omega}^* = [0, |\Omega|]$, by

$$v_*(s) = \operatorname{Inf}\{\theta \in \mathbb{R},\ |v > \theta| \leqq s\} \tag{2}$$

where $|v > \theta| = \operatorname{meas}\{x \in \Omega,\ v(x) > \theta\}$ (for any measurable set E, we denote $|E|$ its measure). The spherical rearrangement of v in $\bar{\tilde{\Omega}}$, which decreases along the radii is

$$\underset{\sim}{v}(x) = v_*(\alpha_N |x|^N)\ , \qquad \text{for } x \in \bar{\tilde{\Omega}}\,, \tag{3}$$

where α_N is the measure of the unit ball of $\mathbb{R}^N$. If v is defined in $(0, T)\times\Omega$, and is measurable with respect to the space variable x of Ω, we consider its rearrangement with respect to x:

$$v_*(t, s) = (v(t,\cdot))_*(s) = \operatorname{Inf}\{\theta \in \mathbb{R},\ |v(t,\cdot) > \theta| \leqq s\}\,, \tag{4}$$

$$\underset{\sim}{v}(t, x) = v_*(t, \alpha_N |x|^N)\ . \tag{5}$$

C. Bandle [2] proved that every *strong* solution u of problems (1) satisfies

$$\forall t \in [0, T],\ \forall s \in \bar{\Omega}^*\,, \qquad \int_0^s u_*(t, \sigma)\, d\sigma \leqq \int_0^s U_*(t, \sigma)\, d\sigma\,, \tag{6}$$

which leads to

$$\forall t \in [0, T],\ \forall r \in [1, \infty]\,, \qquad \|u(t,\cdot)\|_{L^r(\Omega)} \leqq \|U(t,\cdot)\|_{L^r(\tilde{\Omega})}\,. \tag{7}$$

J. L. Vasquez [9] obtained the same result, if u is a weak solution of a degenerate parabolic equation, the equation of porous media:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta\varphi(u) & \text{in } Q = (0, \infty)\times\mathbb{R}^N\,, \\ u(0,\cdot) = u_0 & \text{for } t = 0\,, \end{cases} \tag{8}$$

where $\varphi\colon \mathbb{R}^+ \to \mathbb{R}^+$ is increasing and continuous, $\varphi(0) = 0$. He used the

[1] In the whole paper, we consider the Lebesgue measure.

semigroups theory, and the isoperimetric inequalities for *elliptic* equations (see [7] for example).

In this paper, we give a direct proof of (6), (7), valid for every *weak* solution of problems (1) (see Section 2).

Our method relies on the calculation of the directional derivative of the mapping $u \to u_*$, that is $v_{*u} = \lim_{\lambda \downarrow 0} ((u + \lambda v)_* - u_*)/\lambda$. This calculation was made first by J. Mossino and R. Temam [6], with a direction v in $L^\infty(\Omega)$. In the first section, we extend their result to functions v in $L^p(\Omega)$ $(1 \leqq p \leqq +\infty)$. Moreover we prove that, if u belongs to $H^1(0, T; L^p(\Omega))$, then u_* belongs to $H^1(0, T; L^p(\Omega^*))$ and

$$\frac{\partial u^*}{\partial t} = \left(\frac{\partial u}{\partial t}\right)_{*u}.$$

Besides, $(\partial u/\partial t)(t, \cdot)$ is shown to be constant on every set where $u(t, \cdot)$ is constant. The last formula is a crucial point in Section 2.

1. – Directional derivative of the rearrangement mapping.

In this Section 1, we assume that Ω is a measurable subset of $\mathbb{R}^N$ ($|\Omega| < \infty$, $N \geqq 1$). For the sake of completeness, we first recall some properties of rearrangements (see the proofs in [7] for example), and a result of [6].

1.1. *Properties of rearrangements.*

Let u be a measurable function: $\Omega \to \mathbb{R}$ and u^* be its increasing rearrangement, defined by (2) and

$$u^* = -(-u)_* . \tag{1.1}$$

An essential property of rearrangement is that u and u^* are equi-measurable:

$$\forall \theta \in \mathbb{R}, \qquad |u < \theta| (= \text{meas}\,\{x \in \Omega, u(x) < \theta\}) = |u^* < \theta| ,$$

which implies

$$\int_\Omega F(u)\,dx = \int_{\Omega^*} F(u^*)\,ds , \tag{1.2}$$

for every Borel measurable $F\colon \mathbb{R} \to \mathbb{R}^+$. Here are some other properties of the increasing rearrangement mapping.

(*a*) If u_1, u_2 are two measurable functions such that $u_1 \leqq u_2$ almost everywhere, then $u_1^* \leqq u_2^*$ everywhere.

(*b*) For all constants C, $(u + C)^* = u^* + C$.

(*c*) More generally, if φ is an increasing function from $\mathbf{R}$ into $\mathbf{R}$, then $(\varphi(u))^* = \varphi(u^*)$ almost everywhere.

(*d*) The mapping $u \to u^*$ applies $L^p(\Omega)$ into $L^p(\Omega^*)$ $(1 \leqq p \leqq \infty)$. It is contracting and norm-preserving.

(*e*) If u is in $L^p(\Omega)$, v in $L^{p'}(\Omega)(1/p + 1/p' = 1)$, then

$$\int_{\Omega} uv\, dx \leqq \int_{\Omega^*} u^* v^*\, ds = \left(\int_{\tilde{\Omega}} \tilde{u}\tilde{v}\, dx \right). \tag{1.3}$$

This inequality is due to Hardy and Littlewood.

We shall use a slight extention of (*d*):

LEMMA 1.1. *Let $u\colon \Omega \to \mathbf{R}$ be measurable, v in $L^p(\Omega)$ $(1 \leqq p \leqq \infty)$. Then $(u + v)^* - u^*$ belongs to $L^p(\Omega^*)$ and*

$$\|(u + v)^* - u^*\|_{L^p(\Omega^*)} \leqq \|v\|_{L^p(\Omega)}.$$

This lemma was proved in [7]. For convenience, we reproduce the proof here.

(i) If $p = \infty$, we have

$$u - \|v\|_{L^\infty(\Omega)} \leqq u + v \leqq u + \|v\|_{L^\infty(\Omega)}, \qquad \text{a.e.}.$$

By properties (*a*) and (*b*) above,

$$u^* - \|v\|_{L^\infty(\Omega)} \leqq (u + v)^* \leqq u^* + \|v\|_{L^\infty(\Omega)},$$

that is

$$\|(u + v)^* - u^*\|_{L^\infty(\Omega^*)} \leqq \|v\|_{L^\infty(\Omega)}.$$

(ii) If $p < \infty$, we use the truncation

$$f_n(t) = \begin{cases} -n & \text{if } t \leqq -n, \\ t & \text{if } -n \leqq t \leqq n, \\ n & \text{if } t \geqq n. \end{cases}$$

Then $f_n(u)$ and $f_n(u+v)$ are in $L^\infty(\Omega)$. By (c) and (d), $(f_n(u))^* = f_n(u^*)$, $(f_n(u+v))^* = f_n((u+v)^*)$, these functions are in $L^\infty(\Omega^*)$, and

$$\|f_n((u+v)^*) - f_n(u^*)\|_{L^p(\Omega^*)} = \|(f_n(u+v))^* - (f_n(u))^*\|_{L^p(\Omega^*)} \leqq \|f_n(u+v) - f_n(u)\|_{L^p(\Omega)} \leqq \|v\|_{L^p(\Omega)}$$

(as f_n is contracting). Then, using Fatou lemma,

$$\begin{aligned}\|v\|^p_{L^p(\Omega)} &\geqq \varliminf_{n\to\infty} \int_{\Omega^*} |f_n((u+v)^*) - f_n(u^*)|^p\, ds \\ &\geqq \int_{\Omega^*} \varliminf |f_n((u+v)^*) - f_n(u^*)|^p\, ds \\ &= \int_{\Omega^*} |(u+v)^* - u^*|^p\, ds\,.\end{aligned}$$

1.2. *Directional derivative of the rearrangement mapping. Relative rearrangement.*

First, we shall recall a result due to J. Mossino and R. Temam [6].

Consider a couple of functions (u, v), $u: \Omega \to \mathbf{R}$ is measurable, v is in $L^p(\Omega)$ $(1 \leqq p \leqq \infty)$, and a parameter $\lambda > 0$. By Lemma 1.1, $(u+\lambda v)^* - u^*$ belongs to $L^p(\Omega^*)$, and we can define

$$w_\lambda(s) = \int_0^s \frac{(u+\lambda v)^* - u^*}{\lambda}\, d\sigma\,. \tag{1.4}$$

Thus, $dw_\lambda/ds = ((u+\lambda v)^* - u^*)/\lambda$. By Lemma 1.1.,

$$\left\|\frac{dw_\lambda}{ds}\right\|_{L^p(\Omega^*)} \leqq \|v\|_{L^p(\Omega)}\,. \tag{1.5}$$

We are going to show that dw_λ/ds tends (in the sense of distributions) to dw/ds, where

$$w(s) = \begin{cases} \displaystyle\int_{u<u^*(s)} v\, dx & \text{if } |u = u^*(s)| = 0\,, \\ \displaystyle\int_{u<u^*(s)} v\, dx + \int_0^{s-|u<u^*(s)|} (v|_{P(s)})^*\, d\sigma\,, & \text{otherwise}\,, \end{cases} \tag{1.6}$$

$v|_{P(s)}$ is the restriction of v to $P(s) = \{u = u^*(s)\}$. The following was proved in [6].

THEOREM 1.1. *If u is a measurable function from Ω into $\mathbf{R}$, v is in $L^\infty(\Omega)$, then w is lipschitz,*

$$\left\|\frac{dw}{ds}\right\|_{L^\infty(\Omega^*)} \leqq \|v\|_{L^\infty(\Omega)},$$

and, when λ decreases to zero,

(i) $w_\lambda \to w$ in $\mathfrak{C}^0([0, |\Omega|])$ that is uniformly;

(ii) $\dfrac{dw_\lambda}{ds} = \dfrac{(u+\lambda v)^* - u^*}{\lambda} \to \dfrac{dw}{ds}$ in $L^\infty(\Omega^*)$ weak $*$. □

We shall extend Theorem 1.1 to functions v in $L^p(\Omega)$ $(1 \leqq p \leqq \infty)$.

THEOREM 1.1 bis. *Let u, v be two measurable functions from Ω into $\mathbf{R}$, v in $L^p(\Omega)$ $(1 \leqq p \leqq \infty)$. Then w belongs to $W^{1,p}(\Omega^*)$,*

$$\left\|\frac{dw}{ds}\right\|_{L^p(\Omega^*)} \leqq \|v\|_{L^p(\Omega)}, \tag{1.7}$$

and, when λ decreases to zero

(i) *$w_\lambda \to w$ in $\mathfrak{C}^0([0, |\Omega|])$;*

(ii) *$\dfrac{dw_\lambda}{ds} = \dfrac{(u+\lambda v)^* - u^*}{\lambda} \to \dfrac{dw}{ds}$ in the sense of distributions:*

$$\forall \varphi \in \mathfrak{D}(\Omega^*), \quad \int_{\Omega^*} \frac{dw_\lambda}{ds}\varphi\, ds \to \int_{\Omega^*} \frac{dw}{ds}\varphi\, ds.$$

(In particular, $dw_\lambda/ds \to dw/ds$ in $L^p(\Omega^*)$-weak if $1 < p < \infty$, in $L^\infty(\Omega^*)$-weak $*$ if $p = \infty$). □

PROOF. Consider v_n in $L^\infty(\Omega)$; $w_{\lambda,n}$, w_n are associated to (u, v_n) as in (1.4), (1.6). We have

$$|w_\lambda(s) - w(s)| \leqq |w_\lambda(s) - w_{\lambda,n}(s)| + |w_{\lambda,n}(s) - w_n(s)| + |w_n(s) - w(s)|.$$

By Lemma 1.1,

$$|w_{\lambda,n}(s) - w_\lambda(s)| = \left|\int_0^s \frac{(u+\lambda v_n)^* - (u+\lambda v)^*}{\lambda}\, d\sigma\right| \leqq \|v_n - v\|_{L^1(\Omega)},$$

and, clearly,

$$|w_n(s) - w(s)| \leqq \|v_n - v\|_{L^1(\Omega)}.$$

Then

$$\operatorname*{Sup}_s |w_\lambda(s) - w(s)| \leqq \operatorname*{Sup}_s |w_{\lambda,n}(s) - w_n(s)| + 2\|v_n - v\|_{L^1(\Omega)}.$$

By Theorem 1.1. (i), $w_{\lambda,n}$ tends to w_n in $\mathfrak{C}^0([0, |\Omega|])$. When λ decreases to zero,

$$\overline{\lim_{\lambda \downarrow 0}} \operatorname*{Sup}_s |w_\lambda(s) - w(s)| \leqq 2\|v_n - v\|_{L^1(\Omega)}.$$

We deduce (i). Evidently (ii) follows, as, with φ in $\mathfrak{D}(\Omega^*)$,

$$\int_{\Omega^*} \frac{dw_\lambda}{ds} \varphi\, ds = -\int_{\Omega^*} w_\lambda \frac{d\varphi}{ds}\, ds \to -\int_{\Omega^*} w \frac{d\varphi}{ds}\, ds \qquad \text{(by (i))}$$

$$= \int_{\Omega^*} \frac{dw}{ds} \varphi\, ds.$$

Now, we shall prove that dw/ds is in $L^p(\Omega^*)$, and satisfies (1.7). Taking again φ in $\mathfrak{D}(\Omega^*)$, we have by (1.5)

$$\left|\int_{\Omega^*} w_\lambda \frac{d\varphi}{ds}\, ds\right| = \left|\int_{\Omega^*} \frac{dw_\lambda}{ds} \varphi\, ds\right| \leqq \|v\|_{L^p(\Omega)} \|\varphi\|_{L^{p'}(\Omega^*)}$$

$(1/p + 1/p' = 1)$. From (i), it follows

$$\left|\int_{\Omega^*} w \frac{d\varphi}{ds}\, ds\right| \leqq \|v\|_{L^p(\Omega)} \|\varphi\|_{L^{p'}(\Omega^*)}.$$

If $p > 1$, $L^p(\Omega^*)$ is the dual of $L^{p'}(\Omega^*)$, and we get immediately (1.7). In any case $(p \geqq 1)$, we can use the following argument. Let v_n be a sequence in $L^\infty(\Omega)$. As previously, one can prove that

$$\left\|\frac{dw_m}{ds} - \frac{dw_n}{ds}\right\|_{L^p(\Omega^*)} \leqq \|v_m - v_n\|_{L^p(\Omega)}, \tag{1.8}$$

for any $p > 1$, and, consequently, (passing to the limit) for $p \geqq 1$. Now consider v_1, v_2 in $L^p(\Omega)$ $(p \geqq 1)$, v_{in} $(i = 1, 2)$ in $L^\infty(\Omega)$, $v_{in} \to v_i$ in $L^p(\Omega)$; w_i, w_{in} are associated to (u, v_i) and (u, v_{in}) respectively as in (1.6). By (1.8),

dw_{in}/ds is a Cauchy sequence in $L^p(\Omega^*)$. As $|w_{in}(s) - w_i(s)| \leqq \|v_{in} - v_i\|_{L^1(\Omega)}$, w_{in} tends to w_i in $\mathfrak{C}^0([0, |\Omega|])$, $dw_{in}/ds \to dw_i/ds$ in $L^p(\Omega^*)$, and, by passing to the limit in

$$\left\| \frac{dw_{1n}}{ds} - \frac{dw_{2n}}{ds} \right\|_{L^p(\Omega^*)} \leqq \|v_{1n} - v_{2n}\|_{L^p(\Omega)} ,$$

we get

$$\left\| \frac{dw_1}{ds} - \frac{dw_2}{ds} \right\|_{L^p(\Omega^*)} \leqq \|v_1 - v_2\|_{L^p(\Omega)} . \tag{1.9}$$

With $v_1 = v$, $v_2 = 0$, we get evidently (1.7). □

Relative rearrangement.

DEFINITION. According to J. Mossino and R. Temam [6] the function dw/ds is called *the rearrangement of v with respect to u*, and is denoted by v_u^*.

The usual rearrangement of a function is also the rearrangement of this function with respect to a constant ($u_c^* = u^*$) or with respect to itself ($u_u^* = u^*$). More generally, if a Borel function F: $\mathbb{R} \to \mathbb{R}$, and a measurable function u: $\Omega \to \mathbb{R}$, are such that $F(u)$ is in $L^p(\Omega)$, then $F(u^*)$ is in $L^p(\Omega^*)$ (by (1.2)) and $(F(u))_u^* = F(u^*)$. In fact $(F(u))_u^* = dw/ds$, with

$$w(s) = \begin{cases} \displaystyle\int_{u<u^*(s)} F(u)\, dx & \text{if } |u = u^*(s)| = 0 , \\ \displaystyle\int_{u<\alpha} F(u)\, dx + \int_0^{s-s_\alpha} (F(u)|_{P_\alpha})^*\, d\sigma & \text{otherwise} , \end{cases}$$

with $\alpha = u^*(s)$, $P_\alpha = \{u = \alpha\}$, $|P_\alpha| \neq 0$, $s_\alpha = |u < \alpha|$,

$$= \begin{cases} \displaystyle\int_0^s F(u^*)\, d\sigma & \text{if } |u = u^*(s)| = 0 , \\ \displaystyle\int_0^{s_\alpha} F(u^*)\, ds + F(\alpha)(s - s_\alpha) = \int_0^s F(u^*)\, d\sigma & \text{otherwise} \end{cases}$$

(by (1.2))

$$= \int_0^s F(u^*)\, d\sigma .$$

However, generally, v_u^* is not an increasing function, the property of equimeasurability and properties (c), (e) above, for the usual rearrangement, do not seem to have their analogue for the rearrangement of a function with respect to another one. But we have, if v is in $L^p(\Omega)$ $(1 \leqq p \leqq \infty)$, $u\colon \Omega \to \mathbf{R}$ is measurable

(a') $v_1 \leqq v_2$ a.e. implies $(v_1)_u^* \leqq (v_2)_u^*$ a.e.

In fact, with φ in $\mathfrak{D}(\Omega^*)$, $\varphi \geqq 0$,

$$\int_{\Omega^*}[(v_2)_u^* - (v_1)_u^*]\varphi\, ds = \lim_{\lambda \downarrow 0}\int_{\Omega^*}\varphi\, \frac{(u+\lambda v_2)^* - (u+\lambda v_1)^*}{\lambda}\, ds \geqq 0$$

by property (a).

(b') For all constants C, $(v + C)_u^* = v_u^* + C$.

In fact, with φ in $\mathfrak{D}(\Omega^*)$,

$$\int_{\Omega^*}(v + C)_u^*\varphi\, ds = \lim_{\lambda \downarrow 0}\int_{\Omega^*}\frac{(u+\lambda(v+C))^* - u^*}{\lambda}\,\varphi\, ds$$

$$= \lim_{\lambda \downarrow 0}\int_{\Omega^*}\frac{(u+\lambda v)^* - u^*}{\lambda}\,\varphi\, ds + \int_{\Omega^*}C\varphi\, ds$$

(by property (b))

$$= \int_{\Omega^*}(v_u^* + C)\varphi\, ds\,.$$

(d') If $u\colon \Omega \to \mathbf{R}$ is measurable, the mapping $v \to v_u^*$ is a contraction from $L^p(\Omega)$ into $L^p(\Omega^*)$ $(1 \leqq p \leqq \infty)$ as we have seen in (1.9).

(f') Besides, the mapping $v \to v_u^*$ $(L^1(\Omega) \to L^1(\Omega^*))$ preserves the integral:

$$\int_{\Omega^*}v_u^*\, ds = \int_{\Omega^*}\frac{dw}{ds}\, ds = w(|\Omega|) - w(0) = w(|\Omega|) = \int_{\Omega}v\, dx\,. \quad \square$$

One can also define another rearrangement v_{*u} which is relative to the directional derivative of the mapping $u \to u_*$ (the decreasing rearrangement of u):

$$v_{*u} = \frac{dw}{ds} = \lim_{\lambda \downarrow 0}\frac{dw_\lambda}{ds}$$

(the limit is taken in the sense of distributions),

$$\frac{dw_\lambda}{ds} = \frac{(u + \lambda v)_* - u_*}{\lambda} = \frac{-(-u - \lambda v)^* + (-u)^*}{\lambda}$$

(by (1.1)). Thus

$$v_{*u} = -(-v)^*_{-u} \,. \tag{1.10}$$

1.3. *Symmetrization of a family of functions.*

In this Section, $u\colon [0, T] \times \Omega \to \mathbb{R}$ will be a function defined everywhere in $[0, T]$, and almost everywhere in $\Omega \subset \mathbb{R}^N$. For all t in $[0, T]$, we denote by $u(t)\colon \Omega \to \mathbb{R}$, the function $u(t)(x) = u(t, x)$. (For a fixed t, if no confusion is possible, we shall sometimes write u instead of $u(t)$.). We assume that $u(t)$ is measurable for every t in $[0, T]$. Then, we can define the function $u^*\colon [0, T] \times \bar{\Omega}^* \to \mathbb{R}$, the increasing rearrangement of u with respect to the x variable in Ω, that is:

$$\forall t \in [0, T],\ \forall s \in \bar{\Omega}^* , \qquad u^*(t, s) = (u(t))^*(s)(= u^*(t)(s)) \,. \tag{1.11}$$

We consider now another real function v defined almost everywhere in $Q = (0, T) \times \Omega$, such that, for almost every t in $(0, T)$, $v(t)$ is in $L^p(\Omega)$ $(1 \leqq p \leqq +\infty)$. Then, we can define as in Section 1.2, $(v(t))^*_{u(t)}$, which is in $L^p(\Omega^*)$

$$\|(v(t))^*_{u(t)}\|_{L^p(\Omega^*)} \leqq \|v(t)\|_{L^p(\Omega)} \,. \tag{1.12}$$

We denote by v^*_u the function defined almost everywhere in $Q^* = (0, T) \times \Omega^*$ by

$$\text{a.e. } t \in (0, T),\ \text{a.e. } s \in \Omega^* , \qquad v^*_u(t, s) = (v(t))^*_{u(t)}(s) \,. \tag{1.13}$$

The aim of this Section 1.3 is to study the regularity of u^* with respect to t (assuming a certain regularity of u with respect to t), and to compute $\partial u^*/\partial t$. We have

THEOREM 1.2. *If u belongs to $H^1(0, T; L^p(\Omega))$ $(1 \leqq p \leqq \infty)$, then u^* belongs to $H^1(0, T; L^p(\Omega^*))$, and*

$$\|u^*\|_{H^1(0,T;L^p(\Omega^*))} \leqq \|u\|_{H^1(0,T;L^p(\Omega))} \,. \tag{1.14}$$

Moreover

$$\frac{\partial u^*}{\partial t} = \left(\frac{\partial u}{\partial t}\right)^*_u = \frac{\partial w}{\partial s} \qquad (\textit{in the sense of distributions}) \tag{1.15}$$

where

$$(1.16)\quad w(t,s)=\begin{cases}\displaystyle\int_{u(t)<u(t)^*(s)}\frac{\partial u}{\partial t}\,dx & \text{if } |u(t)=u(t)^*(s)|=0\,,\\[2ex] \displaystyle\int_{u(t)<u(t)^*(s)}\frac{\partial u}{\partial t}\,dx+\int_0^{s-|u(t)<u(t)^*(s)|}\left(\frac{\partial u}{\partial t}\Big|_{u(t)=u(t)^*(s)}\right)d\sigma & \textit{otherwise}\,.\end{cases}$$

PROOF. As $\|u(t)^*\|_{L^p(\Omega^*)}=\|u(t)\|_{L^p(\Omega)}$ (by (1.2)), we have

$$\|u^*\|_{L^2(0,T;\,L^p(\Omega^*))}=\|u\|_{L^2(0,T;\,L^p(\Omega))}\,.$$

Besides, by (1.12), (1.13)

$$\left\|\left(\frac{\partial u}{\partial t}\right)^*_u(t)\right\|_{L^p(\Omega^*)}\leqq\left\|\frac{\partial u}{\partial t}(t)\right\|_{L^p(\Omega)},$$

$$\left\|\left(\frac{\partial u}{\partial t}\right)^*_u\right\|_{L^2(0,T;L^p(\Omega^*))}\leqq\left\|\frac{\partial u}{\partial t}\right\|_{L^2(0,T;\,L^p(\Omega))}.$$

Thus, we have only to prove (1.15), (1.16). Our proof uses the following lemma (see its proof in the Appendix).

LEMMA 1.2. *Let u be in $H^1(0,T;L^p(\Omega))$ $(1\leqq p\leqq\infty)$,*

$$r_h=\frac{u(t+h)-u(t)}{h}-\frac{\partial u}{\partial t}\,.$$

Consider a fixed number $\varepsilon>0$. When h tends to zero, r_h tends to zero in $L^\alpha(Q_\varepsilon)$, with $\alpha=\operatorname{Min}(p,2)$, $Q_\varepsilon=(\varepsilon,T-\varepsilon)\times\Omega$.

Let φ be in $\mathfrak{D}(Q^*)$, and let $\varepsilon>0$ be such that the support of φ is included into $Q^*_\varepsilon=(\varepsilon,T-\varepsilon)\times\Omega^*$. Consider $0<h<\varepsilon$. We have

$$\int_{Q^*}\frac{u(t+h)^*-u(t)^*}{h}\varphi(t)\,ds\,dt=\int_{Q^*}\frac{(u+h(\partial u/\partial t))^*-u^*}{h}\varphi\,ds\,dt$$
$$+\int_{Q^*}\frac{(u+h(\partial u/\partial t+r_h))^*-(u+h(\partial u/\partial t))^*}{h}\varphi\,ds\,dt\,.$$

The first integral in the right hand side is $\int_0^T A_h(t)\,dt$, where

$$\text{a.e. } t\,,\quad A_h(t)=\int_{\Omega^*}\frac{\big(u(t)+h(\partial u/\partial t)(t)\big)^*-u(t)^*}{h}\,\varphi(t)\,ds\ \xrightarrow[(h\to 0)]{}\ \int_{\Omega^*}\left(\frac{\partial u}{\partial t}\right)^*_u(t)\,\varphi(t)\,ds$$

(by Theorem 1.1 bis), and

$$|A_h(t)|\leqq\left\|\frac{\partial u}{\partial t}(t)\right\|_{L^p(\Omega)}\|\varphi(t)\|_{L^{p'}(\Omega^*)}\,,$$

with $1/p+1/p'=1$ (by property (d') above). Using Lebesgue theorem

$$\int_0^T A_h(t)\,dt\ \xrightarrow[(h\to 0)]{}\ \int_{Q^*}\left(\frac{\partial u}{\partial t}\right)^*_u\varphi\,ds\,dt\,.$$

The other integral is majorized by

$$\int_\varepsilon^{T-\varepsilon}\|r_h(t)\|_{L^\alpha(\Omega)}\|\varphi(t)\|_{L^{\alpha'}(\Omega^*)}\,dt$$

with $\alpha=\text{Min}\,(p,2)$, $(1/\alpha+1/\alpha'=1)$

$$\leqq\|r_h\|_{L^\alpha(Q_\varepsilon)}\,\|\varphi\|_{L^{\alpha'}(Q^*_\varepsilon)},$$

which tends to zero with h, by Lemma 1.2. Thus

$$\int_{Q^*}\frac{u(t+h)^*-u(t)^*}{h}\varphi(t)\,ds\,dt\ \xrightarrow[(h\to 0)]{}\ \int_{Q^*}\left(\frac{\partial u}{\partial t}\right)^*_u\varphi\,ds\,dt\,.$$

But, classically,

$$\int_{Q^*}\frac{u(t+h)^*-u(t)^*}{h}\,\varphi(t)\,ds\,dt\to-\int_{Q^*}u^*\frac{\partial\varphi}{\partial t}\,ds\,dt\,.$$

We conclude that, in the sense of distributions,

$$\frac{\partial u^*}{\partial t}=\left(\frac{\partial u}{\partial t}\right)^*_u.\qquad\square$$

A direct consequence of Theorem 1.2 is the following

PROPOSITION. *Assume u belongs to $H^1(0, T; L^1(\Omega))$. Then, for almost every t in $(0, T)$, $\partial u/\partial t(t,\cdot)$ is constant (almost everywhere) on any set where $u(t,\cdot)$ is constant (almost everywhere).*

PROOF. If, in the proof of Theorem 1.2, we consider $h < 0$, we get

$$A_h(t) = \int_{\Omega^*} \frac{(u(t) + h(\partial u/\partial t)(t))^* - u(t)^*}{h} \varphi(t)\, ds = -\int_{\Omega^*} \frac{(u(t) + (-h)(-\partial u/\partial t))^* - u(t)^*}{-h} \varphi(t)\, ds\,,$$

which tends to $-\int_{\Omega^*}(-\partial u/\partial t)^*_u \varphi(t)\, ds$. Thus, one has, in the sense of distributions,

$$\frac{\partial u^*}{\partial t} = \left(\frac{\partial u}{\partial t}\right)^*_u = -\left(-\frac{\partial u}{\partial t}\right)^*_u \quad \left(= \left(\frac{\partial u}{\partial t}\right)_{*-u} \text{ by (1.10)}\right),$$

and

$$\left(\frac{\partial u}{\partial t}\right)^*_u = \frac{\partial w}{\partial s} \quad (w \text{ defined in (1.16)}), \qquad -\left(-\frac{\partial u}{\partial t}\right)^*_u = \frac{\partial w'}{ds}\,,$$

with

$$w'(t,s) = \begin{cases} \displaystyle\int_{u(t)<u(t)^*(s)} \frac{\partial u}{\partial t}\, dx & \text{if } |u(t) = u(t)^*(s)| = 0\,, \\ \displaystyle\int_{u(t)<u(t)^*(s)} \frac{\partial u}{\partial t}\, dx + \int_0^{s-|u(t)<u(t)^*(s)|} -\left(-\frac{\partial u}{\partial t}\bigg|_{u(t)=u(t)^*(s)}\right)^* d\sigma\,, & \text{otherwise}. \end{cases}$$

The last integral is also

$$\int_0^{s-|u(t)<u(t)^*(s)|} \left(\frac{\partial u}{\partial t}\bigg|_{u(t)=u(t)^*(s)}\right)_* d\sigma \quad (\text{by (1.1)})\,.$$

Now, fix t in $(0, T)$, such that $\partial w/\partial s = \partial w'/\partial s$ (in $L^1(\Omega^*)$) (this is true for almost every t in $(0, T)$), and consider a flat region of $u(t)$: $P_\theta(t) = \{u(t) = \theta\}$, $|P_\theta(t)| \neq 0$. Set $s_\theta = |u(t) < \theta|$, $s'_\theta = |u(t) \leqq \theta|$. As $w(t, 0) = 0 = w'(t, 0)$, one has

$$w(t, s) = \int_0^s \frac{\partial w}{\partial s}(t, \sigma)\, d\sigma = \int_0^s \frac{\partial w'}{\partial s}(t, \sigma)\, d\sigma = w'(t, s)\,,$$

for all s in $\bar{\Omega}^*$. Moreover, for all s in $[s_\theta, s'_\theta]$, one has, by definition of w and w',

$$\frac{\partial w}{\partial s}(t, s) = \left(\frac{\partial u}{\partial t}\bigg|_{P_\theta(t)}\right)^*(s - s_\theta)$$

$$= \frac{\partial w'}{\partial s}(t, s) = \left(\frac{\partial u}{\partial t}\bigg|_{P_\theta(t)}\right)_*(s - s_\theta) .$$

In particular,

$$\left(\frac{\partial u}{\partial t}\bigg|_{P_\theta(t)}\right)_*(0) = \operatorname*{Sup\,ess}_{P_\theta(t)} \frac{\partial u}{\partial t}$$

$$= \left(\frac{\partial u}{\partial t}\bigg|_{P_\theta(t)}\right)^*(0) = \operatorname*{Inf\,ess}_{P_\theta(t)} \frac{\partial u}{\partial t} ,$$

that is $(\partial u/\partial t)(t, \cdot)$ is constant almost everywhere on $P_\theta(t)$. □

We shall give now the application to parabolic equations.

2. – Isoperimetric inequalities for linear parabolic equations.

Let us consider first the parabolic equation

$$(2.1)\qquad \begin{cases} \dfrac{\partial u}{\partial t} + \mathfrak{A}_t(u) + cu = f & \text{in } Q = (0, T)\times\Omega , \\ u = 0 & \text{on } \Sigma = (0, T)\times\partial\Omega , \\ u(0, \cdot) = u_0 & \text{in } \Omega , \end{cases}$$

where Ω is a bounded regular open set in $\mathbb{R}^N$,

$$\mathfrak{A}_t(u) = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_j}\left(a_{ij}(t, x)\frac{\partial u}{\partial x_i}\right).$$

We denote by $A(= A(t, x))$ the matrix $(a_{ij}(t, x))$, as well as the bilinear form on $\mathbb{R}^N$ associated with A, and we assume that A satisfies the unifomr (with respect to (t, x)) ellipticity condition:

$$A(\xi, \xi) = \sum_{i,j=1}^{N} a_{ij}(t, x)\xi_i\xi_j \geqq |\xi|^2 , \qquad \forall\xi \in \mathbb{R}^N.$$

Furthermore, we assume that the data satisfy:

c, f, u_0 are non-negative functions; c, a_{ij} are in $L^\infty(Q)$;

$\partial a_{ij}/\partial t$ are continuous in $\bar{Q}$, f is in $L^2(Q)$, and u_0 is in $H_0^1(\Omega)$.

Then, the solution u is in $L^\infty(0, T; H_0^1(\Omega))$, $\partial u/\partial t$ is in $L^2(Q)$ (see [5], pp. 113-114, and [4] if $a_{ij} \neq a_{ji}$).

Let us introduce the problem

$$(\widetilde{2.1}) \qquad \begin{cases} \dfrac{\partial U}{\partial t} - \Delta U = \underset{\sim}{f} & \text{in } \tilde{Q} = (0, T) \times \tilde{\Omega}, \\ U = 0 & \text{on } \tilde{\Sigma} = (0, T) \times \partial\tilde{\Omega}, \\ U(0, \cdot) = \underset{\sim}{u}_0 & \text{in } \tilde{\Omega}, \end{cases}$$

$\tilde{\Omega}$, $\underset{\sim}{f}$, $\underset{\sim}{u}_0$ are as in the Introduction.

We are going to compare the solution u of (2.1) with the solution U of $(\widetilde{2.1})$. More precisely, we have

THEOREM 2.1. *With the assumptions above,*

$$(2.2) \qquad \forall t \in [0, T],\ \forall s \in \bar{\Omega}^*, \qquad \int_0^s u_*(t, \sigma)\, d\sigma \leqq \int_0^s U_*(t, \sigma)\, d\sigma \leqq \int_0^s g(t, \sigma)\, d\sigma$$

where $g(t, s) = \int_0^t f_*(\tau, s)\, d\tau + (u_0)_*(s)$. *We deduce*

$$(2.3) \qquad \forall t \in [0, T],\ \forall r \in [1, \infty], \qquad \|u(t, \cdot)\|_{L^r(\Omega)} \leqq \|U(t, \cdot)\|_{L^r(\tilde{\Omega})} \leqq \|g(t, \cdot)\|_{L^r(\Omega^*)} (\leqq +\infty),$$

PROOF. For a fixed $t \in [0, T]$, we denote for convenience $u = u(t)$, $f = f(t)$ .... We argue as for the elliptic problem (see [8], [7]). By the maximum principle, we have $u \geqq 0$. For any $\theta > 0$, we get from (2.1),

$$(2.4) \qquad \int_\Omega A(\nabla u, \nabla(u - \theta)_+)\, dx = \int_\Omega \left(f - cu - \frac{\partial u}{\partial t}\right)(u - \theta)_+\, dx.$$

Thus, as in [8], [7], a simple derivation gives:

$$(2.5) \qquad -\frac{d}{d\theta} \int_{u>\theta} A(\nabla u, \nabla u)\, dx = \int_{u>\theta} \left(f - cu - \frac{\partial u}{\partial t}\right) dx.$$

The uniform ellipticity condition and the Cauchy-Schwartz inequality lead to

$$\left[-\frac{d}{d\theta}\int_{u>\theta}|\nabla u|\,dx\right]^2 \leqq \mu'(\theta)\frac{d}{d\theta}\int_{u>\theta}A(\nabla u, \nabla u)\,dx$$

where $\mu(\theta) = |u > \theta|$, and, by (2.5),

$$\left[-\frac{d}{d\theta}\int_{u>\theta}|\nabla u|\,dx\right]^2 \leqq -\mu'(\theta)\int_{u>\theta}\left(f - cu - \frac{\partial u}{\partial t}\right)dx\,. \tag{2.6}$$

Using a result of Fleming-Rishel, and the isoperimetric inequality for the perimeter in the sense of De Giorgi, we find

$$N\alpha_N^{1/N}\mu(\theta)^{1-(1/N)} \leqq -\frac{d}{d\theta}\int_{u>\theta}|\nabla u|\,dx\,. \tag{2.7}$$

Hence, combining (2.6), (2.7),

$$N^2\alpha_N^{2/N}\mu(\theta)^{2-(2/N)} \leqq -\mu'(\theta)\int_{u>\theta}\left(f - cu - \frac{\partial u}{\partial t}\right)dx\,. \tag{2.8}$$

By the inequality (1.3) of Hardy-Littlewood,

$$\int_{u>\theta}(f - cu)\,dx \leqq \int_{u>\theta} f\,dx \leqq \int_0^{\mu(\theta)} f_*\,ds = F(t, \mu(\theta)) \tag{2.9}$$

if we set

$$F(t, s) = \int_0^s f_*(t, \sigma)\,d\sigma\,. \tag{2.10}$$

For almost every θ, $|u = \theta| = 0$, and $u_*(\mu(\theta)) = \theta$ because u_* is continuous in $]0, |\Omega|]$ (as u is in $H_0^1(\Omega)$, u non-negative, then $\underset{\sim}{u}$ is in $H_0^1(\tilde{\Omega})$, see [7], for example). By Theorem 1.2

$$\int_{u>\theta}\frac{\partial u}{\partial t}\,dx = \int_{u>u_*(\mu(\theta))}\frac{\partial u}{\partial t}\,dx = w(t, \mu(\theta))\,,$$

with

$$w(t,s)=\int_{u(t)>u(t)_*(s)}\frac{\partial u}{\partial t}\,dx \qquad \text{if } |u(t)=u(t)_*(s)|=0\,,$$

$$\frac{\partial w}{\partial s}=\left(\frac{\partial u}{\partial t}\right)_{*u}=\frac{\partial u_*}{\partial t}\,.$$

Thus

$$\int_{u>\theta}\frac{\partial u}{\partial t}\,dx=\int_0^{\mu(\theta)}\frac{\partial u_*}{\partial t}\,ds=\frac{\partial k}{\partial t}\big(t,\mu(\theta)\big)$$

if we set

$$k(t,s)=\int_0^s u_*(t,\sigma)\,d\sigma \tag{2.11}$$

$\left(\text{as } (\partial k/\partial t)(t,s)=\int_0^s(\partial u_*/\partial t)(t,\sigma)\,d\sigma\right)$.

Thus

$$\int_{u>\theta}\frac{\partial u}{\partial t}\,dx=\frac{\partial k}{\partial t}\big(t,\mu(\theta)\big)\,,\qquad \text{a.e. } \theta>0\,. \tag{2.12}$$

From (2.8), (2.9), (2.12), we get

$$1\leqq -N^{-2}\alpha_N^{-2/N}\mu(\theta)^{(2/N)-2}\left[F\big(t,\mu(\theta)\big)-\frac{\partial k}{\partial t}\big(t,\mu(\theta)\big)\right]\mu'(\theta)\,. \tag{2.13}$$

As $F(t,\cdot)-\partial k/\partial t(t,\cdot)$ is continuous in $\bar{\Omega}^*$, then, the function $H(t,\cdot)$ defined in Ω^* by

$$H(t,s)=s^{(2/N)-2}\left[F(t,s)-\frac{\partial k}{\partial t}(t,s)\right]$$

is continuous in $]0,|\Omega|]$. By integrating (2.13), we get, for any $0\leqq\theta\leqq\theta'$,

$$\theta'-\theta\leqq -N^{-2}\alpha_N^{-2/N}\int_{\mu(\theta)}^{\mu(\theta')}s^{(2/N)-2}\left[F(t,s)-\frac{\partial k}{\partial t}(t,s)\right]ds\,. \tag{2.14}$$

Thus, as in [7], one has for almost every s in Ω^*,

$$0\leqq-\frac{\partial^2 k}{\partial s^2}=-\frac{d}{ds}\big(u(t)\big)_*\leqq N^{-2}\alpha_N^{-2/N}s^{(2/N)-2}\left[F(t,s)-\frac{\partial k}{\partial t}(t,s)\right]. \tag{2.15}$$

Hence, k satisfies

$$(2.16)\quad \begin{cases} \dfrac{\partial k}{\partial t} - N^2 \alpha_N^{2/N} s^{2-(2/N)} \dfrac{\partial^2 k}{\partial s^2} \leqq F & \text{a.e. in } Q^* = (0, T)\times \Omega^* , \\ k(t, 0) = 0 , \quad \dfrac{\partial k}{\partial s}(t, |\Omega|) = 0 , & \forall t \in [0, T] , \\ k(0, s) = \displaystyle\int_0^s (u_0)_* \, d\sigma = k_0(s) , & \forall s \in \bar{\Omega}^* . \end{cases}$$

Let $K(t, s) = \int_0^s U_*(t, \sigma)\, d\sigma$, where U is the solution of $(\widetilde{2.1})$. We are going to show that the equality is achieved in (2.16) for K instead of k.

By the maximum principle, $U(t, \cdot)$ decreases along the radii in $\tilde{\Omega}$, and (2.1) can be written

$$\frac{\partial U_*}{\partial t} - \frac{\partial}{\partial s}\left(N^2 \alpha_N^{2/N} s^{2-(2/N)} \frac{\partial U_*}{\partial s}\right) = f_* \quad \text{in } \Omega^* .$$

By integrating between 0 and s, using the fact that $s^{2-(2/N)}(\partial U_*/\partial s) = O(s)$ when s tends to zero (see the remark below) we obtain

$$\frac{\partial K}{\partial t} - N^2 \alpha_N^{2/N} s^{2-(2/N)} \frac{\partial^2 k}{\partial s^2} = F \quad \text{in } Q^* .$$

REMARK 2.1. Using Cauchy-Schwartz inequality in the first line of (2.16), we get

$$0 \leqq - s^{2-(2/N)} \frac{\partial u_*}{\partial s} \leqq N^{-2} \alpha_N^{-(2/N)} s^{1/2} \left[\|f(t)\|_{L^2(\Omega)} + \left\| \frac{\partial u}{\partial t}(t) \right\|_{L^2(\Omega)} \right] . \quad \square$$

Now, setting $\chi = k - K$,

$$(2.17)\quad \begin{cases} \dfrac{\partial \chi}{\partial t} - N^2 \alpha_N^{2/N} s^{2-(2/N)} \dfrac{\partial^2 \chi}{\partial s^2} \leqq 0 & \text{a.e. in } Q^* , \\ \chi(t, 0) = 0 , \quad \dfrac{\partial \chi}{\partial s}(t, |\Omega|) = 0 & \forall t \in [0, T] , \\ \chi(0, s) = 0 , \quad \forall s \in \bar{\Omega}^* . \end{cases}$$

The first inequality in (2.2) will result from a maximum principle for χ:

LEMMA 2.1. *Let* $\chi(t,s) = (k-K)(t,s) = \int_0^s (u_* - U_*)(t,\sigma)\,d\sigma$. *One has* $\chi \leqq 0$ *everywhere in* $\overline{Q^*}$.

PROOF OF LEMMA 2.1. Multiplying the inequality in (2.17) by $s^{(2/N)-2}\chi_+$ we get

$$s^{(2/N)-2}\frac{\partial\chi}{\partial t}\chi_+ \leqq N^2\alpha_N^{2/N}\frac{\partial^2\chi}{\partial s^2}\chi_+ \qquad \text{a.e. in } Q^* . \tag{2.18}$$

For fixed t, we shall denote $u, \chi \ldots$, for simplicity, instead of $u(t), \chi(t) \ldots$. We shall also denote by [ ] a function of t, independent of s. First we prove that $(\partial^2\chi/\partial s^2)\chi_+$ is in $L^1(\Omega^*)$. In fact, by Remark 2.1,

$$\left|\frac{\partial u_*}{ds}\right| \leqq [\ ]\ s^{(2/N)-(3/2)},$$
$$\left|\frac{\partial^2\chi}{\partial s^2}\right| \leqq \left|\frac{\partial u_*}{\partial s}\right| + \left|\frac{\partial U_*}{\partial s}\right| \leqq [\ ]\ s^{(2/N)-(3/2)} . \tag{2.19}$$

On the other hand

$$|k| \leqq \int_0^s |u_*|\,d\sigma \leqq s^{1/2}\|u\|_{L^2(\Omega)} = [\ ]s^{1/2}$$
$$|\chi_+| \leqq |\chi| \leqq |k| + |K| \tag{2.20}$$
$$|\chi_+| \leqq [\ ]s^{1/2} .$$

Thus,

$$\left|\frac{\partial^2\chi}{\partial s^2}\chi_+\right| \leqq [\ ]s^{(2/N)-1},$$

which belongs to $L^1(\Omega^*)$. By integrating by parts, we are going to prove that $\int_{\Omega^*}(\partial^2\chi/\partial s^2)\chi_+\,ds$ is non-positive. For $a > 0$, as χ belongs to $W^{2,\infty}(a, |\Omega|)$ by (2.19), the following integration by parts is justified

$$\int_a^{|\Omega|}\frac{\partial^2\chi}{\partial s^2}\chi_+\,ds = -\int_a^{|\Omega|}\left(\frac{\partial\chi_+}{\partial s}\right)^2 ds - \frac{\partial\chi}{\partial s}(a)\,\chi_+(a) \tag{2.21}$$

(we used the fact that $(\partial\chi/\partial s)(|\Omega|) = 0$ by (2.17)). When a tends to zero, the two integrals tend respectively to $\int_{\Omega^*}(\partial^2\chi/\partial s^2)\chi_+\,ds$ and $\int_{\Omega^*}(\partial\chi_+/\partial s)^2\,ds$ (χ_+, as χ, belongs to $H^1(\Omega^*)$). Now we prove that $(\partial\chi/\partial s)(a)\chi_+(a)$ tends to zero with a. One has

$$\left|\frac{\partial\chi}{\partial s}(a)\right| = \left|\int_{\Omega|}^{a}\left(\frac{\partial u_*}{\partial s} - \frac{\partial U_*}{\partial s}\right) ds\right| \leqq [\ ]\left|a^{(2/N)-(1/2)} - |\Omega|^{(2/N)-(1/2)}\right| \qquad \text{(by (2.19))}.$$

By (2.20),

$$\left|\frac{\partial\chi}{\partial s}(a)\,\chi_+(a)\right| \leqq [\ ]\left|a^{2/N} - |\Omega|^{(2/N)-(1/2)} a^{1/2}\right|$$

which tends to zero with a. From (2.21), we get

$$\int_{\Omega^*}\frac{\partial^2\chi}{\partial s^2}\,\chi_+\,ds = -\int_{\Omega^*}\left(\frac{\partial\chi_+}{\partial s}\right)^2 ds \leqq 0\,.$$

From (2.18),

$$0 \geqq 2\int_0^t\int_{\Omega^*} s^{(2/N)-2}\frac{\partial\chi}{\partial t}\,\chi_+\,ds\,d\tau = \int_0^t\int_{\Omega^*} s^{(2/N)-2}\frac{\partial}{\partial t}(\chi_+^2)\,ds\,d\tau = \int_{\Omega^*} s^{(2/N)-2}\chi_+^2\,ds\,.$$

It follows $\chi_+ \equiv 0$ in $\bar{Q}^*$. □

Now we shall prove the second inequality in (2.2). Let us consider the equation satisfied by K in Q^*:

$$F - \frac{\partial K}{\partial t} = -N^2\alpha_N^{2/N} s^{2-(2/N)}\frac{\partial U_*}{\partial s} \geqq 0\,.$$

Thus

$$\int_0^s f_*(t,\sigma)\,d\sigma \geqq \frac{\partial}{\partial t}\int_0^s U_*(t,\sigma)\,d\sigma.$$

By integration, we find

$$\int_0^s U_*(t,\sigma)\,d\sigma - \int_0^s u_{0*}\,d\sigma \leqq \int_0^s d\sigma\int_0^t f_*(\tau,\sigma)\,d\tau\,. \quad \square$$

Now, (2.3) is a simple consequence of a lemma in [2] (p. 174), for all r in $[1, \infty[$, and then for $r = \infty$.

REMARK 2.2. If $f_*(t)$ is absolutely continuous in $[0, |\Omega|]$, for almost every t in $(0, T)$, then we can obtain an isoperimetric energy inequality:

we get from (2.1)

$$\begin{aligned}\int_{\Omega}\left(\frac{\partial u}{\partial t}u + A(\nabla u, \nabla u) + cu^2\right)dx &= \int_{\Omega} fu\, dx\\ &\leqq \int_{\Omega^*} f_* u_*\, ds \quad \text{(by Hardy-Littlewood inequality)}\\ &= -\int_{\Omega^*}\frac{\partial f_*}{\partial s}k\, ds + f_*(|\Omega|)k(|\Omega|)\\ &\leqq -\int_{\Omega^*}\frac{\partial f_*}{\partial s}K\, ds + f_*(|\Omega|)K(|\Omega|) \quad \text{(by Theorem 2.1)}\\ &= \int_{\tilde{\Omega}^*} f_* U_*\, ds\\ &= \int_{\tilde{\Omega}}\left(\frac{\partial U}{\partial t}U + |\nabla U|^2\right)dx\,.\end{aligned}$$

Using the uniform ellipticity condition, we have

$$\int_{\Omega}\left(\frac{\partial u}{\partial t}u + |\nabla u|^2\right)dx \leqq \int_{\tilde{\Omega}}\left(\frac{\partial U}{\partial t}U + |\nabla U|^2\right)dx\,,$$

and, by integration

$$\frac{1}{2}\int_{\Omega} u(T)^2\, dx + \int_{Q}|\nabla u|^2\, dx\, dt \leqq \frac{1}{2}\int_{\tilde{\Omega}} U(T)^2\, dx + \int_{\tilde{Q}}|\nabla U|^2\, dx\, dt\,.$$

Appendix.

In the proof of Lemma 1.2, we shall use the following lemma, whose proof is easy (see [1] for example).

LEMMA A. *Let v in $W^{1,\alpha}(0, T)$ $(1 \leqq \alpha \leqq \infty)$. If $0 < |h| < \varepsilon$, we have*

$$\int_{\varepsilon}^{T-\varepsilon}\left|\frac{v(t+h) - v(t)}{h}\right|^{\alpha} dt \leqq \left\|\frac{dv}{dt}\right\|^{\alpha}_{L^{\alpha}(\varepsilon-|h|, T-\varepsilon+|h|)}\,.$$

PROOF OF LEMMA 1.2. If u belongs to $H^1(0, T; L^p(\Omega))$, then u and $\partial u/\partial t$ belong to $L^2(0, T; L^p(\Omega)) \subset L^\alpha(Q)$;

$$\text{a.e. } x\,, \qquad u(x) \qquad \text{and} \qquad \frac{\partial u}{\partial t}(x) \in L^\alpha(0, T)\,,$$

that is

$$\text{a.e. } x\,, \qquad u(x) \in W^{1,\alpha}(0, T)\,.$$

We can apply Lemma A, with $v = u(x)$. For $0 < |h| < \varepsilon$, we have, with $q_h(t, x) = \big(u(t+h) - u(t)\big)/h$,

$$\text{a.e. } x\,, \qquad \int_\varepsilon^{T-\varepsilon} |q_h(t, x)|^\alpha\, dt \leqq \left\|\frac{\partial u}{\partial t}\right\|^\alpha_{L^\alpha(\varepsilon-|h|, T-\varepsilon+|h|)}.$$

By integrating over Ω, we get

$$\overline{\lim_{h\to 0}}\, \|q_h\|_{L^\alpha(Q_\varepsilon)} \leqq \left\|\frac{\partial u}{\partial t}\right\|_{L^\alpha(Q_\varepsilon)}. \tag{A.1}$$

1) If $p > 1$ (then $\alpha > 1$), it is easy to prove that $q_h \to \partial u/\partial t$ in $L^\alpha(Q_\varepsilon)$, weakly. Then, classically, by (A.1), $q_h \to \partial u/\partial t$ in $L^\alpha(Q_\varepsilon)$ for the strong topology.

2) If $p = 1$, then $\alpha = 1$. There exists a sequence u_n in $H^1(0, T; L^\infty(\Omega))$ such that $u_n \to u$ in $H^1(0, T; L^1(\Omega))$. Let

$$r_{hn} = \frac{u_n(t+h) - u_n(t)}{h} - \frac{\partial u_n}{\partial t}\,.$$

We have, with the $L^1(Q_\varepsilon)$ norms,

$$\|r_h\| \leqq \|r_{hn}\| + \|r_{hn} - r_h\|\,.$$

We have just seen (case $p > 1$) that $r_{hn} \to 0$ in $L^2(Q_\varepsilon)$ (and consequently in $L^1(Q_\varepsilon)$). Besides, by Lemma A,

$$\left\|\frac{(u_n - u)(t+h) - (u_n - u)(t)}{h}\right\|_{L^1(Q_\varepsilon)} \leqq \left\|\frac{\partial(u_n - u)}{\partial t}\right\|_{L^1(Q)}.$$

Thus

$$\|r_{hn} - r_h\|_{L^1(Q_\varepsilon)} \leqq 2\left\|\frac{\partial(u_n - u)}{\partial t}\right\|_{L^1(Q)}$$

which tends to zero with n. It follows that $r_h \to 0$ in $L^1(Q_\varepsilon)$, and Lemma 1.2 is proved.

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