Annali della
Scuola Normale Superiore di Pisa
Classe di Scienze

Franc Forstnerič

On the boundary regularity of proper mappings

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 13, nº 1 (1986), p. 109-128

<http://www.numdam.org/item?id=ASNSP_1986_4_13_1_109_0>

On the Boundary Regularity of Proper Mappings.

FRANC FORSTNERIČ (*)

1. – Statement of the results.

There exist well-known results on smooth extensions of proper holomorphic maps between certain classes of smoothly bounded domains in $\mathbb{C}^n$ [2, 5]. On the other hand, very little is known about proper holomorphic maps into domains in higher dimensional spaces. Suppose that $D \subset \mathbb{C}^n$ and $\Omega \subset \mathbb{C}^N$ $(N > n)$ are bounded domains and that $f: D \to \Omega$ is a proper holomorphic map. What can be said about the boundary regularity of the image subvariety $f(D)$ in Ω and about the boundary regularity of f in terms of the regularity of bD and $b\Omega$?

It has been proved recently that, unlike in the equidimensional case $N = n$, the map f needs not extend continuously to $\bar{D}$ even if bD and $b\Omega$ are smooth or real analytic [10]. Therefore additional hypotheses are needed. In this paper we shall prove some results under the assumption that the nontangential boundary values of f at bD, which exist almost everywhere on bD with respect to the surface measure on bD, lie in a smooth submanifold M of dimension $2n - 1$ of $\mathbb{C}^N$ contained in $b\Omega$. Our first main result is the following.

1.1. THEOREM. *Let $D \subset \mathbb{C}^n$ and $\Omega \subset \mathbb{C}^N$ $(N > n)$ be bounded domains of class C^2, let $b\Omega$ be strictly pseudoconvex, and let M be a compact connected real submanifold of $\mathbb{C}^n$ of class C^r $(r \geq 2)$ and of dimension $2n - 1$ that is contained in the boundary of Ω. If f is a proper holomorphic map of D into Ω such that for almost every point $p \in bD$ with respect to the Lebesgue measure on bD the nontangential limit $f^*(p)$ of f at p lies in M, then the following hold:*

(*) Research supported in part by a Sloan Foundation Predoctoral Fellowship.
Pervenuto alla Redazione il 14 Febbraio 1985.

(i) *the closure* $\overline{V}$ *of the subvariety* $V=f(D)$ *of* Ω *is* $V\cup M$, *and the pair* (V, M) *is a local* C^r *manifold with boundary in a neighborhood of each point* $q\in M$. *In particular, the singular variety* V_{sing} *is finite*;

(ii) *the map* f *extends to a continuous map on* $\overline{D}$ *which satisfies the Hölder condition with exponent* $\frac{1}{2}-\varepsilon$ *for every* $\varepsilon>0$;

(iii) *if* D *is also strictly pseudoconvex, then the restriction*

$$f\colon \overline{D}\setminus f^{-1}(V_{\text{sing}})\to\overline{V}\setminus V_{\text{sing}}$$

is a finite covering projection that is Hölder-continuous with the exponent $\frac{1}{2}$.

Note that if a proper map $f\colon D\to\Omega$ exists, then D is necessarily pseudoconvex. Using a local extension theorem for biholomorphic maps due to Lempert [20, p. 467] we obtain the following corollary.

1.2. Corollary. *Let* $f\colon D\to\Omega$ *and* $M\subset b\Omega$ *be as in Theorem* 1.1, *and assume that both* D *and* Ω *are strictly pseudoconvex. If* bD *and* M *are of class* C^r *for some* $r\geqslant 6$, *then* f *extends to a* C^{r-4} *map on* $\overline{D}$. *In particular, if* bD *and* M *are* C^∞ *on* $\overline{D}$, *and if* bD *and* M *are real-analytic, then* f *extends holomorphically to a neighborhood of* $\overline{D}$.

Note. In the case when bD and M are real-analytic, Corollary 1.2 above can be considered to be a generalization of the reflection principle [21, 23, 33] to maps into higher dimensional spaces. Certain generalizations for this kind of maps have been obtained earlier by Lewy [21, p. 8] and Webster [33].

A similar result holds if M is only an immersed submanifold of $b\Omega$, provided that the set of its self-intersections is not too large. In the next theorem we assume that $D\subset\mathbf{C}^n$ and $\Omega\subset\mathbf{C}^N$, $N>n$, are bounded C^2 strictly pseudoconvex domains.

1.3. Theorem. *Let* M^{2n-1} *be a compact connected* C^r *manifold,* $r\geqslant 2$, *and let* $i\colon M\to\mathbf{C}^N$ *be an immersion of class* C^r, *with the image* $i(M)$ *contained in* $b\Omega$. *Denote by* S *the set of points* $q\in i(M)$ *at which* $i(M)$ *is not a manifold. Assume that*

(*a*) $i(M)\setminus S$ *is connected, and*

(*b*) $\mathcal{H}^{2n-1}(S)=0$, *where* $\mathcal{H}^k$ *denotes the* k*-dimensional Hausdorff measure.*

If $f\colon D\to\Omega$ *is a proper holomorphic map with* $f^*(p)\in i(M)$ *for almost every point* p *in* bD, *then the following hold.*

(i) *Each point $q \in M$ has a neighborhood U in $\mathbf{C}^N$ such that*

$$U \cap M \quad = M_1 \cup M_2 \cup \ldots \cup M_s \,,$$
$$U \cap f(D) = V_1 \cup V_2 \cup \ldots \cup V_s \,,$$

and $V_j \cup M_j$ is a C^k manifold with boundary M_j for each $j = 1, \ldots, s$. In particular, the singular locus of the variety $V = f(D)$ is finite;

(ii) *f extends to a Hölder continuous map on $\bar{D}$, and its branching locus consists of at most finitely many points of D;*

(iii) *if $r \geqslant 6$, then f extends to a C^{r-4} map on $\bar{D}$.*

Remark 1. Since the map f is bounded on D, the generalized theorem of Fatou [29, p. 13] asserts that there exists a set $E \subset bD$ whose complement $bD \setminus E$ has surface measure 0 such that f has a nontangential limit $f^*(p)$ at every point $p \in E$. One of our hypotheses is that this limit lies in M for almost every point $p \in E$.

Remark 2. The regularity of the subvariety $f(D)$ at the boundary of Ω can also be deduced from the work of Harvey and Lawson [14, Theorems 4.7, 4.8 and 10.3]. Their methods include the structure theorems for certain types of currents. Our proof of Theorem 1.1 is perhaps more elementary. However, the hypothesis that $b\Omega$ be strictly pseudoconvex is essential in our proof of Theorem 1.1.

Remark 3. In the case $n = 1$ our Theorem 1.1 follows from a more general result of Čirka [4, p. 293] which states that if $f\colon \Delta \to \mathbf{C}^N$ is a holomorphic map on the unit disk $\Delta \subset \mathbf{C}$ such that all of its boundary values on an open arc $\gamma \subset b\Delta$ lie in a totally real submanifold $M \subset \mathbf{C}^N$ of class C^r, $r \geqslant 2$, then f is of class $C^{r-1,\alpha}$ on $\Delta \cup \gamma$ for all $0 < \alpha < 1$. If D is a domain of class C^r in $\mathbf{C}$, then we can find for every point $p \in bD$ a simply connected domain $U \subset D$ with bU of class C^r such that $\bar{U} \cap bD$ contains an open arc γ and $p \in \gamma$. If $f\colon D \to \Omega$ is as in Theorem 1.1 above and if all boundary values of f lie in a C^r curve M contained in $b\Omega$, then the theorem of Čirka implies that f is of class $C^{r-1,\alpha}$ on $\bar{D}$. If ϱ is a strictly plurisubharmonic defining function for Ω, then $\varrho \circ f$ is a negative subharmonic function on D that vanishes on bD. The Hopf lemma implies $d(\varrho \circ f) \neq 0$ on bD. It follows that $df \neq 0$ on bD, and $f(\bar{D})$ intersects $b\Omega$ transversely. From the proof of part (i) of Theorem 1.1 we shall be able to see that the set $f(\bar{D})$ is in fact of class C^r near its boundary $f(bD) = M$.

My sincere thanks go to Professor Edgar Lee Stout.

2. – Boundary regularity of the image variety.

In this section we shall give a self-contained proof of Theorem 1.1 in the case when $n \geq 2$. The first part of the proof applies also to the case $n = 1$.

By an embedding theorem of Fronæss and Khenkin [9, 17] we may assume that Ω is strictly *convex*. The maximum modulus principle for functions in $\boldsymbol{H}^\infty(D)$ implies that $f(D)$ lies in the polynomially convex hull $\hat{M}$ of M. Since Ω is strictly convex, we have $\hat{M} \cap b\Omega = M$ and hence

$$\overline{f(D)} \subset f(D) \cup M ,$$

i.e., *all* limiting values of f at bD lie in M.

We shall first prove that $\overline{f(D)}$ is a $\boldsymbol{C}^r$ manifold with boundary in a small neighborhood of each point $p \in \overline{f(D)} \cap M$ at which the following condition holds:

$$T_p M \not\subset T_p^{\mathbf{C}} b\Omega . \tag{2.1}$$

Here, $T_p^{\mathbf{C}} b\Omega$ denotes the maximal complex subspace of the tangent space $T_p b\Omega$. By translating to the origin we may assume that $p = 0$. The assumption (2.1) implies that $W = T_0 M \cap T_0^{\mathbf{C}} b\Omega$ is a real $(2n-2)$-dimensional vector subspace of $\mathbf{C}^N$.

We claim that we can find a complex $(n-1)$-dimensional subspace Σ' of $\mathbf{C}^N$ such that the orthogonal projection $\pi'\colon \mathbf{C}^N \to \Sigma'$ maps W bijectively onto Σ'. This is equivalent to finding a complex subspace Σ'' of $\mathbf{C}^N$ such that $W \oplus \Sigma'' = \mathbf{C}^N$, since we may then take for Σ' the orthogonal complement of Σ'' in $\mathbf{C}^N$. If $W = \{(x, y) \in \mathbf{C}^2 \colon x, y \text{ real}\}$, we may take $\Sigma'' = \mathbf{C}\cdot(1, i)$. In general, if we choose coordinates correctly, we have

$$W = \mathbf{C}^m \oplus (\mathbb{R}^2)^l \oplus \{0\} \subset \mathbf{C}^N ,$$

where each copy of $\mathbb{R}^2$ is embedded as the standard totally real plane in $\mathbf{C}^2$, and $m + l = n - 1$. For each copy of $\mathbb{R}^2$ in the above sum we take $\Sigma_j'' = \mathbf{C}\cdot(1, i)$ as above. The complex subspace

$$\Sigma'' = \{0\} \oplus \Sigma_1'' \oplus \ldots \oplus \Sigma_l'' \oplus \mathbf{C}^{N-n+1}$$

has the required property $W \oplus \Sigma'' = \mathbf{C}^N$, and we take Σ' to be the orthogonal complement of Σ'' in $\mathbf{C}^N$.

Let Σ be the complex n-dimensional subspace of $\mathbf{C}^N$ spanned by Σ' and by the normal vector to $b\Omega$ at 0. We denote by π the orthogonal projection of $\mathbf{C}^N$ onto Σ. The restriction $\pi\colon T_0M \to \Sigma$ is one-to-one by the choice of Σ, and therefore $\pi\colon M \to \Sigma$ is a C^r embedding near 0.

We will show that $\pi(M) \subset \Sigma$ is a strictly *convex* hypersurface near the point $0 \in \Sigma$. By a unitary change of coordinates at 0 we may assume that

$$\Sigma = \{z \in \mathbf{C}^N\colon z_{n+1} = \ldots = z_N = 0\}$$

and that in some neighborhood U of 0 the domain Ω is given by

$$\Omega \cap U = \{z \in U\colon x_1 + Q(z) + o(|z|^2) < 0\}\,,$$

where $z_j = x_j + iy_j$ and $Q(z)$ is a real positive definite quadratic form in z. Let

$$c = \tfrac{1}{2}\inf\{Q(z)\colon |z| = 1\} > 0\,.$$

For all sufficiently small $\varepsilon > 0$ we have

$$\Omega \cap \{x_1 > -\varepsilon\} \subset \{z \in \mathbf{C}^N\colon x_1 + c|z|^2 < 0\} = B_c$$

and therefore

$$\pi(\Omega \cap \{x_1 > -\varepsilon\}) \subset B_c \cap \Sigma\,.$$

In particular, $\pi(M \cap \{x_1 > -\varepsilon\})$ is a hypersurface in the ball $B_c \cap \Sigma$ that is internally tangent to the sphere $bB_c \cap \Sigma$ at 0, and therefore $\pi(M)$ is strictly convex near 0 as claimed.

Let G be the domain in Σ bounded by $\pi(M) \cap \{x_1 > -\varepsilon\}$ and by $\{z_1 = -\varepsilon\}$. For each sufficiently small $\varepsilon > 0$ we have

$$\widehat{\pi(M)} \cap \{x_1 \geqslant -\varepsilon\} = \bar{G}\,,$$

where $\widehat{\pi(M)}$ is the polynomially convex hull of $\pi(M)$. The maximum maximum modulus principle for H^∞ implies

$$(\pi \circ f)(D) \subset \widehat{\pi(M)}\,.$$

It follows that

$$\pi(f(D) \cap \{x_1 > -\varepsilon\}) = \pi(f(D)) \cap \{x_1 > -\varepsilon\} \subset \widehat{\pi(M)} \cap \{x_1 > -\varepsilon\} \subset \bar{G}\,.$$

By the maximum principle for varieties [22, p. 54] we have

$$\pi(f(D) \cap \{x_1 > -\varepsilon\}) \subset G\,.$$

The variety $V = f(D) \cap \{x_1 > -\varepsilon\}$ is closed in $\pi^{-1}(G)$, and the restriction $\pi|_V: V \to G$ maps V properly and holomorphically into G. Hence the pair $(V, \pi|_V)$ is an analytic cover [13, p. 101] of multiplicity λ for some integer λ.

We claim that $\lambda = 1$. The following is the crucial observation about V:

If $\{w_\nu\} \subset V$ is a sequence for which $\{\pi(w_\nu)\}$ converges to a point

$$q \in (M) \cap \{x_1 > -\varepsilon\}\,,$$

then $\{w_\nu\}$ converges to the unique point $\tilde{q} \in M$ for which $\pi(\tilde{q}) = q$.

Intuitively this says that all sheets of the analytic cover $\pi: V \to G$ are glued together along M, and will show that as a consequence there is only one sheet.

After a unitary change of coordinates $z_{n+1}, \ldots, z_N$ we can assume that for some $z \in G$ there are λ distinct points $w^{(1)}(z), \ldots, w^{(\lambda)}(z)$ in $\pi^{-1}(z) \cap V$ with distinct N-th coordinates $w_N^{(1)}(z), \ldots, w_N^{(\lambda)}(z)$. The same is then true for every point z outside a proper subvariety $L \subset G$, and each $w_N^{(j)}$ is locally a holomorphic function of z. However, these function need not be well-defined globally.

Consider the polynomial $P(t, z) \in O(G \setminus L)[t]$ in the variable t defined by

$$P(t, z) = \prod_{j=1}^{\lambda} \left(t - w_N^{(j)}(z)\right) = t^\lambda + a_1(z)\, t^{\lambda-1} + \ldots + a_\lambda(z)\,, \qquad z \in G \setminus L\,.$$

The coefficients $a_j(z)$ are elementary symmetric polynomials in the $w_N^{(j)}(z)$'s, and hence they are well-defined bounded holomorphic functions on $G \setminus L$ that extend to bounded holomorphic functions on G. The same is then true for the discriminant $\Delta(z)$ of P. By the generalized theorem of Fatou [29, p. 13] there is a set E contained in $\pi(M) \cap \{x_1 > -\varepsilon\} = S$, E being of full measure in S, such that all coefficients $a_j(z)$ and $\Delta(z)$ have non-tangential limits at all points of E. Since Δ is not identically zero on G by the construction of P, the boundary uniqueness theorem [27] implies that $\Delta(e) \neq 0$ for some $e \in E$ (in fact $\Delta \neq 0$ almost everywhere on E). Hence the polynomial $P(t, e)$ has λ distinct complex roots $t_1, \ldots, t_\lambda$.

In order to reach a contradiction we assume that $\lambda > 1$, and let $t_1 \neq t_2$ be two distinct roots of $P(t, e)$. Since the roots of a polynomial depend continuously on its coefficients, we can find a sequence of points $\{z_\nu\}$ in G converging nontangentially to e, and we can find roots $t_1(z_\nu)$, $t_2(z_\nu)$ of $P(t, z_\nu)$ such that

$$\lim_{\nu\to\infty} t_1(z_\nu) = t_1 \quad \text{and} \quad \lim_{\nu\to\infty} t_2(z_\nu) = t_2\,.$$

By the definition of $P(t, z)$ there exist points $w_\nu^{(1)}$ and $w_\nu^{(2)}$ in $V \cap \pi^{-1}(z_\nu)$ with the N-th coordinates equal to $t_1(z_\nu)$ and $t_2(z_\nu)$, respectively. Clearly the sequences $\{w_\nu^{(1)}\}$ and $\{w_\nu^{(2)}\}$ cannot both converge to the same point $\tilde{e} = M \cap \pi^{-1}(e)$. This contradicts the observation about V that we have made above.

Therefore $\lambda = 1$ as claimed. Hence the map $\pi|_V \colon V \to G$ is one-to-one and therefore it is a biholomorphism of V onto G. Its inverse is of the form

$$z \to (z, \sigma(z)), \qquad z \in G,$$

where $\sigma \colon G \to \mathbf{C}^{N-n}$ is a holomorphic map on G. Our observation about V implies that σ extends continuous to $G \cup S$, where $S = \pi(M) \cap \{x_1 > -\varepsilon\}$, and the map $z \to (z, \sigma(z))$, $z \in S$, is the inverse of $\pi|_M$ on S. Since $\pi|_M$ is a C^r diffeomorphism onto S, $\sigma|_S$ is of class C^r by the inverse mapping theorem. The regularity theorem [14, Theorem 5.6] implies that σ is of class C^r on $G \cup S$.

This proves that $\overline{f(D)} \cap M$ is a C^r manifold with boundary near every point $p \in f(D) \cap M$ at which the condition (2.1) holds. In particular, M is maximally complex near every such point p, and a neighborhood of p in M is contained in $\overline{f(D)}$. It remains to show that (2.1) holds for every point $p \in \overline{f(D)} \cap M$. In the case $n = 1$ we refer to the theorem of Čirka [4]. (See Remark 3 in Section 1). We shall give a self-contained proof in the case $n \geqslant 2$.

Define the subsets C and E of M by

$$C = \{p \in M \mid T_p M \subset T_p^{\mathbf{C}} b\Omega\}, \tag{2.2}$$

$$E = M \cap \overline{f(D)}. \tag{2.3}$$

We have seen above that $E \setminus C$ is an open subset of $M \setminus C$, and M is maximally complex at each point of $E \setminus C$. Since E is closed, $E \setminus C$ is also closed in $M \setminus C$ and therefore it is a union of connected components of $M \setminus C$. We want to show that $C = \emptyset$ and hence $E = M$.

We will first show that the set $E \setminus C$ is not empty. Suppose on the contrary that $E \subset C$, i.e., the transversality condition (2.1) does not hold at any point of E. Extending Ω to a strictly convex domain in $\mathbf{C}^{N'}$ for a $N' \geqslant N$ we may assume that $\dim b\Omega \geqslant 2 \dim M + 1$.

The strictly pseudoconvex hypersurface $b\Omega$ is a contact manifold with the contact form $\eta = i(\bar\partial - \partial)\varrho$ whose kernel is $\ker \eta = T^{\mathbf{C}} b\Omega$, where ϱ is a defining function for Ω [31]. (For the general theory of contact manifolds see [3].) Let $\iota \colon M \hookrightarrow b\Omega$ be the inclusion of M into $b\Omega$. We have $\iota^* \eta = 0$

on the set C. By an argument of Duchamp [7] every point $p \in M$ has an open neighborhood $U \subset M$ and a C^1 embedding $\tilde{\iota}$: $U \to b\Omega$ such that $\tilde{\iota} = \iota$ on the set $C \cap U$, and $\iota^* \eta = 0$ on U. Then $\tilde{\iota}$: $U \to b\Omega$ is an *interpolation manifold* [31], and by a theorem of Rudin [26] each compact subset of $\tilde{\iota}(U)$ is a peak-interpolation set for the algebra $A(\Omega)$. It follows that E is a local peak-interpolation set and hence a peak-interpolation set [30, Chapter 4]. If $h \in A(\Omega)$ is a peak function on E, then $h \circ f$ is a nonconstant bounded holomorphic function on D whose boundary values equal 1 almost everywhere on bD. This is a contradiction which implies that $E \setminus C \neq \emptyset$.

The following lemma implies that the set C is empty, thereby concluding the proof of part (i) of Theorem 1.1.

2.1. Lemma. *Let S be a strictly pseudoconvex hypersurface of class C^2 in $\mathbf{C}^N$ and let M be a C^2 submanifold of S of dimension $2m+1$ for some $m \geqslant 1$. If M is maximally complex at every point of an open subset $U \subset M$, then we have for every $p \in \overline{U}$*

$$T_p M \not\subset T_p^{\mathbf{C}} S\,. \tag{2.4}$$

Assume the validity of Lemma 2.1 for a moment. Let $S = b\Omega$ and $U = E \setminus C$. If $C \neq \emptyset$, then there exists a point $p \in C \cap \overline{U}$. By Lemma 2.1 the condition (2.4) holds at p which is a contradiction with the definition (2.2) of the set C. Hence $C = \emptyset$ and Theorem 1.1 is proved provided that Lemma 2.1 holds.

Proof of Lemma 2.1. Let η be a contact form on S with kernel $T^{\mathbf{C}}S$. If X is a C^1 vector field on S that is tangent to $T^{\mathbf{C}}S$, then the vector field JX is also tangent to $T^{\mathbf{C}}S$. (Here J denotes the almost complex structure on $T^{\mathbf{C}}S$.) By virtue fo the strict pseudoconvexity of S we have

$$-\langle d\eta, (X + iJX) \otimes (X - iJX)\rangle_p \neq 0 \tag{2.5}$$

at every point p where $X_p \neq 0$. By the Cartan formula (2.5) is equal to

$$\begin{aligned} -\langle \eta, [X + iJX, X - iJX]\rangle_p &= -\langle \eta, -2i[X, JX]\rangle_p \\ &= 2i\langle \eta, [X, JX]\rangle_p\,. \end{aligned} \tag{2.6}$$

Hence the continuous vector field $Y = [X, JX]$ satisfies $Y_p \notin T_p^{\mathbf{C}} S$ if $X_p \neq 0$. This shows that (2.4) holds at each point $p \in U$. We need to prove that (2.4) also holds on the boundary of U.

Fix a point $p_0 \in \overline{U} \setminus U$ and choose real functions $r_1, \ldots, r_s$ of class C^2

on $\mathbf{C}^N$ such that near p_0 the manifold M is defined by the equations

$$r_1(z) = \ldots = r_s(z) = 0 .$$

Let $\theta_j = i\partial r_j$ for $1 \leqslant j \leqslant s$. Each θ_j is a complex 1-form of class C^1 which is real-valued on TM. Moreover, we have

$$T_p^{\mathbf{C}} M = \bigcap_{j=1}^{s} (\ker \theta_j)_p$$

for every p near p_0. Since M is odd dimensional, $T_{p_0}^{\mathbf{C}} M \neq T_{p_0} M$, and hence one of the forms, say θ_{j_0}, does not vanish on $T_{p_0} M$. Hence the restriction of θ_{j_0} to TM defines a C^1 distribution of codimension 1 on TM near p_0. Since M is assumed to be maximally complex at every point of U, it follows that

$$T_p M \cap (\ker \theta_{j_0})_p = T_p^{\mathbf{C}} M$$

for each $p \in U$ near p_0.

Choose a C^1 vector field X' on M near p_0, $X'_{p_0} \neq 0$, such that

$$\langle \theta_{j_0}, X' \rangle \equiv 0 .$$

Since $\eta = 0$ on $T^{\mathbf{C}} M$, we have

$$\langle \eta, X' \rangle_p = 0 \qquad \text{for } p \in U .$$

We claim that there is a C^1 vector field X on a neighborhood of p_0 in S such that

$$X_p = X'_p \qquad \text{for } p \in U \text{ and } \langle \eta, X \rangle \equiv 0 .$$

The problem is local near p_0. Choose local coordinates such that $p_0 = 0$, $M = \mathbf{R}^{2m+1}$, $S = \mathbf{R}^{2N-1}$, U is an open subset of M with $0 \in \overline{U}$,

$$\eta(x) = \sum_{j=1}^{2N-1} a_j(x)\, dx_j \quad \text{ and } \quad X'(x) = \sum_{j=1}^{2N-1} b_j(x)(\partial/\partial x_j) \quad \text{ for } x \in \mathbf{R}^{2m+1} .$$

One of the coefficients a_j is nonzero at 0, say $a_1(0) \neq 0$. We have

$$\langle \eta, X' \rangle_x = \sum_{j=1}^{2N-1} a_j(x) b_j(x) = 0$$

for $x \in U$. Rewrite this as

$$b_1(x) = -\frac{1}{a_1(x)} \sum_{j=2}^{2N-1} a_j(x) b_j(x) \tag{2.7}$$

for x near 0 in U. We extend the functions $b_2, \ldots, b_{2N-1}$ smoothly to a neighborhood of 0 in $\mathbb{R}^{2N-1}$, and we let $b_1(x)$ be defined by (2.7). This gives us a vector field $X(x) = \sum_{j=1}^{2N-1} b_j(x)(\partial/\partial x_j)$ on $\mathbb{R}^{2N-1}$ with the required properties.

The C^0 vector field $Y = [X, JX]$ defined on S near $p_0 = 0$ is tangent to M on the set U. By the continuity it follows that $Y_0 \in T_0 M$. Moreover, the strict pseudoconvexity of S implies that $\langle \eta, Y \rangle_0 \neq 0$ (see (2.5) and (2.6)). Together these imply that $T_0 M \not\subset T_0^{\mathbb{C}} S$ and Lemma 2.1 is proved.

3. – Continuous extension to the boundary.

In this section we shall conclude the proof of Theorem 1.1. Following an idea of Khenkin [16] we first prove that the map f in Theorem 1.1 extends continuously to $\bar{D}$.

3.1. LEMMA. *Let $f\colon D \to \Omega$ be as in Theorem 1.1. Denote by $d_D(z)$ the Euclidean distance fo a point $z \in D$ to bD, and similarly for d_Ω. Then there exist constants $c_1, c_2 > 0$ and $0 < \varepsilon < 1$ such that the inequality*

$$c_1 d_D(z) \leqslant d_\Omega(f(z)) \leqslant c_2 d_D(z)^\varepsilon \tag{3.1}$$

holds for all $z \in D$. If D is also strictly pseudoconvex, we may take $\varepsilon = 1$ in (3.1).

PROOF. Let r_D and r_Ω be C^2 defining functions for D resp. Ω. Since Ω is strictly pseudoconvex, we may take r_Ω to be plurisubharmonic on Ω. Hence $r_\Omega \circ f$ is plurisubharmonic on D, it is negative and tends to 0 as we approach bD. By the Hopf lemma [15] there is a constant $c_1 > 0$ such that

$$r_\Omega(f(z)) \leqslant c_1 r_D(z)\,, \qquad z \in D\,.$$

Since the function $-r_D$ is proportional to d_D on D and similarly $-r_\Omega$ is proportional to d_Ω on Ω, the above is equivalent to the left estimate in (3.1).

To prove the right estimate in (3.1) we choose by [6] an ε in $(0, 1)$

such that the function

$$r' = -(-r_D)^\varepsilon$$

is plurisubharmonic on D. If D is strictly pseudoconvex, we may assume that r_D is plurisubharmonic and hence $\varepsilon = 1$ would do. There is a proper subvariety V' of $V = f(D)$ such that $V \setminus V'$ is regular and the restriction

$$f\colon D \setminus f^{-1}(V') \to V \setminus V'$$

is a finite unbranched covering projection. We define a function φ on V by

$$\varphi(w) = \max\{r'(z)\colon z \in D \text{ and } f(z) = w\}\,.$$

Locally on $V \setminus V'$ the function φ is the maximum of a finite number of plurisubharmonic functions and hence it is itself plurisubharmonic. Since φ is clearly continuous on V, it is plurisubharmonic on all of V according to [12, Satz 3]. Moreover, φ is negative on V and tends to 0 as we approach $bV = M$. Since $\overline{V}$ is transversal to $b\Omega$ by the proof of part (i) of Theorem 1.1, we have

$$d(r_\Omega|_{\overline{V}})(q) \neq 0\,, \qquad q \in M = \overline{V} \cap b\Omega\,.$$

The Hopf lemma implies

$$c_2\varphi(w) \leqslant r_\Omega(w)\,, \qquad w \in V$$

for some constant $c_2 > 0$. Taking the absolute values we have

$$c_2|r'(z)| \geqslant |r_\Omega(f(z))|$$

for $z \in D$. By the definition of r' we have $|r'(z)| = |r_D(z)|^\varepsilon$, and hence

$$|r_\Omega(f(z))| \leqslant c_2|r_D(z)|^\varepsilon\,.$$

This is equivalent to the right estimate in (3.1) and Lemma 3.1 is proved.

Using Lemma 3.1 and the properties of the infinitesimal Kobayashi metric we can prove that f extends to a Hölder continuous map with the exponent $\varepsilon/2$ on $\overline{D}$, where ε is as in (3.1). The idea of this proof is due to Khenkin [16].

If N is an arbitrary complex manifold, $z \in N$ and $X \in T_z^{1,0}N$ is a complex tangent vector to N at z, the Kobayashi metric $K_N(z, X)$ is given by

$$\begin{aligned} K_N(z, X) &= \inf\{\alpha > 0 \mid \text{there is a holomorphic } f\colon \Delta \to M \text{ with} \\ &\qquad f(0) = z \text{ and } f'(0) = \alpha^{-1}X\}\,, \\ &= \inf\{r^{-1} \mid \text{there is a holomorphic } f\colon \Delta_r \to M \text{ with} \\ &\qquad\qquad f(0) = z \text{ and } f'(0) = X\}\,. \end{aligned}$$

(Here Δ_r denotes the disk of radius r centered at 0 in $\mathbf{C}$.) For further details concerning the Kobayashi metric see [18].

If $D \subset \mathbf{C}^n$ is a bounded domain, then

$$K_D(s, X) \leqslant |X|/d_D(z)\,, \tag{3.2}$$

where $|X|$ is the Euclidean length of X. If D is strictly pseudoconvex, then

$$K_D(z, X) \geqslant c|X|/d(z)^{\frac{1}{2}} \tag{3.3}$$

for some constant $c > 0$ [11]. Finally, if $f\colon D \to \Omega$ is a holomorphic map, then

$$K_\Omega(f(z), f_*X) \leqslant K_D(z, X)\,,$$

where $f_*X = df(z)X$. These properties together imply

$$c|f_*X|/d_\Omega(f(z))^{\frac{1}{2}} \leqslant K_\Omega(f(z), f_*X) \leqslant K_D(z, X) \leqslant |X|/d_D(z)\,.$$

If $X \neq 0$, Lemma 3.1 implies

$$|f_*X|/|X| \leqslant c d_D(z)^{-1+\varepsilon/2}\,, \qquad X \in T_z^{1,0}(D)\,.$$

From this it follows by a simple integration argument that f is Hölder continuous of the exponent $\varepsilon/2$ on D, and hence it extends continuously to $\bar{D}$.

Once we know that f is continuous on $\bar{D}$, we can improve our result by using the local plurisubharmonic exhaustion functions on D constructed in [6, Theorem 3]. In particular it follows that Lemma 3.1 above holds for *every* $0 < \varepsilon < 1$, and hence f is Hölder continuous on $\bar{D}$ of the exponent α for every $0 < \alpha < \frac{1}{2}$. If D is strictly pseudoconvex, we may take $\alpha = \frac{1}{2}$. This proves part (ii) of Theorem 1.1.

We shall use the idea of Pinčuk [24] to show that the map f is unbranched in a neighborhood of each point $p \in bD$ at which bD is strictly pseudoconvex. We need the following local version of the result of Pinčuk:

3.2. THEOREM. *Let D^j $(j = 1, 2)$ be bounded strictly pseudoconvex domains in $\mathbb{C}^m$ with C^2 boundaries and let $p^j \in bD^j$. Suppose that U^j is an open subset of D^j such that for some small $\varepsilon > 0$ we have*

$$D^j \cap B_\varepsilon(p^j) \subset U^j , \qquad j = 1, 2 ,$$

where $B_\varepsilon(p)$ is the ball of radius ε centered at p. Let $f\colon U^1 \to U^2$ be a proper holomorphic map that extends continuously to $\overline{U}^1$ and $f(p^1) = p^2$. Then the branching locus of f avoids a neighborhood of p^1 in D^1.

NOTE. The difference between Theorem 3.2 and [24] is that in our case the map f is only defined on an open subset of D^j.

PROOF. We recall the proof of Pinčuk given in [24]. Assume that there is a sequence of points $\{p_k\} \subset U^1$ converging to p^1 such that each p_k is a branch point of f. Pinčuk constructed a sequence of domains D^j_k $(k = 1, 2, \ldots)$ such that $\overline{D}^j_k$ is biholomorphically equivalent to $\overline{D}^j$ for each $k \in \mathbb{Z}_+$, the point $p_k \in D^1$ (resp. $f(p_k) \in D^2$) corresponds to the point $(0, \ldots, 0, -1) \in D^1_k$ (resp. $(0, \ldots, 0, -1) \in D^2_k$), and as $k \to \infty$ the sequence of domains D^j_k converges uniformly on compact subsets of $\mathbb{C}^m$ to the domain

$$B = \left\{ z \in \mathbb{C}^m \mid 2 \operatorname{Re} z_m + \sum_{s=1}^{m-1} |z_s|^2 < 0 \right\}$$

for $j = 1, 2$. The domain B is biholomorphically equivalent to the unit ball $\mathbb{B}^m$ [25, p. 31], and the map f gives rise to a proper holomorphic map $F\colon B \to B$ such that $F(0, \ldots, -1) = (0, \ldots, 0, -1)$, and F is branched at the point $(0, \ldots, 0, -1)$. A theorem of Alexander [1, 25, p. 316] implies that F is an automorphism of B. This contradicts the fact that F is branched at $(0, \ldots, 0, -1) \in B$, and hence the original map f is unbranched in a neighborhood of the point p^1.

To prove the local version of the theorem as stated above we perform the same construction of domains D^j_k. (See Lemma 1 in [24].) Let $U^j_k \subset D^j_k$ be the subset that corresponds to $U^j \subset D^j$ under the given biholomorphism of D^j onto D^j_k. It follows from the construction in [24] that the sequence U^j_k converges to B as $k \to \infty$ and the map $F\colon B \to B$ can still be constructed, thus yielding a contradiction exactly as above. For the details we refer the reader to [24].

To apply Theorem 3.2 we choose a point $p^1 \in bD$ and let $p^2 = f(p^1) \in M$. Let Σ be a complex n-plane through p^2 such that the corresponding orthogonal projection $\pi\colon \mathbf{C}^N \to \Sigma$ maps a neighborhood of p^2 in V biholomorphically onto a strictly pseudoconvex domain $D^2 \subset \Sigma$ with C^2 boundary. Let $U^2 = D^2$ and $U^1 = (\pi\circ f)^{-1}(D^2) \subset D = D^1$. By Theorem 3.2 the map $\pi\circ f$ is not branched near p^1 and hence f is not branched near p^1.

This proves that the branching locus of f stays away from the strictly pseudoconvex boundary points of D. In particular, if D is strictly pseudoconvex, then the branching locus of f is compactly contained in D and hence it is finite.

It remains to prove the part (iii) of Theorem 1.1. The restriction

$$f\colon D\diagdown f^{-1}(V_{\mathrm{sing}}) \to V\diagdown V_{\mathrm{sing}} \tag{3.4}$$

is a proper holomorphic map of n-dimensional complex manifolds, and hence its branching locus is either empty or else it is a subvariety of $D\diagdown f^{-1}(V_{\mathrm{sing}})$ of pure dimension $n-1$. Since the second case is excluded by what we have just said above, the map (3.4) is unbranched.

Consider now the extended map

$$f\colon \overline{D}\diagdown f^{-1}(V_{\mathrm{sing}}) \to \overline{V}\diagdown V_{\mathrm{sing}}\,. \tag{3.5}$$

We fix a point $q \in M = \overline{V}\diagdown V$ and choose a simply connected subset $V_0 \subset V\diagdown V_{\mathrm{sing}}$ with C^2 strictly pseudoconvex boundary such that for some $\varepsilon > 0$ we have

$$B_\varepsilon(q) \cap V \subset V_0\,. \tag{3.6}$$

Since (3.4) is a covering projection, the inverse image $f^{-1}(V_0)$ is a disjoint union of k connected open subsets $D_1, D_2, \ldots, D_k$ of D such that the restriction of f to D_j is a biholomorphism of D_j onto V_0 for each $j = 1, \ldots, k$. Let D_0 be any of the sets D_j, and denote by $g\colon V_0 \to D_0$ the inverse of $f\colon D_0 \to V_0$. If V_0 is chosen sufficiently small, then V_0 is very close to its projection onto the complex plane $T_q\overline{V}$, and hence property (3.2) of the Kobayashi metric gives an estimate

$$K_{V_0}(w, X) \leqslant |X|/d(w, bV_0) \tag{3.7}$$

for $X \in T_w^{1,0}V_0$. Since $\overline{V}$ is transversal to $b\Omega$ at q, we have

$$d(w, bV_0) \geqslant c_1 d(w, b\Omega) \tag{3.8}$$

for each $w \in V_0$ sufficiently close to q. The estimates (3.7) and (3.8) together imply

$$K_{V_0}(w, X) \leqslant c_2 |X| / d(w, b\Omega) \tag{3.9}$$

for each $w \in V_0$ close to q and $X \in T_w^{1,0} D$. Hence

$$c_3 |g_* X| / d(g(w), bD)^{\frac{1}{2}} \leqslant K_D(g(w), g_* X) \leqslant K_{V_0}(w, X) \leqslant c_2 |X| / d(w, b\Omega) .$$

From this and Lemma 3.1 we obtain an estimate

$$\|dg(w)\| \leqslant c_5 / d(w, b\Omega)^{\frac{1}{2}}$$

on the norm of the derivative $dg = g_*$ at the points $w \in V_0$ close to q. This implies that g is Hölder-continuous with the exponent $\frac{1}{2}$ on V_0 near q [8, p. 74] and hence it extends to a Hölder-continuous map on $\overline{V}_0$ near q.

This is true for each local inverse $g_j\colon V_0 \to D_j$. By shrinking V_0 if necessary we may assume that $g_j\colon \overline{V}_0 \to \overline{D}_j$ is a Hölder continuous map that is the inverse of $f\colon \overline{D}_j \to \overline{V}_0$.

Let $V_1 = \overline{V}_0 \cap B_\varepsilon(q)$, where ε is as in (3.6). We claim that

$$f^{-1}(V_1) = \bigcup_{j=1}^{k} g_j(V_1) . \tag{3.10}$$

To prove this, suppose that $f(z)$ lies in V_1 for some $z \in \overline{D}$. Pick a sequence $\{z_\nu\} \subset D$ such that $\lim_{\nu \to \infty} z_\nu = z$. By the continuity of f we have $\lim_{\nu \to \infty} f(z_\nu) = f(z)$. There is a ν_0 such that $f(\nu_0) \in V_0$ for each $\nu \geqslant \nu_0$. Since

$$f^{-1}(V_0) = \bigcup_{j=1}^{k} g_j(V_0) ,$$

it follows that

$$z_\nu = g_j(f(z_\nu)) \tag{3.11}$$

for some $j = j(\nu) \in \{1, \dots, k\}$. One j has to appear infinitely many times as $\nu \to \infty$. Passing to a subsequence we may assume that (3.11) holds for all ν, with j fixed. Hence

$$z = \lim_{\nu \to \infty} z_\nu = \lim_{\nu \to \infty} g_j(f(z_\nu)) = g_j(\lim_{\nu \to \infty} f(z_\nu)) = g_j(f(z))$$

which implies $z \in g_j(V_0)$. This proves (3.10). Since q was an arbitrary point of M, it follows that (3.5) is a topological covering projection. This completes the proof of Theorem 1.1.

4. – Smooth extension to the boundary.

In this section we shall prove Corollary 1.2 and Theorem 1.3. We will use a local extension theorem for biholomorphic mappings due to Lempert [20, p. 467]:

THEOREM. *Let Ω_1 and Ω_2 de domains in $\mathbb{C}^n$, let $f\colon \Omega_1 \to \Omega_2$ be a biholomorphic map, and let p_j be a point in $b\Omega_j$ for $j = 1, 2$. Assume that*

$$\lim_{\substack{z\in\Omega_1\\ z\to p_1}} = p_2 \quad \text{and} \quad \lim_{\substack{w\in\Omega_2\\ w\to p_2}} = p_1 .$$

If the boundaries $b\Omega_j$ $(j = 1, 2)$ are of class C^r and strictly pseudoconvex in some neighborhood of the points p_1 resp. p_2 and if $r \geqslant 6$, then the map f extends to a C^{r-4} map on a neighborhood of p_1 in $\overline{\Omega}_1$.

Assuming this theorem we shall now prove Corollary 1.2. Suppose that the map $f\colon D \to \Omega$ is as in Theorem 1.1. Recall that f extends continuously to $\overline{D}$ by the part (ii) of Theorem 1.1. Choose a point $p_1 \in bD$ and let $p_2 = f(p_1) \in M$. Since M is of class C^r and $f(D) \cup M$ is a C^r manifold with boundary near p_2, we can find a simply connected domain $\Omega_2 \subset f(D)$ with C^r boundary such that

$$B_\varepsilon(p_2) \cap f(D) \subset \Omega_2$$

for some small $\varepsilon > 0$. We may choose Ω_2 so small that the orthogonal projection of $\mathbb{C}^N$ onto the complex n-plane $T_{p_2}\overline{f(D)}$ maps Ω_2 onto a C^r strictly pseudoconvex domain.

We have seen in Section 3 above that the map f has a local inverse g on Ω_2 that is continuous on $\overline{\Omega}_2$ and sends p_2 to p_1. If we let $\Omega_1 = g(\Omega_2) \subset D$, then the continuity of f on $\overline{D}$ implies that

$$B_\delta(p_1) \cap D \subset \Omega_1$$

for some small $\delta > 0$. In particular, a part of $b\Omega_1$ near p_1 coincides with bD, and hence $b\Omega_1$ is of class C^r and strictly pseudoconvex near p_1. The theorem of Lempert implies that f is of class C^{r-4} on $\bar{D}$ near the point p_1. Since $p_1 \in bD$ was chosen arbitrarily, f is of class C^{r-4} on $\bar{D}$.

NOTE. The same conclusion applies to each local inverse of f near M, and hence the map

$$f\colon \bar{D}\setminus f^{-1}(V_{\text{sing}}) \to \bar{V}\setminus V_{\text{sing}}$$

is a C^{r-4} covering projection (here $V = f(D)$).

PROOF OF THEOREM 1.3. Recall that $S \subset i(M)$ is the set of non-smooth points of $i(M)$. Let $V = f(D)$. We have seen in the proof of Theorem 1.1 that $\bar{V} \subset V \cup i(M)$. We claim that $\bar{V}$ cannot be contained in $V \cup S$. Since $\mathcal{H}^{2n-1}(S) = 0$, the assumption $\bar{V} \subset V \cup S$ would imply that $\bar{V}$ is a complex subvariety of $\mathbf{C}^N$ according to a theorem of Shiffman [28, p. 11]. Since $\bar{V}$ is compact, this is a contradiction. Hence the set $\bar{V} \cap i(M)\setminus S$ is not empty, and the proof of part (i) of Theorem 1.1 shows that $V \cup i(M)$ is a local C^r manifold with boundary near each point $p \in i(M)\setminus S$. Moreover, the set $\bar{V} \cap i(M)\setminus S$ is open and closed in $i(M)\setminus S$. Since $i(M)\setminus S$ is assumed to be connected, it follows that $i(M)\setminus S \subset \bar{V}$, and the immersion i is maximally complex at each point $x \in M$ for which $i(x) \notin S$. Further, because of $\mathcal{H}^{2n-1}(S) = 0$ the set S is nowhere dense in $i(M)$, hence by Lemma 2.1 the immersion i is maximally complex on all of M and we have $\bar{V} = V \cup i(M)$.

It remains to consider the structure of $\bar{V}$ at the points of S. Fix a point $p \in S$ and choose local coordinates in $\mathbf{C}^N$ near p such that $p = 0$, $b\Omega$ is strictly convex near 0, $T_0 b\Omega = \{x_1 = 0\}$ and $\Omega \subset \{x_1 < 0\}$. If we choose a sufficiently small $\varepsilon > 0$ and let $U = \{x_1 > -\varepsilon\}$, then

$$i(M) \cap U = M_1 \cup \ldots \cup M_s\,, \tag{4.1}$$

where each M_j is a closed connected submanifold of U. Since $b\Omega$ is strictly convex and each M_j is a maximally complex submanifold of $b\Omega$, we can choose ε so small that each M_j bounds a closed irreducible complex subvariety V_j of $U \cap \Omega$, and $V_j \cup M_j$ is a C^r manifold with boundary M_j. At every point $q \in M_j\setminus S$ the manifold M_j also bounds the variety V. It follows that V_j is an irreducible component of $V \cap U$, and hence

$$V_1 \cup V_2 \cup \ldots \cup V_s \subset V\,.$$

We claim that

$$V_1 \cup V_2 \cup \ldots \cup V_s = f(D) \cap U\,. \tag{4.2}$$

Suppose that there is another irreducible component V_0 of $V \cap U$. If $\bar{V}_0 \cap U$ contains a point $q \in M_j \setminus S$ for some $j = 1, \ldots, s$, then we have $V_0 = V_j$ which is a contradiction. Hence $\bar{V}_0 \cap U$ is contained in $V_0 \cup S$. The theorem of Shiffman [28, p. 111] implies that $\bar{V}_0 \cap U$ is a closed complex subvariety of U. Since $U = \{x_1 > -\varepsilon\}$, the plurisubharmonic function x_1 assumes its maximum on $\bar{V}_0$ which is a contradiction to the maximum principle [12]. This proves (4.2) and hence part (i) of Theorem 1.3.

The proof that we have given in Section 3 above shows that f extends to a Hölder continuous map on $\bar{D}$. Fix a point $p \in bD$. We will show that f is not branched in a neighborhood of p in D. Let $q = f(p) \in M$. Choose a neighborhood U of q in $\mathbb{C}^N$ such that (4.1) and (4.2) hold. The preimage $f^{-1}(U) \subset D$ has exactly one connected component D_1 such that $B_\delta(p) \cap D \subset D_1$ for some $\delta > 0$. The restriction $f\colon D_1 \to U \cap \Omega$ is a proper map and hence (4.2) implies that $f(D_1) = V_j$ for some j. If we apply Theorem 3.2 to the proper map

$$f\colon D_1 \to V_j,$$

we conclude that f is not branched near the point p. This proves the part (ii) of Theorem 1.3.

If we choose the set U in (4.2) sufficiently small, then the map (4.3) is a biholomorphism, and we can see the same way as in Section 3 above that the local inverse

$$g = f^{-1}\colon V_j \to D_1$$

extends to a Hölder continuous map on $\bar{V}_j$ near q. If $r \geqslant 6$, the theorem of Lempert implies that the map (4.3) is of class C^{r-4} on a neighborhood of p in $\bar{D}$. Since the point $p \in bD$ was arbitrary, f is of class C^{r-4} on $\bar{D}$ and Theorem 1.3 is proved.

REFERENCES

[1] H. Alexander, *Holomorphic mappings from the ball and polydisk*, Math. Ann., **209** (1974), pp. 249-256.

[2] St. Bell - D. Catlin, *Boundary regularity of proper holomorphic mappings*, Duke Math. J., **49** (1982), pp. 358-369.

[3] D. E. Blair, *Contact Manifolds in Riemannian Geometry*, Lecture Notes in Math., **509**, Springer-Verlag, Berlin, Heidelberg, New York, 1976.

[4] E. M. Čirka, *Regularity of boundaries of analytic sets*, Mat. Sb. (N.S.), **117** (159) (1982), n. 3, pp. 291-334; English translation in Math. USSR-Sb., **45** (1983), n. 3, pp. 291-336.

[5] K. DIEDERICH - J. E. FORNÆSS, *Boundary regularity of proper holomorphic mappings*, Invent. Math., **67** (1982), pp. 363-384.

[6] K. DIEDERICH - J. E. FORNÆSS, *Pseudoconvex domains: Bounded strictly plurisubharmonic exhaustion functions*, Invent. Math., **39** (1977), pp. 129-141.

[7] T. DUCHAMP, *The classification of Legendre immersions*, preprint.

[8] P. L. DUREN, *The Theory of H^p Spaces*, Academic Press, New York, London, 1970.

[9] J. E. FORNÆSS, *Embedding strictly pseudoconvex domains in convex domains*, Amer. J. Math., **98** (1976), pp. 529-569.

[10] F. FORSTNERIČ, *Embedding strictly pseudoconvex domains into balls*, Trans. Amer. Math. Soc., **295** (1986), pp. 347-368.

[11] I. GRAHAM, *Boundary behavior of the Caratheodory and Kobayashi metrics on strongly pseudoconvex domains in $\mathbf{C}^n$ with smooth boundary*, Trans. Amer. Math. Soc., **207** (1975), pp. 219-240.

[12] H. GRAUERT - R. REMMERT, *Plurisubharmonische Funktionene in Komplexen Räumen*, Math. Z., **65** (1956), pp. 175-194.

[13] R. C. GUNNING - H. ROSSI, *Analytic Functions of Several Complex Variables*, Prentice Hall, Englewood Cliffs, 1965.

[14] F. R. HARVEY - H. B. LAWSON, *On boundaries of complex analytic varieties - I*, Ann. of Math., **102** (1975), pp. 223-290.

[15] B. N. HIMČENKO, *The behavior of superharmonic functions near the boundary of a region of type $A^{(1)}$*, Differential Equations, **5** (1969), pp. 1371-1377.

[16] G. M. KHENKIN, *An analytic polyhedron is not biholomorphically equivalent to a strictly pseudoconvex domain*, (Russian), Dokl. Akad. Nauk SSSR, **210** (1973), n. 5, pp. 858-862; English translation in Math. USSR Dokl., **14** (1973), n. 3, pp. 858-862.

[17] G. M. KHENKIN - E. M. ČIRKA, *Boundary properties of holomorphic functions of several compelx variables*, (Russian), Sovremeni Problemi Mat., **4**, pp. 13-142, Moskva 1975; English translation in Soviet Math. J., **5** (1976), n. 5, pp. 612-687.

[18] S. KOBAYASHI, *Hyperbolic Manifolds and Holomorphic Mappings*, Marcel Dekker, New York, 1970.

[19] L. LEMPERT, *Imbedding strictly pseudoconvex domains into balls*, Amer. J. Math., **104** (1982), n. 4, pp. 901-904.

[20] L. LEMPERT, *La métrique de Kobayashi et la représentation des domaines sur la boule*, Bull. Soc. Math. France, **109** (1981), n. 4, pp. 427-474.

[21] H. LEWY, *On the boundary behavior of holomorphic mappings*, Accademia Nazionale dei Lincei, **35** (1977).

[22] R. NARASIMHAN, *Introduction to the Theory of Analytic Spaces*, Springer-Verlag, Berlin, Heidelberg, New York, 1966.

[23] S. I. PINČUK, *On the analytic continuation of holomorphic maps*, (Russian), Mat. Sb. (N.S.), **98** (140) (1975), n. 3, pp. 416-435; English translation in Math. USSR Sb. (N.S.), **27** (1975), n. 3, pp. 375-392.

[24] S. I. PINČUK, *Proper holomorphic mappings of strictly pseudoconvex domains*, (Russian), Dokl. Akad. Nauk SSSR, **241** (1978), n. 1; English translation in Math. USSR Dokl., **19** (1978), n. 1, pp. 804-807.

[25] W. RUDIN, *Function Theory on the Unit Ball of $\mathbf{C}^n$*, Springer-Verlag, New York, 1980.

[26] W. RUDIN, *Peak-interpolation sets of class C^1*, Pac. J. Math., **75** (1978), pp. 267-279.

[27] A. SADULLAEV, *A boundary uniqueness theorem in $\mathbf{C}^n$*, (Russian), Mat. Sb., **101** (143) (1976), n. 4, pp. 568-583; English translation in Math. USSR Sb. (N.S.), **30** (1976), pp. 501-514.

[28] B. SHIFFMAN, *On the removal of singularities of analytic sets*, Michigan Math. J., **15** (1968), pp. 111-120.
[29] E. M. STEIN, *Boundary Bebavior of Analytic Functions of Several Complex Variables*, Princeton University Press, Princeton, New Jersey, 1972.
[30] E. L. STOUT, *The Theory of Uniform Algebras*, Bogden and Quigley, Tarrytown on Hudson, 1971.
[31] E. L. STOUT, *Interpolation manifolds, recent developments in several complex variables*, Ann. Math. Studies, **100**, Princeton, 1981.
[32] S. M. WEBSTER, *On the reflection principle in several complex variables*, Proc. Amer. Math. Soc., **71** (1978), n. 1, pp. 26-28.
[33] S. M. WEBSTER, *Holomorphic mappings of domains with generic corners*, Proc. Amer. Math. Soc., **86** (1982), n. 2, pp. 236-240.

University of Washington
Department of Mathematics GN-50
Seattle, Washington 98195

Current address:
Institute of mathematics,
physics and mechanics
19 Jadranska
61000 Ljubljana, Yugoslavia