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# Twistorial Construction of Harmonic Maps of Surfaces into Four-Manifolds.

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*Dedicated to Professor Nicolaas H. Kuiper*

## 0. – Introduction.

Twistorial constructions of harmonic maps were first made by Calabi [$C_2$] who gave an effective parametrization of isotropic harmonic maps of Riemann surfaces into a real projective space. More than a decade later, analogous constructions were produced for maps into a complex projective space [$EW_2$]. Harmonic maps are the solutions of the Euler-Lagrange equation of the energy functional

$$E(\varphi) = \tfrac{1}{2}\int |d\varphi|^2\, dx$$

(see [EL]). Local minima of $E$ may not exist; indeed, there is only a fragmentary existence theory for harmonic maps of surfaces. It is quite significant therefore to discover that harmonic maps can sometimes be constructed explicitly via their twistor transforms.

The study of maps into a 4-dimensional manifold received a large impetus from the work of Bryant [Br], who proved that any compact Riemann surface can be conformally and harmonically immersed in the 4-sphere $S^4$. This was achieved by applying Calabi's techniques to the Penrose fibration $\mathbb{C}P^3 \to S^4$. In the present work we examine the 4-dimensional case in a more general context by considering conformal harmonic

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maps from a Riemann surface $M$ into an arbitrary oriented Riemannian 4-manifold $N$. Although conformal harmonic maps $M \to N$ are the same as minimal branched immersions, we study the consequences of the conformal and harmonic properties separately. Our approach is then based upon a parametrization of such maps announced in [ES].

In the early sections, we consider in detail the fibre bundles $\mathcal{S}_\pm$ over $N$ consisting of unit eigenvectors of the Hodge $*$ operator acting on $\Lambda^2 TN$. These total spaces admit a natural almost complex structure $J_1$, which was shown by Atiyah, Hitchin and Singer [AHS] to be integrable if $N$ is $\mp$ selfdual. Our parametrization involves a different almost complex structure $J_2$ obtained from $J_1$ by reversing orientation along the fibres:

COROLLARY 5.4. *There is a bijective correspondence between nonconstant conformal harmonic maps* $\varphi\colon M \to N$ *and nonvertical* $J_2$ *holomorphic curves* $\psi\colon M \to \mathcal{S}_\pm$.

This correspondence is achieved by taking $\psi$ to be the natural « Gauss lift » of $\varphi$, whose value at a point is determined by the 1-jet of $\varphi$ at that point. Unlike $J_1$, the almost complex structure $J_2$ is never integrable, so its relevance may come as a surprise. However in homogeneous cases $J_2$ has made previous appearances, namely in the description of invariant almost complex structures by Borel and Hirzebruch [BH], and in the classification of 3-symmetric spaces by Wolf and Gray [WG].

Having obtained a twistorial description for all conformal harmonic maps, it is an easy matter to distinguish special classes. If $\psi$ is both $J_1$ and $J_2$ holomorphic, then it is horizontal, and as explained in section 6, its projection $\varphi$ is a real isotropic harmonic map. Such maps include Bryant's superminimal immersions in $S^4$, most of the known minimal surfaces in the complex projective plane $\mathbb{C}P^2$ [$\mathrm{EW}_2$], and by work of Micallef [M] many stable minimal surfaces in Euclidean space $\mathbb{R}^4$. Indeed, we explain that twistor methods are most valuable when the target manifold is selfdual and Einstein.

Spinor terminology is used for the first time in section 8, to introduce the twistor degrees of a conformal harmonic map $\varphi$ of a compact Riemann surface into a 4-manifold; these are used to relate analytical and topological properties of $\varphi$. The twistor bundle $\mathbb{C}P^3$ of $S^4$ is considered in detail in sections 9 and 10, first from the viewpoint of 3-symmetric spaces, and then as a Riemannian submersion. This enables us to obtain examples of harmonic maps into $\mathbb{C}P^3$, and more generally to introduce the theory of harmonic maps into Kähler manifolds.

This theory is in turn applied to study maps into Kähler surfaces, for which we pay special attention to the notions of real and complex isotropy.

Calculation of twistor degrees allows us to extend the validity of a formula of Eschenburg, Tribuzy and Guadalupe [ETG]. Conformal harmonic maps $M \to \mathbb{C}P^2$ are interpreted in terms of $J_2$ holomorphic curves in a flag manifold. This implies that such maps into $\mathbb{C}P^2$ really come in triples, and explains the existence of associated harmonic maps.

Applications of our techniques to 3-dimensional domains appear in [E$_2$], and to higher dimensional domains in [S$_3$]. There will also be analogous results for maps of surfaces into pseudoRiemannian manifolds. For background information concerning harmonic maps we recommend [EL], and for 4-dimensional Riemannian geometry [S$_2$].

Starting with the construction in [Br], Friedrich [F] studied $J_1$ holomorphicity of Gauss lifts into the twistor space of a 4-manifold, with examples.

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## 1. – Harmonic maps.

Throughout $M$ denotes a Riemann surface, i.e. a connected complex 1-dimensional manifold. If $z = x + iy$ is a local complex coordinate on $M$, then

$$\partial/\partial z = \tfrac{1}{2}(\partial/\partial x - i\,\partial/\partial y)$$
$$\partial/\partial \bar{z} = \tfrac{1}{2}(\partial/\partial x + i\,\partial/\partial y)$$

span respectively the space $T^{1,0}M$ of complexified tangent vectors of type $(1,0)$, and the conjugate space $T^{0,1}M$ of $(0,1)$ vectors. The complex structure of $M$ is equally well defined by a conformal class of Riemannian metrics together with an orientation, in accordance with the isomorphism $GL(1,\mathbb{C}) \cong \mathbb{R}^+ \times SO(2)$. A Riemannian metric $g$ belongs to the conformal class of $M$ iff

$$g(\partial/\partial x, \partial/\partial y) = 0\,, \qquad g(\partial/\partial x, \partial/\partial x) = g(\partial/\partial y, \partial/\partial y)\,.$$

Letting $g$ also denote the complex symmetric extension of the metric, conformality is therefore equivalent to the vanishing of the quadratic differential

$$g^{2,0} = g(\partial/\partial z, \partial/\partial z)\,dz^2\,.$$

Let $N$ be a Riemannian manifold with metric $h$, and consider a mapping $\varphi: M \to N$. Its differential $\varphi_*: TM \to TN$ defines a section

$$d\varphi \in \Gamma(M, T^*M \otimes \varphi^{-1}TN)\,,$$

where $\varphi^{-1}TN$ denotes the pulled-back bundle. Fixing a compatible metric $g$ on $M$, and using $\nabla$ to denote the Levi-Civita connections on both $TM$ and $\varphi^{-1}TN$, enables one to compute the tensor $\nabla d\varphi$. To this end, take a complex coordinate $z$ on an open set of $M$ and let

$$\delta = \nabla_{\partial/\partial z}\,, \qquad \bar{\delta} = \nabla_{\partial/\partial \bar{z}}$$

be the corresponding covariant derivatives acting on $\varphi^{-1}TN$. For completeness we also define $\delta\varphi$, $\bar{\delta}\varphi$ by the formula

$$d\varphi = dz \otimes \delta\varphi + d\bar{z} \otimes \bar{\delta}\varphi\,.$$

In the sequel we shall frequently identify the fibres of $\varphi^{-1}TN$ and $TN$ in order to write $\delta\varphi = \varphi_*(\partial/\partial z)$, $\bar{\delta}\varphi = \varphi_*(\partial/\partial\bar{z})$.

Because the Levi-Civita connection on $TN$ has zero torsion,

$$\delta\bar{\delta}\varphi - \bar{\delta}\delta\varphi = [\varphi_*\,\partial/\partial z, \varphi_*\,\partial/\partial\bar{z}] = \varphi_*[\partial/\partial z, \partial/\partial\bar{z}] = 0\,,$$

and the quantity $\delta\bar{\delta}\varphi = \bar{\delta}\delta\varphi$ is real. Moreover on $M$, $\nabla dz$ is proportional to $dz \otimes dz = dz^2$, so

$$\nabla d\varphi = (dz \vee d\bar{z}) \otimes \delta\bar{\delta}\varphi + \text{terms in } dz^2, d\bar{z}^2$$

is a section of $S^2T^*M \otimes \varphi^{-1}TN$. If $tr$ denotes contraction with $g$, the tensor $\tau_\varphi = tr(\nabla d\varphi)$ with values in $\varphi^{-1}TN$ is called the *tension field* and represents an invariantly defined Laplacian of the mapping $\varphi$. Thus $\varphi$ is said to be *harmonic* if $\tau_\varphi = 0$ on $M$, or equivalently if $\delta\bar{\delta}\varphi = 0$. In particular this notion depends solely on the conformal class of $M$.

If $\varphi: M \to (N, h)$ is harmonic,

$$\partial/\partial\bar{z}\, h(\delta\varphi, \delta\varphi) = 2h(\bar{\delta}\delta\varphi, \delta\varphi) = 0\,, \tag{1.1}$$

so $(\varphi^*h)^{2,0} = h(\delta\varphi, \delta\varphi)\, dz^2$ is a holomorphic quadratic differential. Now we shall call a mapping $\varphi$ which satisfies $(\varphi^*h)^{2,0} = 0$ *conformal*; this means that away from the zeros of $\varphi_*$, the pullback $\varphi^*h$ is a Riemannian metric compatible with the conformal structure of $M$. For example since a Riemann surface of genus 0 admits no holomorphic differentials, any harmonic map

$\varphi\colon S^2 \to N$ with domain the 2-sphere is automatically conformal. Similarly if $\varphi\colon M \to N$ is harmonic with the restriction $\varphi|_{\mathcal{U}}$ conformal for some nonempty open subset $\mathcal{U} \subset M$, then $\varphi$ is conformal on all of $M$.

Now suppose that $\varphi$ is a conformal immersion, and work exclusively with the induced metric $g = \varphi^* h$ on $M$. In this case for any vector field $X$ on $M$,

$$\nabla(\varphi_* X) = \varphi_*(\nabla X) + \nabla^{\perp}(\varphi_* X)\,,$$

where $\nabla^{\perp}$ denotes the covariant derivative on $TN$ followed by projection to the orthogonal complement $(\varphi_* TM)^{\perp}$ in $TN$. It follows that

$$(\nabla d\varphi)(X) = \nabla^{\perp}(\varphi_* X)\,,$$

and $\nabla d\varphi$ can be identified with the second fundamental form whose trace $\tau_\varphi$ equals the mean curvature of the immersion. *A nonconstant conformal harmonic map $\varphi\colon M \to (N, h)$ is then the same thing as a minimal branched immersion.* Indeed the zeros of $\varphi_*$ are isolated and if $z$ is chosen so that $z = 0$ is one of them, there exist local coordinates $x_1, \ldots, x_n$ on $N$ for which $\varphi$ is given by

$$x^1 + ix^2 = cz^k + o(|z|^k)$$

$$x^r = o(|z|^k)\,, \qquad 3 \leqslant r \leqslant n\,.$$

[GOR]. This fact will enable us to handle the zeros of the differential of $\varphi$ without special problems. Any holomorphic or antiholomorphic map of a Riemann surface into a Kähler manifold is easily seen to be both conformal and harmonic. We will see two important generalizations of this in the sequel (proposition 3.3, theorem 5.3).

Some additional definitions can be given under the continued assumption that $\varphi\colon M \to (N, h)$ is conformal and $g = \varphi^* h$. First $\varphi$ is said to have *constant mean curvature* if $\nabla^{\perp}\tau_\varphi = 0$. The condition complementary to $\tau_\varphi = 0$ is that $\nabla d\varphi$ have no trace-free component; in this case $\varphi$ is said to be *totally umbilic*. Using $h(\delta^2\varphi, \delta\varphi) = 0$, this is equivalent to the equation $\delta\varphi \wedge \delta^2\varphi = 0$. Finally $\varphi$ is *totally geodesic* if $\nabla d\varphi = 0$.

## 2. – Gauss lifts.

Let $\tilde{G}_2(\mathbb{R}^n)$ denote the Grassmannian of real oriented 2-dimensional subspaces of $\mathbb{R}^n$. Each $V \in \tilde{G}_2(\mathbb{R}^n)$ may be identified with the simple 2-vector $\sigma = e_1 \wedge e_2$, where $\{e_1, e_2\}$ is any oriented orthonormal basis of $V$. Therefore

$$\tilde{G}_2(\mathbb{R}^n) = \{\sigma \in \Lambda^2(\mathbb{R}^n)\colon \|\sigma\| = 1,\ \sigma \text{ simple}\}\,.$$

Alternatively, one can associate to $V$ the complex projective class $[v] \in \mathbb{C}P^{n-1}$, where $v = e_1 + ie_2 \in \mathbb{C}^n$. If $h$ denotes the complex symmetric metric on $\mathbb{C}^n$, then $h(v, v) = 0$, and this construction identifies $\tilde{G}_2(\mathbb{R}^n)$ with the quadric hypersurface

$$Q_{n-2} = \left\{\sum_0^{n-1} z_i^2 = 0\right\} \subset \mathbb{C}P^{n-1} .$$

The action of $SO(n)$ on $\mathbb{R}^n$ gives a Riemannian symmetric space description

$$\tilde{G}_2(\mathbb{R}^n) \cong Q_{n-2} \cong \frac{SO(n)}{SO(2)\times SO(n-2)} .$$

Any immersion $\varphi\colon M \to \mathbb{R}^n$ defines a Gauss map

$$\gamma_\varphi\colon M \to \tilde{G}_2(\mathbb{R}^n) ,$$

where $\gamma_\varphi(m)$ is the real 2-plane $\varphi_*(T_m M)$ translated to the origin. If $\varphi$ is conformal, then $h(\partial\varphi/\partial\bar{z}, \partial\varphi/\partial\bar{z}) = 0$ in $\mathbb{C}^n$, and in terms of the quadric identification $\gamma_\varphi$ is the projective class $[\partial\varphi/\partial\bar{z}]$. It follows that a conformal immersion $\varphi\colon M \to \mathbb{R}^n$ is harmonic iff its Gauss map $\gamma_\varphi\colon M \to Q_{n-2}$ is antiholomorphic, a result due to Chern [Ch$_1$]. More generally a theorem of Ruh-Vilms [RV] states that a conformal immersion $\varphi\colon M \to \mathbb{R}^n$ has constant mean curvature iff $\gamma_\varphi$ is harmonic.

For an arbitrary manifold $N$, there is no way of associating a Gauss map in the traditional sense to an immersion $\varphi\colon M \to N$. However there is a related concept involving the Grassmann bundle $\tilde{G}_2(TN)$ over $N$ whose fibre at $x \in N$ is the space $\tilde{G}_2(T_x N)$ of real oriented 2-subspaces in $T_x N$. For each $m \in M$ with $\varphi(m) = x$, the oriented subspace $\varphi_*(T_m M)$ is an element of $\tilde{G}_2(T_x N)$, and in this way we obtain the *Gauss lift*

$$\tilde{\varphi}\colon M \to \tilde{G}_2(TN) \qquad \text{of } \varphi .$$

Obviously $\pi\circ\tilde{\varphi} = \varphi$, where $\pi$ is the bundle projection. In the case $N = \mathbb{R}^n$ there is a canonical isomorphism $\tilde{G}_2(TN) \cong \mathbb{R}^n \times \tilde{G}_2(\mathbb{R}^n)$, and $\gamma_\varphi = \pi_2\circ\tilde{\varphi}$, where $\pi_2$ denotes projection to the second factor.

When $N$ is 3-dimensional, $\tilde{G}_2(TN)$ can be identified, via orthogonal complementation, with the sphere bundle $S(TN)$ of unit tangent vectors. In fact

$$\tilde{G}_2(\mathbb{R}^3) \cong \frac{SO(3)}{SO(2)} \cong S^2 ,$$

and there is the star operator

$$*: \Lambda^2(\mathbb{R}^3) \xrightarrow{\cong} \Lambda^1(\mathbb{R}^3)\,.$$

Now in 4 dimensions the star operator defines an endomorphism of $\Lambda^2(\mathbb{R}^4)$ with $*^2 = 1$ and $\pm 1$-eigenspaces $\Lambda^2_\pm$ say. If $\{e_1, e_2, e_3, e_4\}$ is an oriented orthonormal basis of $\mathbb{R}^4$ then (with appropriate conventions) $\Lambda^2_\pm$ has an oriented orthonormal basis

$$\{e_1\wedge e_2 \pm e_3\wedge e_4,\ e_1\wedge e_3 \pm e_4\wedge e_2,\ e_1\wedge e_4 \pm e_2\wedge e_3\}\,.$$

The action of $SO(4)$ on each eigenspace gives rise to a double covering

$$SO(4) \to SO(3)\times SO(3)\,, \tag{2.1}$$

and

$$\tilde{G}_2(\mathbb{R}^4) \cong \frac{SO(4)}{SO(2)\times SO(2)} \cong \frac{SO(3)}{SO(2)} \times \frac{SO(3)}{SO(2)} \cong S(\Lambda^2_+)\times S(\Lambda^2_-)$$

is a product of spheres.

Now let $N$ be an oriented Riemannian 4-manifold. Associated to the principal $SO(4)$-bundle of oriented orthonormal frames is a vector bundle for each representation of $SO(4)$. In particular the eigenspaces of $*$ give rise to a decomposition

$$\Lambda^2 TN = \Lambda^2_+ TN \oplus \Lambda^2_- TN\,; \tag{2.2}$$

let

$$\mathcal{S}_\pm = S(\Lambda^2_\pm TN)$$

be the corresponding 2-sphere bundles of unit vectors. Then fibrewise $\tilde{G}_2(TN)$ is the product of $\mathcal{S}_+$ with $\mathcal{S}_-$, and there are projections

$$p_\pm\colon \tilde{G}_2(TN) \to \mathcal{S}_\pm\,.$$

We define subsidiary Gauss lifts

$$\tilde{\varphi}_\pm = p_\pm \circ \tilde{\varphi}\,.$$

Let $F \in \mathcal{S}_+$ with $\pi(F) = x \in N$. By choosing an appropriate oriented orthonormal basis $\{e_1, e_2, e_3, e_4\}$ of $T_xN$, it is possible to write

$F = e_1 \wedge e_2 + e_3 \wedge e_4$. Then under the isomorphism

$$\Lambda^2 TN \hookrightarrow TN \otimes TN \cong T^* N \otimes TN = \mathrm{End}\,(TN)$$

defined by the Riemannian metric $h$, $F$ corresponds to the almost complex structure $J$ on $T_x N$ with

$$Je_1 = e_2\,, \quad Je_3 = e_4\,, \quad J^2 = -1\,.$$

The 2-vector $F$ is in fact dual to the so-called fundamental 2-form

$$\omega(X, Y) = h(JX, Y)\,, \quad X, Y \in T_x N\,.$$

The same argument shows that the disjoint union

$$(\mathcal{S}_+)_x \cup (\mathcal{S}_-)_x \cong \frac{O(4)}{U(2)}$$

parametrizes all the almost complex structures on $T_x N$ compatible with $h$, i.e. such that $h(JX, JY) = h(X, Y)$. Those in $\mathcal{S}_+$ are oriented consistently with $N$, those in $\mathcal{S}_-$ are oriented contrariwise.

Suppose that $\varphi: M \to N$ is a conformal immersion. Fix $m \in M$ with $x = \varphi(m)$, and choose a complex coordinate $z$ on $M$ such that

$$\delta\varphi(m) = \varphi_*(\partial/\partial z) = e_1 - ie_2\,,$$

where $e_1, e_2$ extend to an oriented orthonormal basis $\{e_1, e_2, e_3, e_4\}$ of $T_x N$. Then

$$\tilde{\varphi}_\pm(m) = e_1 \wedge e_2 \pm e_3 \wedge e_4 = \frac{1}{2i}(1 \pm *)(\delta\varphi \wedge \bar{\delta}\varphi)\,, \tag{2.3}$$

where $\frac{1}{2}(1 \pm *)$ equals the projection to $\Lambda^2_\pm TN$. The almost complex structures corresponding to $\tilde{\varphi}_\pm(m)$ are uniquely determined by the requirement that $\varphi_*(m)$ be complex linear. Using the word holomorphic for this pointwise property, we have

PROPOSITION 2.1. *A conformal immersion $\varphi: M \to N$ is holomorphic with respect to both of its Gauss lifts $\tilde{\varphi}_+, \tilde{\varphi}_-$.*

EXAMPLE. To investigate the Gauss lifts of a map into the sphere $S^4$, it is convenient to identify $S^4$ with the quaternionic projective line $\mathbb{H}P^1$

as follows. The group $Sp(2)$ of $2\times 2$ quaternionic unitary matrices acts naturally on the space $\mathbb{H}^2$ of column vectors; let $U$ denote the underlying complex 4-dimensional vector space. Then $Sp(2)$ leaves invariant a skew form $\omega\in\Lambda^2 U$, and its action on the orthogonal complement of $\omega$ in $\Lambda^2 U$ defines a 2:1 homomorphism $Sp(2)\to SO(5)$. Restricting to the subgroup of diagonal matrices defines another double covering

$$Sp(1)\times Sp(1)\to SO(4) \tag{2.4}$$

that coincides with (2.1) at the Lie algebra level (for more details see for example [$S_2$]). Since $Sp(1)\times Sp(1)$ is the isotropy for the transitive action of $Sp(2)$ on $\mathbb{H}P^1$, there is an isomorphism

$$S^4=\frac{SO(5)}{SO(4)}\cong\frac{Sp(2)}{Sp(1)\times Sp(1)}=\mathbb{H}P^1$$

of Riemannian symmetric spaces.

Using (2.1) and (2.4) it is now an easy matter to identify

$$\begin{cases}\tilde{G}_2(TS^4)\cong\dfrac{SO(5)}{SO(2)\times SO(2)}\cong\dfrac{Sp(2)}{U(1)\times U(1)}\\ \mathcal{S}_+\cong\dfrac{SO(5)}{U(2)}\cong\dfrac{Sp(2)}{U(1)\times Sp(1)},\end{cases} \tag{2.5}$$

and these isomorphisms are compatible with the projection $p_+$: Furthermore the isotropy subgroup $Sp(1)\times U(1)$ of $Sp(2)$ defining $\mathcal{S}_-$ is conjugate to $U(1)\times Sp(1)$, so $\mathcal{S}_+$ and $\mathcal{S}_-$ are isomorphic as homogeneous spaces. Indeed both are isomorphic to the complex projective space $P(U)=\mathbf{C}P^3$.

In addition to $p_\pm$: $\tilde{G}_2(TS^4)\to\mathcal{S}_\pm$, there are distinct projections $p_1, p_2$ of $\tilde{G}_2(TS^4)$ to

$$\tilde{G}_2(\mathbb{R}^5)\cong\frac{SO(5)}{SO(2)\times SO(3)}\cong\frac{Sp(2)}{U(2)},$$

such that $p_1\circ\tilde{\varphi}$ is the Gauss map $\gamma_\Phi$ of the composition

$$\Phi\colon M\xrightarrow{\varphi}S^4\hookrightarrow\mathbb{R}^5,$$

and $p_2\circ\tilde{\varphi}$ is the Obata normal Gauss map $*(\Phi\wedge\gamma_\Phi)$ [O, $E_1$]. Since $\mathbf{C}P^3$ and $\tilde{G}_2(\mathbb{R}^5)\cong Q_3$ are complex 3-manifolds with certain similarities (for example the same additive cohomology) one might expect the Gauss lifts $\tilde{\varphi}_+, \tilde{\varphi}_-$

and the Gauss map $\gamma_\Phi$ to have similar properties. If $\varphi$ is conformal and harmonic, then $\Phi$ is pseudo-umbilic with constant mean curvature which implies that $\gamma_\Phi$ is also conformal and harmonic (for more details see [O, RV]). We shall see that the same is true for $\tilde{\varphi}_+$, $\tilde{\varphi}_-$ (corollary 9.2).

## 3. – Almost Hermitian manifolds.

In this section we review some standard facts concerning $U(n)$-structures, but from a new point of view which will be developed in the sequel.

Let $N$ be an almost Hermitian manifold of real dimension $2n$. This means that $N$ has a Riemannian metric $h$ and an almost complex structure $J$ satisfying

$$h(JX, JY) = h(X, Y)\,, \qquad X, Y \in TN\,,$$

and the tensors $h$ and $J$ taken together reduce the structure group of the tangent bundle $TN$ to $SO(2n) \cap GL(n, \mathbb{C}) = U(n)$. As is customary, we write

$$(TN)^{\mathbb{C}} = T^{1,0}N \oplus T^{0,1}N$$

where $T^{1,0}N = \{X - iJX \colon X \in TN\}$ is the $i$-eigenbundle of $J$, and $T^{0,1}N = \overline{T^{1,0}N}$ its conjugate. Then $h$ has type $(1, 1)$, so $T^{1,0}N$ is totally isotropic and $\alpha \to h(\alpha, \cdot)$ defines an isomorphism

$$T^{1,0}N \cong (T^{0,1}N)^*\,. \tag{3.1}$$

If $\{\alpha_1, \dots, \alpha_n\}$ is a unitary basis of $T^{1,0}N$ so that $h(\alpha_k, \bar{\alpha}_l) = \delta_{kl}$, we shall call

$$F = -i \sum_1^n \alpha_k \wedge \bar{\alpha}_k$$

the *fundamental* 2-*vector* of $N$; it is invariantly defined and dual to the 2-form $\omega(X, Y) = h(JX, Y)$.

For the type decomposition of the exterior algebra of $TN$, we use the notation

$$(\Lambda^r TN)^{\mathbb{C}} = \bigoplus_{p+q=r} T^{p,q}N$$

where

$$T^{p,q}N \cong \Lambda^p(T^{1,0}N) \otimes \Lambda^q(T^{0,1}N)\,.$$

Note that $T^{p,p}N$ and $T^{p,q}N \oplus T^{q,p}N$ are both (the complexifications of) real vector bundles. Let $\nabla$ denote the Levi-Civita connection on $TN$, uniquely determined by the Riemannian metric $h$. Fix a real vector $X \in T_x N$, and a unitary basis $\{\alpha_k\}$ of $T^{1,0}N$ in a neighbourhood of $x$. If we set

$$h(\nabla_X \alpha_k, \bar{\alpha}_l) = \lambda_{kl} = -h(\alpha_k, \nabla_X \bar{\alpha}_l)\,,$$

then

$$\nabla_X F = -i \sum_k [(\nabla_X \alpha_k)\wedge \bar{\alpha}_k + \alpha_k \wedge (\nabla_X \bar{\alpha}_k)] \tag{3.2}$$

has $(1,1)$-component $\sum_{l,k} [\lambda_{lk}\alpha_l \wedge \bar{\alpha}_k - \lambda_{kl}\alpha_k \wedge \bar{\alpha}_l] = 0$. Thus

PROPOSITION 3.1. *For any* $X \in TN$, $\nabla_X F \in T^{2,0}N \oplus T^{0,2}N$.

The tensor $\nabla F$ is a convenient measure of the torsion of the unitary structure on $N$. More precisely if $\tilde{\nabla}$ is any $U(n)$-connection on $TN$ then $\tilde{\nabla} F = 0$, and $\nabla F = (\nabla - \tilde{\nabla})F$ can be identified with an invariant component of the torsion of $\tilde{\nabla}$. By proposition 3.1, at each point $\nabla F$ belongs to

$$(T^{2,0}N \oplus T^{0,2}N) \otimes (T^* N)^{\mathbf{C}} = \mathfrak{S}_1 \oplus \mathfrak{S}_2\,,$$

where

$$\mathfrak{S}_1 \cong (T^{2,0}N \otimes T^{1,0}N) \oplus (T^{0,2}N \otimes T^{0,1}N)$$

and $\mathfrak{S}_2 \cong T^{2,1}N \oplus T^{1,2}N$. Here we have made use of the isomorphism (3.1). Set

$$\nabla F = D_1 F + D_2 F\,, \qquad D_a F \in \mathfrak{S}_a\,. \tag{3.3}$$

Then $D_1 F$ and $D_2 F$ represent the irreducible real components of $\nabla F$ relative to $GL(n, \mathbf{C})$.

THEOREM 3.2. $D_1 F = 0$ *iff the almost complex structure* $J$ *is integrable, whereas* $D_2 F = 0$ *iff the 3-form* $d\omega$ *has no component of type* $(1, 2)$.

PROOF. Take a local unitary basis $\{\alpha_k\}$ of $T^{1,0}N$ and set

$$\mu_{jkl} = h(\nabla_{\alpha_j}\alpha_k, \alpha_l)\,,$$

so that $\mu_{jkl} = -\mu_{jlk}$. Replacing $X$ by $\alpha_j$ in (3.2) gives

$$\begin{aligned} D_1 F = 0 &\Leftrightarrow \nabla_{\alpha_j}\alpha_k \in T^{1,0}\,, \qquad \forall j, k \\ &\Leftrightarrow \mu_{jkl} = 0\,, \qquad \forall j, k, l\,. \end{aligned}$$

On the other hand, by the Newlander-Nirenberg theorem [NN], $J$ is integrable iff $T^{1,0}N$ is closed under Lie bracket. This is the case iff

$$h([\alpha_j, \alpha_k], \alpha_l) = h(\nabla_{\alpha_j}\alpha_k - \nabla_{\alpha_k}\alpha_j, \alpha_l) = \mu_{jkl} - \mu_{kjl}$$

vanishes for all $j, k, l$. Therefore $D_1 F = 0$ implies that $J$ is integrable. Conversely if $J$ is integrable,

$$\mu_{jkl} = \mu_{kjl} = -\mu_{klj} = -\mu_{lkj} = \mu_{ljk} = \mu_{jlk} = -\mu_{jkl}$$

vanishes, and $D_1 F = 0$.

As for $d\omega$, we have

$$d\omega(\alpha_j, \alpha_k, \bar{\alpha}_l) = (\nabla_{\alpha_j}\omega)(\alpha_k, \bar{\alpha}_l) + (\nabla_{\alpha_k}\omega)(\bar{\alpha}_l, \alpha_j) + (\nabla_{\bar{\alpha}_l}\omega)(\alpha_j, \alpha_k)$$
$$= (\nabla_{\bar{\alpha}_l}\omega)(\alpha_j, \alpha_k) .$$

Now $\omega$ is dual to $F$ with respect to the covariant constant metric $h$, so $(\nabla_{\bar{\alpha}_l}\omega)(\alpha_j, \alpha_k)$ equals the component of $\nabla_{\bar{\alpha}_l}F$ parallel to $\alpha_j \wedge \alpha_k$. It follows that $D_2 F$ can be identified with the real part of $(d\omega)^{1,2}$. $\square$

The significance of the vanishing of $D_1 F$ needs no comment. Now if the non-degenerate 2-form $\omega$ is closed, $N$ is called symplectic; accordingly when $D_2 F = 0$ we shall say that $N$ is $(1, 2)$-*symplectic*. Observe however that when $N$ is 4-dimensional, the prefix $(1, 2)$ is redundant. By definition $N$ is a Kähler manifold iff $\nabla F = 0$; this means that the (restricted linear) holonomy group lies in $U(n)$. The expression « quasi-Kähler » has been used (for instance in [WG]) for $(1, 2)$-symplectic, but this terminology is somewhat contrary to our viewpoint in which Kähler should be thought of as the intersection of complex and $(1, 2)$-symplectic.

In order to demonstrate the relevance of $D_2 F$ to the theory of harmonic mappings we first recall a definition. A map $\varphi\colon (M, J^M) \to (N, J^N)$ between almost complex manifolds is said to be *holomorphic* if its differential is complex linear, i.e. $\varphi_* \circ J^M = J^N \circ \varphi_*$. This is the same as saying that $\varphi_*$ preserves types, but does not require integrability. The following result is due to Lichnerowicz [Li, section 16] (see also Gray [$G_1$]).

PROPOSITION 3.3. *Let $\varphi\colon M \to N$ be a holomorphic map from a Riemann surface to an almost Hermitian $(1, 2)$-symplectic manifold. Then $\varphi$ is harmonic.*

PROOF. Take a local coordinate $z$ on $M$ and a local unitary basis $\{\alpha_k\}$ of $T^{1,0}N$ and set $\delta\varphi = \varphi_*(\partial/\partial z) = \lambda^j \alpha_j$ (summation). Replacing $X$ by $\bar{\alpha}_j$ in (3.2),

$$D_2 F = 0 \iff \nabla_{\bar{\alpha}_j}\alpha_k \in T^{1,0}N , \quad \forall j, k .$$

Hence $D_2 F = 0$ implies that

$$\bar{\delta}\delta\varphi = \bar{\lambda}^j \nabla_{\bar{\alpha}_j}(\lambda^k \alpha_k)$$

belongs to $T^{1,0}N$. Since $\bar{\delta}\delta\varphi = \delta\bar{\delta}\varphi$ is real, it must vanish. $\square$

EXAMPLE. Consider the manifold $N = S^1 \times S^3$ with the product metric. There is a Riemannian fibration $p\colon N \to S^2$ formed by following the projection to $S^3$ by the Hopf fibration $h\colon S^3 \to S^2$. Let $X$ be a unit vector field on $S^1$ and $Y$ a unit vertical field relative to $h$. An orthogonal almost complex structure $J = (J^h, J^v)$ can then be defined on $N$ by letting $J^h$ be the horizontal lift of the complex structure on $S^2$, and setting $J^v(X) = Y$, $J^v(Y) = -X$. Then $J$ is integrable and gives $N$ the structure of a Hopf surface; $p$ is a holomorphic fibration whose fibres are elliptic curves [Be, Exp. VII].

Now $N$ cannot admit a Kähler metric, so the above structure is not (1, 2)-symplectic. Despite this every holomorphic map $M \to (S^1 \times S^3, J)$ goes into a fibre of $p$ [K]; and is therefore conformal and harmonic.

On the other hand, an example of A. Gray [EL, § 9.11] shows that the hypotheses of proposition 3.3 cannot be weakened. More to the point, when $\dim N \geqslant 6$ maps satisfying the hypotheses of proposition 3.3 will not generally minimize energy. (See the remark after corollary 9.2.)

Suppose finally that the almost Hermitian manifold $N$ has real dimension 4. If $T_0^{1,1}N$ denotes the space of *primitive* (1, 1) vectors, i.e. those orthogonal to the fundamental 2-vector $F$, then

$$\Lambda^2 TN = (T^{2,0}N \oplus T^{0,2}N) \oplus \mathbb{R}F \oplus T_0^{1,1}N\,.$$

At each point the right hand side is the direct sum of three real $U(2)$-modules of dimension 2, 1, 3 respectively. But at the same time there is the direct sum (2.2) relative to the larger group $SO(4)$. Choosing the standard orientation for $N$ so that $F$ is a section of $\Lambda_+^2 TN$, we must have

PROPOSITION 3.4.

$$\Lambda_+^2 TN = (T^{2,0}N \oplus T^{0,2}N) \oplus \mathbb{R}F$$
$$\Lambda_-^2 TN = T_0^{1,1}N\,.$$

The Levi-Civita connection certainly preserves the subbundle $\Lambda_+^2 TN$ of $\Lambda^2 TN$, so for any $X \in TN$, $\nabla_X F \in \Lambda_+^2 TN$. Furthermore the fact that $F$ has constant norm implies that $\nabla_X F$ is orthogonal to $F$. In 4 dimensions one therefore recovers proposition 3.1 from proposition 3.4.

## 4. – Twistor spaces.

The results of the last section can be understood more fully by considering the sphere bundles $\mathcal{S}_+$, $\mathcal{S}_-$ over an oriented Riemannian 4-manifold $N$, no longer assumed to admit a global almost complex structure. For ease of notation and consistency with [AHS] we concentrate on $\mathcal{S}_- \subset \Lambda^2_- TN$, although with a change of orientation everything will hold for $\mathcal{S}_+$. Suppose that $\mathcal{U}$ is an open set of $N$, and that $s\colon \mathcal{U} \to \mathcal{S}_-$ is a smooth section. Reserving the symbol $s$ for the mapping, let $\sigma$ denote the corresponding 2-form defined on $\mathcal{U}$. Thus $\sigma$ is the fundamental 2-vector of some almost complex structure $J$ on $\mathcal{U}$, relative to the Riemannian metric $h$. We shall relate the geometry of the submanifold $s(\mathcal{U})$ in $\mathcal{S}_-$ with properties of the almost Hermitian manifold $(\mathcal{U}, h, J)$.

Fix $x \in \mathcal{U}$, and consider the tangent space $T_y\mathcal{S}_-$ to $\mathcal{S}_-$ at $y = s(x)$. This has a distinguished subspace $V_y$ consisting of vertical vectors, that is those tangent to the fibre $(\mathcal{S}_-)_x$. The latter is a 2-sphere, and its tangent space at $y$ is the orthogonal complement of the corresponding 2-vector in $\Lambda^2_- T_xN$. From proposition 3.4 with signs reversed, we obtain an isomorphism

$$T^{2,0}_x N \oplus T^{0,2}_x N \xrightarrow{\cong} (V_y)^{\mathbf{C}} \tag{4.1}$$

which we denote by $\alpha \to \alpha^v$. The type decomposition in (4.1) is relative to the almost complex structure $J$ on $\mathcal{U}$, but in fact depends only on the point $y$ that represents the value of $J$ at $x$. Consequently there exist well defined vertical subbundles $(T^{2,0})^v$, $(T^{0,2})^v$ of $(T\mathcal{S}_-)^{\mathbf{C}}$ which detect the complex structure of each fibre $S^2 \cong \mathbf{C}P^1$.

Proposition 3.1 expresses the fact that the Levi-Civita connection $\nabla$ reduces to the bundle $\mathcal{S}_-$. Given $X \in T_x\mathcal{U}$, the vector

$$X^h = s_* X - (\nabla_X \sigma)^v \in T_{s(x)}\mathcal{S}_- \tag{4.2}$$

depends only on $x$ and $s(x)$, not on neighbouring values of $s$. Indeed

$$H_{s(x)} = \{X^h\colon X \in T_xN\}$$

is the so-called horizontal subspace which is defined by $\nabla$ at any $y \in \mathcal{S}_-$, and $\nabla_X\sigma$ represents the vertical component of $s_*X$ in accordance with the direct sum

$$T_y\mathcal{S}_- = H_y \oplus V_y\,. \tag{4.3}$$

For any $y \in \pi^{-1}(x)$, $X \to X^h = (\pi_*|_{H_y})^{-1}X$ defines an isomorphism $T_xN \cong H_y$, and so a subspace $(T_x^{1,0})^h$ of $(H_y)^{\mathbf{C}}$ consisting of (1, 0)-vectors relative to $y$. In this way we obtain a horizontal subbundle $(T^{1,0})^h$ of $(T\mathcal{S}_-)^{\mathbf{C}}$.

Using (4.2) it is now possible to define *two very distinct almost complex structures* $J_1, J_2$ *on the total space* $\mathcal{S}_-$. It suffices to give the respective bundles of (1, 0)-vectors which are

$$T^{1,0}\mathcal{S}_- = \begin{cases} (T^{1,0})^h \oplus (T^{2,0})^v & \text{for } J_1 \\ (T^{1,0})^h \oplus (T^{0,2})^v & \text{for } J_2 . \end{cases} \tag{4.4}$$

In other words at $y \in \mathcal{S}_-$, $J_1$ and $J_2$ consist of the direct sum of the almost complex structure on $H_y \cong T_xN$ defined by $y$, and plus or minus a standard almost complex structure on $V_y$. The section $s$ now becomes a mapping between two almost complex manifolds, namely $(\mathcal{U}, J)$ and $(\mathcal{S}_-, J_a)$ for $a = 1$ or 2.

PROPOSITION 4.1. $s: (\mathcal{U}, J) \to (\mathcal{S}_-, J_a)$ *is holomorphic iff* $D_a\sigma = 0$.

PROOF. If $\alpha \in T\mathcal{U}$ is a (1, 0) vector relative to $J$, then on $s(\mathcal{U})$ the horizontal vector $\alpha^h$ is automatically of type (1, 0) relative to both $J_1$ and $J_2$. Thus $s$ is holomorphic iff $(\nabla_\alpha\sigma)^v = \alpha^h - s_*\alpha$ has type (1, 0) relative to $J_a$. The result follows from the definitions (3.3) and (4.4). □

For brevity we shall call a holomorphic map into $(\mathcal{S}_-, J_a)$ $J_a$ *holomorphic*. If the section $s$ above is $J_2$ holomorphic and the induced almost complex structure $J$ is integrable, then theorem 3.2 and proposition 4.1 imply that $\nabla\sigma = 0$ and $s$ is horizontal. As a consequence, we can conclude that $(\mathcal{S}_-, J_2)$ *is never a complex manifold* [$S_3$, proposition 3.4.]

To sum up: the almost complex structures $J_1, J_2$ are characterized by certain non-linear differential operators $D_1, D_2$ acting on sections of $\mathcal{S}_-$: The relationship between $D_1$ and $D_2$, while not at all obvious in theorem 3.2, corresponds to a reversal of the orientation of the fibres of $\mathcal{S}_-$.

Properties of $J_1$ and $J_2$ will depend on the Riemannian curvature tensor $R$ of $N$, which is a section of $S^2(\Lambda^2TN)$. Now

$$R = tA + B + W_+ + W_- , \tag{4.5}$$

where $t$ is the scalar curvature, $A$ is an invariant, $B$ represents the trace-free Ricci curvature, and $W_\pm \in S^2(\Lambda^2TN)$ are the two halves of the Weyl conformal curvature (we use the notation of [$S_2$]). The manifold $N$ is said to be *Einstein* if $B = 0$, and $\pm$ *selfdual* if $W_\mp = 0$. The curvature of $\mathcal{S}_-$ is then determined by $tA + B + W_-$, and guided by the Penrose twistor programme, Atiyah, Hitchin & Singer [AHS] showed that $J_1$ is integrable iff $W_- = 0$. In this case the holomorphic structure of $(\mathcal{S}_-, J_1)$ depends only on the conformal class of $h$, and $\mathcal{S}_-$ has become known as the twistor space of $N$. Equally important for us is the following result (essentially [$\mathrm{S}_2$, theorem 10.1]) which brings the Riemannian structure of $N$ more into the picture.

THEOREM 4.2. *Let $N$ be selfdual, so that $(\mathcal{S}_-, J_1)$ is a complex manifold. Then $(T^{1,0})^h$ is a complex analytic subbundle of $T^{1,0}\mathcal{S}_-$ iff $N$ is Einstein.*

Now suppose that $N$ is an Einstein selfdual 4-manifold. The resulting short exact sequence

$$0 \to (T^{1,0})^h \to T^{1,0}\mathcal{S}_- \to (T^{2,0})^v \to 0$$

defines a complex analytic 1-form on $\mathcal{S}_-$ with values in the line bundle $(T^{2,0})^v$. Provided $t \neq 0$, this makes $\mathcal{S}_-$ into a complex contact manifold. With the same hypotheses, Rawnsley [R, section 10] shows that there exists a Riemannian metric $k$ on $\mathcal{S}_-$ making $(\mathcal{S}_-, k, J_2)$ almost Hermitian and $(1, 2)$ symplectic. The most important cases are those in which $N$ is a Riemannian symmetric space, and these are covered by [WG, theorem 8.13] (see section 9).

PROBLEM. Let $\varphi\colon M \to (N, h)$ be a conformal map into a Riemannian 4-manifold. Bearing in mind the conformal invariance of $J_1$, when is there a conformal map $\varphi_1$ regularly homotopic to $\varphi$ and a metric $h_1$ conformally equivalent to $h$, with respect to which $\varphi_1$ is harmonic?

## 5. – $J_2$ holomorphicity.

From now on $N$ will denote an oriented Riemannian 4-manifold and $M$ a Riemann surface. In general $N$ will not admit a global almost complex structure, so it makes no sense to talk about holomorphic maps $\varphi\colon M \to N$. However associated to $N$ there are always the twistor spaces $\mathcal{S}_+$, $\mathcal{S}_-$, and an immersion $\varphi\colon M \to N$ produces the Gauss lifts $\tilde{\varphi}_\pm\colon M \to \mathcal{S}_\pm$ described in section 2. For example, at each point $x \in \varphi(M)$, $\tilde{\varphi}_-$ defines an almost complex structure on $T_xN$. Thus $\tilde{\varphi}_-$ gives rise to an almost complex structure on the fibres of $\varphi^{-1}TN$, and so a decomposition

$$(\varphi^{-1}TN)^{\mathbf{C}} = T^{1,0}_- \oplus T^{0,1}_-; \tag{5.1}$$

similarly for $\tilde{\varphi}_+$. If $\{e_1, e_2, e_3, e_4\}$ is a local oriented orthonormal basis of

$\varphi^{-1}TN$ with $e_1\wedge e_2$ spanning $\varphi(M)$, then

$$T^{1,0}_{+} = \mathbf{C}(e_1 - ie_2) \oplus \mathbf{C}(e_3 - ie_4) \tag{5.2}$$
$$T^{1,0}_{-} = \mathbf{C}(e_1 - ie_2) \oplus \mathbf{C}(e_3 + ie_4)\,.$$

We shall continue to work with $\mathcal{S}_-$, although all the results of this section are equally valid with all signs reversed.

LEMMA 5.1. *$\varphi^{-1}\mathcal{S}_-$ is naturally isomorphic to the complex projective bundle $P[T^{1,0}_{+}]$.*

PROOF. Given a 2-vector $\sigma \in \varphi^{-1}\mathcal{S}_-$, $\pi(\sigma) = m$, there exists an oriented orthonormal basis $\{e_1, e_2, e_3, e_4\}$ of $T^{1,0}_{+}N$ with

$$\tilde{\varphi}_+(m) = e_1\wedge e_2 + e_3\wedge e_4\,, \qquad \sigma = e_1\wedge e_2 - e_3\wedge e_4$$

(cf. (2.1)). Then $\tilde{\varphi}_+(m) + \sigma = 2e_1\wedge e_2$ represents a 2-plane in $T_{\varphi(m)}N$ invariant by the almost complex structure $\tilde{\varphi}_+(m)$. Identifying $\sigma$ with the projective class $[e_1 - ie_2] \in P(T^{1,0}_{+})$ gives the required isomorphism. $\square$

Given any map $\varphi\colon M \to N$, the vector bundle $(\varphi^{-1}TN)^{\mathbf{C}}$ over the Riemann surface $M$ has a natural complex analytic structure described by Koszul-Malgrange [KM]. Local complex analytic sections $s$ are characterized by the equation

$$\bar{\delta}s = \nabla_{\partial/\partial\bar{z}}s = 0\,. \tag{5.3}$$

Indeed the complex structure on the total space of $(\varphi^{-1}TN)^{\mathbf{C}}$ is obtained by « adding » the almost complex structures of the base and fibre, using the splitting determined by the Levi-Civita connection $\nabla$. This makes the zero section a complex submanifold and its normal bundle, isomorphic to $(\varphi^{-1}TN)^{\mathbf{C}}$ itself, is complex analytic. It follows from (5.3) and is well known that $\varphi$ is harmonic iff $\delta\varphi$ is a local complex analytic section of $(\varphi^{-1}TN)^{\mathbf{C}}$. The next result is an extension of this fact.

THEOREM 5.2. *A conformal immersion $\varphi\colon M \to N$ is harmonic iff $T^{1,0}_{+}$ is a complex analytic subbundle of $(\varphi^{-1}TN)^{\mathbf{C}}$.*

PROOF. Since $\varphi$ is conformal, $\delta\varphi$ spans $T^{1,0}_{+} \cap T^{1,0}_{-}$ (see (5.2)). Now $T^{1,0}_{+}$ is complex analytic iff it is closed under the application of $\bar{\delta}$. If this is the case, then $\bar{\delta}\delta\varphi \in T^{1,0}_{+}$, so by reality $\bar{\delta}\delta\varphi = 0$ and $\varphi$ is harmonic.

Conversely suppose that $\varphi$ is conformal harmonic, and take a local unitary basis $\{\alpha, \beta\}$ of $T^{1,0}_{+}$ with $\alpha = \delta\varphi/\|\delta\varphi\|$. Then both $\bar{\delta}\alpha$, $\bar{\delta}\beta$ belong to

$T_+^{1,0}$, the second because

$$h(\alpha, \bar{\delta}\beta) = -h(\bar{\delta}\alpha, \beta) = 0$$

$$h(\beta, \bar{\delta}\beta) = \tfrac{1}{2}\bar{\delta}h(\beta, \beta) = 0\,.$$

Thus $\bar{\delta}(T_+^{1,0}) \subset T_+^{1,0}$. $\square$

Given an immersion $\varphi$, there is a natural map

$$i\colon P(T_+^{1,0}) \cong \varphi^{-1}\mathcal{S}_- \to \mathcal{S}_-\,.$$

Moreover if $\varphi$ is conformal, $i$ maps the canonical section $[\delta\varphi]$ of $P(T_+^{1,0})$ onto the Gauss lift $\tilde{\varphi}_-$. The following result is then a re-interpretation of theorem 5.2:

THEOREM 5.3. *An immersion $\varphi\colon M \to N$ is conformal and harmonic iff $\tilde{\varphi}_-$ is $J_2$ holomorphic.*

PROOF. Let $\sigma \in \Gamma(M, \varphi^{-1}\Lambda_-^2 TN)$ be the fundamental 2-vector defined by $\tilde{\varphi}_-$; $\sigma$ has type $(1,1)$ relative to (5.1). Exactly as in proposition 3.1, we have

$$\delta\sigma = \nabla_{\partial/\partial z}\sigma \in T_-^{2,0} \oplus T_-^{0,2}\,,$$

where $T_-^{2,0} = \Lambda^2(T_-^{1,0})$, and in analogy to (3.3) we can write

$$\delta\sigma = \delta_1\sigma + \delta_2\sigma\,,$$

where $\delta_1\sigma = (\delta\sigma)_-^{0,2}$, $\delta_2\sigma = (\delta\sigma)_-^{2,0}$. Here and in the sequel $(\ )_-^{p,q}$ denotes the component of type $(p, q)$ relative to (5.1). Now

$$(\tilde{\varphi}_-)_*(\partial/\partial z) = (\varphi_*\,\partial/\partial z)^h + (\delta\sigma)^v$$

(cf. (4.2)), so as a combination of propositions 2.1, 4.1 we deduce that $\tilde{\varphi}_-$ is $J_a$ holomorphic iff (i) $\varphi_*$ is holomorphic relative to $\tilde{\varphi}_-$, and (ii) $\delta_a\sigma = 0$.

Condition (i) is satisfied iff $\varphi$ is conformal. Suppose this is the case. By (2.3),

$$\sigma = -ic(1 - *)(\delta\varphi \wedge \bar{\delta}\varphi)\,,$$

where $c$ is a positive normalizing factor. Hence

$$\delta\sigma = (c^{-1}\delta c)\,\sigma - ic(1 - *)(\delta^2\varphi \wedge \bar{\delta}\varphi + \delta\varphi \wedge \delta\bar{\delta}\varphi)\,,$$

and by proposition 3.4,

$$\begin{aligned}\delta_1\sigma &= -2ic(\delta^2\varphi)^{0,1}_{-}\wedge\bar\delta\varphi\\ \delta_2\sigma &= -2ic\,\delta\varphi\wedge(\delta\bar\delta\varphi)^{1,0}_{-}\,.\end{aligned}\tag{5.4}$$

Since $h(\bar\delta\varphi, (\delta\bar\delta\varphi)^{1,0}_{-}) = h(\bar\delta\varphi, \delta\bar\delta\varphi) = 0$, $\delta_2\sigma = 0$ iff $(\delta\bar\delta\varphi)^{1,0}_{-} = 0$. This holds iff the real quantity $\delta\bar\delta\varphi$ vanishes, i.e. $\varphi$ is harmonic. □

When $N = \mathbb{R}^4$ is Euclidean space with its flat metric, $\mathcal{S}_\pm \cong \mathbb{R}^4\times S^2$ and the traditional Gauss map of a conformal immersion $\varphi\colon M\to\mathbb{R}^4$ is given by

$$\gamma_\varphi = (\pi_2\circ\tilde\varphi_+,\ \pi_2\circ\tilde\varphi_-)\tag{5.5}$$

(see section 2). By convention the projection $\pi_2\colon(\mathcal{S}_\pm, J_2)\to S^2$ is antiholomorphic, so one recovers from theorem 5.3 the result that $\varphi$ is harmonic iff $\gamma_\varphi$ is antiholomorphic [Ch$_1$].

As another special case, suppose that $\varphi$ is a holomorphic mapping from a Riemann surface $M$ into an almost Hermitian (1, 2) symplectic manifold $N$. Then there exists a global section $F\in\Gamma(N, \mathcal{S}_+)$ with $D_2F = 0$, and by proposition 4.1, $\tilde\varphi_+ = F\circ\varphi$ is $J_2$ holomorphic. Proposition 3.3 then becomes a corollary of theorem 5.3. In the general situation a conformal map $\varphi$ is « rendered holomorphic » by the almost complex structure $\tilde\varphi_\pm$ on $\varphi^{-1}TN$.

We shall call a smooth $J_a$ holomorphic map from a Riemann surface into $\mathcal{S}_+$ or $\mathcal{S}_-$ a $J_a$ *holomorphic curve.* Given a nonconstant conformal harmonic map $\varphi\colon M\to N$, its Gauss lifts $\tilde\varphi_+, \tilde\varphi_-$ are both $J_2$ holomorphic curves; by remarks in section 1 this is true even at the isolated points where $\varphi_* = 0$. Conversely given a $J_2$ holomorphic curve $\psi\colon M\to\mathcal{S}_\pm$, its projection $\varphi = \pi\circ\psi$ is conformal. Provided $\psi$ is not vertical, i.e. not contained in a fibre, we must have $\psi = \tilde\varphi_\pm$ so by theorem 5.3 again $\varphi$ is also harmonic.

COROLLARY 5.4. *The assignment $\varphi\to\psi$ is a bijective correspondence between nonconstant conformal harmonic maps $\varphi\colon M\to N$ and nonvertical $J_2$ holomorphic curves $\psi\colon M\to\mathcal{S}_\pm$.*

As a corollary one therefore obtains in addition a bijective correspondence between the $J_2$ holomorphic curves in $\mathcal{S}_+$ and those in $\mathcal{S}_-$. This is significant since the manifolds $\mathcal{S}_+$ and $\mathcal{S}_-$ are generally distinct. A local existence theorem for holomorphic curves in an arbitrary almost complex manifold has been given by Nijenhuis & Woolf [NW]. See also [Gr].

PROBLEM. For compact manifolds the regular homotopy classes of smooth immersions (two being equivalent iff they are homotopic through immersions) have been classified by Hirsch and Smale [Hi; Sm]. One can define analogously the regular homotopy classes of branched immersions $\varphi\colon M \to N$ of a surface. *When does such a class contain a minimal representative, i.e. a conformal harmonic map?*

Not always. If $M$ has genus 2, there is no nonconstant conformal harmonic map $\varphi\colon M \to T^4$ into a flat torus. For its Gauss map $\gamma_\varphi$ would determine two meromorphic functions on $M$ of degree 1, in violation of the Riemann-Roch formula. By way of contrast, every homotopy class of maps $M \to T^4$ has a harmonic representative minimizing energy.

## 6. – $J_1$ holomorphicity.

Up to now orientation has not played an important role; we have been able to formulate results by working with just one of the twistor spaces $\mathcal{S}_+$, $\mathcal{S}_-$. For example given a map $\varphi\colon M \to N$ from a Riemann surface into an oriented Riemannian 4-manifold, it follows from theorem 5.3 that both or neither of $\tilde\varphi_+$, $\tilde\varphi_-$ are $J_2$ holomorphic. As will become apparent in the proof of the following result, the situation for $J_1$ is very different.

PROPOSITION 6.1. *A conformal immersion $\varphi\colon M \to N$ is totally umbilic iff both $\tilde\varphi_+$, $\tilde\varphi_-$ are $J_1$ holomorphic.*

PROOF. From the proof of theorem 5.3 and in particular (5.4), $\tilde\varphi_-$ is $J_1$ holomorphic iff $(\delta^2\varphi)_-^{0,1}\wedge\bar\delta\varphi = 0$. Now

$$h\big((\delta^2\varphi)_-^{0,1}, \delta\varphi\big) = h(\delta^2\varphi, \delta\varphi) = \tfrac{1}{2}\,\delta h(\delta\varphi, \delta\varphi) = 0\,, \tag{6.1}$$

so $(\delta^2\varphi)_-^{0,1}$ has no component proportional to $\bar\delta\varphi$, and $(\delta^2\varphi)_-^{0,1}\wedge\bar\delta\varphi = 0$ iff $(\delta^2\varphi)_-^{0,1} = 0$. Similarly for $\tilde\varphi_+$, so at a given point

$$\tilde\varphi_\pm \text{ is } J_1 \text{ holomorphic iff } \delta^2\varphi \in T_\pm^{1,0}\,. \tag{6.2}$$

Both $\tilde\varphi_+$, $\tilde\varphi_-$ are $J_1$ holomorphic iff $\delta^2\varphi \in T_+^{1,0} \cap T_-^{1,0} = \mathbb{C}\,\delta\varphi$. The latter defines totally umbilic for a conformal map. $\square$

A mapping $\varphi\colon M \to N$ from a Riemann surface into a Riemannian manifold with metric $h$ is said to be *real isotropic* if

$$h(\delta^r\varphi, \delta^s\varphi) = 0\,, \qquad \forall r, s \geqslant 1 \tag{6.3}$$

relative to any complex coordinate on $M$. This just means that the differentials $\delta^r\varphi$, $r \geqslant 1$, span an isotropic subspace of $(TN)^{\mathbf{C}}$, and for $N = \mathbb{R}P^n$ coincides with [EW$_2$, definition 5.6; R, section 6]. Putting $r = s = 1$ shows that $\varphi$ is conformal, so real isotropy may be regarded as a generalization of conformality.

EXAMPLE. For $N = S^n$ or $\mathbb{C}P^n$, it is known that any harmonic map $\varphi\colon S^2 \to N$ must be real isotropic. For $S^n$ this was proved by Calabi [C$_1$] who called real isotropic harmonic maps « pseudo-holomorphic ». The crucial point is that a Riemann surface of genus 0 admits no holomorphic differentials, and it suffices to show inductively that $\bar{\delta}h(\delta^r\varphi, \delta^s\varphi) = 0$. For $\mathbb{C}P^n$, $\varphi$ actually satisfies a stronger condition called complex isotropy which will be studied in section 10. If $N$ is a real space form and $\varphi\colon M \to N$ a harmonic map which is real isotropic on an open set $\mathcal{U} \subset M$, then $\varphi$ is real isotropic on all $M$ [Bu].

PROPOSITION 6.2. *A conformal immersion $\varphi\colon M \to N$ is real isotropic iff either $\tilde{\varphi}_+$ or $\tilde{\varphi}_-$ is $J_1$ holomorphic at each point.*

PROOF. Suppose that $\varphi$ is real isotropic; so,

$$h(\delta\varphi, \delta\varphi) = 0 = h(\delta^2\varphi, \delta^2\varphi)\,. \tag{6.4}$$

Then $\{\delta\varphi, \delta^2\varphi\}$ span an isotropic subspace, and at a given point $m \in M$ either $\delta^2\varphi \in T_+^{1,0}$ or $\delta^2\varphi \in T_-^{1,0}$. By (6.2), $\tilde{\varphi}_+$ or $\tilde{\varphi}_-$ respectively is $J_1$ holomorphic at $m$.

Conversely if either $\tilde{\varphi}_+$ or $\tilde{\varphi}_-$ is $J_1$ holomorphic at each point, then (6.4) must be satisfied everywhere. Let $\mathcal{U}$ be the open subset of $M$ where $\delta\varphi \wedge \delta^2\varphi \neq 0$. Then on $M \setminus \overline{\mathcal{U}}$, $\varphi$ is totally umbilic and so clearly isotropic. It is therefore enough to prove that (6.3) holds at each $m \in \mathcal{U}$. At $m$, $\delta\varphi$ and $\delta^2\varphi$ span a maximal isotropic subspace $W$ of $(T_{\varphi(m)}N)^{\mathbf{C}}$. Differentiating (6.4) gives

$$h(\delta^r\varphi, \delta\varphi) = 0 = h(\delta^r\varphi, \delta^2\varphi)$$

for $r = 3$, and thereafter for all $r$ by induction. Hence $\delta^r\varphi \in W$ at $m$, and (6.3) holds. $\square$

If $\varphi\colon M \to N$ is conformal and harmonic, then by theorem 5.2 the line bundle

$$L = T_+^{1,0} \cap T_-^{1,0}$$

is complex analytic in both $T_+^{1,0}$ and $T_-^{1,0}$. The second fundamental form of $L$ in $T_\pm^{1,0}$ is determined by the component of $(\delta^2\varphi)_\pm^{1,0}$ orthogonal to $\delta\varphi$, or more invariantly by the form

$$\mu_\pm = dz^3 \otimes \left(\delta\varphi \wedge (\delta^2\varphi)_\pm^{1,0}\right) \in \Gamma(M, K^3 \otimes T_\pm^{2,0})\,. \tag{6.5}$$

Here $K$ denotes the canonical bundle $(T^{1,0}M)^*$ of $M$ and as before, $T_\pm^{2,0} = \Lambda^2(T_\pm^{1,0})$. Note that $\mu_\pm$ does not depend upon the choice of coordinate $z$ on $M$.

LEMMA 6.3. *An immersion $\varphi\colon M \to N$ has $\mu_\pm = 0$ iff $\tilde\varphi_\mp$ is $J_1$ holomorphic.*

PROOF. Consider $\tilde\varphi_-$ for definiteness. Let $\beta$ be a non-zero vector in $T_-^{0,1} \cap T_+^{1,0}$, so that $T_-^{0,1} = \mathbf{C}\,\bar\delta\varphi \oplus \mathbf{C}\beta$. (See (5.2).) Using (6.1) and (6.2),

$$\tilde\varphi_- \text{ is } J_1 \text{ holomorphic iff } h\left((\delta^2\varphi)_-^{0,1}, \bar\beta\right) = 0\,.$$

But

$$h\left((\delta^2\varphi)_-^{0,1}, \bar\beta\right) = h(\delta^2\varphi, \bar\beta) = h\left((\delta^2\varphi)_+^{1,0}, \bar\beta\right)\,,$$

and the last term vanishes iff $\mu_+ = 0$. $\square$

A mapping $\psi\colon M \to \mathbb{S}_\pm$ is said to be *horizontal* if each image $\psi_*(T_m M)$ is contained in the horizontal distribution $H$ defined by (4.3). Since $J_1$ and $J_2$ coincide on $H$, if $\psi$ has two of the properties $J_1$ holomorphic, $J_2$ holomorphic, horizontal, then it has the third. In this case one can say that $\psi$ is horizontal holomorphic, the distinction between $J_1$ and $J_2$ being unnecessary. For example, by theorem 5.3 and lemma 6.3, a conformal harmonic map $\varphi\colon M \to N$ has $\mu_\pm = 0$ iff $\tilde\varphi_\mp$ is horizontal holomorphic. A similar approach was adopted by Poon [P]. On the other hand,

PROPOSITION 6.4. *If $\varphi\colon M \to N$ is a conformal harmonic map with $N$ $\pm$ self-dual and Einstein, then $\mu_\pm$ is complex analytic in the sense of theorem* 5.2.

PROOF. It is necessary to show that

$$\nabla_{\partial/\partial\bar z}\left(\delta\varphi \wedge (\delta^2\varphi)_\pm^{1,0}\right) = 0\,,$$

where $\nabla$ denotes the Levi-Civita connection on $(\varphi^{-1}TN)^{\mathbf{C}}$. Since $\varphi$ is harmonic, $\bar\delta\delta\varphi = 0$. Put $\beta = (\delta^2\varphi)_\pm^{0,1}$. Then from the definition (6.5) of $\mu_\pm$ it remains to check that relative to the almost complex structure $\tilde\varphi_\pm$,

$$\bar\delta\left((\delta^2\varphi)_\pm^{1,0}\right) = \bar\delta\delta^2\varphi - \bar\delta\beta$$

has $(1,0)$ component proportional to $\delta\varphi$. First note that this is true for the second term on the right hand side: the equations $h(\beta, \delta\varphi) = 0$,

$h(\beta, \bar\delta\beta) = 0$ imply that $(\bar\delta\beta)^{1,0}_{\pm}$ is proportional to $\delta\varphi$. Now.

$$\bar\delta\delta^2\varphi = -(\delta\bar\delta - \bar\delta\delta)(\delta\varphi) = -(\varphi^*R)(\partial/\partial z, \partial/\partial\bar z)(\delta\varphi) = -R(\delta\varphi, \bar\delta\varphi)(\delta\varphi),$$

where $R = tA + W_{\pm}$ is the curvature tensor of $N$ (see (4.5)). The constant curvature operator $A$ is essentially the identity, so certainly $A(\delta\varphi, \bar\delta\varphi)(\delta\varphi)$ is proportional to $\delta\varphi$. Now we can use proposition 3.4 to deduce that

$$W_{\pm}(\delta\varphi, \bar\delta\varphi) \in \Lambda^2_{\pm} TN$$

has no primitive $(1,1)$-component (i.e. no component in $T^{1,1}_0$ relative to $\tilde\varphi_{\pm}$). Consequently $(W_{\pm}(\delta\varphi, \bar\delta\varphi)(\delta\varphi))^{1,0}_{\pm}$ is also proportional to $\delta\varphi$. □

Suppose in addition to the hypotheses of proposition 6.4 that $\varphi$ is real isotropic. Then by proposition 6.2, if $\mu_{\mp}(m) \neq 0$ for some $m \in M$, $\mu_{\pm} = 0$ on a neighbourhood of $m$. It follows that either $\mu_+ = 0$ on $M$ or $\mu_- = 0$ on $M$. Let $\mathcal{H}_{\mp} = \mathcal{H}_{\mp}(M, N)$ denote the set of horizontal holomorphic curves $M \to \mathcal{S}_{\mp}$, and $\sim$ the equivalence relation that identifies two curves in $\mathcal{H}_+ \cup \mathcal{H}_-$ having the same projection $\varphi$ in $N$. If the curves are distinct then one is $\tilde\varphi_+$, the other is $\tilde\varphi_-$, and by theorem 5.3 and proposition 6.1, $\varphi$ is totally geodesic. Combining corollary 5.4 with the above remarks yields

THEOREM 6.5. *Let $N$ be $\pm$ selfdual and Einstein. There is a bijective correspondence between real isotropic harmonic maps $\varphi$: $M \to N$ and the set $(\mathcal{H}_+ \cup \mathcal{H}_-)/\sim$.*

Euclidean space $N = \mathbb{R}^4$ has twistor space $\mathcal{S}_+ = \mathbb{C}P^3 - \mathbb{C}P^1$; $\tilde\varphi_+$ or $\tilde\varphi_-$ is horizontal iff $\varphi$ is holomorphic with respect to some orthogonal complex structure on $\mathbb{R}^4$. This is equivalent to saying that the Gauss map $\gamma_\varphi$: $M \to S^2 \times S^2$ (5.5) is constant in one of its factors. Micallef [M] has proved that a wide class of oriented complete (area) *stable* minimal surfaces in $\mathbb{R}^4$, including those of finite total curvature, have this isotropy property. Similar remarks hold for $N = T^4$, a flat torus. In this case $(\mathcal{S}_+, J_1)$ is integrable (Blanchard's variety [AHS]) but not $(1,2)$-symplectic, whereas $(\mathcal{S}_+, J_2)$ is $(1,2)$-symplectic but not integrable.

For the sphere $N = S^4$, real isotropic harmonic maps are called *superminimal* by Bryant [Br], and such a map $\varphi$ is said to have positive or negative spin according as $\tilde\varphi_{\pm}$ is horizontal. Indeed the forms $\mu_-$, $\mu_+$ are closely related to $\sigma_1$, $\sigma_2$ in [Br], and the prefix « super » refers to the vanishing of the quartic differential $h(\delta^2\varphi, \delta^2\varphi)\, dz^4$. The case of $S^4$ will be examined in more detail in section 8.

More generally, if $N$ is $\pm$ selfdual and Einstein, it follows from theorem 4.2 that there is no local obstruction to the existence of horizontal holomorphic curves in $\mathcal{S}_\mp$. The analyticity of $\mu_\pm$ is then a reflection of the complex contact structure of $\mathcal{S}_\mp$. Conversely if $N$ has $\mathcal{H}_\mp \neq \emptyset$, the curvature tensors $B$ and $W_\mp$ are severely restricted. However there are instances of 4-manifolds $N$ not satisfying the hypotheses of theorem 6.5, but for which $\mathcal{H}_\mp \neq \emptyset$ (see the example following proposition 3.3).

REMARK. The following characterization has been given by Friedrich [F, proposition 5]. A map $\varphi\colon M \to N$ has $\tilde{\varphi}_\pm$ horizontal holomorphic iff for any path $c_t$ $(0 \leqslant t \leqslant 1)$ in $M$ the compositions

$$\varphi_*(T_{c_0}M) \xrightarrow{\tau} T_{\varphi(c_1)}N \xrightarrow{p} \varphi_*(T_{c_1}M),\ (\varphi_*(T_{c_1}M))^\perp$$

of parallel translation $\tau$ along $\varphi(c_t)$ with the orthogonal projections are conformal.

## 7. – The case of 3-manifolds.

In this section we shall state without proof a theorem which is based upon results of the two preceding sections. For more details we refer the reader to [$E_2$, $S_3$].

Suppose first that $N$ is an oriented 4-dimensional Riemannian manifold containing an embedded 3-manifold $N'$. As usual set

$$\mathcal{S}_\pm = S(\Lambda^2_\pm TN),$$

and in addition let $\mathcal{S} = S(TN')$ denote the bundle of unit tangent vectors to $N'$. If $\nu$ denotes the oriented unit normal to $N'$ in $N$, then any $X \in \mathcal{S}$ defines elements

$$p_\pm(X\wedge\nu) = \tfrac{1}{2}(1\pm *)(X\wedge\nu) \in \mathcal{S}_\pm .$$

In this way we can identify $\mathcal{S}$ with the restriction of both $\mathcal{S}_+$ and $\mathcal{S}_-$ to $N'$. Moreover if $\varphi\colon M \to N'$ is an immersed surface, then

$$\tilde{\varphi} = \tilde{\varphi}_+ = \tilde{\varphi}_-,$$

where $\varphi\colon M \to \mathcal{S}$ is the Gauss lift defined by the normal to $M$ in $N'$.

In the above situation the almost complex structures $J_1, J_2$ on $\mathcal{S}_\pm$ induce almost $CR$ structures on $\mathcal{S}$, which may therefore be treated as a type of twistor space in its own right. We shall explain this more carefully in the abstract setting in which $N'$ is an arbitrary oriented Riemannian 3-manifold. In this case we use the Levi-Civita connection of $N'$ to give a splitting

$$T_y \mathcal{S} = H_y \oplus V_y$$

as in (4.3). The horizontal space $H_y$ contains a distinguished 2-dimensional subspace $H'_y$ equal to the orthogonal complement of the vector $y$, and we define a distribution $\Pi$ on $\mathcal{S}$ by

$$\Pi_y = H'_y \oplus V_y \,.$$

Since $H'_y$, $V_y$ are oriented 2-planes, they admit natural almost complex structures, and in analogy to (4.4) there exist tensors

$$J_a \in \Gamma(\mathcal{S}, \operatorname{End} \Pi)\,, \qquad a = 1, 2\,,$$

satisfying $J_a^2 = -1$.

Given a Riemann surface $M$ and an immersion $\varphi\colon M \to N'$, we shall say that its Gauss lift is $J_a$ holomorphic if $\tilde{\varphi}_*(TM) \subset \Pi$ and $J_a\, \delta\tilde{\varphi} = i\, \delta\tilde{\varphi}$. Associating $\varphi$ with $\tilde{\varphi}$ then yields

THEOREM 7.1. *There is a bijective correspondence between*

(i) *nonconstant conformal totally umbilic maps* $M \to N'$ *and nonvertical* $J_1$ *holomorphic curves* $M \to \mathcal{S}$;

(ii) *nonconstant conformal harmonic maps* $M \to N'$ *and nonvertical* $J_2$ *holomorphic curves* $M \to \mathcal{S}$.

In this set-up the Gauss lift $\tilde{\varphi}$ is horizontal iff $\varphi$ is totally geodesic. Moreover in the situation of an embedding $i\colon N' \subset N$, the natural map $\mathcal{S} \to \mathcal{S}_\pm$ is $J_2$ holomorphic in the obvious sense iff $i$ is totally geodesic. This ensures that harmonic maps into $N'$ remain harmonic into $N$.

The subbundle

$$\Pi^{1,0} = \{\alpha \in \Pi^{\mathbf{C}}\colon J_1 \alpha = i\alpha\} \subset (T\mathcal{S})^{\mathbf{C}}$$

is always closed under Lie bracket; curvature provides no obstruction because the horizontal part of $\Pi$ has only 1 complex dimension. This means that $(\mathcal{S}, J_1)$ is an integrable *CR* manifold; that was in fact exploited by LeBrun [Le] in a conformally invariant setting to furnish examples of nonrealizable *CR* manifolds. Although twistor *CR* manifolds exist in higher dimensions [S3], a 3-dimensional base guarantees that $\Pi$ has real codimension 1. When $N' = \mathbb{R}^3$ is Euclidean space, $\mathcal{S}$ fibres over the twistor space $T^{1,0}\mathbf{C}P^1$ used by Hitchin to study monopoles [H3].

Although $(\mathcal{S}, J_2)$ is not integrable, theorem 7.1 does show that $J_2$ holomorphic curves exist in abundance. The distinction between $J_1$ and $J_2$ in this regard is best appreciated in terms of the Levi form. Fix a nowhere zero 1-form

$$\gamma = \sum_{i=1}^{3} a_i \pi^* dx^i$$

on $\mathcal{S}$ annihilating the distribution $\Pi$; $x^i$ are local coordinates on $N'$ and $a_i$ are suitable fibre coordinates on $TN$. The Levi form of $(\mathcal{S}, J_1)$ is the Hermitian form

$$L(\alpha, \beta) = d\gamma(\alpha, \bar{\beta}) , \qquad \alpha, \beta \in \Pi^{1,0} ,$$

and is nondegenerate with signature $(+1, -1)$. This imposes stringent conditions on the existence of $J_1$ holomorphic curves. For example such a curve $\psi \colon M \to \mathcal{S}$ satisfies $\psi^* \gamma = 0$, and so has a null tangent vector:

$$L(\delta\psi, \delta\psi) = 0 . \tag{7.1}$$

On the other hand $d\gamma$ has no component of type $(1, 1)$ relative to $J_2$, so (7.1) is no extra condition on the tangent vector of a $J_2$ holomorphic curve.

## 8. – Spinors and degrees.

We resume work with an oriented Riemannian 4-manifold $N$. A section of $\mathcal{S}_+$ over an open set $\mathcal{U} \subset N$ determines an isomorphism of the twistor space $\mathcal{S}_-|_{\mathcal{U}}$ with the associated complex projective bundle $P(T_+^{1,0}\mathcal{U})$, by the proof of lemma 5.1. However for a more invariant description of this type one must resort to the so-called spinor bundles.

The homomorphism (2.4) exhibits $Sp(1) \times Sp(1)$ as the universal covering Spin (4) of $SO(4)$. Regarding the quaternions $\mathbb{H}$ as a right $\mathbb{H}$-module and a left $\mathbb{H}$-module respectively gives the isomorphism of modules $\mathbb{H} \otimes_{\mathbb{H}} \mathbb{H} \cong \mathbb{R}^4$ corresponding to (2.4). To avoid the confusion between left and right, it is more convenient to treat $\mathbb{H}$ as a complex 2-dimensional vector space. In this case we may write

$$\mathbb{H} \otimes_{\mathbb{C}} \mathbb{H} \cong (\mathbb{R}^4)^{\mathbb{C}} , \tag{8.1}$$

where the right-hand side is the complexification of the basic $SO(4)$-module. Complex conjugation on the left is given by $\overline{a \otimes b} = ja \otimes jb$, and a typical real element in (8.1) can be represented by complex matrices as

$$\begin{pmatrix}1\\0\end{pmatrix}\otimes(z_1, z_2) + \begin{pmatrix}0\\1\end{pmatrix}\otimes(-\bar{z}_2, \bar{z}_1) = \begin{pmatrix} z_1 & z_2\\ -\bar{z}_2 & \bar{z}_1\end{pmatrix}.$$

This is the decomposition of a vector (right) into the product of spinors (left); for more details see [$S_2$].

Now suppose that $N$ is a spin manifold, which means that its structure group can be lifted from $SO(4)$ to $Sp(1)\times Sp(1)$. Then associated to (8.1) are complex rank 2 vector bundles $\Delta_+$, $\Delta_-$ such that

$$\Delta_+\otimes_{\mathbf{C}}\Delta_-\cong(TN)^{\mathbf{C}}. \tag{8.2}$$

Given $u\in\Delta_-$, $u\neq 0$, $\pi(u)=x$,

$$W_u=(\Delta_+)_x\otimes\mathbf{C}ju \tag{8.3}$$

defines a maximal isotropic subspace of $(T_xN)^{\mathbf{C}}$ which must be the space of $(1, 0)$-vectors for some almost Hermitian structure on $T_xN$. With the correct orientation convention, the corresponding fundamental 2-form belongs to $\mathcal{S}_-$. Since $W_u$ depends only upon the projective class $[u]$, this argument establishes

$$\mathcal{S}_-\cong P(\Delta_-), \tag{8.4}$$

and similarly with the opposite signs. Taking exterior powers of (8.2) gives related isomorphisms $(\Lambda^2_\pm TN)^{\mathbf{C}}\cong S^2\Delta_\pm$.

The pullback $\pi^{-1}\Delta_-$ over the total space $P(\Delta_-)$ contains a tautologous complex line subbundle $\zeta$ whose fibre at $[u]\in P(\Delta_-)$ is the subspace $\mathbf{C}u$. The fibre of its conjugate or dual $\bar{\zeta}\cong\zeta^{-1}$ can be identified with the complementary subspace $\mathbf{C}ju$. It follows from (8.3) that the horizontal bundle of $(1, 0)$-vectors on the almost complex manifold $(\mathcal{S}_-, J_a)$ ($a=1$ or $2$) is

$$(T^{1,0})^h\cong\pi^*\Delta_+\otimes\bar{\zeta}.$$

Consequently

$$(T^{2,0})^v\cong\Lambda^2(\pi^*\Delta_+\otimes\bar{\zeta})\cong\bar{\zeta}^2,$$

and these expressions can now be substituted into (4.4). Using the same symbol $\zeta$ to denote the tautologous line bundle over $\mathcal{S}_+$, and omitting $\otimes$, $\pi^{-1}$ for ease of notation

$$T^{1,0}\mathcal{S}_\pm\cong\begin{cases}\bar{\zeta}\Delta_\mp\oplus\bar{\zeta}^2 & \text{for } J_1\\ \bar{\zeta}\Delta_\mp\oplus\zeta^2 & \text{for } J_2.\end{cases} \tag{8.5}$$

Even if $N$ is not a spin manifold, (8.5) can be established by choosing a spin structure locally. In particular the bundles $\bar{\zeta}^2$, $\bar{\zeta}\Delta_\mp$ are unambiguously defined over $\mathcal{S}_\pm$ and independent of the choice of spin structure. We shall give (8.5) a group theoretic interpretation in the next section.

PROPOSITION 8.1. *For any oriented Riemannian 4-manifold $N$, the first Chern class of the almost complex manifold $(\mathcal{S}_\pm, J_2)$ is zero.*

PROOF. By general principles [Hir, theorem 4.4.3], the first Chern class of $T^{1,0}\mathcal{S}_\pm$ relative to $J_2$ equals that of

$$\Lambda^3(T^{1,0}\mathcal{S}_\pm) \cong \Lambda^3(\bar{\zeta}\Delta_\mp \oplus \zeta^2) \cong \bigoplus_{r=0}^{3} \Lambda^r(\bar{\zeta}\Delta_\mp) \otimes \Lambda^{3-r}(\zeta^2) \cong \Lambda^2(\bar{\zeta}\Delta_\mp) \otimes \zeta^2$$
$$\cong \bar{\zeta}^2 \otimes \zeta^2 \cong \mathbf{C}\,. \qquad \square$$

An identical argument gives

$$c_1(\mathcal{S}_\pm, J_1) = c_1(\bar{\zeta}^4)\,;$$

the right hand side is four or two times an integral cohomology class according as $N$ is spin or not.

For the remainder of this section we suppose that $\varphi\colon M \to N$ is a non-constant conformal harmonic map from a compact Riemann surface into the 4-manifold $N$. Define the *twistor degrees* $d_+$, $d_-$ of $\varphi$ by

$$2d_\pm = (\tilde{\varphi}_\pm^* c_1(\bar{\zeta}^2))[M]\,,$$

$\bar{\zeta}^2$ being the bundle of (1, 0)-vectors tangent to the fibres in either $\mathcal{S}_+$ or $\mathcal{S}_-$ relative to $J_1$. Thus $d_\pm$ is an integer whenever $N$ is spin. Identifying $H^2(M, \mathbb{Z}) \cong \mathbb{Z}$, we have

LEMMA 8.2. $2d_\pm = c_1(T_\pm^{1,0}) = c_1(T_\pm^{2,0})$.

PROOF. The second equality follows immediately since $T_\pm^{2,0} = \Lambda^2(T_\pm^{1,0})$ is the determinant bundle of $T_\pm^{1,0}$. From proposition 8.1,

$$0 = \tilde{\varphi}_\pm^* c_1(\zeta^2 \oplus (T^{1,0})^h) = -\tilde{\varphi}_\pm^* c_1(\bar{\zeta}^2) + c_1(\tilde{\varphi}_\pm^{-1}(T^{1,0})^h)\,.$$

By definition of $\tilde{\varphi}_\pm$ and (5.1),

$$\tilde{\varphi}_\pm^{-1}(T^{1,0})^h \cong T_\pm^{1,0}\,.$$

The lemma follows. $\square$

From theorem 5.2 and its proof,

$$L = T_+^{1,0} \cap T_-^{1,0}$$

is a complex analytic subbundle of both $T_+^{1,0}$ and $T_-^{1,0}$ relative to the Koszul-Malgrange structure on $(\varphi^{-1}TN)^{\mathbb{C}}$. Given a complex coordinate $z$ on $M$, $L$ is spanned generically by $\delta\varphi = \varphi_*(\partial/\partial z)$, and the globally defined form

$$\lambda = dz \otimes \delta\varphi$$

is complex analytic. Thus $\lambda \in H^0(K \otimes L)$ where $K$ is the canonical bundle of $M$. If $p$ is the genus of $M$, $\chi = 2 - 2p$ the Euler characteristic, and

$$r = |\lambda| = \sum_{x:\lambda(x)=0} \operatorname{ord}_x(\lambda)$$

the ramification index of $\varphi$, then

$$0 \leqslant r = c_1(K \otimes L) = -\chi + c_1(L) .$$

Now $c_1(L) = c_1(T_\pm^{1,0}) - c_1(Q_\pm)$, where

$$Q_\pm = T_\pm^{1,0}/L$$

is the quotient. Note that $Q_- \cong \bar{Q}_+$ (e.g. by (5.2)) and the real vector bundle underlying $Q_+$ or $Q_-$ is the normal bundle of $\varphi(M)$ in $N$, so $e = c_1(Q_+) = -c_1(Q_-)$ is its Euler number. From lemma 8.2 we obtain

PROPOSITION 8.3. *The twistor degrees of a conformal harmonic map $\varphi\colon M \to N$ from a compact Riemann surface satisfy*

$$d_\pm = 1 - p + \tfrac{1}{2}r \pm \tfrac{1}{2}e .$$

When the 4-manifold $N$ is $\pm$ selfdual there is by proposition 6.4 in addition to $\lambda$, a complex analytic differential

$$\mu_\pm = dz^3 \otimes \left(\delta\varphi \wedge (\delta^2\varphi)_\pm^{1,0}\right) \in H^0(M, K^3 \otimes T_\pm^{2,0}) .$$

If $x \in M$ is a zero of $\delta\varphi$ of order $k$, then $\delta\varphi \wedge \delta^2\varphi$ has a zero of order $2k$ at $x$; therefore

$$|\mu_\pm| = 2r + s_\pm$$

for some integer $s_\pm \geqslant 0$. Provided $\tilde{\varphi}_\mp$ is not horizontal, $\mu_\pm$ is not identically zero and by lemma 8.2,

$$2r + s_\pm = c_1(K^3 \otimes T^{2,0}_\pm) = -3\chi + 2d_\pm .$$

Combining this with

$$d_+ + d_- = 2 - 2p + r$$

(proposition 8.3) and picking one sign for definiteness gives

PROPOSITION 8.4. *If $\varphi\colon M \to N$ is conformal harmonic, with $N$ selfdual and Einstein, and $\tilde\varphi_-$ not horizontal, then*

$$d_+ = 3 - 3p + r + \tfrac{1}{2}s_+ , \qquad d_- = p - 1 - \tfrac{1}{2}s_+ .$$

REMARK. If $N$ is selfdual and Einstein, and $\tilde\varphi_-$ is horizontal holomorphic, then

$$d_- = \frac{1}{48\pi} t^N A(\varphi) ,$$

where $t^N$ is the (constant) scalar curvature of $N$ and $A(\varphi)$ is the area of the map $\varphi$ [F, P]. This follows from examining the curvature of the line bundle $(T^{2,0})^v \cong \bar\zeta^2$ over $\mathcal{S}_-$ used to define $d_-$.

## 9. – Maps into $S^4$.

The spinor description of the last section is best appreciated by considering the twistor space

$$\mathcal{S}_+ \cong \frac{Sp(2)}{U(1)\times Sp(1)} \cong \mathbb{C}P^3$$

of $S^4 \cong \mathbb{H}P^1$ (see (2.5)). If $\zeta \cong \mathbb{C}$, $\Delta_- \cong \mathbb{C}^2$ now denote the basic complex representations of $U(1)$ and $Sp(1)$, then the basic representation decomposes as

$$\zeta \oplus \bar\zeta \oplus \Delta_- \cong \mathbb{C}^4$$

under the action of the isotropy subgroup $U(1)\times Sp(1)$. The tangent space $T_0\mathcal{S}_+$ at the identity coset can be read off from the isomorphisms

$$(9.1)\qquad sp(2) \cong S^2(\zeta \oplus \bar\zeta \otimes \Delta_-) \cong S^2(\zeta \oplus \bar\zeta) \oplus S^2\Delta_- \oplus (\zeta \oplus \bar\zeta)\Delta_-$$
$$\cong u(1) \oplus sp(1) \oplus (\zeta\Delta_- \oplus \bar\zeta\Delta_-) \oplus (\zeta^2 \oplus \bar\zeta^2) .$$

Thus $T_0 \mathrm{S}_+ = H \oplus V$, where $H^{\mathbf{C}} \cong \zeta\Delta_- \oplus \bar{\zeta}\Delta_-$ and $V^{\mathbf{C}} \cong \zeta^2 \oplus \bar{\zeta}^2$ are the horizontal and vertical subspaces.

Because the isotropy representation $H \oplus V$ has precisely two irreducible real components, there are exactly $2^2 = 4$ $Sp(2)$-invariant almost complex structures (see [BH]). These are of course $\pm J_1$ and $\pm J_2$ as defined in (8.5), and $J_1$ is the standard complex structure of $\mathbf{C}P^3$. The integrability of $J_1$ follows from the formula

$$\Lambda^2(\bar{\zeta}\Delta_-) \cong \bar{\zeta}^2$$

which implies that the subspace $\bar{\zeta}\Delta_- \oplus \bar{\zeta}^2$ of $sp(2)$ is closed under Lie bracket.

There are also important Lie bracket relations concerning $J_2$; namely if

$$a_0 = u(1) \oplus sp(1)$$

$$a_1 = \bar{\zeta}\Delta_- \oplus \zeta^2$$

$$a_2 = \zeta\Delta_- \oplus \bar{\zeta}^2\,,$$

then

$$[a_i, a_j] \subset a_k\,, \qquad k = i + j \pmod 3)\,.$$

Thus if $w$ denotes a primitive cube root of unity, the transformation $\theta$ of $sp(2)$ which acts on $a_i$ as $w^i$ times the identity is a Lie algebra automorphism with $\theta^3 = 1$. Observe that on the tangent space $(T_0 \mathrm{S}_+)^{\mathbf{C}} = a_1 \oplus a_2$,

$$\theta = -\frac{1}{2} + \frac{\sqrt{3}}{2} J_2\,. \tag{9.2}$$

A homogeneous space $G/H$ for which the Lie algebra $g = h \oplus m$ admits an automorphism $\theta$ with $\theta^3 = 1$ and $h = \ker(\theta - 1)$ is called *3-symmetric* [$G_2$]. Therefore $\mathbf{C}P^3$ has a 3-symmetric structure arising from $J_2$.

Let $\pi$ denote the projection $\mathbf{C}P^3 \cong \mathrm{S}_+ \to S^4$. It is also convenient to identify $\mathrm{S}_-$ with $\mathbf{C}P^3$; in this case the projection $\mathrm{S}_- \to S^4$ equals $a \circ \pi$, where $a$ is the orientation reversing antipodal map of $S^4$ [Br]. Any invariant almost Hermitian metric on $\mathbf{C}P^3$ (relative to $J_1$ or $J_2$) is a constant multiple of

$$k_t = \pi^* h + t h^v,$$

where $t > 0$ is a constant, and $h^v$ is the induced metric on the vertical space $(T^{2,0} \oplus T^{0,2})^v$. The next theorem then follows from results of Wolf & Gray [WG, theorems 8.13, 9.4].

THEOREM 9.1. 1) *For a unique value* $t = t_0$, $(\mathbf{C}P^3, k_{t_0}, J_1)$ *is a Kähler manifold.*

2) *For any* $t > 0$, *the almost Hermitian manifold* $(\mathbf{C}P^3, k_t, J_2)$ *is* $(1, 2)$-*symplectic.*

Part 2 is related to proposition 8.1 which implies that $\mathbf{C}P^3$ has an $SU(3)$-structure. Indeed if

$$\omega_{2,t}(X, Y) = k_t(J_2 X, Y)$$

denotes the fundamental 2-form for $J_2$, the $(3, 0)$ component of $d\omega_{2,t}$ trivializes the canonical bundle for $J_2$. This phenomenon also occurs with the sphere $S^6 \cong G_2/SU(3)$ which is another 3-symmetric space. More generally, 3-symmetric spaces are classified in [WG; theorem 7.10]; as explained in [$S_3$], many consist of twistor bundles over Riemannian symmetric spaces. Note that an invariant almost complex structure can be defined on any 3-symmetric space by using (9.2).

Combining proposition 3.3 and theorem 5.3, we obtain

COROLLARY 9.2. *If* $\varphi: M \to S^4$ *is conformal and harmonic, so are both Gauss lifts* $\tilde{\varphi}_+, \tilde{\varphi}_-: M \to (\mathbf{C}P^3, k_t)$ *for any* $t > 0$.

REMARK. Not every holomorphic map $\psi: M \to N$ into an almost Hermitian manifold is a minimum of the energy functional $E$ even if $N$ is $(1, 2)$ symplectic. For instance, take $N = (\mathbf{C}P^3, J_2)$ and let $\psi$ be the Gauss lift of a conformal harmonic map $M \to S^4$ which is not real isotropic. Then $\psi$ is $J_2$ holomorphic but not $J_1$ holomorphic; we conclude from [$EW_1$] that $\psi$ is not an $E$-minimum. In particular, in corollary 9.2 unless $\tilde{\varphi}_\pm$ is horizontal its energy is not an $E$-minimum.

Any harmonic map $\varphi: S^2 \to S^4$ being real isotropic necessarily has $\tilde{\varphi}_+$ or $\tilde{\varphi}_-$ horizontal holomorphic (theorem 6.5). We illustrate this with Veronese surfaces.

Regard the quaternions $\mathbf{H}$ as a right $\mathbf{H}$-module, and let the group $Sp(1)$ of unit quaternions act by left multiplication. The complex 2-dimensional vector space $W$ underlying $\mathbf{H}$ then admits an antilinear endomorphism $j$ satisfying $j^2 = -1$ and commuting with the action of $Sp(1) \cong SU(2)$. The subspace $S^3 W$ of $W \otimes_{\mathbf{C}} W \otimes_{\mathbf{C}} W$ of totally symmetric tensors is also a right $\mathbf{H}$-module with $(w \otimes w \otimes w) j = wj \otimes wj \otimes wj$. Consider the quaternionic basis of $S^3 W$ consisting of

$$e_+ = 1 \otimes 1 \otimes 1$$

$$e_- = \frac{1}{\sqrt{3}} (j \otimes 1 \otimes 1 + 1 \otimes j \otimes 1 + 1 \otimes 1 \otimes j)\,.$$

Identifying $e_+$, $e_-$ with the row vectors $(1, 0)$, $(0, 1)$ defines an isomorphism $S^3 W \cong \mathbb{H}^2$ relative to which $q = \alpha + j\beta \in Sp(1)$ acts as the matrix

$$X = \begin{pmatrix} \alpha^3 + j\beta^3 & \sqrt{3}(\alpha^2\beta - j\alpha\beta^2) \\ \sqrt{3}(-\alpha^2\bar{\beta} + j\bar{\alpha}\beta^2) & (3|\alpha|^2 - 2)\alpha + j(1 - 3|\alpha|^2)\beta \end{pmatrix},$$

where $\alpha, \beta \in \mathbb{C}$, $|\alpha|^2 + |\beta|^2 = 1$. Since $\bar{X}X^T = 1$, we obtain an explicit monomorphism

$$i\colon Sp(1) \to Sp(2)\,.$$

The 7-dimensional homogeneous space $Sp(2)/i(Sp(1))$ has irreducible isotropy, and has strictly positive sectional curvature [B].

Given any element $e \in S^3 W$, let $N_e$ denote the subgroup of $Sp(2)$ which preserves $e$ up to a complex multiple, so that $Sp(2)/N_e \cong \mathbb{C}P^3$ Next consider the subgroup $U(1) = \{\alpha \in \mathbb{C}\colon |\alpha| = 1\}$ of $Sp(1)$; since $Sp(1)/U(1) \cong \mathbb{C}P^1$, $i$ induces a map $\psi_e\colon \mathbb{C}P^1 \to \mathbb{C}P^3$ for any $e$ such that $i(U(1)) \subset N_e$. But $\alpha \in U(1)$ acts on $S^3 W$ with eigenvalues $\alpha^3, \bar{\alpha}^3, \bar{\alpha}, \alpha$ and respective eigenvectors $e_+$, $e_+ j$, $e_-$, $e_- j$; thus $e$ must be proportional to one of the latter Taking $e = e_\pm$ gives embeddings

$$\psi_\pm\colon \mathbb{C}P^1 \to \mathbb{C}P^3$$

which are defined explicitly by the rows of $X$. In terms of the inhomogeneous coordinate $z = \beta\alpha^{-1}$ on $\mathbb{C}P^1$ and homogeneous coordinates on $\mathbb{C}P^3$,

$$\begin{aligned} \psi_+(z) &= \left(1, z^3, \sqrt{3}z, -\sqrt{3}z^2\right) \\ \psi_-(z) &= \left(-\sqrt{3}\bar{z}, \sqrt{3}z^2, 1 - 2|z|^2, \left(|z|^2 - 2\right)z\right). \end{aligned} \tag{9.3}$$

LEMMA 9.3. *The curve $\psi_+$ is horizontal holomorphic, whereas $\psi_-$ is only $J_2$ holomorphic.*

PROOF. If $\eta$ denotes the standard representation of $U(1) \subset Sp(1)$, then

$$sp(1) \cong S^2(\eta \oplus \bar{\eta}) \cong u(1) \oplus (\eta^2 \oplus \bar{\eta}^2)\,.$$

The differential $(\psi_+)_*\colon T_0\mathbb{C}P^1 \to T_0\mathbb{C}P^3$ is determined by the substitution

$$\zeta = \eta^3\,, \quad \Delta_- = \eta \oplus \bar{\eta}$$

in (9.1), so that

$$(\psi_+)_*(\bar{\eta}^2) \subset \bar{\zeta}\Delta_- \subset H^{\mathbb{C}}\,.$$

Decreeing $T_0^{1,0}\mathbf{C}P^1 \cong \bar{\eta}^2$ and using (8.5), we see that $\psi_+$ is horizontal holomorphic. A similar argument holds for $\psi_-$. $\square$

By construction,

$$\varphi = \pi\circ\psi_+ = a\circ\pi\circ\psi_-: \mathbf{C}P^1 \to \mathbf{H}P^1\,,$$

where $\pi$: $(z_0, z_1, z_2, z_3) \to (z_0 + jz_1, z_2 + jz_3)$, and $a$ is the antipodal map sending a quaternionic line to its orthogonal complement. Consequently $\varphi$ is isotropic harmonic and we can identify $\psi_\pm = \tilde{\varphi}_\pm$.

Just as $\psi_+$ is essentially the Veronese curve in $\mathbf{C}P^3$, so $\varphi$ is the Veronese surface in $S^4$ [EW$_2$, section 8]; that interrelationship has been studied in detail by Rawnsley. Lemma 9.3 generalizes to give various harmonic maps $\mathbf{C}P^1 \to \mathbf{H}P^n$, $n \geqslant 2$, [S$_3$].

From theorem 4.2 the distribution $(T^{1,0})^h$ of horizontal (1, 0) vectors in $\mathbf{C}P^3$ is complex analytic. Indeed, Bryant shows in [Br] that given meromorphic functions $f$, $g$ on a Riemann surface $M$,

$$\psi = \left(1,\, f - \frac{1}{2}\, g\,\frac{df}{dg},\; g,\, \frac{1}{2}\,\frac{df}{dg}\right) \tag{9.4}$$

is horizontal holomorphic, and that $f$, $g$ can be chosen to make $\psi$ an embedding when $M$ is compact. The projection $\varphi = \pi\circ\psi$: $M \to \mathbf{H}P^1$ is then an isotropic harmonic or « superminimal » immersion. Taking $f = -2z^3$, $g = \sqrt{3}z$ on $\mathbf{C}P^1$ gives $\psi = \psi_+$ (9.3).

The canonical bundle of $(\mathbf{C}P^3, J_1)$ is isomorphic to $\zeta^4$ (cf. proposition 8.1), so $c_1(\zeta)$ is the positive generator of $H^2(\mathbf{C}P^2, \mathbf{Z})$. It follows that the twistor degrees of a conformal harmonic map $\varphi$: $M \to S^4$ from a compact Riemann surface are the ordinary topological degrees

$$d_\pm = \deg \tilde{\varphi}_\pm\,.$$

For an immersion without double points, the Euler characteristic $e = c(Q_+)$ of the normal bundle equals twice the self-intersection number (cf. [LS]).

PROPOSITION 9. 4. *If $\varphi$: $M \to S^4$ is a real isotropic harmonic immersion which is not totally geodesic, then $|e| \geqslant 2p + 4$.*

PROOF. Reversing orientation if necessary, we may assume that $\tilde{\varphi}_+$ is horizontal holomorphic. If $\tilde{\varphi}_+$ lies in a hyperplane $\mathbf{C}P^2$, one can deduce from (9.4) that $\tilde{\varphi}_+$ actually lies in a $\mathbf{C}P^1$ and that $\varphi$ maps to an equatorial

$S^2$. Therefore $\tilde{\varphi}_+$ is full and from proposition 8.3,

$$3 \leqslant d_+ = 1 - p + \tfrac{1}{2} e . \quad \square$$

In the above proof, the twistor degree $d_+$ can be identified with $1/4\pi$ times the area of $\varphi(M)$ [Br, proposition 2.4]. Taking $p = 0$ we retrieve a theorem of Ruh [Ru] which asserts that any minimal immersion $S^2 \to S^4$ with a trivial normal bundle is totally geodesic. See also Gauduchon and Lawson [GL]. By the Hirsch-Smale theory, the set $\mathfrak{R}[S^2, S^4]$ of regular homotopy classes (i.e. immersions $S^2 \to S^4$ homotopic through immersions) is parametrized by $e$ with all even $e$ realized [Hi]. In particular *the class* $e = 2$ *has no harmonic representative.*

PROBLEM. For any 4-manifold $N$ two immersions $S^2 \to N$ are regularly homotopic iff they are homotopic as maps and have the same normal class in $H^2(S^2, \mathbb{Z})$ [Hi]. Which classes can be realized by minimal immersions?

## 10. – Riemannian submersion formulae.

This section is a continuation of the last; by studying the fibration $\pi\colon \mathbb{C}P^3 \to S^4$ we shall introduce the notion of complex isotropy for maps of surfaces into Kähler manifolds. Accordingly we focus attention on the Kähler metric $k = k_{t_0}$ on $\mathbb{C}P^3$.

Let $\tilde{\nabla}$ denote the Levi-Civita connection on $\mathbb{C}P^3$ corresponding to $k$, and as usual $\nabla$ denotes the Levi-Civita connection on $\mathbb{H}P^1$. We shall generally use $\alpha, \alpha'$ for basic vector fields on $\mathbb{C}P^3$ and $\beta, \beta'$ for vertical fields; the former are $\pi$-related to vector fields on $\mathbb{H}P^1$ which we denote by the same symbol. Thus

$$\alpha = \operatorname{hor}(\alpha + \beta) , \qquad \beta = \operatorname{ver}(\alpha + \beta)$$

are the components of $\alpha + \beta$ corresponding to $T\mathbb{C}P^3 = H \oplus V$ (4.3). Since the fibres of $\mathbb{C}P^3$ are totally geodesic, the Riemannian submersion equations of O'Neill [ON] give

$$(10.1)\quad \begin{aligned} &1)\ \operatorname{hor}(\tilde{\nabla}_\alpha \alpha') = \nabla_\alpha \alpha' , \qquad \operatorname{ver}(\tilde{\nabla}_\alpha \alpha') = A_\alpha \alpha' \\ &2)\ \operatorname{hor}(\tilde{\nabla}_\alpha \beta) = \tilde{\nabla}_\beta \alpha = A_\alpha \beta \\ &3)\ \operatorname{hor}(\tilde{\nabla}_\beta \beta') = 0 , \end{aligned}$$

where $A$ is the tensor defined by

$$A_\alpha \alpha' = \tfrac{1}{2}\,\mathrm{ver}\,[\alpha, \alpha'] \,, \qquad k(A_\alpha \beta, \alpha') + k(\beta, A_\alpha \alpha') = 0 \,.$$

Now $A$ measures the obstruction to the integrability of the horizontal distribution $H$, and is essentially the curvature of $\mathbf{C}P^3$ as a fibre bundle. Actually $A_\alpha \beta$ defines a $U(2)$-invariant element of $\Lambda^2 H^* \otimes V$ (cf. [S$_1$, proposition 3.2]), but we shall only need

LEMMA 10.1. *If $\alpha, \alpha'$ are tangent vectors belonging to $(T^{1,0})^h \subset H^{\mathbf{C}}$, then*

$$A_{\bar\alpha} \alpha' = 0 \,, \qquad A_\alpha \alpha' \in (T^{2,0})^v \,.$$

PROOF. Extend $\alpha, \alpha'$ to basic vector fields so that

$$\mathrm{ver}\,(\tilde\nabla_{\bar\alpha} \alpha') = A_{\bar\alpha} \alpha' = -A_{\alpha'} \bar\alpha = -\,\mathrm{ver}\,(\tilde\nabla_{\alpha'} \bar\alpha) \,.$$

For any $X \in T\mathbf{C}P^3$, $\tilde\nabla_X$ preserves the $J_1$ type decomposition. Since the extensions are of type $(1, 0)$ to first order along horizontal directions,

$$A_{\bar\alpha} \alpha' \in (T^{2,0})^v \cap (T^{0,2})^v = \{0\} \,.$$

Similarly $A_\alpha \alpha' \in (T^{2,0})^v$. □

Suppose that $\varphi\colon M \to S^4$ is an immersion, and let $\psi\colon M \to \mathbf{C}P^3$ denote either of the Gauss lifts $\tilde\varphi_+, \tilde\varphi_-$. Set

$$\delta\psi = \alpha + \bar\beta \,, \qquad \alpha \in H^{\mathbf{C}},\ \beta \in V^{\mathbf{C}} \,.$$

Thus $\alpha, \beta$ are really sections of $\psi^{-1} T\mathbf{C}P^3$, but in order to use (10 1) it is convenient to extend $\alpha$ to a basic vector field and $\beta$ to a vertical field on $\mathbf{C}P^3$. We can then write $\alpha = \delta\varphi$, and $\varphi$ is conformal iff $\alpha \in (T^{1,0})^h$. By theorem 5.3 $\varphi$ is also harmonic iff $\bar\beta \in (T^{0,2})^v$, and in this case $\alpha$ and $\beta$ are both $(1, 0)$-vectors relative to $J_1$.

It is instructive to prove corollary 9.2 directly for the case $t = t_0$. For the remainder of this section we shall use superscripts to indicate type decompositions relative to the standard complex structure $J_1$ on $\mathbf{C}P^3$. First suppose that $\varphi$ is conformal so that $\alpha^{0,1} = 0$ (if $\psi = \tilde\varphi_\pm$, $\alpha^{0,1} = \alpha_\pm^{0,1}$ in previous notation); then

$$(\bar\delta\delta\psi)^{0,1} = (\tilde\nabla_{\bar\alpha+\beta}(\alpha + \bar\beta))^{0,1} = \tilde\nabla_{\bar\alpha+\beta}(\bar\beta^{0,1}) \,.$$

Hence hor $(\bar{\delta}\delta\psi)^{0,1} = A_{\bar{\alpha}}(\bar{\beta}^{0,1})$ which vanishes by lemma 10.1, and $\psi$ is « horizontally harmonic » in the sense that its tension field $\tau_\psi$ is vertical. If $\varphi$ is conformal and harmonic,

$$\text{ver}\,(\bar{\delta}\delta\psi)^{1,0} = \text{ver}\,(\tilde{\nabla}_{\bar{\alpha}+\beta}\alpha) = A_{\bar{\alpha}}\alpha = 0$$

which proves that $\psi$ is harmonic relative to $k$.

THEOREM 10.2. *A conformal harmonic map* $\varphi\colon M \to S^4$ *is real isotropic iff the Gauss lift* $\psi$ *satisfies*

$$k(\delta^r\alpha, \delta^s\bar{\beta}) = 0\,, \qquad \forall r, s \geqslant 0\,. \tag{10.2}$$

PROOF. Firstly,

$$\delta\alpha = \tilde{\nabla}_{\alpha+\bar{\beta}}\alpha = \nabla_\alpha\alpha + A_\alpha\bar{\beta} = \delta^2\varphi + A_\alpha\bar{\beta}$$

is a $(1, 0)$-vector for $J_1$, so belongs to $(T^{1,0})^h$. But $A_\alpha\bar{\beta} \in (T^{0,1})^h$ which implies that

$$A_\alpha\bar{\beta} = -(\delta^2\varphi)^{0,1}$$

thus

$$\text{hor}\,(\delta^2\psi) = \text{hor}\,(\tilde{\nabla}_{\alpha+\bar{\beta}}(\alpha + \bar{\beta})) = \delta^2\varphi + 2A_\alpha\bar{\beta} = (\delta^2\varphi)^{1,0} - (\delta^2\varphi)^{0,1}\,.$$

Now $\delta^2\psi = \delta\alpha + \delta\bar{\beta}$ with $\delta\alpha$ horizontal, so if (10.2) holds then

$$0 = k(\text{hor}\,\delta^2\psi, \text{hor}\,\delta^2\psi) = -2k\big((\delta^2\varphi)^{1,0}, (\delta^2\varphi)^{0,1}\big) = -h(\delta^2\varphi, \delta^2\varphi)\,,$$

and $\varphi$ is real isotropic.

Conversely suppose that $\varphi$ is real isotropic, so that at least one of $\tilde{\varphi}_+$, $\tilde{\varphi}_-$ is horizontal (theorem 6.5). If $\beta = 0$, (10.2) is immediate, so we may assume that $\delta\varphi \wedge (\delta^2\varphi)^{1,0} = 0$ (see (6.5)). Thus

$$\delta\alpha = \tilde{\nabla}_{\alpha+\bar{\beta}}\alpha = (\delta^2\varphi)^{1,0}$$

is proportional to $\alpha$, and at each point of $\mathbb{C}P^3$,

$$\text{span}\,\{\delta^r\alpha\colon r \geqslant 0\} = \mathbb{C}\alpha\,.$$

Moreover

$$\text{span}\,\{\delta^s\bar{\beta}\colon s \geqslant 0\} = \mathbb{C}A_\alpha\bar{\beta} \oplus \mathbb{C}\bar{\beta}\,,$$

$A_\alpha \bar\beta$ being the horizontal component of $\delta\bar\beta$, and (10.2) follows from the orthogonality condition

$$k(A_\alpha \bar\beta, \alpha) = -k(\bar\beta, A_\alpha \alpha) = 0 . \quad \square$$

Given a mapping $\psi: M \to N$ from a Riemann surface $M$ to a Kähler manifold $N$ with

$$\delta\psi = \alpha + \bar\beta , \qquad \alpha, \beta \in T^{1,0}N ,$$

the notation

$$(D')^r \psi = (\nabla')^r \psi = \delta^{r-1}\alpha$$
$$(D'')^s \psi = (\nabla'')^s \psi = \bar\delta^{s-1}\beta$$

is often used. If $\psi$ satisfies (10.2) which means that the $D'$ and $D''$ osculating spaces are orthogonal, then $\psi$ is said to be *complex isotropic* [$\mathrm{EW}_2$, definition 5.5; R, section 8]. Since (10.2) implies (6.3) (with $\psi$ in place of $\varphi$), complex isotropy implies real isotropy. Twistor space interpretations of various types of isotropy are given in [$\mathrm{S}_3$, section 6].

Suppose that $\varphi: M \to S^4$ is a real isotropic harmonic map with $\tilde\varphi_+$ horizontal holomorphic. Then like $\tilde\varphi_+$, $\tilde\varphi_-$ is a complex isotropic harmonic map. Such maps have been classified [$\mathrm{EW}_2$] and consist of the complex analytic maps together with certain associated curves or « transforms ». One can show that $\tilde\varphi_-$ is the first associated curve of $\tilde\varphi_+$ in the sense of Eells and Wood [$\mathrm{EW}_2$, remark 6.10 (iv), $r = 1$]; for instance this applies to (9.3). From proposition 8.3,

$$\deg \tilde\varphi_- = 2 - 2p + r - \deg \tilde\varphi_+ .$$

Any harmonic map $S^2 \to \mathbb{C}P^n$, $n \geqslant 1$, is necessarily complex isotropic, as is any harmonic map $T^2 \to \mathbb{C}P^n$ of degree $\neq 0$ from a Riemann surface of genus 1 [$\mathrm{EW}_2$, section 7].

PROPOSITION 10.3. *If $\varphi: M \to S^4$ is conformal, harmonic, but not real isotropic, then*

$$-3(p-1) \leqslant \deg \tilde\varphi_\pm \leqslant p - 1 .$$

PROOF. Since $S^4$ is conformally flat and neither $\tilde\varphi_+$, $\tilde\varphi_-$ is horizontal, proposition 8.4 is applicable with either sign. $\square$

The only known examples satisfying the hypotheses of proposition 10.3 are minimal immersions in $S^3$ [$\mathrm{L}_1$] composed with the totally geodesic inclusion $S^3 \subset S^4$. In this case $\tilde{\varphi}_+$, $\tilde{\varphi}_-$ can be identified (see section 7), and will not be horizontal unless $\varphi$ is totally geodesic. Except in the latter case, $\varphi$ is not real isotropic, and $\tilde{\varphi}_\pm$ is far from being complex isotropic in the sense that its $D'$ and $D''$ osculating spaces each have their maximum possible 3 dimensions. Putting $r = 0 = e$ in proposition 8.3 gives $\deg \tilde{\varphi}_\pm = 1 - p$. In particular any branched minimal immersion of a surface of genus zero in $S^3$ must be a great 2-sphere [A; C₂].

The techniques of this section are applicable to other Riemannian manifolds $N$ admitting a Kähler twistor space. If $N$ is compact and 4-dimensional, the only possibilities are $S^4$ and $\mathbb{C}P^2$ [H₂]. The complex projective plane $\mathbb{C}P^2$ is selfdual, and the corresponding fibration $S_- \to \mathbb{C}P^2$ will be discussed in section 12. More generally one can take $N$ to be a $4n$-dimensional quaternionic Kähler symmetric space with positive scalar curvature (there is one for each compact simple Lie group [W; S₁; S₃]). Provided $\varphi\colon M \to N$ is *inclusive* in the sense that each $\varphi_*(T_m M)$ lies in a quaternionic line, then $\varphi$ has a well defined Gauss lift $\psi\colon M \to Z$, where $Z$ is a homogeneous Kähler manifold of complex dimension $2n + 1$.

## 11. – Kähler surfaces.

Throughout this section, we suppose that the Riemannian 4-manifold $N$ is Kähler, so that it has an orthogonal covariant constant complex structure $J$. Given a mapping $\varphi\colon M \to N$ from a Riemann surface, we adopt the notation of section 10 to write

$$\delta\varphi = \alpha + \bar{\beta}\,, \qquad \alpha, \beta \in T^{1,0}N\,.$$

Hence $\varphi$ is conformal iff $h(\alpha, \bar{\beta}) = 0$. Because the Levi-Civita connection preserves types and $\bar{\delta}\delta\varphi$ is real, $\bar{\delta}\alpha = \delta\beta$, and $\varphi$ is harmonic iff $\bar{\delta}\alpha = 0$. In this case $dz \otimes \bar{\beta}$, $dz \otimes \alpha$ are complex analytic forms which vanish iff $\varphi$ is holomorphic or antiholomorphic respectively.

LEMMA 11.1. *A conformal map $\varphi\colon M \to N$ is real isotropic iff $h(\delta\alpha, \delta\bar{\beta}) = 0$, and complex isotropic iff $h(\delta\alpha, \bar{\beta}) = 0$.*

PROOF. The first assertion follows from the characterization (6.4) of real isotropy. Now suppose that $h(\delta\alpha, \bar{\beta}) = 0$. Differentiating $h(\alpha, \bar{\beta}) = 0$ gives also $h(\alpha, \delta\bar{\beta}) = 0$. But $\alpha, \beta$ form an orthogonal basis of $T^{1,0}N$ where they are both non-zero, so

$$\operatorname{span}\{\delta^r\alpha\colon r\geqslant 0\} = \mathbf{C}\alpha$$
$$\operatorname{span}\{\bar\delta^s\beta\colon s\geqslant 0\} = \mathbf{C}\beta\,,$$

and $\varphi$ is complex isotropic. □

From section 6 we know that, roughly speaking, real isotropic harmonic maps $\varphi\colon M\to N$ are those for which one of the two Gauss lifts $\tilde\varphi_+$, $\tilde\varphi_-$ is horizontal. To assess the possibilities, we first express the twistor spaces of the Kähler surface $N$ as complex projective bundles. Below, $\underline{\mathbf{C}}$ denotes the trivial complex line bundle, and $\|\beta\|^2 = h(\beta,\bar\beta)$.

THEOREM 11.2. *There are isomorphisms*

$$\begin{aligned} &\mathcal{S}_+\cong P(T^{2,0}N\oplus\underline{\mathbf{C}})\\ &\mathcal{S}_-\cong P(T^{1,0}N)\end{aligned}\tag{11.1}$$

*relative to which the Gauss lifts of a conformal immersion $\varphi\colon M\to N$ are*

$$\begin{aligned} &\tilde\varphi_+ = [\alpha\wedge\beta,\ \|\beta\|^2]\\ &\tilde\varphi_- = [\alpha]\,.\end{aligned}\tag{11.2}$$

PROOF. First note that (11.2) makes sense even at points where one of $\alpha,\beta$ vanish. For $\tilde\varphi_+$ is taken to be the projective class $[1,0]$ where $\beta=0$, and $\tilde\varphi_-$ is the line orthogonal to $[\beta]$ where $\alpha=0$.

The fundamental 2-vector $F$ of $N$ is a horizontal section of the twistor space $\mathcal{S}_+$. The proof of lemma 5.1 then gives $\mathcal{S}_-\cong P(T^{1,0}N)$. Indeed, given $\sigma\in\mathcal{S}_-$, $\pi(\sigma)=x$, we have

$$F = e_1\wedge e_2 + e_3\wedge e_4\,,\qquad \sigma = e_1\wedge e_2 - e_3\wedge e_4\,,$$

in some orthonormal basis, and $\sigma$ corresponds to the projective class $[e_1 - ie_2]$. By (2.3) and proposition 3.4,

$$\tilde\varphi_- = -ic(\alpha\wedge\bar\alpha - \beta\wedge\bar\beta)_0\,,$$

where $c$ is a positive normalizing factor, and the subscript 0 denotes the primitive component of the (1, 1) vector. Since $F+\tilde\varphi_-$ is proportional to $\alpha\wedge\bar\alpha$, $\tilde\varphi_-$ corresponds to $[\alpha]$.

If $\tau \in T^{2,0}N$ and $u \in \mathbf{C}$, the projective class $[\tau, u]$ determines an element

$$i(u\bar{\tau} - \tau\bar{u}) + \tfrac{1}{2}(\|\tau\|^2 - |u|^2)F \in \Lambda^2_+ TN$$

(see proposition 3.4 again), which after normalization lies in $\mathcal{S}_+$. We leave the reader to verify that this induces an isomorphism $P(T^{2,0}N \oplus \mathbf{C}) \cong \mathcal{S}_+$. Putting $\tau = \alpha\wedge\beta$, $u = \|\beta\|^2$, gives a 2-vector proportional to

$$2i(\alpha\wedge\beta - \bar{\alpha}\wedge\bar{\beta}) + (\|\alpha\|^2 - \|\beta\|^2)F = i(1 + *)\,\delta\varphi\wedge\bar{\delta}\varphi$$

whose normalization equals $\tilde{\varphi}_+$. $\square$

The isomorphisms (11.1) are related via (8.4) to the representation theoretic formulae

$$\Delta_+ = L \oplus \bar{L}$$
$$\Delta_- = L \otimes T^{1,0}N\,, \qquad \bar{L}^2 = T^{2,0}N\,,$$

for the spinor bundles of an almost Hermitian 4-manifold (see for example [$S_2$, section 7]).

Using (11.2) it is easy to check properties of the Gauss lifts. For example $\tilde{\varphi}_-$ is $J_2$ holomorphic iff the line $[\alpha]$ is stable under $\bar{\delta}$; i.e. $\alpha\wedge\bar{\delta}\alpha = 0$. This is actually equivalent to the equation $\bar{\delta}\alpha = 0$, thereby illustrating theorem 5.3. Moreover

COROLLARY 11.3. *A conformal harmonic map $\varphi\colon M \to N$ into a Kähler surface is complex isotropic iff $\tilde{\varphi}_-$ is horizontal, or $\varphi$ is holomorphic or antiholomorphic.*

PROOF. The argument is reminiscent of the proof of proposition 6.2. For $\varphi$ is complex isotropic iff $h(\delta\alpha, \bar{\beta}) = -h(\alpha, \delta\bar{\beta})$ vanishes. At each point this implies $\beta = 0$ or $\alpha\wedge\delta\alpha = 0$, and since $dz \otimes \bar{\beta}$ is complex analytic, one of these possibilities must hold everywhere. If $\alpha\wedge\delta\alpha$ vanishes identically but $\alpha$ does not, then $\tilde{\varphi}_- = [\alpha]$ is $J_1$ holomorphic and so horizontal. The converse is similar. $\square$

COROLLARY 11.4. *A conformal harmonic map $\varphi\colon M \to N$ has $\tilde{\varphi}_+$ horizontal iff $\|\alpha\|/\|\beta\|$ is constant. If $N$ is Kähler-Einstein with nonzero Ricci tensor, that constant is $0, 1$, or $\infty$.*

PROOF. Since $h(\delta\alpha, \bar{\alpha}) = \delta(\|\alpha\|^2)$, we have

$$\delta(\alpha\wedge\beta) = \delta\alpha\wedge\beta = \|\alpha\|^{-2}\delta(\|\alpha\|^2)\,\alpha\wedge\beta\,.$$

Therefore $[\alpha\wedge\beta, \|\beta\|^2]$ is stable under $\delta$ iff

$$\|\alpha\|^{-2}\delta(\|\alpha\|^2) = \|\beta\|^{-2}\delta(\|\beta\|^2)\ ,$$

which is equivalent to $\delta(\|\alpha\|/\|\beta\|) = 0$.

If $\|\alpha\|/\|\beta\|$ is some constant other than 0 or $\infty$, then some multiple of $\tau = \alpha\wedge\beta$ is covariant constant and

$$R(\delta\varphi, \bar\delta\varphi)(\tau) = 0\ , \tag{11.3}$$

where $R \in S^2(T^{1,1}N)$ is the curvature tensor of the Kähler manifold $N$. The space $T_0^{1,1}N$ of primitive vectors acting as a derivation annihilates $T^{2,0}N$, whereas $F$ acts as the identity. Consequently in (11.3) $R$ can be replaced by the Ricci form $\varrho$ to yield $\varrho(\alpha, \bar\alpha) = \varrho(\beta, \bar\beta)$. If $\varrho$ is a non-zero multiple of the Kähler form $\omega$, then $\|\alpha\|^2 = \|\beta\|^2$. □

The values $\infty, 0, 1$, of $\|\alpha\|/\|\beta\|$ correspond to $\varphi$ holomorphic, antiholomorphic, and totally real respectively. In the latter case, $J(\varphi_* T_m M)$ is orthogonal to $\varphi_* T_m M$. We remark that one can define [CW, ETG] the Kähler angle $\theta_\varphi$ at a point where $\varphi_* \neq 0$ by

$$\cot\left(\tfrac{1}{2}\theta_\varphi\right) = \|\alpha\|/\|\beta\|\ , \qquad 0 \leqslant \theta_\varphi \leqslant \pi\ .$$

That plays the same role as the basic function $k_\varphi\colon M \to \mathbb{R}$ of Lichnerowicz [Li, section 17], defined by $k_\varphi = \langle \omega^M, \varphi^*\omega^N\rangle = \cos\theta_\varphi$, where $\omega^M, \omega^N$ are the Kähler forms of $M, N$ (as in section 3). If $M$ is compact, then

$$\int k_\varphi\, dx$$

is a homotopy invariant of the map $\varphi$.

It may be that no multiple of $\omega^N$ represents an integral cohomology class. In contrast, the Ricci form $\varrho$ represents $2\pi$ times the first Chern class $c_1(T^{1,0}N)$ of $N$. Accordingly given a map $\varphi\colon M \to N$ from a compact Riemann surface, we consider the integer

$$c_\varphi = c_1(\varphi^{-1}T^{1,0}N)[M] = \frac{1}{2\pi}\int_M \varphi^*\varrho\ .$$

The next result appears in [ETG] for immersions into $\mathbf{C}P^2$; in this case $c_\varphi = 3\deg\varphi$ since the first Chern class of $\mathbf{C}P^2$ is 3 times the positive generator of $H^2(\mathbf{C}P^2, \mathbb{Z})$. The more general proof below is due to J. H. Rawnsley, and is based upon a calculation of the twistor degree $d_-$ using lemma 8.2 and proposition 8.3.

THEOREM 11.5. *If $\varphi$: $M \to N$ is a conformal harmonic map into a Kähler surface which is neither holomorphic nor antiholomorphic, then*

$$|c_\varphi| + e \leqslant 2p - 2 \, .$$

PROOF. The Gauss lift $\tilde{\varphi}_-$ determines a splitting

$$(\varphi^{-1} TN)^{\mathbf{C}} = T_-^{1,0} \oplus T_-^{0,1}$$

(5.2), in addition to the type decomposition

$$(\varphi^{-1} TN)^{\mathbf{C}} = \varphi^{-1} T^{1,0} N \oplus \varphi^{-1} T^{0,1} N$$

relative to $J$. Generically the components $\alpha, \bar{\beta}$ of $\delta\varphi$ span a negatively oriented maximal isotropic subspace containing $\delta\varphi$, which is therefore $T_-^{1,0}$ Hence $\alpha, \beta$ generate line bundles $[\alpha]$, $[\beta]$ satisfying

$$\varphi^{-1} T^{1,0} N = [\alpha] \oplus [\beta] \, , \qquad T_-^{1,0} = [\alpha] \oplus \overline{[\beta]} \, .$$

Furthermore

$$a = |dz \otimes \alpha| = -\chi + c_1[\alpha] \geqslant r \geqslant 0$$

$$b = |dz \otimes \bar{\beta}| = -\chi - c_1[\beta] \geqslant r \geqslant 0 \, ,$$

and so

$$\chi + r - e = 2d_- = c_1(T_-^{1,0}) = 2\chi + a + b \, , \qquad c_1(\varphi^{-1} T^{1,0} N) = a - b \, .$$

Therefore

$$\chi + e + c_\varphi = (r - b) - b \leqslant 0$$

$$\chi + e - c_\varphi = (r - a) - a \leqslant 0 \, ,$$

and putting $\chi = 2 - 2p$ gives the result. □

The last three lines also give

$$2d_+ = \chi + r + e \leqslant 0 \, ;$$

this inequality was proved independently by Webster [We, theorem 3], and used to deduce that any embedded minimal 2-sphere in $\mathbf{C}P^2$ is a complex line or conic. See also [ETG, theorem 4.7; GL, theorem D].

EXAMPLE. The case of $\mathbf{C}P^2$ will be covered in depth in the next section, but first consider at the other extreme a Kähler surface $N$ with zero Ricci form. This means that the bundle $\mathcal{S}_+$ is flat, and in particular that $N$ is $-$selfdual. If we also suppose that $N$ is simply connected and compact, then it is a $K3$ surface endowed with the Yau-Calabi metric [$\mathrm{H}_1$]. In this case $\mathcal{S}_+$ is naturally a product $N\times S^2$, and $N$ is said to have a hyperKähler structure because each point $aI + bJ + cK \in S^2$ $(a^2+b^2+c^2=1)$ defines a Kähler metric.

Since $c_1(T^{1,0}N)=0$, any conformal harmonic map $\varphi\colon M\to N$ satisfies

$$e \leqslant 2p-2 .$$

This was obtained independently by Poon [P], using integral representations of the characteristic classes. In analogy to (5.5), $\varphi$ has an antiholomorphic Gauss map $\gamma_\varphi = \pi_2\circ\tilde{\varphi}_+$, where $\pi_2$ is the projection $\mathcal{S}_+\to S^2$. Thus $\gamma_\varphi$ is constant iff $\varphi$ is holomorphic relative to one of the complex structures $aI+bJ+cK$, in which case one may say that $\varphi$ is *hyperholomorphic.*

PROBLEM. Is every real isotropic harmonic map $\varphi\colon M\to N$ hyperholomorphic? This amounts to asking whether $\tilde{\varphi}_-$ can ever be horizontal, and requires knowledge of the curvature of the Yau metric.

PROBLEM. Any map $\varphi\colon M\to N$ between compact Riemannian manifolds can be deformed to a harmonic map, provided that the sectional curvature of $N$ is nonpositive [ESa]. We doubt that such an assertion would be valid if the curvature restriction is replaced by one involving Ricci curvature. A search for the simplest counterexample leads us to the following question:

Let $N$ be a $K3$ surface endowed with a Ricci-flat Kähler metric. Does every homotopy class of maps $\varphi\colon S^2\to N$ have a harmonic representative? We know that there are such $N$ for which every holomorphic map $\varphi\colon S^2\to N$ is constant. On the other hand, there is a set of generators of $\pi_2(N)$ consisting of conformal harmonic maps [SU].

## 12. – Maps into $\mathbf{C}P^2$.

The projective holomorphic tangent bundle $P(T^{1,0}\mathbf{C}P^2)$ of $\mathbf{C}P^2$ can be identified with the flag manifold

$$F = U(3)/U(1)\times U(1)\times U(1) .$$

Labelling the $U(1)$ factors in the isotropy by the digits 0, 1, 2, let $\zeta_0, \zeta_1, \zeta_2$

denote the corresponding complex 1-dimensional $U(1)$ modules. Thus

$$u(3) \cong (\zeta_0 \oplus \zeta_1 \oplus \zeta_2) \otimes (\bar{\zeta}_0 \oplus \bar{\zeta}_1 \oplus \bar{\zeta}_2) \cong u(1) \oplus u(1) \oplus u(1) \oplus m ,$$

where

$$m \cong (\zeta_1 \bar{\zeta}_2 \oplus \bar{\zeta}_1 \zeta_2) \oplus (\zeta_2 \bar{\zeta}_0 \oplus \bar{\zeta}_2 \zeta_0) \oplus (\zeta_0 \bar{\zeta}_1 \oplus \bar{\zeta}_0 \zeta_1)$$

can be identified with the tangent space to $F$ at the identity coset. The three real subspaces of $m$ are the vertical spaces of respective projections $\pi_0, \pi_1, \pi_2$: $F \to \mathbf{C}P^2$. To be more precise, define a nonintegrable almost complex structure $\tilde{J}$ on $F$ by

$$T^{1,0}F = \bar{\zeta}_1 \zeta_2 \oplus \bar{\zeta}_2 \zeta_0 \oplus \bar{\zeta}_0 \zeta_1 \quad \text{for } \tilde{J} . \tag{12.1}$$

For each $i$ there is an identification $\mathcal{S}_- \cong F$ for which $\pi_i$ is the twistor projection that identifies $\tilde{J}$ with the almost complex structure $J_2$ of (4.4).

By general principles $F$ admits $2^3 = 8$ invariant almost complex structures and exactly $3! = 6$ of these are integrable [BH, chapter 4]. For example, there exists a unique complex structure $J_{02}$ for which $\pi_0$ is holomorphic and $\pi_2$ is antiholomorphic:

$$T^{1,0}F = \bar{\zeta}_1 \zeta_2 \oplus \zeta_2 \bar{\zeta}_0 \oplus \bar{\zeta}_0 \zeta_1 \quad \text{for } J_{02} . \tag{12.2}$$

In fact $J_{02}$ coincides with the structure $J_1$ of (4.4) with respect to $\pi_1$: By contrast $\tilde{J}$ is up to sign the only invariant structure which is not integrable, and makes $F$ a 3-symmetric space, as explained in section 9.

Let $\psi$: $M \to F$ be a map, and in terms of a local coordinate $z$ on $M$, suppose that $\psi(z)$ is the flag corresponding to an orthogonal sum

$$\mathbf{C}^3 = \varphi_0(z) \oplus \varphi_1(z) \oplus \varphi_2(z) ,$$

where $\varphi_i(z) = \pi_i \circ \psi(z)$ is a line in $\mathbf{C}^3$. Let $\partial = \partial/\partial z$ denote componentwise differentiation in $\mathbf{C}^3$.

LEMMA 12.1. *$\psi$ is $\tilde{J}$ holomorphic iff*

$$\partial(\varphi_0) \subset \varphi_0 \oplus \varphi_1 , \qquad \partial(\varphi_1) \subset \varphi_1 \oplus \varphi_2 , \qquad \partial(\varphi_2) \subset \varphi_2 \oplus \varphi_0 .$$

PROOF. The component of $\partial(\varphi_i)$ in $\varphi_j$ $(i \neq j)$ can be identified with the component of the tangent vector $\psi_*(\partial/\partial z)$ in $\bar{\zeta}_i \zeta_j \subset m$. □

The inclusions in lemma 12.1 can be represented by a triangle:

$$\begin{tikzcd} & \varphi_0 \arrow[dl, "\partial"'] & \\ \varphi_1 \arrow[rr, "\partial"'] & & \varphi_2 . \arrow[ul, "\partial"'] \end{tikzcd} \tag{12.3}$$

Equivalently one could reverse the direction of the arrows, and replace $\partial$ by $\bar{\partial} = \partial/\partial\bar{z}$.

By corollary 5.4, there is an essentially bijective correspondence between $\tilde{J}$ holomorphic curves in $F$ and conformal harmonic maps into $\mathbf{C}P^2$. Given a $\tilde{J}$ holomorphic curve $\psi\colon M \to F$, the three projections

$$\varphi_i = \pi_i \circ \psi\colon M \to \mathbf{C}P^2$$

will all be conformal and harmonic. Conversely if we start with a conformal harmonic map $\varphi = \varphi_0\colon M \to \mathbf{C}P^2$, its Gauss lift $\psi = \tilde{\varphi}_-$ via $\pi_0$ is $\tilde{J}$ holomorphic; thus:

THEOREM 12.2. *A conformal harmonic map $\varphi\colon M \to \mathbf{C}P^2$ has two «transforms» $\varphi_1, \varphi_2\colon M \to \mathbf{C}P^2$ which are also conformal harmonic.*

Take a local lift $f\colon \mathcal{U} \to \mathbf{C}^3$, $\mathcal{U} \subset M$, of $\varphi$, so that on $U$, $\varphi = [f]$. If $\langle . , . \rangle$ denotes the standard inner product on $\mathbf{C}^3$, antilinear in the second factor, then

$$\begin{aligned} \varphi_1 &= [\langle f, f\rangle\, \partial f - \langle \partial f, f\rangle f] \\ \varphi_2 &= [\langle f, f\rangle\, \bar{\partial} f - \langle \bar{\partial} f, f\rangle f]\,, \end{aligned} \tag{12.4}$$

provided $\varphi$ is not antiholomorphic or holomorphic respectively (cf. [$\mathrm{EW}_2$, remark 6.10 (iv)]).

The above theory is well known if $\varphi = \varphi_0$ happens to be holomorphic, so that $\bar{\partial}(\varphi_0) \subset \varphi_0$. In this case (12.3) breaks up into a linear sequence

$$\begin{tikzcd} \varphi_0 \arrow[r, shift left, "\partial"] & \varphi_1 \arrow[l, shift left, "\bar{\partial}"] \arrow[r, shift left, "\partial"] & \varphi_2 . \arrow[l, shift left, "\bar{\partial}"] \end{tikzcd}$$

Consequently $\varphi_2$ is antiholomorphic, whereas $\varphi_1$ is a conformal harmonic map which in general is neither holomorphic nor anitholomorphic. In these circumstances $\varphi_1$ is said to be *associated* (to the holomorphic curve $\varphi_0$) [ETG; $\mathrm{EW}_2$]. Referring to (12.2), $\psi$ must be the natural $J_{02}$ holomorphic lift of $\varphi_0$ to $P(T^{1,0}\mathbf{C}P^2) = F$. But $\psi$ is also $\tilde{J}$ holomorphic, so it is actualy horizontal for the projection $\pi_1$ and by corollary 11.3, $\varphi_1$ is complex isotropic. Conversely if $\varphi_1\colon M \to \mathbf{C}P^2$ is any complex isotropic harmonic map which is neither holomorphic nor antiholomorphic, then its Gauss lift $\psi$ via $\pi_1$ is horizontal. Therefore $\varphi_0 = \pi_0 \circ \psi$ is holomorphic and $\varphi_1$ is associated. Letting $\mathcal{O}$ denote the set of full holomorphic maps $M \to \mathbf{C}P^2$ (those not contained in some $\mathbf{C}P^1$) one obtains the following special case of the classification theorem [$\mathrm{EW}_2$, theorem 6.9]:

COROLLARY 12.3. *There is a bijective correspondence between full complex isotropic harmonic maps $M \to \mathbf{C}P^2$ and the set $\mathcal{O} \times \{0, 1, 2\}$.*

Combining the above facts with theorem 6.5, we conclude that *the real isotropic harmonic maps into $\mathbf{C}P^2$ consist of the holomorphic, antiholomorphic, associated, and totally real ones.* The latter are yet to be classified.

The fact that $\mathbf{C}P^2$ has constant holomorphic curvature implies that globally defined forms

$$dz^3 \otimes (\alpha \wedge \delta\alpha)\,, \qquad dz^3 \otimes (\bar\beta \wedge \delta\bar\beta)$$

associated to a harmonic map $\varphi\colon M \to \mathbf{C}P^2$ are complex analytic. The contraction of their product equals $-2P^2$, where

$$P = h(\delta\alpha, \bar\beta)\, dz^3$$

is the cubic differential introduced in [$\mathrm{EW}_2$, section 7] and studied in detail in [CW] and [Wo]. (Chern and Wolfson call $\varphi$ superminimal if $P$ vanishes; in our language this is equivalent to $\varphi$ being complex isotropic.) If $\varphi\colon M \to \mathbf{C}P^2$ is a conformal harmonic map which is not complex isotropic, then proposition 8.5 is applicable giving

$$e = 4 - 4p + (r + s_+) \geqslant 4 - 4p\,.$$

On the other hand, proposition 11.5 yields

$$e + 3|\deg \varphi| \leqslant 2p - 2\,,$$

and in particular

$$|\deg \varphi| \leqslant 2p - 2 \tag{12.5}$$

(cf. [$\mathrm{EW}_2$, proposition 7.8]).

EXAMPLES. Consider the parametrization

$$\varphi(t, u) = [e^{it}, e^{iu}, 1]$$

of the Clifford torus in $\mathbb{C}P^2$. Take a complex vector $\partial/\partial z = \partial/\partial t + \omega\partial/\partial u$ on $M = T^2$, where $\omega$ is to be determined. Applying (12.4) gives

$$\begin{aligned} \varphi_1 &= [(\omega - 2)e^{it}, (1 - 2\omega)e^{iu}, 1 + \omega] \\ \varphi_2 &= [(\bar\omega - 2)e^{it}, (1 - 2\bar\omega)e^{iu}, 1 + \bar\omega], \end{aligned} \tag{12.6}$$

and the orthogonality of $\varphi_1, \varphi_2$ (equivalent to the conformality of $\varphi$) implies that $\omega^2 - \omega + 1 = 0$. Thus $\omega$ is a primitive 6-th root of unity, which ensures the 3-fold symmetry:

$$\begin{aligned} \varphi_1 &= [\omega^2 e^{it}, -\omega e^{iu}, 1] \\ \varphi_2 &= [-\omega e^{it}, \omega^2 e^{iu}, 1]. \end{aligned}$$

These transforms give essentially the same map. The harmonicity of $\varphi$ now follows from the formula $\bar\partial\varphi_1 \subset \varphi_0 \oplus \varphi_1$. The above representatives of $\varphi_1, \varphi_2$ correspond to the components $\alpha, \bar\beta$ of the tangent vector $\delta\varphi$. Since $\|\alpha\| = \|\beta\|$, $\varphi$ is totally real. See Naitoh [N] for similar examples.

PROBLEM. Does there exist a conformal harmonic map $T^2 \to \mathbb{C}P^2$ which is not real isotropic? Such a map has degree 0 by (12.5), and $r = 0 = s_+$ by proposition 8.4 and theorem 11.5.

The Veronese map $\varphi: \mathbb{C}P^1 \to \mathbb{C}P^2$ is given by

$$\begin{aligned} \varphi = \varphi_0 &= [1, z, z^2] \\ \varphi_1 &= [\bar z(1 + 2|z|^2), |z|^4 - 1, -z(2 + |z|^2)] \\ \varphi_2 &= [\bar z^2, \bar z, 1], \end{aligned}$$

and $\varphi_1$ has degree 0 [$\mathrm{EW}_2$, example 8.1]. More generally, in the context of theorem 12.2 we have

PROPOSITION 12.4. *If $\varphi: M \to \mathbb{C}P^2$ is conformal harmonic with twistor degree $d_- = 1 - p + \frac{1}{2}r - \frac{1}{2}e$, then*

$$\begin{aligned} 2 \deg \varphi_1 &= -\deg \varphi - 2d_- \\ 2 \deg \varphi_2 &= -\deg \varphi + 2d_-. \end{aligned}$$

PROOF. Put $\varphi = \varphi_0$ and $d_i = \deg \varphi_i$. There are complex line bundles $\zeta_0, \zeta_1, \zeta_2$ on $F$ such that $\zeta_0 \oplus \zeta_1 \oplus \zeta_2$ is trivial, and

$$d_i = -(\psi^* c_1(\zeta_i))[M].$$

By (12.1) the vertical bundle $(T^{2,0})^v$ for the projection $\pi_0$ equals $\zeta_1\bar{\zeta}_2$, so by lemma 8.2

$$2d_- = -d_1 + d_2 .$$

The result now follows from the equation $d_0 + d_1 + d_2 = 0$. $\square$

Finally, if $\varphi$ is also holomorphic then by considering the horizontal lift $\tilde{\varphi}_+$, it is easily shown that $2d_+ = 3\deg\varphi$, so that in this case

$$\deg\varphi_1 = \deg\varphi + 2p - 2 - r$$

$$\deg\varphi_2 = -2\deg\varphi - 2p + 2 + r .$$

REMARK. The ideas of this section are also applicable for studying maps $M \to \mathbf{C}P^n$ for arbitrary $n$. In place of $F$ one can first consider the flag manifold

$$U(n+1)/U(1)\times U(1)\times U(n-1) ,$$

which also has an almost complex structure $\tilde{J}$ for which lemma 12.1 holds, except that $\varphi_2$ is now a map into the Grassmannian $G_{n-1}(\mathbf{C}^{n+1}) \cong G_2(\mathbf{C}^{n+1})$. Once again, $\tilde{J}$ holomorphic curves correspond to conformal harmonic maps into $\mathbf{C}P^n$, but they also correspond to inclusive conformal harmonic maps in the quaternionic Kähler manifold $G_2(\mathbf{C}^{n+1})$ (see the end of section 10).

PROBLEM. Let $N$ be a compact simply connected 4-manifold. A theorem of Thom asserts that every homology class $\alpha \in H_2(N)$ can be represented by an embedded closed oriented surface. The minimum genus of such a surface is not yet determined; however, not every class can be represented by an embedded 2-sphere. Given Riemannian structures on $M$ and $N$, which classes $\alpha$ can be represented by branched minimal immersions? Is there a lower bound for genus $M$ in terms of the twistor degrees? For instance, if $N = \mathbf{C}P^2$ and $\alpha = a\gamma$ where $\gamma$ generates $H_2(\mathbf{C}P^2)$, then is genus $M \geqslant (a-1)(a-2)/2$?

## REFERENCES


[A] ALMGREN F. J., *Some interior regularity theorems for minimal surfaces and an extension of Bernstein's theorem*, Ann. of Math., **84** (1966), pp. 277-292.

[AHS] ATIYAH M. F. - HITCHIN N. J. - SINGER I. M.: *Self-duality in four-dimensional Riemannian geometry*, Proc. Roy. Soc. Lond. Ser. A, **362** (1978), pp. 425-461.

[B] Berger M., *Les variétés riemanniennes homogènes normales simplement connexes à courbure strictement positive*, Ann. Scuola Norm. Sup. Pisa Cl. Sci., **15** (1961), pp. 179-246.

[Be] Sém. A. Besse (1978-79), *Géométrie riemannienne en dimension* 4, Cedic/F. Nathan (1981).

[BH] Borel A. - Hirzebruch F., *Characteristic classes and homogeneous spaces I*, Amer. J. Math., **80** (1958), pp. 458-538.

[Br] Bryant R. L., *Conformal and minimal immersions of compact surfaces into the* 4*-sphere*, J. Differential Geom., **17** (1982), pp. 455-473.

[Bu] Burstall F. E., *Non-linear functional analysis and harmonic maps*, Ph. D. Thesis, University of Warwick (1984).

[$C_1$] Calabi E., *Minimal immersions of surfaces in Euclidean spheres*, J. Differential Geom., **1** (1967), pp. 111-125.

[$C_2$] Calabi E., *Quelques applications de l'analyse complexe aux surfaces d'aire minima. Topics in Complex Manifolds*, Univ. de Montréal (1967), pp. 59-81.

[$Ch_1$] Chern S. S., *Minimal surfaces in Euclidean space of N dimensions*, in *Differential and Combinatorial Topology: Symposium in honor of Marston Morse*, Princeton University Press (1965), pp. 187-198.

[$Ch_2$] Chern S. S., *On the minimal immersions of the two-sphere in a space of constant curvature. Problems in Analysis*, Princeton University Press (1970), pp. 27-40.

[CW] Chern S. S. - Wolfson J. G., *Minimal surfaces by moving frames*, Amer. J. Math., **105** (1983), pp. 59-83.

[$E_1$] Eells J., *Gauss maps of surfaces*, in *Perspectives in Mathematics*, Anniversary of Oberwolfach 1984, Birkhäuser.

[$E_2$] Eells J., *Minimal branched immersions into three-manifolds*, to appear.

[EL] Eells J. - Lemaire L., *A report on harmonic maps*, Bull. London Math. Soc., **10** (1978), pp. 1-68.

[ES] Eells J. - Salamon S., *Constructions twistorielles des applications harmoniques*, C. R. Acad. Sci. Paris, **296** (1983), pp. 685-687.

[ESa] Eells J. - Sampson J. H., *Harmonic mappings of Riemannian manifolds*, Amer. J. Math., **86** (1964), pp. 109-160.

[$EW_1$] Eells J. - Wood J. C., *Maps of minimum energy*, J. London Math. Soc., **23** (1981), pp. 303-310.

[$EW_2$] Eells J. - Wood J. C., *Harmonic maps from surfaces to complex projective spaces*, Adv. in Math., **49** (1983), pp. 217-263.

[ETG] Eschenburg J. H. - Tribuzy R. de A. - Guadalupe I. V., *The fundamental equations of minimal surfaces in* $CP^2$, Math. Ann., **270** (1985), pp. 571-598.

[F] Friedrich T., *On surfaces in four-spaces*, Ann. Glob. Analysis and Geometry, **2** (1984), pp. 257-287.

[$G_1$] Gray A., *Minimal varieties and almost Hermitian manifolds*, Michigan Math. J., **12** (1965), pp. 273-287.

[$G_2$] Gray A., *Riemannian manifolds with geodesic symmetries of order* 3, J. Differential Geom., **7** (1972), pp. 343-369.

[GOR] Gulliver R. D. - Osserman R. - Royden H. L., *A theory of branched immersions of surfaces*, Amer. J. Math., **95** (1973), pp. 750-812.

[Hi] Hirsch M. W., *Immersions of manifolds*, Trans. Amer. Math. Soc., **93** (1959), pp. 242-276.

[Hir] HIRZEBRUCH F., *Topological Methods in Algebraic Geometry*, 3rd edition, Springer, 1966.

[$H_1$] HITCHIN N. J., *Compact four-dimensional Einstein manifolds*, J. Differential Geometry, **9** (1974), pp. 435-441.

[$H_2$] HITCHIN N. J., *Kählerian twistor spaces*, Proc. London Math. Soc., **43** (1981), pp. 133-150.

[$H_3$] HITCHIN N. J., *Monopoles and geodesics*, Comm. Math. Phys., **83** (1982), pp. 579-602.

[K] KODAIRA K., *On the structure of compact complex analytic surfaces II*, Amer. J. Math., **88** (1966), pp. 682-721.

[KM] KOSZUL J.-L. - MALGRANGE B., *Sur certaines structures fibrées complexes*, Arch. Math., **9** (1958), pp. 102-109.

[LS] LASHOF R. - SMALE S., *On the immersion of manifolds in Euclidean space*, Ann. Math., **68** (1958), pp. 562-583.

[$L_1$] LAWSON H. B., *Complete minimal surfaces in $S^3$*, Ann. of Math., **92** (1970), pp. 335-374.

[$L_2$] LAWSON H. B., *Lectures on Minimal Submanifolds*, 2nd edition, Publish or Perish, 1980.

[Le] LEBRUN C. R., *Twistor CR manifolds and 3-dimensional conformal geometry*, Trans. Amer. Math. Soc. **284** (1984), pp, 601-616.

[Li] LICHNEROWICZ A., *Applications harmoniques et variétés kählériennes*, Symposia Mathematica, **3** (1970), pp. 341-402.

[M] MICALLEF M. J., *Stable minimal surfaces in Euclidean space*, J. Differential Geom., **19** (1984), pp. 57-84.

[N] NAITOH H., *Isotropic submanifolds with parallel second fundamental form in $P^m(c)$*, Osaka J. Math., **18** (1981), pp. 427-464.

[NN] NEWLANDER A. - NIRENBERG L., *Complex analytic coordinates in almost complex manifolds*, Ann. of Math., **65** (1957), pp. 391-404.

[NW] NIJENHUIS A. - WOOLF W. B., *Some integration problems in almost-complex and complex manifolds*, Ann. of Math., **77** (1963), pp. 424-489.

[O] OBATA M., *The Gauss map of immersions of Riemannian manifolds in spaces of constant curvature*, J. Differential Geom., **2** (1968), pp. 217-223.

[ON] O'NEILL B., *The fundamental equations of a submersion*, Michigan Math. J., **13** (1966), pp. 459-469.

[P] POON Y. S., *Minimal surfaces in four-dimensional manifolds*, M. Sc. Thesis, University of Oxford, 1983.

[R] RAWNSLEY J. H., *f-structures, f-twistor spaces and harmonic maps*, in *Geometry Seminar Luigi Bianchi* II, to appear.

[Ru] RUH E. A., *Minimal immersions of 2-spheres in $S^4$*, Proc. Amer. Math. Soc., **28** (1971), pp. 219-222.

[RV] RUH E. A. - VILMS J., *The tension field of the Gauss map*, Trans. Amer. Math. Soc., **149** (1970), pp. 569-573.

[$S_1$] SALAMON S., *Quaternionic Kähler manifolds*, Invent. Math., **67** (1982), pp. 143-171.

[$S_2$] SALAMON S., *Topics in four-dimensional Riemannian geometry*, in *Geometry Seminar Luigi Bianchi*, Lecture Notes in Mathematics 1022, Springer (1983), pp. 33-124.

[$S_3$] SALAMON S., *Harmonic and holomorphic maps*, in *Geometry Seminar Luigi Bianchi* II, to appear.

[Sm] Smale S., *A survey of some recent developments in differential topology*, Bull. Amer. Math. Soc., **69** (1963), pp. 131-145.

[SU] Sacks J. - Uhlenbeck K., *The existence of minimal immersions of two-spheres*, Ann. of Math., **113** (1981), pp. 1-24.

[W] Wolf J. A., *Complex homogeneous contact manifolds and quaternionic symmetric spaces*, J. Math. Mech., **14** (1965), pp. 1033-1047.

[WG] Wolf J. A. - Gray A., *Homogeneous spaces defined by Lie group automorphisms I, II*, J. Differential Geom., **2** (1968), pp. 77-159.

[Wo] Wolfson J. G., *Minimal surfaces in complex manifolds*, Ph. D. Thesis, University of California Berkley (1982).

The text also contains last minute references to the following relevant preprints, which have recently appeared:

[GL] Gauduchon P. - Lawson H. B., *Topologically nonsingular minimal cones.*

[Gr] Gromov, M., *Pseudo holomorphic curves in symplectic manifolds.*

[We] Webster S. M., *Minimal surfaces in a Kähler surface.*

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