J. BLAND
IAN GRAHAM

On the Hausdorff measures associated to the Carathéodory and Kobayashi metrics

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On the Hausdorff Measures Associated to the Carathéodory and Kobayashi Metrics.

J. BLAND - IAN GRAHAM (*)

1. – Introduction and statement of results.

The purpose of this paper is to give explicit expressions (in some cases) and estimates for the Radon-Nikodym derivative with respect to Lebesgue measure in local coordinates of the top-dimensional Hausdorff measure associated to the Carathéodory and Kobayashi metrics (Theorem 1). In conjunction with the results of [8], this gives a new criterion for the biholomorphic equivalence of a bounded strictly convex domain in $\mathbf{C}^n$ with the unit ball (Theorem 2), thus answering a question posed in [8]. Our main tool for the proof of Theorem 1 is the work of H. Busemann on Hausdorff measure and Finsler metrics [3]. A second result of Busemann and Mayer [4] is used to deal with the fact that the indicatrix of the infinitesimal Kobayashi-Royden metric is in general not convex.

To state our results, let M be a complex manifold of dimension n and let $p \in M$. Choose local coordinates $z_1, \ldots, z_n$ at p and let λ^{2n} denote Lebesgue measure in these coordinates. Let B^n denote the unit ball in $\mathbf{C}^n$. Let $\mathcal{H}_C^{2n}$ (respectively $\mathcal{H}_K^{2n}$) denote the $2n$-dimensional Hausdorff measure associated to the Carathéodory (respectively Kobayashi) distance. Let $\mathcal{S}_C^{2n}$ (respectively $\mathcal{S}_K^{2n}$) denote the $2n$-dimensional spherical measure associated to the Carathéodory (respectively Kobayashi) distance. Let $I_C(p)$ (respectively $I_K(p)$) denote the indicatrix of the Carathéodory (respectively Kobayashi-Royden) metric at p. Let $\hat{I}_K(p)$ denote the convex hull of the latter. Let $C_n(p; \partial/\partial z_1 \wedge \ldots \wedge (\partial/\partial z_n)(p))$ (respectively $E_n(p; \partial/\partial z_1 \wedge \ldots \wedge (\partial/\partial z_n)(p))$) denote the Carathéodory (respectively Eisenman-Kobayashi) volume density at p.

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THEOREM 1. *With the notations as in the previous paragraph, we have the following*:

(*a*) *If M is Carathéodory-hyperbolic and $p \in M$ then*

$$\frac{d\mathcal{H}_C^{2n}}{d\lambda^{2n}}(p) = \frac{d\mathcal{S}_C^{2n}}{d\lambda^{2n}}(p) = \frac{\operatorname{vol}(B^n)}{\operatorname{vol}(I_C(p))}.$$

(*All volumes are Euclidean volumes.*)

(*b*) *If M is taut* (*more generally if M is hyperbolic and the infinitesimal Kobayashi-Royden metric is continuous*) *then*

$$\frac{d\mathcal{H}_K^{2n}}{d\lambda^{2n}}(p) = \frac{d\mathcal{S}_K^{2n}}{d\lambda^{2n}}(p) = \frac{\operatorname{vol}(B^n)}{\operatorname{vol}(\hat{I}_K(p))}.$$

(*c*) *If M is hyperbolic then*

$$\frac{d\mathcal{H}_K^{2n}}{d\lambda^{2n}}(p) \leqslant \frac{d\mathcal{S}_K^{2n}}{d\lambda^{2n}}(p) \leqslant \frac{\operatorname{vol}(B^n)}{\operatorname{vol}(\hat{I}_K(p))}.$$

Parts (*a*) and (*b*) of Theorem 1 enable us to answer a question posed in [8].

THEOREM 2. *Let Ω be a strictly convex bounded domain in $\mathbf{C}^n$. Let M be an n-dimensional complex manifold. Let p denote either a point of Ω or a point of M, and let $z_1, \ldots, z_n$ denote either global holomorphic coordinates on $\mathbf{C}^n \supset \Omega$ or local holomorphic coordinates in a neighborhood of $p \in M$.*

(*a*) *If there exists a point $p \in \Omega$ such that*

$$\frac{d\mathcal{H}_C^{2n}}{d\lambda^{2n}}(p) = C_n\left(p; \frac{\partial}{\partial z_1}\wedge\cdots\wedge\frac{\partial}{\partial z_n}(p)\right)$$

then Ω is biholomorphic to B^n.

(*b*) *If there exists a point $p \in \Omega$ such that*

$$\frac{d\mathcal{H}_K^{2n}}{d\lambda^{2n}}(p) = E_n\left(p; \frac{\partial}{\partial z_1}\wedge\cdots\wedge\frac{\partial}{\partial z_n}(p)\right)$$

then Ω is biholomorphic to B^n.

(c) *If M is Carathéodory-hyperbolic and there exists a point $p \in M$ such that*

$$\frac{d\mathcal{H}_C^{2n}}{d\lambda^{2n}}(p) = C_n\left(p;\ \frac{\partial}{\partial z_1}\wedge\ldots\wedge\frac{\partial}{\partial z_n}(p)\right)$$

then there exists a holomorphic map $f\colon M \to B^n$ such that f is an isometry relative to the Carathéodory metric at p, i.e. $df\colon T_pM \to T_0B^n$ is Carathéodory length-preserving.

(d) *If M is taut and there exists a point $p \in M$ such that*

$$\frac{d\mathcal{H}_K^{2n}}{d\lambda^{2n}}(p) = E_n\left(p;\ \frac{\partial}{\partial z_1}\wedge\ldots\wedge\frac{\partial}{\partial z_n}(p)\right)$$

and if $I_K(p)$ is convex then there exists a holomorphic map $f\colon B^n \to M$ such that $f(0) = p$ and f is an isometry relative to the Kobayashi metric at 0.

2. – Preliminaries.

For the moment let M denote a real m-dimensional differentiable manifold and let d be a distance function on M which induces the usual topology of M. Thus $d\colon M\times M \to \mathbb{R}$, and if p, q, r denote points of M then $d(p, q)\geqslant 0$, $d(p, q) = 0$ iff $p = q$, $d(p, q) = d(q, p)$, and $d(p, r)\leqslant d(p, q) + d(q, r)$.

We recall the definition of the top-dimensional Hausdorff and spherical measures associated to d. Let α_m denote the Euclidean volume of the unit ball B_m in $\mathbb{R}^m$. Let A be a subset of M. We first let

$$\Phi_\varepsilon^m(A) = \inf\left\{\alpha_m\sum_{j=1}^{\infty}\left(\frac{\operatorname{diam} A_j}{2}\right)^m \,\middle|\, A\subset\bigcup_{j=1}^{\infty}A_j \text{ and } \operatorname{diam} A_j<\varepsilon\right\} \tag{1}$$

and then define (noting that $\Phi_\varepsilon^m(A)$ increases as $\varepsilon\downarrow 0$) the m-dimensional Hausdorff outer measure of A by

$$\mathcal{H}^m(A) = \lim_{\varepsilon\to 0}\Phi_\varepsilon^m(A)\,. \tag{2}$$

If in (1) we require the sets $\{A_j\}_{j=1}^{\infty}$ to be balls in the metric d, and then take the limit as in (2), we obtain the top-dimensional spherical outer measure $\mathcal{S}^m(A)$. In either case all Borel subsets of M are measurable [6, p. 170]. In $\mathbb{R}^m$ with the Euclidean distance, $\mathcal{H}^m$ and $\mathcal{S}^m$ coincide with each other and with Lebesgue measure [6, p. 197]. On a Riemannian manifold,

$\mathcal{H}^m$ and $\mathcal{S}^m$ coincide with the measure defined by the Riemannian volume element [6, p. 281]. (The situation in lower dimensions is more complicated [6].)

Busemann [3] studied the top-dimensional spherical and Hausdorff measures associated to a (real) Finsler metric on a differentiable manifold. Precisely, the properties assumed by Busemann are the following:

DEFINITION. A Finsler metric on M is a nonnegative real-valued function F on the tangent bundle TM such that

$$F \text{ is continuous} \tag{3a}$$

$$F(p, \xi) = 0 \quad \text{iff } \xi = 0 \tag{3b}$$

$$F(p, c\xi) = |c| F(p, \xi) \quad \text{if } c \in \mathbb{R} \tag{3c}$$

$$F(p, \xi_1 + \xi_2) \leqslant F(p, \xi_1) + F(p, \xi_2)\,. \tag{3d}$$

The condition (3d) means that the restriction of F to each fibre satisfies the triangle inequality. This property is equivalent to requiring convexity of the indicatrix

$$I(p) = \{\xi \in T_p M \mid F(p, \xi) < 1\}$$

at each point $p \in M$. A Finsler metric on $\mathbb{R}^m$ which is independent of $p \in \mathbb{R}^m$ is called a Minkowski metric.

Busemann [3, § 6] showed that the top-dimensional Hausdorff and spherical measures defined by the distance function d associated to F coincide, and that their Radon-Nikodym derivative with respect to Lebesgue measure in local coordinates is given by $\operatorname{vol}(B^n)/\operatorname{vol}(I(p))$. (Note that this expression gives the Riemannian volume density on a Riemannian manifold.)

The idea behind Busemann's results is roughly as follows: for a Minkowski metric in $\mathbb{R}^m$ a form of the Vitali covering theorem immediately gives the Radon-Nikodym derivative of S^m with respect to Lebesgue measure; that $\mathcal{S}^m = \mathcal{H}^m$ is a direct consequence of the fact that for a Minkowski metric the subsets of $\mathbb{R}^m$ which maximize Euclidean volume for a fixed Minkowski diameter are precisely the Minkowski balls. Busemann's result for a general Finsler metric is then obtained by using the continuity of the Finsler metric to locally approximate it by a Minkowski metric. Convexity of the indicatrix plays an important role, for only with this assumption (and not with continuity of F alone) can one locally approximate balls in the distance d by scalar multiples of the indicatrix.

Convexity of $I(p)$ has the following additional consequence [4, p. 186]: suppose that F satisfies only properties (3a)-(3c) and that γ: $[0, 1] \to M$ is a piecewise C^1 (absolutely continuous would suffice) parametrized curve on M. There are then two possible ways to define the length of γ, namely

$$l_1(\gamma) = \int_0^1 F(\gamma(t), \gamma'(t))\, dt \tag{4a}$$

and

$$l_2(\gamma) = \sup \sum_{j=1}^{k} d(\gamma(t_{j-1}), \gamma(t_j)) \tag{4b}$$

where the sup is taken over all partitions of $[0, 1]$. These definitions coincide for all curves iff $I(p)$ is convex at all points $p \in M$ [4, p. 186].

A key result of Busemann and Mayer [4, p. 184] which applies to the Kobayashi metric on a taut manifold is the following: if we replace a given metric F satisfying (3a)-(3c) with the Finsler metric $\hat{F}$ whose indicatrix at each point is precisely the convex hull of the indicatrix of F, then the distance d associated to F coincides with the distance $\hat{d}$ associated to $\hat{F}$. (On referring to (4a) and (4b) we see that $l_1(\gamma)$ will change in general when we replace F by $\hat{F}$ but $l_2(\gamma)$ will not.)

Turning now to the case of an n-dimensional complex manifold M, we recall the definition of the Carathéodory and Kobayashi pseudometrics. Let D denote the unit disc in $\mathbb{C}$. The Carathéodory distance between two points p and q of M is defined by

$$C(p, q) = \sup \varrho(f(p), f(q))$$

where the sup is taken over all holomorphic maps f: $M \to D$ and ϱ denotes the Poincaré distance on D, i.e.

$$\varrho(a, b) = \frac{1}{2} \ln\left(1 + \left|\frac{a-b}{1-\bar{a}b}\right|\right) - \frac{1}{2} \ln\left(1 - \left|\frac{a-b}{1-\bar{a}b}\right|\right).$$

M is called Carathéodory-hyperbolic if $C(\, , \,)$ is a genuine distance, i.e. if $C(p, q) = 0$ iff $p = q$. The infinitesimal Carathéodory pseudometric is defined by

$$C(p; \xi) = \sup |f_*(\xi)|$$

where the sup is taken over all holomorphic maps f: $M \to D$ such that $f(p) = 0$, and $|f_*(\xi)|$ indicates Euclidean length. The indicatrix $I_C(p)$ of the Carathéodory metric at a point $p \in M$ is easily seen to be convex, and $C(\, ; \,)$

is known to be continuous [15]. Reiffen [15] showed that on a Carathéodory-hyperbolic manifold the two possible definitions of the Carathéodory length of a piecewise C^1 parametrized curve $\gamma\colon [0,1] \to M$, namely

$$l_1^C(\gamma) = \int_0^1 C(\gamma(t), \gamma'(t))\, dt \tag{5a}$$

and

$$l_2^C(\gamma) = \sup \sum_{j=1}^{k} C(\gamma(t_{j-1}), \gamma(t_j)) \tag{5b}$$

where the sup is taken over all partitions of $[0,1]$, coincide. This result does not follow immediately from the convexity of $I_C(p)$ using the theorem of Busemann and Mayer [4, p. 186] because the Carathéodory distance is not an inner distance. That is, if we define $\tilde{C}(p, q) = \inf l_1^C(\gamma) = \inf l_2^C(\gamma)$, where the inf is taken over all piecewise C^1 curves joining p to q, then $\tilde{C}(p, q) \neq C(p, q)$ in general. A counterexample appears in [1]. Busemann's theorem [3] applies directly to the Hausdorff and spherical measures associated to $\tilde{C}$ on a Carathéodory-hyperbolic manifold; however it is not hard to see that these measures are the same as those associated to C.

LEMMA 1. *If M is a Carathéodory-hyperbolic manifold then given a point $p_0 \in M$ and a number $\varepsilon > 0$, then there exists a neighborhood U of p_0 such that for all $p, q \in U$*

$$C(p, q) \leqslant \tilde{C}(p, q) \leqslant (1+\varepsilon)\, C(p, q)\,. \tag{6}$$

PROOF. This result is essentially contained in Reiffen's proof [15] of the equivalence of (5a) and (5b).

COROLLARY. *The top-dimensional Hausdorff and spherical measures associated to C and $\tilde{C}$ on a Carathéodory-hyperbolic manifold all coincide.*

PROOF. Refer to the definitions of Hausdorff and spherical measure (eqs. (1) and (2)) noting that $\Phi_\varepsilon^m(A)$ increases as $\varepsilon \downarrow 0$.

The infinitesimal Kobayashi-Royden metric is defined [16] by

$$K(p;\xi) = \inf\big\{|\alpha| \;\big|\; \alpha \in T_0 D \text{ and there exists } f\colon D \to M \text{ such that } f(0) = p \text{ and } f_*(\alpha) = \xi\big\};$$

again we may take the Euclidean length of α. The associated distance function obtained by integrating $K(\;;\;)$ along curves is the Kobayashi distance

$K(\ ,\ )$; see [9] for the original definition of the Kobayashi distance and [16] for the proof that it coincides with the distance function associated to $K(\ ;\ )$. M is said to be hyperbolic if $K(\ ,\ )$ is a genuine distance. The indicatrix $I_K(p)$ is in general not convex. Also $K(\ ;\ )$ is in general only upper semicontinuous [16]. A sufficient condition that $K(p; \xi)$ be continuous and nonzero whenever $\xi \neq 0$ (and in fact that M be hyperbolic) is that M be taut [16, Proposition 5]. That is, the set of holomorphic mappings from the unit disc into M should form a normal family.

Finally we recall the definitions of the Carathéodory and Eisenman-Kobayashi volume densities [5]. Letting $z_1, \dots, z_n$ be local coordinates at $p \in M$, the former is defined by

$$(7a)\qquad C_n\left(p; \frac{\partial}{\partial z_1}\wedge\dots\wedge\frac{\partial}{\partial z_n}(p)\right)$$
$$= \sup\{|Jf(p)|^2 \mid \text{there exists a holomorphic map } f\colon M\to B^n \text{ satisfying } f(p)=0\}$$

and the latter by

$$(7b)\qquad E_n\left(p; \frac{\partial}{\partial z_1}\wedge\dots\wedge\frac{\partial}{\partial z_n}(p)\right)$$
$$= \inf\{|Jf(0)|^{-2} \mid \text{there exists a holomorphic map } f\colon B^n\to M \text{ satisfying } f(0)=p\}.$$

Our notation arises from the fact that these quantities are values of certain intrinsic norms on $\Lambda^n TM$, the n-th exterior power of the holomorphic tangent bundle of M (see [7, 8]).

On a Carathéodory-hyperbolic (respectively hyperbolic) manifold we let $\mathcal{H}_C^{2n}$ (respectively $\mathcal{H}_K^{2n}$) denote the top-dimensional Hausdorff measure associated to the Carathéodory (respectively Kobayashi) distance. It is easy to see that these measures are absolutely continuous with respect to Lebesgue measure in local coordinates (compare with the Carathéodory or Kobayashi Hausdorff measure on a small ball). Also the inequalities

$$(8a)\qquad \frac{d\mathcal{H}_C^{2n}}{d\lambda^{2n}}(p) \geq C_n\left(p; \frac{\partial}{\partial z_1}\wedge\dots\wedge\frac{\partial}{\partial z_n}(p)\right)$$

and

$$(8b)\qquad \frac{d\mathcal{H}_K^{2n}}{d\lambda^{2n}}(p) \leq E_n\left(p; \frac{\partial}{\partial z_1}\wedge\dots\wedge\frac{\partial}{\partial z_n}(p)\right)$$

are consequences of standard facts about intrinsic measures [5, 14]. Namely, letting θ_n denote the Euclidean volume form $(i/2)^n\, dz_1\wedge d\bar{z}_1\wedge\dots\wedge dz_n\wedge d\bar{z}_n$, if we equip the unit ball B^n with the volume form $(1-\|z\|^2)^{-n+1}\theta_n$, then

$$C_n\left(p;\frac{\partial}{\partial z_1}\wedge\ldots\wedge\frac{\partial}{\partial z_n}\right)\theta_n \qquad \left(\text{respectively } E_n\left(p;\frac{\partial}{\partial z_1}\wedge\ldots\wedge\frac{\partial}{\partial z_n}\right)\theta_n\right)$$

is the smallest (respectively largest) volume form on M such that all holomorphic mappings from M to B^n (respectively all holomorphic mappings from B^n to M) are volume-decreasing.

3. – Proofs of the Theorems.

PROOF OF THEOREM 1. (*a*) This statement follows from Busemann's theorem [3, § 6] and the Lemma in § 2 of the present paper.

(*b*) This follows from Busemann's theorem [3, § 6] together with the theorem of Busemann and Mayer [4, Theorem 1, p. 184] which implies that $K(\ ;\ )$ may be replaced by the Finsler metric $\hat{K}(\ ;\ )$ whose indicatrix at each point p is $\hat{I}_K(p)$ without changing the Kobayashi distance.

(*c*) We first show

LEMMA 2. *If M is a real m-dimensional differentiable manifold and F is a pseudometric on M (i.e. a nonnegative real-valued function on the tangent bundle TM satisfying $F(p, c\xi)=|c|F(p,\xi)$ whenever $c\in\mathbb{R}$) which is upper semi-continuous, then there exists a sequence $\{F_j\}_{j=1}^{\infty}$ of continuous pseudometrics on M such that $F_j\downarrow F$. Furthermore if M is a complex manifold and F satisfies $F(p,c\xi)=|c|F(p,\xi)$ whenever $c\in\mathbb{C}$ then the F_j's inherit this property.*

PROOF. It suffices to do the construction locally and then patch using a partition of unity. Thus let V be an open subset of M whose closure is contained in a coordinate neighbourhood. We restrict F to the unit sphere bundle (the unit sphere in the Euclidean metric in the coordinate neighborhood) over V, which we denote $\Sigma(V)$. We observe that $F|_{\Sigma(V)}$ is bounded above, and that the standard procedure for constructing a sequence of decreasing continuous functions which converge to a given upper semi-continuous function [11, Proposition 2.1.2] depends only on the existence of a distance function. Since the tangent bundle $T(V)$ is trivial, we may use the Euclidean product metric in $V\times\mathbb{R}^m$. We thus obtain a sequence of continuous functions f_j on $\Sigma(V)$ such that $f_j\downarrow F|_{\Sigma(V)}$. We now define, for $p\in V$,

$$F_j(p,\xi)=f_j\left(p,\frac{\xi}{|\xi|}\right)|\xi| \quad \text{if} \quad 0\neq\xi\in T_pM\,, \quad \text{and} \quad F_j(p,0)=0\,.$$

To prove the last statement of the lemma (which we do not need) we observe that the regularization procedure could be carried out on the complex projective bundle obtained from the tangent bundle over V, using the Fubini-Study metric on the fibre and the Euclidean metric on the base.

CONTINUATION OF THE PROOF OF (c). Let F_j be a sequence of continuous pseudo-metrics on M such that $F_j \downarrow K(\ ;\ )$. Let d_j be the distance function associated to F_j; it follows from the monotone convergence theorem that $d_j \downarrow K(\ ,\ )$ (pointwise, as functions on $M \times M$). Let $\mathcal{H}_j^{2n}$ (respectively $\mathcal{S}_j^{2n}$) denote the Hausdorff (respectively spherical) measure on M associated to d_j; of course $\mathcal{H}_j^{2n} = \mathcal{S}_j^{2n}$ via the theorem of Busemann and Mayer [4, p. 184] since F_j is continuous. Let $I_j(p)$ denote the indicatrix of F_j at p and $\hat{I}_j(p)$ its convex hull. By the already-quoted theorems of Busemann [3] and Busemann-Mayer [4] we have

$$\frac{d\mathcal{H}_j^{2n}}{d\lambda^{2n}}(p) = \frac{d\mathcal{S}_j^{2n}}{d\lambda^{2n}}(p) = \frac{\operatorname{vol}(B^n)}{\operatorname{vol}(\hat{I}_j(p))}. \tag{9}$$

It is not hard to see that $I_j(p) \uparrow I_K(p)$ and hence $\hat{I}_j(p) \uparrow \hat{I}_K(p)$. Also clearly

$$\mathcal{H}_j^{2n} \geqslant \mathcal{H}_K^{2n} \quad \text{and} \quad \mathcal{S}_j^{2n} \geqslant \mathcal{S}_K^{2n} \quad \text{for all } j.$$

Thus letting $j \to \infty$ in (9) we obtain

$$\frac{d\mathcal{H}_K^{2n}}{d\lambda^{2n}}(p) \leqslant \frac{d\mathcal{S}_K^{2n}}{d\lambda^{2n}}(p) \leqslant \frac{\operatorname{vol}(B^n)}{\operatorname{vol}(\hat{I}_K(p))}. \tag{10}$$

(The left-hand inequality is clear from the definition of Hausdorff and spherical measures.)

REMARK. In some ways a lower bound for the Radon-Nikodym derivatives would be more interesting. We already know that

$$\frac{d\mathcal{H}_K^{2n}}{d\lambda^{2n}}(p) \leqslant \frac{d\mathcal{S}_K^{2n}}{d\lambda^{2n}}(p) \leqslant E_n\left(p;\ \frac{\partial}{\partial z_1}\wedge\cdots\wedge\frac{\partial}{\partial z_n}(p)\right).$$

PROOF OF THEOREM 2. The following facts are contained in [8]; we formulate them here explicitly as a lemma.

LEMMA 3. (i) *Suppose that M is Carathéodory-hyperbolic and there is a point $p \in M$ such that, choosing local coordinates $z_1, \ldots, z_n$ at p,*

$$C_n\left(p;\ \frac{\partial}{\partial z_1}\wedge\cdots\wedge\frac{\partial}{\partial z_n}(p)\right) = \frac{\operatorname{vol}(B^n)}{\operatorname{vol}(I_C(p))}. \tag{11a}$$

Then the holomorphic map $f\colon M \to B^n$ such that $f(p) = 0$ which realizes the supremum in the definition of $C_n(p;\, \partial/\partial z_1 \wedge \ldots \wedge (\partial/\partial z_n)(p))$ must be Carathéodory-isometric at p.

(ii) *Suppose that M is taut and there is a point $p \in M$ such that, choosing local coordinates $z_1, \ldots, z_n$ at p,*

$$E_n\left(p;\, \frac{\partial}{\partial z_1} \wedge \ldots \wedge \frac{\partial}{\partial z_n}(p)\right) = \frac{\operatorname{vol}(B_n)}{\operatorname{vol}\big(I_K(p)\big)}\,. \tag{11b}$$

Then the holomorphic map $f\colon B^n \to M$ such that $f(0) = p$ which realizes the infimum in the definition of $E_n(p;\, (\partial/\partial z_1) \wedge \ldots \wedge (\partial/\partial z_n)(p))$ must be Kobayashi-isometric at 0.

PROOF. (i) The holomorphic map in question must satisfy both $f_*(I_C(p)) \subset \subset B^n$ (by the distance-decreasing property) and $|Jf(p)|^2 = \operatorname{vol}(B^n)/\operatorname{vol}\big(I_C(p)\big)$. From this it follows that $f_*(I_C(p)) = B^n$. (By abuse of notation we are writing B^n for the unit ball in T_0B^n.)

(ii) The extremal holomorphic map in question (which exists because M is taut) must satisfy both $f_*(B^n) \subset I_K(p)$ and $1/|Jf(0)|^2 = \operatorname{vol}(B^n)/\operatorname{vol}\big(I_K(p)\big)$. From this it follows that $\operatorname{vol}\big(f_*(B^n)\big) = \operatorname{vol}\big(I_K(p)\big)$. Since $I_K(p)$ has a continuous boundary (because of the tautness of M) it now follows that $f_*(B^n) = I_K(p)$.

We are now in a position to complete the proof of Theorem 2. To prove (a), we note that from Lemma 1 and Busemann's theorem [3, § 6], the assumption implies that (11a) holds. Hence by Lemma 3 there is a holomorphic map $f\colon \Omega \to B^n$ such that $f(p) = 0$ which is Carathéodory-isometric at p. We now invoke Theorem 2 of [8]: such a map must be biholomorphic. (This result uses properties of the Lempert map [12].)

To prove (b), we note that because the Carathéodory and Kobayashi metrics coincide on a strictly convex domain [13, 17], $I_K(p)$ must be convex. Hence the assumption together with Busemann's theorem [3, § 6] implies that (11b) holds. Hence from Lemma 3, noting that Ω is taut [2], it follows that there exists a holomorphic map $f\colon B^n \to \Omega$ such that $f(0) = p$ and f is Kobayashi-isometric at 0. By Theorem 2 of [8] it follows that f must be $1-1$ and onto. (Again we make use of properties of the Lempert map·)

To prove (c) we use the first part of the argument from the proof of (a).

To prove (d) we use the first part of the argument from the proof of (b).

REMARK. The inequality

$$C_n\left(p;\ \frac{\partial}{\partial z_1}\wedge\cdots\wedge\frac{\partial}{\partial z_n}(p)\right)\leqslant\frac{\operatorname{vol}(B^n)}{\operatorname{vol}(I_C(p))}$$

at a point p in a Carathéodory-hyperbolic manifold can be deduced in two different ways: (i) from Theorem 1 (a) and the inequality ($8a$); (ii) from the distance-decreasing property of holomorphic maps $f\colon M\to B^n$, as in Lemma 3.

Similarly, the inequality

$$E_n\left(p;\ \frac{\partial}{\partial z_1}\wedge\cdots\wedge\frac{\partial}{\partial z_n}(p)\right)\geqslant\frac{\operatorname{vol}(B^n)}{\operatorname{vol}(I_K(p))}$$

at a point p in a taut manifold where $I_K(p)$ is convex, can be deduced in two ways: (i) from Theorem 1 (b) and the inequality ($8b$), or (ii) from the distance-decreasing property of holomorphic maps $f\colon B^n\to M$, as in Lemma 3. (The latter argument of course works without the assumptions of tautness or convexity.)

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Department of Mathematics
Tulane University
New Orleans, Louisiana 70118

Department of Mathematics
University of Toronto
Toronto, Canada
M5S 1A1