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**Elliptic differential operators on noncompact manifolds**

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# Elliptic Differential Operators on Noncompact Manifolds.

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## 0. – Introduction.

Suppose $A\colon C^\infty(E) \to C^\infty(F)$ is an elliptic differential operator with $C^\infty$-coefficients and order $m$ between sections of two vector bundles $E$ and $F$ of the same fibre dimension over a $C^\infty$-manifold $X$ of dimension $n$. If $X$ is compact then it is well-known that on $L^p$-Sobolev spaces

$$A\colon H^p_{s+m}(E) \to H^p_s(F) \tag{0.1}$$

is Fredholm for every $1 < p < \infty$ and $s \in \mathbb{N}$ (the nonnegative integers). If $X$ is noncompact, the ellipticity of $A$ is no longer sufficient to ensure that (0.1) is Fredholm. Even when $X = \mathbb{R}^n$, $\dim E = \dim F = 1$, and the coefficients of $A$ are bounded on $\mathbb{R}^n$ with all derivatives vanishing as $|x| \to \infty$, an ellipticity condition on $A$ is required at infinity; for example if $\sigma_A(x, \xi)$ denotes the total symbol of $A$, and

$$|\sigma_A(x, \xi)| > c > 0 \tag{0.2}$$

for all $\xi \in \mathbb{R}^n$ and $x \in \mathbb{R}^n \setminus K$ where $K$ is some compact set, then (0.1) is Fredholm.

This type of result has been obtained by many authors (generalized to pseudo-differential operators in $\mathbb{R}^n$ by Beals [4], Cordes and Herman [9], Illner [12], Kumano-go [15], and to certain manifolds by McOwen [21] and Rabinović [25]). Unfortunately, it does not apply to the Laplacian or any constant coefficient, homogeneous, elliptic operator

$$A_\infty = \sum_{|\alpha| = m} a_\alpha D_x^\alpha \qquad (D_x = -i(\partial/\partial x))$$

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in $\mathbb{R}^n$ since (0.2) fails to hold. However, Nirenberg and Walker [24] were able to show that (0.1) at least has a finite-dimensional nullspace for perturbations of such operators of the form

$$A = A_\infty + \sum_{|\alpha|\leqslant m} b_\alpha(x) D_x^\alpha \tag{0.3}$$

where $A$ is elliptic and the $b_\alpha(x)$ satisfy

$$\lim_{|x|\to\infty} \left| |x|^{m-|\alpha|+|\gamma|} D_x^\gamma b_\alpha(x)\right| = 0 \tag{0.4}$$

for all $|\gamma|\leqslant s$.

This work suggested replacing (0.1) by the mapping

$$A\colon\ W^p_{s+m,\delta}(\mathbb{R}^n, dx_e) \to W^p_{s,\delta+m}(\mathbb{R}^n, dx_e) \tag{0.5}$$

where $dx_e$ denotes Euclidean measure, and $u \in W^p_{s,\delta}(\mathbb{R}^n, dx_e)$ if $u \in H^p_{s,\mathrm{loc}}(\mathbb{R}^n)$ and

$$\sum_{|\alpha|\leqslant s} \int_{\mathbb{R}^n} |(1+|x|)^{\delta+|\alpha|} D_x^\alpha u(x)|^p \, dx_e < \infty\,.$$

The authors of the present article showed in [16] and [22] that (0.5) is Fredholm whenever: $\delta > -n/p$ and $\delta + m - n/p' \notin \mathbb{N}$, or $\delta < -n/p$ and $-\delta - n/p \notin \mathbb{N}$. In fact, these results are true for « classically elliptic » systems (cf. [22]), and were later generalized to systems « elliptic in the sense of Douglis-Nirenberg » in [17] (cf. Section 9 in this paper). Partial results along these lines were found in [5] and [8] (1).

In this paper we study a much larger class of elliptic operators $A\colon C^\infty(E) \to C^\infty(F)$ over a non-compact manifold $X$ which, outside of a compact set, has topologically $L$ cylinders $\Omega_l \times (1,\infty)$ where $\Omega_l$ is compact. For the purposes of this introduction, we will assume $L=1$, $E = X\times \mathbf{C} = F$, and the local coordinate $\omega$ on $\Omega$ will be treated as if globally defined on $\Omega$. (See Section 1 for a rigorous treatment.) To define Sobolev spaces on $X$ we must specify a measure. It is natural to take $dx$ to be a positive $C^\infty$ measure on $X$ with $dx = r^{-1}\,dr\,d\omega$ in $\Omega\times(1,\infty)$, where $r$ denotes the coordinate in $(1,\infty)$ and $d\omega$ is a positive $C^\infty$ measure on $\Omega$. We then

(1) We have recently been made aware of earlier and more general results in $\mathbb{R}^n$ by Bagirov and Kondrat'ev [28] which are very similar to those in Section 9 below.

define $W^p_{s,\delta}(X)$ to be the space of $u \in H^p_{s,\mathrm{loc}}(X)$ such that

$$\sum_{q+|\beta|\leq s} \int_{r>1} |r^{\delta+q} D^q_r D^\beta_\omega u(\omega, r)|^p r^{-1}\, dr\, d\omega < \infty .$$

Note that if $X = \mathbb{R}^n$ and $r = |x|$ then

$$W^p_{s,\delta}(X) = W^p_{s,\delta-n/p}(\mathbb{R}^n, dx_e) .$$

Now suppose $A$: $C^\infty(X) \to C^\infty(X)$ is elliptic of order $m$ and define $\tilde{A} = \varrho^m A$ where $\varrho \in C^\infty(X)$ is positive with $\varrho = r$ for $r > 1$. Assume that for $r > 1$

$$\tilde{A} = \sum_{q+|\beta|\leq m} \tilde{a}_{q\beta}(\omega, r)(rD_r)^q D^\beta_\omega$$

satisfies

$$\lim_{r\to\infty} \left|r^h D^h_r D^\gamma_\omega(\tilde{a}_{q\beta}(\omega, r) - \overset{\infty}{a}_{q\beta}(\omega))\right| = 0 \tag{0.6}$$

for all $h + |\gamma| \leq s$ and $q + |\beta| \leq m$, where the $\overset{\infty}{a}_{q\beta}(\omega)$ are functions on $\Omega$ defining the operator

$$\tilde{A}_\infty = \sum_{q+|\beta|\leq m} \overset{\infty}{a}_{q\beta}(\omega)(rD_r)^q D^\beta_\omega \tag{0.7}$$

which satisfies the ellipticity condition

$$\sum_{q+|\beta|=m} \overset{\infty}{a}_{q\beta}(\omega)\, \lambda^q \xi^\beta \neq 0 \tag{0.8}$$

for $\omega \in \Omega$ and $(\lambda, \xi) \in \mathbb{R}^n \setminus \{0\}$. (For example, if $X = \mathbb{R}^n$, $\Omega = S^{n-1}$, and $A$ is of the form (0.3) then (0.7) is just the expression for $\tilde{A}_\infty = r^m A_\infty$ in spherical coordinates, and (0.6) expresses (0.4).)

Under these hypotheses,

$$A\colon W^p_{s+m,\delta}(X) \to W^p_{s,\delta+m}(X) \tag{0.9}$$

is bounded for all $\delta \in \mathbb{R}$. One purpose of this paper is to determine when (0.9) is Fredholm and provide some information on its Fredholm index, $i_\delta(A)$.

As we shall see, the behavior of (0.9) depends on the asymptotic behavior of $A$ as $r \to \infty$, *i.e.*, on the operator (0.7). To be more precise, for $\lambda \in \mathbb{C}$ consider the elliptic operator on $\Omega$

$$\tilde{A}_\infty(\lambda) = \sum_{q+|\beta|\leq m} \overset{\infty}{a}_{q\beta}(\omega)\, \lambda^q D^\beta_\omega .$$

Using the results of [2], the operator $A_\infty(\lambda)\colon H^q_{s+m}(\Omega) \to H^p_s(\Omega)$ is an isomorphism for all $\lambda \in \mathbb{C}\setminus\mathcal{C}_A$ where $\mathcal{C}_A$ is discrete and finite in any complex strip $\varepsilon_1 < \operatorname{Im}\lambda < \varepsilon_2$; furthermore, for $\lambda \in \mathcal{C}_A$ there are $d(\lambda) < \infty$ linearly independent solutions of

$$\tilde{A}_\infty(\omega, D_z, D_\omega)u = \sum_{q+|\beta|<m} \tilde{a}^{\infty}_{q\beta}(\omega) D^q_z D^\beta_\omega u = 0$$

in $z = \ln r > 0$ of the form $\exp[i\lambda z]p(\omega, z)$ where $p(\omega, z)$ is a polynomial in $z$ with coefficients in $C^\infty(\Omega)$. Let $\mathcal{D}_A = \{\delta = \operatorname{Im}\lambda \in \mathbb{R}\colon \lambda \in \mathcal{C}_A\}$ and, for $\delta_1, \delta_2 \in \mathbb{R}\setminus\mathcal{D}_A$ with $\delta_1 \leqslant \delta_2$, let

$$N(\delta_1, \delta_2) = \sum \{d(\lambda)\colon \lambda \in \mathcal{C}_A \text{ with } \delta_1 < \operatorname{Im}\beta < \delta_2\}.$$

The results of this paper (cf. Theorem 6.2) assert that (0.9) is Fredholm if and only if $\delta \in \mathbb{R}\setminus\mathcal{D}_A$; furthermore, if $\delta_1, \delta_2 \in \mathbb{R}\setminus\mathcal{D}_A$ with $\delta_1 \leqslant \delta_2$ then the change in the Fredholm index is given by

$$i_{\delta_1}(A) - i_{\delta_2}(A) = N(\delta_1, \delta_2). \tag{0.10}$$

Of course (0.9) is Fredholm if and only if

$$\tilde{A}\colon W^p_{s+m,\delta}(X) \to W^p_{s,\delta}(X) \tag{0.11}$$

is Fredholm. In particular,

$$\tilde{A}\colon W^p_{s+m,0}(X) \to W^p_{s,0}(X) \tag{0.12}$$

is Fredholm if and only if

$$\mathcal{C}_A \text{ contains no } \lambda \text{ with } \operatorname{Im}\beta = 0. \tag{0.13}$$

However, if we use the change of coordinates $z = \ln r$ for $r > 1$ then (0.12) becomes analogous to (0.1):

$$A\colon H^p_{s+m}(X) \to H^p_s(X) \tag{0.14}$$

where $H^p_s(X)$ denotes the $u \in H^p_{s,\mathrm{loc}}(X)$ such that

$$\sum_{q+|\beta|\leqslant s} \int_{z>0} |D^q_z D^\beta_\omega u(\omega, z)|^p \, d\omega\, dz < \infty.$$

But $\mathbb{C}_A$ is determined by $\tilde{A}_\infty$, so (0.13) can be considered as an ellipticity condition on $A$ at infinity required for (0.14) to be Fredholm, similar to (0.2) for (0.1) to be Fredholm.

This paper also deals with boundary-value problems on $X$ by considering $(A, B)$: $C^\infty(E|X^+) \to C^\infty(F|X^+) \times C^\infty(G|\Gamma)$ where $E$, $F$, and $G$ are vector bundles over $X = X^+ \cup X^-$ where $X^\pm$ are $C^\infty$-manifolds with boundary $\partial X^\pm = \Gamma$ (not necessarily connected), $X^-$ is compact, $X^\pm$ contains the $L$ cylinders $\Omega_l \times (1, \infty)$, $A$: $C^\infty(E) \to C^\infty(F)$ is elliptic, and $B$: $C^\infty(E|X^+) \to C^\infty(G|\Gamma)$ is a boundary operator which satisfies the Lopatinski-Shapiro conditions on $\Gamma$. Again for this introduction let us assume that $L = 1$, $E = X \times \mathbb{C} = F$, and locally $Bu = (B_1 u, \ldots, B_{m/2} u)$ where order $(B_j) = m_j < m =$ order $(A)$. If $A$ satisfies (0.6) in $r > 1$ then the results of this paper (cf. Theorem 6.3) show that

$$(0.15) \qquad (A, B)\colon W^p_{s+m,\delta}(X^+) \to W^p_{s,\delta+m}(X^+) \times \prod_{j=1}^{m/2} H^p_{s+m-m_j-1/p}(\Gamma)$$

is Fredholm if and only if $\delta \in \mathbb{R} \setminus \mathfrak{D}_A$, where $\mathfrak{D}_A$ is the same discrete set as before (which depends only on the asymptotic operator $\tilde{A}_\infty$). In fact the Fredholm index $i_\delta(A, B)$ of (0.15) changes exactly as in (0.10): if $\delta_1, \delta_2 \in \mathbb{R} \setminus \mathfrak{D}_A$ with $\delta_1 \leqslant \delta_2$ then

$$(0.16) \qquad i_{\delta_1}(A, B) - i_{\delta_2}(A, B) = N(\delta_1, \delta_2)\,.$$

Thus (0.10) and (0.16) assert that (0.9) and (0.15) have Fredholm indices differing by an integer independent of $\delta$, a fact observed for $\Delta$ in $\mathbb{R}^n$ in [20].

The paper is divided into two parts. In the first part we analyze operators $A$: $C^\infty(E) \to C^\infty(F)$ between sections of vector bundles $E$ and $F$ over a manifold $X$ with one cylindrical end. If the vector bundles decompose as direct sums then we can take the generalized notion of ellipticity provided by Douglis and Nirenberg [10]. To determine when these operators are Fredholm on the appropriate weighted Sobolev spaces we follow [1], [3], [14], and [18]. To determine how the index changes with $\delta$ we construct multiple layer potentials using a Fredholm inverse. The formula (0.10) is obtained in two steps: i) using multiple-layer potentials to show $i_{\delta_1}(A) - i_{\delta_2}(A) = \dim\,(K^+_{\delta_1}(A)/K^+_{\delta_2}(A))$ where $K^+_\delta(A) = \{u \in W^p_{s+m,\delta}(X^+)\colon Au = 0$ in $X^+ = \Omega \times [1, \infty)\}$, and ii) using classical asymptotic expansions in $X^+ = \Omega \times \mathbb{R}^+$ to show $\dim\,(K^+_{\delta_1}(A)/K^+_{\delta_2}(A)) = N(\delta_1, \delta_2)$. Similarly the formula (0.16) is obtained by comparing $i_\delta(A, B)$ with $i_\delta(A)$. Finally, we show that the exact value of $i_\delta(A)$ can be determined when $A$ has self-adjoint principal part.

In the second part of this paper we give some generalizations and applications of the theory. If the manifold $X$ has multiple ends, then we may introduce separate weights on each end and again obtain Fredholm theorems and a change of index formula. This is done in Section 8. In the final two sections we apply our results first to $X = \mathbb{R}^n$ with Euclidean measure to generalize the results of [16], [17], [19], [20], and [22], and secondly to manifolds with conic singularities to discuss when $L^2$-harmonic forms a e closed and co-closed. (The latter application is not much more than an interpretation of the calculations of Cheeger [6] in terms of weighted Sobolev spaces.)

Finally, we should mention that R. Melrose and G. Mendoza [23] have independently obtained similar results for $p = 2$ generalized to pseudo-differential operators.

# I. ANALYSIS ON MANIFOLDS WITH ONE CYLINDRICAL END

## 1. – Notation and results for translation invariant operators.

Suppose $X$ is an $n$-dimensional noncompact $C^\infty$-manifold without boundary containing $X_0$, a compact submanifold with boundary satisfying

$$X \setminus X_0 = \Omega \times \mathring{\mathbb{R}}^+ = \{(\omega, z) \colon \omega \in \Omega,\ 0 < z < \infty\}$$

where $\Omega = \partial X_0$ is an $(n-1)$-dimensional compact $C^\infty$-manifold. We choose a positive smooth measure on $\Omega$ which we denote by $d\omega$.

Given a vector bundle $E$ over $X$, $d = \dim E$, let $C^\infty(E)$ denote the smooth sections of $E$ and $C_0^\infty(E)$ denote those sections with compact support. Using a finite cover $\Omega_1, \ldots, \Omega_N$ of coordinate patches for $\Omega$, let $\mathring{X}_\nu = \Omega_\nu \times (0, \infty)$ and extend this to a finite cover $\mathring{X}_1, \ldots, \mathring{X}_N, \ldots, \mathring{X}_{N+M}$ of coordinate patches for $X$. If $u \in C_0^\infty(E)$ has support in $X_\nu$, let $u_1, \ldots, u_d$ denote its components in some fixed trivialization of $E|X_\nu$, and for $1 < p < \infty$ and $s \in \mathbb{N}$ (the nonnegative integers) let

$$\|u\|_{H^p_s(E|\mathring{X}_\nu)} = \sum_{|\alpha| \leqslant s} \sum_{\tau=1}^{d} \|D^\alpha u_\tau\|_{L^p(\mathring{X}_\nu)} \tag{1.1}$$

where we use measure $d\omega\, dz$ if $\nu = 1, \ldots, N$. Letting $\varphi_1, \ldots, \varphi_{N+M}$ denote a $C^\infty$ partition of unity subordinate to the cover $\mathring{X}_1, \ldots, \mathring{X}_{N+M}$ we define a norm on $C_0^\infty(E)$ by

$$\|u\|_{H_s^p} = \sum_{\nu=1}^{N+M} \|\varphi_\nu u\|_{H_s^p(E|\mathring{X}_\nu)} \tag{1.2}$$

and let $H_s^p(E)$ denote the closure of $C_0^\infty(E)$ in this norm. We can generalize these spaces by adding a weight at infinity, namely we replace (1.1) by

$$\|u\|_{W_{s,\delta}^p(E|\mathring{X}_\nu)} = \sum_{|\alpha|\leqslant s} \sum_{\tau=1}^{d} \|\exp[\delta z] D^\alpha u_\tau\|_{L^p(\mathring{X}_\nu)} \tag{1.3}$$

where $\delta \in \mathbf{R}$ and $\nu = 1, \ldots, N$. We denote by $W_{s,\delta}^p(E)$ the closure of $C_0^\infty(E)$ under the norm

$$\|u\|_{W_{s,\delta}^p} = \sum_{\nu=1}^{N} \|\varphi_\nu u\|_{W_{s,\delta}^p(E|\mathring{X}_\nu)} + \sum_{\nu=N+1}^{N+M} \|\varphi_\nu u\|_{H_s^p(E|\mathring{X}_\nu)} . \tag{1.4}$$

Next suppose $F$ is another vector bundle over $X$ of the same fiber dimension $d = \dim E$, and suppose $A\colon C_0^\infty(E) \to C_0^\infty(F)$ is a differential operator of order $m$ with $C^\infty$-coefficients. We require $A$ to be translation invariant (with respect to the fixed trivializations of $E|\mathring{X}_\nu$ and $F|\mathring{X}_\nu$) in $z > 0$. If $d = 1$ this means that

$$A_\nu|X = \sum_{q=0}^{m} A^{m-q}(\omega, D_\omega)\, D_z^q \tag{1.5}$$

where $\nu = 1, \ldots, N$ and $A^{m-q}(\omega, D_\omega)$ is a differential operator of order $m - q$ in $\omega \in \Omega_\nu$. For $d > 1$ local coordinates define $A$ as a $d \times d$ matrix of differential operators of order $m$, each of which must be of the form (1.5). Clearly $A$ extends to a bounded operator

$$A\colon\ W_{s+m,\delta}^p(E) \to W_{s,\delta}^p(F) . \tag{1.6}$$

We are interested in those cases when this operator is Fredholm.

If the vector bundles $E$ and $F$ decompose into direct sums

$$E = \bigoplus_{j=1}^{J} E_j , \qquad F = \bigoplus_{i=1}^{I} F_i , \tag{1.7}$$

we can generalize (1.6) following [10] and [11]. Let $t = (t_1, \ldots, t_J)$ and

$s = (s_1, \ldots, s_I)$ be sets of nonnegative integers and define

$$(1.8) \qquad W^p_{t,\delta}(E) = \bigoplus_{j=1}^{J} W_{t_j,\delta}(E_j) , \qquad W^p_{s,\delta}(F) = \bigoplus_{i=1}^{I} W^p_{s_i,\delta}(F_i) .$$

A differential operator $A\colon C_0^\infty(E) \to C_0^\infty(F)$ decomposes into $A_{ij}\colon C_0^\infty(E) \to C_0^\infty(F_i)$. If each $A_{ij}$ is of order $t_j - s_i$ (where $t_j - s_i < 0$ implies $A_{ij} = 0$) then $(t, s)$ is called a *system of orders for* $A$. Since $(t, s)$ may be changed by adding a constant to each term, we may assume that each $t_j > 0$. Assuming that $A$ is translation invariant in $z > 0$ (i.e. each $A_{ij}$ is of the form (1.5) with $m = t_j - s_i$) we find that

$$(1.9) \qquad A\colon W^p_{t,\delta}(E) \to W^p_{s,\delta}(F)$$

is a bounded operator, and again we are interested in whether it is Fredholm or not.

For each nonzero covector $(x, \xi)$ the principal symbol of $A$ is a linear mapping on the fibers, $A^0(x, \xi)\colon E_x \to F_x$, and is obtained by replacing each $A_{ij}$ with its $t_j - s_i$ principal symbol (matrix). In local coordinates the determinant $L(x, \xi) = \det A^0(x, \xi)$ is a homogeneous polynomial of $\xi$. We say $A$ is *elliptic with respect to* $(t, s)$ if $L(x, \xi) \neq 0$ for any nonzero $(x, \xi)$; this requires that $L(x, \xi)$ has even homogeneity degree $2\mu$ if $n \geqslant 3$, which we also assume if $n = 2$. In the next section we prove the following.

THEOREM 1.1. *If $A$ is elliptic with respect to $(t, s)$ and it is translation invariant in $z > 0$, then there is a discrete set $\mathfrak{D}_A \subset \mathbb{R}$ such that* (1.9) *is Fredholm if and only if $\delta \in \mathbb{R} \setminus \mathfrak{D}_A$.*

NOTATION. For $\delta \in \mathbb{R} \setminus \mathfrak{D}_A$ let $i_\delta(A)$ denote the Fredholm index of (1.9).

As in the introduction, the set $\mathfrak{D}_A$ is described by taking the Fourier transform of the equation

$$(1.10) \qquad A(\omega, D_\omega, D_z) u(\omega, z) = 0 \qquad (z > 0)$$

to obtain a « generalized eigenvalue problem » for $\lambda \in \mathbb{C}$:

$$A(\omega, D_\omega, \lambda) \hat{u}(\omega, \lambda) = 0 .$$

The results of [2] show that the eigenvalue problem has a nontrivial solution for $\lambda \in \mathcal{C}_A$ where $\mathcal{C}_A \subset \mathbb{C}$ is a discrete set which is finite in any complex strip $\varepsilon_1 < \operatorname{Im} \lambda < \varepsilon_2$. If $\lambda \in \mathcal{C}_A$ let $d(\lambda)$ denote the dimension of all solutions of (1.10) of the form

$$(1.11) \qquad \exp[i\lambda z]\, p(\omega, z)$$

where $p(\omega, z)$ is a polynomial in $z$ with coefficients in $C^\infty(E|\Omega)$. Then

$$\mathfrak{D}_A = \{\delta = \operatorname{Im} \lambda \in \mathbb{R}: \lambda \in \mathfrak{C}_A\}$$

and, for $\delta_1, \delta_2 \notin \mathfrak{D}_A$ with $\delta_1 < \delta_2$, let

$$N(\delta_1, \delta_2) = \sum \{d(\lambda): \lambda \in \mathfrak{C}_A \text{ with } \delta_1 < \operatorname{Im} \lambda < \delta_2\} .$$

In Section 5 below we show that $i_\delta(A)$ changes as $\delta$ crosses points in $\mathfrak{D}_A$ as follows.

THEOREM 1.2. *If the hypotheses of Theorem* 1.1 *are satisfied and* $\delta_1, \delta_2 \in \mathbb{R}\setminus\mathfrak{D}_A$ *with* $\delta_1 < \delta_2$, *then* $i_{\delta_1}(A) - i_{\delta_2}(A) = N(\delta_1, \delta_2)$.

In order to consider the boundary-value problems let us introduce some additional notation. Let $\Gamma$ be an $(n-1)$-dimensional compact $C^\infty$-submanifold without boundary which is contained in $\mathring{X}_0$, the interior of $X_0$. Let $X = X^- \cup X^+$ where $\partial X^\pm = \Gamma$ and $X^-$ is compact. Consider the restrictions of $E$ and $F$ to $X^\pm$, $E^\pm$ and $F^\pm$, and let $C_0^\infty(E^\pm)$ and $C_0^\infty(F^\pm)$ be the smooth sections of $E^\pm$ and $F^\pm$ respectively with compact support in $X^\pm$. (Note that since $\Gamma \subset X^\pm$, sections in $C_0^\infty(E^\pm)$ and $C_0^\infty(F^\pm)$ need not vanish on $\Gamma$, but all derivatives extend continuously to $\Gamma$.) Define $W^p_{t,\delta}(E^\pm)$ as the closure of $C_0^\infty(E^\pm)$ in the norm (1.4) (with $\mathring{X}_{N+1}, \ldots, \mathring{X}_{N+M}$ chosen so that $\mathring{X}_1, \ldots, \mathring{X}_{N+M}$ forms a coordinate cover of $\mathring{X}^\pm$).

Suppose $G$ is a vector bundle on $\Gamma$ with $\dim G = \mu d$ and which decomposes as

$$G = \bigoplus_{k=1}^{K} G_k .$$

We further suppose that $B$: $C_0^\infty(E^+) \to C^\infty(G)$ is a differential boundary operator of order $(t, r)$ where $r = (r_1, \ldots, r_k)$, i.e. each $B_{kj}$: $C_0^\infty(E_j^+) \to C^\infty(G_k)$ is of order $\leq t_j - r_k$. Hence if we let

$$H^p_{r-1/p}(G) = \bigoplus_{k=1}^{K} H^p_{r_k-1/p}(G_k)$$

where $H^p_{r_k-1/p}(G_k)$ denotes the standard Sobolev space (since $\Gamma$ is compact), then

$$(A, B): W^p_{t,\delta}(E^+) \to W^p_{s,\delta}(F^+) \times H^p_{r-1/p}(G) \tag{1.12}$$

is a bounded operator.

The desired behaviour of (1.12) near $\Gamma$ is partly expressed by the a priori inequality

$$\|u\|_{W^p_{t,\delta}} \leqslant C(\|Au\|_{W^p_{s,\delta}} + \|Bu\|_{H^p_{r-1/p}} + \|u\|_{W^p_{t',\delta}}) \tag{1.13}$$

where $u \in C_0^\infty(E^+)$ vanishes for $z \geqslant 1$ and $t'_j < t_j$ for all $j$. Since the weights are only felt as $z \to \infty$, the inequality (1.13) will hold if $(A, B)$ satisfies the standard elliptic or Lopatinski-Shapiro conditions (which may be found for example in [1]). These conditions also imply the existence of a (right) parametrix

$$P\colon\ C_0^\infty(F^+) \times C^\infty(G) \to C^\infty(E^+) \tag{1.14}$$

which extends to a bounded map

$$P\colon\ H^p_{s,\mathrm{comp}}(F^+) \times H^p_{r-1/p}(G) \to H^p_{t,\mathrm{loc}}(E^+)$$

and satisfies $(A, B)\ P = I + S$ where $S\colon C_0^\infty(F^+) \times C^\infty(G) \to C^\infty(F^+) \times C^\infty(G)$ is infinitely smoothing. We *define* $(A, B)$ to be *elliptic with respect to* $(t, s, r)$ if (1.13) holds and the parametrix (1.14) exists. In the next section we prove the following

THEOREM 1.3. *If $A$ satisfies the hypotheses of Theorem* 1.1 *and* $(A, B)$ *is elliptic with respect to* $(t, s, r)$ *in* $X^+$, *then* (1.12) *is Fredholm if and only if* $\delta \in \mathbb{R} \setminus \mathfrak{D}_A$.

NOTATION. For $\delta \in \mathbb{R} \setminus \mathfrak{D}_A$ let $i_\delta(A, B)$ denote the Fredholm index of (1.12).

The following result, giving the relationship between $i_\delta(A)$ and $i_\delta(A, B)$, is proved in Section 4.

THEOREM 1.4. *If the hypotheses of Theorem* 1.3 *are satisfied, then, for all* $\delta \in \mathbb{R} \setminus \mathfrak{D}_A$, $i_\delta(A, B) = i_\delta(A) + i$ *where $i$ is independent of* $\delta$.

Combining this with Theorem 1.2 we obtain:

COROLLARY 1.5. *If the hypotheses of Theorem* 1.3 *are satisfied and* $\delta_1, \delta_2 \in \mathbb{R} \setminus \mathfrak{D}_A$ *with* $\delta_1 \leqslant \delta_2$ *then* $i_{\delta_1}(A, B) - i_{\delta_2}(A, B) = N(\delta_1, \delta_2)$.

REMARK 1.6. It may be observed that the boundary value problem (1.12) makes sense without requiring that $A$ be the restriction to $X^+$ of an elliptic operator defined on the manifold without boundary $X$. In fact the proof in Section 2 does not require the extendability of $A$ to $X$, so (1.12) is Fredholm for $\delta \in \mathbb{R} \setminus \mathfrak{D}_A$ (note $\mathfrak{D}_A$ only depends on $A|\Omega \times \mathbb{R}^+$). The change of index formula in Corollary 1.5, however, requires the comparison in Theorem 1.4 of the index of $A$ with that of $(A, B)$ provided by the multiple layer potentials in Section 3. If $A$ is not known to extend to such a manifold $X$ then another comparison must be used. Let us mention 2 possibilities. First, if $A$ extends to an elliptic operator on the double $2X^+$ of $X^+$, then $i_\delta(A, B)$ may be compared with the index on $2X^+$ discussed in Part II of this paper. Second, if a Dirichlet problem exists for $A$ in $\Omega\times\mathbb{R}^+$, then this may be used for comparison with $i_\delta(A, B)$. In each of these cases the formula $i_{\delta_1}(A, B) - i_{\delta_2}(A, B) = N(\delta_1, \delta_2)$ is obtained, however a rigorous treatment of each requires reformulating the multiple layer potentials using a different Fredholm inverse; so we shall not discuss this generalization further.

## 2. – Fredholm theorems.

Let $\tilde{X} = \Omega\times\mathbb{R}$ be the full cylinder and suppose

$$\tilde{E} = \bigoplus_{j=1}^{J} \tilde{E}_j , \qquad \tilde{F} = \bigoplus_{i=1}^{I} \tilde{F}_i ,$$

are vector bundles on $\tilde{X}$ with the same fiber dimension. Suppose

$$A = A(\omega, D_\omega, D_z)\colon\; C_0^\infty(\tilde{E}) \to C_0^\infty(\tilde{F})$$

is a translation invariant differential operator of order $(t, s)$ which is elliptic with respect to $(t, s)$. Then $A$ defines a bounded operator

$$A\colon\; \tilde{W}^p_{t,\delta}(\tilde{E}) \to \tilde{W}^p_{s,\delta}(\tilde{F}) \tag{2.1}$$

where the weights in $\tilde{W}^p_{t,\delta}(\tilde{E})$ and $\tilde{W}^p_{s,\delta}(\tilde{F})$ are extended over all $z\in\mathbb{R}$; namely replace $\mathring{X}_\nu$ in (1.3) by $\tilde{X}_\nu = \Omega_\nu\times\mathbb{R}$, let $\varphi_1, \ldots, \varphi_N$ be a partition of unity subordinate to $\Omega_1, \ldots, \Omega_N$, and omit the summation $N+1\leqslant\nu \leqslant N+M$ in (1.4).

Ellipticity and analiticity in $\lambda$ may be used as in [2] or [3] to show that

$$A(\omega, D_\omega, \lambda)\colon\; H^p_t(\tilde{E}|\Omega) \to H^p_s(\tilde{F}|\Omega) \tag{2.2}$$

is an isomorphism (onto) whenever $\lambda\in\mathbb{C}\diagdown\mathbb{C}_A$. Let $R_A(\lambda)$ denote the inverse of (2.2), and for $f\in C_0^\infty(F)$ let

$$\hat{f}(\omega, \lambda) = \int_{-\infty}^{\infty} \exp\,[-i\lambda z]\, f(\omega, z)\, dz\,.$$

If $\delta \in \mathbb{R} \setminus \mathfrak{D}_A$ then, as in [14] and [18], the operator

$$(2.3) \qquad A_\delta^{-1} f(w, z) = \frac{1}{2\pi} \int_{\operatorname{Im}\lambda=\delta} \exp[i\lambda z] R_A(\lambda) \hat{f}(\omega, \lambda)\, d\lambda$$

extends to a bounded operator $\tilde{W}^p_{s,\delta}(\tilde{F}) \to \tilde{W}^p_{t,\delta}(\tilde{E})$ which inverts (2.1). Hence (2.1) is an isomorphism for $\delta \in \mathbb{R} \setminus \mathfrak{D}_A$.

Returning to the vector bundles $E$ and $F$ over $X$, we double their restrictions to $\Omega \times \mathbb{R}^+$ to define $\tilde{E}$ and $\tilde{F}$ on $\tilde{X}$.

The a priori inequality

$$(2.4) \qquad \|u\|_{W^p_{t,\delta}} \leqslant C(\|Au\|_{W^p_{s,\delta}} + \|u\|_{W^p_{t',\delta}}) \qquad (t'_j < t_j)$$

is established for all $\delta \in \mathbb{R}$ by standard parametrix techniques. Though useful for establishing regularity of solutions, (2.4) cannot be used for Fredholm theory as $W^p_{t,\delta}(X) \to W^p_{t',\delta}(X)$ is not a compact map ($X$ is non-compact). To derive an appropriate inequality for Fredholm theory let $X_1 = X_0 \cup \{(\omega, z)\colon \omega \in \Omega,\ 0 < z \leqslant 1\}$, $\varphi_1 \in C_0^\infty(X_1)$ with $\varphi_1 = 1$ on $X_0$, and let $\varphi_2 = 1 - \varphi_1$. For $u \in W^p_{t,\delta}(E)$ we apply (2.4) to obtain

$$(2.5) \qquad \|\varphi_1 u\|_{W^p_{t,\delta}} \leqslant C(\|A\varphi_1 u\|_{W^p_{s,\delta}} + \|\varphi_1 u\|_{W^p_{t',\delta}})$$

and, thinking of $A$ and $\varphi_2$ as defined on $\tilde{X}$, for $\delta \in \mathbb{R} \setminus \mathfrak{D}_A$

$$\|\varphi_2 u\|_{W^p_{t,\delta}} \leqslant C\|A\varphi_2 u\|_{W^p_{s,\delta}}$$

since (2.1) is an isomorphism. Combining these we find that for $\delta \in \mathbb{R} \setminus \mathfrak{D}_A$

$$(2.6) \qquad \|u\|_{W^p_{t,\delta}} < C(\|\varphi_2 Au\|_{W^p_{s,\delta}} + \|\varphi_1 Au\|_{W^p_{s,\delta}} + \|[\varphi_1, A]u\|_{W^p_{s,\delta}} + \|[\varphi_2, A]u\|_{W^p_{s,\delta}} + \|\varphi_1 u\|_{W^p_{t',\delta}})$$

where $[\varphi, A] = \varphi A - A\varphi$. But $[\varphi_1, A]$ and $[\varphi_2, A]$ are compact $W^p_{t,\delta}(E) \to W^p_{s,\delta}(F)$, and $\varphi_1\colon W^p_{t,\delta}(E) \to W^p_{t',\delta}(E)$ is compact (since $t_j > t'_j$), so (2.6) implies that (1.9) has finite-dimensional nullspace and closed range.

To show (1.9) has finite codimensional range let $P_1$ be a parametrix for $A$ in $X_1$, let $\psi_1 \in C_0^\infty(X)$ with $\psi_1 = 1$ on $\operatorname{supp}\varphi_1$, and let $\psi_2 \in C^\infty(X)$ with $\operatorname{supp}\psi_2 \subset \Omega \times \mathbb{R}^+$ and $\psi_2 = 1$ on $\operatorname{supp}\varphi_2$. Define

$$T\colon W^p_{s,\delta}(F) \to W^p_{t,\delta}(E)$$

by

$$Tf = \psi_1 P_1(\varphi_1 f) + \psi_2 A_\delta^{-1}(\varphi_2 f) .$$

Then $AT = I + K$ where $K: W^p_{s,\delta}(F) \to W^p_{s,\delta}(F)$ is compact, so $AT(W^p_{s,\delta}(F))$ has finite codimension, implying the same for $A(W^p_{t,\delta}(E))$. Thus $\delta \in \mathbb{R} \setminus \mathfrak{D}_A$ implies that (1.9) is Fredholm as claimed in Theorem 1.1.

On the other hand, if $\delta \in \mathfrak{D}_A$ then pick $\varepsilon > 0$ so that

$$[\delta - \varepsilon, \delta) \cup (\delta, \delta + \varepsilon] \subset \mathbb{R} \setminus \mathfrak{D}_A .$$

Let $\zeta \in C^\infty(X)$ with $\zeta > 0$ on $X$ and $\zeta = e^z$ for $z > 1$. Then $\zeta^\sigma: W^p_{t,\delta}(E) \to W^p_{t,\delta-\sigma}(E)$ and $\zeta^\sigma: W^p_{s,\delta}(F) \to W^p_{s,\delta-\sigma}(F)$ are isomorphisms for every $\sigma \in \mathbb{R}$. Define the 1-parameter family of operators $A(\tau) = \zeta^{-\tau\varepsilon} A \zeta^{\tau\varepsilon}$ for $-1 \leqslant \tau \leqslant 1$. Then $A(\tau): W^p_{t,\delta}(E) \to W^p_{s,\delta}(F)$ is Fredholm if and only if $A: W^p_{t,\delta-\tau\varepsilon}(E) \to W^p_{s,\delta-\tau\varepsilon}(F)$ is Fredholm and the indices are equal. Thus *if* (1.9) were Fredholm, then $A(\tau)$ would be a 1-parameter family of Fredholm operators so $i_{\delta-\varepsilon}(A) - i_{\delta+\varepsilon}(A) = i_\delta(A(-1)) - i_\delta(A(1)) = 0$. But since $N(\delta - \varepsilon, \delta + \varepsilon) \neq 0$ this contradicts the change of index formula in Theorem 1.2 (proved below). This completes the proof of Theorem 1.1.

If we replace (2.5) by (1.13) and the interior parametrix $P_1$ by the parametrix (1.14) for the boundary problem, the same arguments show that (1.12) is Fredholm if and only if $\delta \in \mathbb{R} \setminus \mathfrak{D}_A$ as claimed in Theorem 1.3.

## 3. – Multiple layer potentials.

In this section we assume (1.9) is elliptic with respect to $(t, s)$ and let $m_j = \max\{t_j - s_i : 1 \leqslant i \leqslant I\}$.

For $u_j \in C_0^\infty(E_j)$ let $Ru = (R_1 u_1, \ldots, R_J u_J)$ where $R_j u_j$ denotes the Cauchy data of order $< m_j$ for $u_j$ on $\Gamma$. Letting

$$CD^p_t(E|\Gamma) = \bigoplus_{j=1}^{J} \bigoplus_{q=0}^{m_j-1} H^{p}_{t_j-q-1/p}(E_j|\Gamma)$$

we obtain a bounded map for any $\delta \in \mathbb{R}$

$$R: W^p_{t,\delta}(E) \to CD^p_t(E|\Gamma) \tag{3.1}$$

(note that $t_j \geqslant m_j$ for each $j$). In fact, choosing a normal coordinate $\nu$ near $\Gamma$ so that $\Gamma \times (-1, 1) = \{(\gamma, \nu): \gamma \in \Gamma, -1 < \nu < 1\}$ forms a neighborhood of $\Gamma$

in $X$ with $(\gamma, \nu) \in X^{\pm}$ if $\pm\nu > 0$, we may let $R_{(\nu)}u$ denote the Cauchy data on $\Gamma \times \{\nu\}$ and define

$$R^{\pm} u = \lim_{\nu \to \pm 0} R_{(\nu)} u$$

to obtain bounded maps for any $\delta \in \mathbb{R}$

$$R^{\pm}\colon\ W^p_{t,\delta}(E^{\pm}) \to CD^p_t(E|\Gamma)$$

(here and below we let $W^p_{t,\delta}(E^-) = H^p_t(E^-)$ since $X^-$ is compact).

Let $N^p_\delta(A)$ denote the nullspace of (1.9) and define

$$N^{p\#}_\delta(A) = \{u|X^+\colon\ u \in N^p_\delta(A) \text{ and } Ru = 0\}\,.$$

We also consider the nullvectors for $A$ in $X^{\pm}$:

$$K^+_\delta(A) = \{u \in W^p_{t,\delta}(E^+)\colon\ Au = 0 \text{ in } X^+\}\,,$$

$$K^-(A) = \{u \in H^p_t(E^-)\colon\ Au = 0 \text{ in } X^-\}\,.$$

Finally we define their restrictions to $\Gamma$:

$$H^+_\delta = \{U = R^+ u\colon\ u \in K^+_\delta(A)\}\,,$$

$$H^- = \{U = R^- u\colon\ u \in K^-(A)\}\,,$$

which are subsets of $CD^p_t(E|\Gamma)$.

Choose Hermitian structure on the $E_j$ and $F_i$ which are « translation-invariant » with respect to the trivializations in $\Omega \times (0, \infty)$. (This means, for example, that if $u, w \in C^\infty(E_i|\Omega_\nu \times (0, \infty))$ with $u(\omega, z)$ and $w(\omega, z)$ independent of $z \in (0, \infty)$, then $\langle u(\omega, z), w(\omega, z)\rangle_{E_j}$ is independent of $z \in (0, \infty)$. This is easily done using the partition of unity $\varphi_1, \ldots, \varphi_N$ in Section 1.) These induce translation-invariant Hermitian structures on $E$ and $F$ which we denote $\langle\ ,\ \rangle_E$ and $\langle\ ,\ \rangle_F$ respectively. If $v \in W^{p'}_{0,-\delta}(F)$ where $p' = p/(p-1)$, then

$$f \mapsto \int_X \langle f, v\rangle_F\, dx$$

defines a continuous linear functional on $W^p_{s,\delta}(F)$ where $dx$ is a positive $C^\infty$ measure on $X$ with $dx = d\omega\, dz$ in $\Omega \times \mathbb{R}^+$. This suggests letting $W^{p'}_{-s,-\delta}(F)$ denote the dual space of $W^p_{s,\delta}(F)$, so $W^{p'}_{0,\delta}(F) \subset W^{p'}_{-s,\delta}(F)$. Similarly we define $W^{p'}_{-t,-\delta}(E)$ as the dual space of $W^p_{t,\delta}(E)$ using $\langle\ ,\ \rangle_E$.

We define the adjoint $A^*$ of $A$ using these Hermitian structures: for $u \in C_0^\infty(E)$ and $v \in C_0^\infty(F)$ define $A^*v$ by

$$\int_X \langle u, A^*v\rangle_E \, dx = \int_X \langle Au, v\rangle_F \, dx . \tag{3.2}$$

Then $A^*\colon C_0^\infty(F) \to C_0^\infty(E)$ is a differential operator which is translation invariant in $z > 0$ and elliptic with respect to a system of orders $(s^*, t^*)$ satisfying $s_i^* - t_j^* = t_j - s_i$. Using (3.2), $A^*$ also defines a continuous mapping

$$A^*\colon W^{p'}_{-s,-\delta}(F) \to W^{p'}_{-t,-\delta}(E) . \tag{3.3}$$

For $u \in C_0^\infty(E)$ and $v \in C_0^\infty(F)$ we may integrate by parts near $\Gamma$ to find

$$\int_{X^\pm} \langle u, A^*v\rangle_E \, dx - \int_{X^\pm} \langle Au, v\rangle_F \, dx = \pm \int_\Gamma \langle \mathcal{A}Ru, Rv\rangle_F \, d\sigma \tag{3.4}$$

where $\mathcal{A}$ is a matrix of differential operators $C^\infty(E|\Gamma) \to C^\infty(F|\Gamma)$. Checking the orders involved we find that

$$\mathcal{A}\colon CD_t^p(E|\Gamma) \to \bigoplus_{i=1}^{I} \bigoplus_{q=0}^{\overline{m}-1} H^p_{s_i+q+1-1/p}(F_i|\Gamma) \tag{3.5}$$

is a bounded map, where $\overline{m} = \max\{m_j\colon 1 \leqslant j \leqslant J\}$; however, unlike the situation in [27], (3.5) need not be invertible. Let $\overline{R}u$ denote the Cauchy data of order $< \overline{m}$ on $\Gamma$, so

$$\overline{R}\colon \bigoplus_{i=1}^{I} W^{p'}_{l-s_i,-\delta}(F_i) \to \bigoplus_{i=1}^{I} \bigoplus_{q=0}^{\overline{m}-1} H^{p'}_{l-s_i-q-1/p'}(F_i|\Gamma)$$

is bounded provided $l \in \mathbf{N}$ satisfies

$$l - s_i \geqslant \overline{m} \qquad \text{for } 1 \leqslant i \leqslant I . \tag{3.6}$$

Since $H^p_{s_i+q+1-1/p} \subset H^p_{s_i+q+1-1/p-l}$ we see that

$$\mu_A = \overline{R}^* \mathcal{A}\colon CD_t^p(E|\Gamma) \to W^p_{s-l,\delta}(F) \tag{3.7}$$

is bounded provided (3.6) holds, and in this case we may express (3.4) as

$$\int_X \langle \mu_A(U), v\rangle_F \, dx = \pm \Big( \int_{X^\pm} \langle u, A^*v\rangle \, dx - \int_{X^\pm} \langle Au, v\rangle \, dx \Big) \tag{3.8}$$

where $U = Ru$, $u \in W^p_{t,\delta}(E)$, and $v \in W^{p'}_{l-s,-\delta}(F)$. Notice that $v \in C_0^\infty(F)$ with $Rv = 0$ implies $\int_X \langle \mu_A(U), v\rangle_F \, dx = 0$ so supp $\mu_A(U) \subset \Gamma$.

Now let us fix $l \in \mathbb{N}$ satisfying (3.6); in particular we have $l \geqslant s_i$ and $l \geqslant t_j$ for all $i$ and $j$. Since $A^*$ is elliptic with respect to $(s^*, t^*)$ we have

$$A^*\colon W^{p'}_{l-s,-\delta}(F) \to W^{p'}_{l-t,-\delta}(F) \tag{3.9}$$

bounded for all $\delta \in \mathbb{R}$. By elliptic regularity the nullspace of (3.3) equals that of (3.9) which we denote by $N^{p'}_{-\delta}(A^*)$. Define

$$CD^p_{t,\delta}(E|\Gamma) = \left\{ U \in CD^p_t(E|\Gamma)\colon \int_X \langle \mu_A(U), v\rangle_F \, dx = 0 \text{ for all } v \in N^{p'}_{-\delta}(A^*) \right\}:$$

which is clearly a closed subspace of $CD^p_t(E|\Gamma)$ of finite codimension. The following is an immediate consequence of (3.8).

PROPOSITION 3.1. $H^+_\delta, H^- \subset CD^p_{t,\delta}(E|\Gamma)$.

Next we construct a specific Fredholm inverse $Q^*_\delta$ for (3.9). Fix $\delta \in \mathbb{R} \setminus \mathfrak{D}_A$ and let $\{u_1, \ldots, u_M, \ldots, u_N\}$ be a basis for $N^p_\delta(A)$ with $u_1, \ldots, u_M$ linearly independent on $X^-$ and $u_{M+1}|X^- = \ldots = u_N|X^- = 0$. Thus we have $M$ linearly independent linear functionals on $C_0^\infty(E^-)$

$$g \to \int_{X^-} \langle g, u_j\rangle_E \, dx \qquad (j = 1, \ldots, M)$$

so let us choose $g_1, \ldots, g_M \in C_0^\infty(E^-)$ satisfying

$$\int_{X^-} \langle g_i, u_j\rangle_E \, dx = \delta_{ij} \qquad (i, j = 1, \ldots, M) \tag{3.10}$$

where $\delta_{ij}$ denotes the Kronecker $\delta$. Similarly w echoose $g_{M+1}, \ldots, g_N \in C_0^\infty(E^+)$ so that

$$\int_{X^+} \langle g_i, u_j\rangle_E \, dx = \delta_{ij} \qquad (i, j = M+1, \ldots, N)\,. \tag{3.11}$$

Extending $g_i$ by zero to all of $X$ we may consider $\{g_i\}_{i=1}^N \subset C_0^\infty(E)$, and let $W \subset W^{p'}_{l-t,-\delta}(E)$ be their linear span. Let $V$ be a linear complement of $N^{p'}_{-\delta}(A^*)$ in $W^p_{l-s,-\delta}(F)$, i.e., $W^{p'}_{l-s,-\delta}(F) = N^{p'}_{-\delta}(A^*) \oplus V$. Then (3.10) and (3.11) show that the $g_i$ are linearly independent mod $A^*(V)$, so $W$ is a linear complement of $A^*(V)$ in $W^{p'}_{l-t,-\delta}(E)$ i.e. $W^{p'}_{l-t,-\delta}(E) = A^*(V) \oplus W$.

Now define

$$Q_\delta^*\colon W^{p'}_{l-t,-\delta}(E) \to W^{p'}_{l-s,-\delta}(F) \qquad \text{by } Q_\delta^*(A^*v_1 + w) = v_1\,,$$

where $v_1 \in V$ and $w \in W$. Notice that if $g \in W^{p'}_{l-t,-\delta}(E)$ and $v \in W^{p'}_{l-s,-\delta}(F)$ then

$$(3.12)\qquad \begin{cases} A^*Q_\delta^*\, g = g - \sum_{i=1}^{N}\left(\int_X \langle g, u_i\rangle_E\, dx\right) g_i\,, \\ Q_\delta^* A^* v = v - \Pi v\,, \end{cases}$$

where $\Pi$ denotes the projection along $V$ onto $N^{p'}_{-\delta}(A^*)$. Furthermore, the adjoint $Q_\delta \equiv Q_\delta^{**}\colon W^p_{s-l,\delta}(F) \to W^p_{t-l,\delta}(E)$ is bounded. Thus the composition $Q_\delta\,\mu_A$ satisfies

$$(3.13)\qquad \|Q_\delta\,\mu_A(U)\|_{W^p_{t-l,\delta}} \leqslant \|U\|_{CD^p_t}$$

for $U \in CD^p_t(Et\Gamma)$ provided (3.6) holds.

If $U \in CD^p_{t,\delta}(E|\Gamma)$ then for any $v \in C_0^\infty(F)$

$$(3.14)\qquad \int_X \langle AQ_\delta\mu_A(U), v\rangle_F\, dx = \int_X \langle \mu_A(U), Q_\delta^* A^* v\rangle_F\, dx$$

$$= \int_X \langle \mu_A(U), v\rangle_F\, dx - \int_X \langle \mu_A(U), \Pi v\rangle_F\, dx = \int_X \langle \mu_A(U), v\rangle_F\, dx\,.$$

Taking $v$ supported in $\mathring{X}^\pm$ we find that $AQ_\delta\mu_A(U) = 0$ in $\mathring{X}^\pm$. For $U \in CD^p_{t,\delta}(E|\Gamma)$ define the multiple layer potentials

$$(3.15)\qquad M_\delta^\pm U = Q_\delta\,\mu_A(U)|_{\mathring{X}^\pm}\,.$$

PROPOSITION 3.2. *If $\delta \in \mathbb{R}\setminus\mathfrak{D}_A$ and $U \in CD^p_{t,\delta}(E|\Gamma)$ then $M_\delta^\pm U$ is in $C^\infty(\mathring{E}^\pm)$, satisfies $AM_\delta^\pm U = 0$ in $\mathring{X}^\pm$, and*

$$\|M_\delta^\pm U\|_{W^p_{t,\delta}(E^\pm)} \leqslant C\|U\|_{CD^p_+}\,.$$

PROOF. By (3.14) and elliptic regularity we need only to verify the estimate. Let $\psi \in C_0^\infty(X)$ with $\psi \equiv 1$ near $\Gamma$. Then by (3.14)

$$A\psi Q_\delta\,\mu_A(U) = \psi A Q_\delta\,\mu_A(U) + f_U\,,$$

$$= \mu_A(U) + f_U\,,$$

where $f_U \in C_0^\infty(F)$ with $f_U \equiv 0$ near $\Gamma$. Now A admits a parametrix $T\colon C_0^\infty(F) \to C^\infty(E)$ such that each $T_{ji}$ is a classical pseudo-differential operator (cf. [11], Lemma 1.0.2′). Thus we may write

$$(TA - I)\psi Q_\delta \mu_A(U) = T\mu_A(U) + Tf_U - \psi Q_\delta \mu_A(U)\,.$$

Since $TA - I$ is infinitely smoothing this implies

$$\psi Q_\delta \mu_A(U) = T\mu_A(U) + f'_U \tag{3.16}$$

where $f'_U \in C^\infty(F)$. However, by [11], Theorem 2.1.4, the limits $R^\pm T\mu_A(U)$ exist and define pseudo-differential operators on $\Gamma$ so that

$$\|R^\pm T\mu_A(U)\|_{CD_t^p} \leqslant C\|U\|_{CD_t^p}\,.$$

Using (3.16) this implies the limits $R^\pm M_\delta^\pm U$ exist and

$$\|R^\pm M_\delta^\pm U\|_{CD_t^p} \leqslant C\|U\|_{CD_t^p}\,. \tag{3.17}$$

By standard parametrix techniques we have the « a priori » inequality

$$\|u\|_{W^p_{t,\delta}(E^\pm)} \leqslant C(\|Au\|_{W^p_{s,\delta}(F^\pm)} + \|R^\pm u\|_{CD_t^p} + \|u\|_{W^p_{t',\delta}(E^\pm)}) \tag{3.18}$$

for all $\delta \in \mathbb{R}$ (cf. (2.4)). Letting $t' = t - l$ where $l \in \mathbb{N}$ satisfies (3.6) and applying (3.18) to $u = M_\delta^+ U$, the desired estimate for $M_\delta^+$ follows from (3.13) and (3.17). A similar argument applies to $M_\delta^-$, completing the proof.

The next two propositions show to what extent we can recover null-vectors in $X^\pm$ from their Cauchy data.

PROPOSITION 3.3. *Suppose $u \in K_\delta^+(A)$. Then on $X^+$ we have $u - M_\delta^+ R^+ u \in N_\delta^{p\#}(A)$, and on $X^-$ we have $M_\delta^- R^+ u = 0$.*

PROOF. For $g \in C_0^\infty(E)$, (3.8) and (3.12) imply

$$\int_X \langle Q_\delta \mu_A(R^+u), g\rangle_E\,dx = \int_X \langle \mu_A(R^+u), Q_\delta^* g\rangle_F\,dx = \int_{X^+} \langle u, A^* Q_\delta^* g\rangle_F\,dx$$
$$= \int_{X^+} \langle u, g\rangle_E\,dx - \sum_{i=1}^N \int_X \langle u_i, g\rangle_E\,dx \int_{X^+} \langle u, g_i\rangle_E\,dx\,.$$

But $g_i = 0$ on $X^+$ for $1 \leqslant i \leqslant M$, and $u_i \in N_\delta^{p\#}(A)$ for $M < i \leqslant N$, so

$$\int_X \langle Q_\delta \mu_A(R^+u), g\rangle_E\,dx = \int_{X^+} \langle u - \bar{u}, g\rangle_E\,dx \tag{3.19}$$

with

$$\bar{u} = \sum_{i=M+1}^{N} \left( \int_{X^+} \langle u, g_i \rangle_E \, dx \right) u_i$$

in $N_\delta^{p\#}(A)$. Taking $g$ in (3.19) with support in $X^\pm$ proves the two statements of the proposition.

PROPOSITION 3.4. *Suppose* $u \in K^-(A)$. *Then on* $X^-$ *we have* $u + M_\delta^- \cdot R^- u = \tilde{u}$, *and on* $X^+$ *we have* $M_\delta^+ R^- u = \tilde{u}$ *where* $\tilde{u} \in N_\delta^p(A)$.

PROOF. For $g \in C_0^\infty(E)$ we find as in the preceding proof

$$\int_X \langle Q_\delta \mu_A(R^- u), g \rangle_E \, dx = -\int_{X^-} \langle u, g \rangle_E \, dx + \sum_{i=1}^{N} \int_X \langle u_i, g \rangle_E \, dx \int_{X^-} \langle u, g_i \rangle_E \, dx .$$

But now $g_i \equiv 0$ in $X^-$ for $M < i \leqslant N$, so we may let

$$\tilde{u} = \sum_{i=1}^{M} \left( \int \langle u, g_i \rangle_E \, dx \right) u_i$$

to find

$$\int_X \langle Q_\delta \mu_A(R^- u), g \rangle_E \, dx = \int_X (\tilde{u}, g \rangle_E \, dx - \int_{X^-} \langle u, g \rangle_E \, dx . \tag{3.20}$$

Again, taking $g$ in (3.20) with support in $X^\pm$ proves the proposition.

These propositions imply the following relations on the Cauchy data.

COROLLARY 3.5. *If* $U \in H_\delta^+$ *then* $U = R^+ M_\delta^+ U$. *If* $u \in H^-$ *then*

$$U = -R^- M_\delta^- U + R^- \tilde{u}$$

*where* $\tilde{u} \in N_s^p(A)$.〉

Now consider two weights $\delta_1 \leqslant \delta_2$ so that $K_{\delta_1}^+(A) \supset K_{\delta_2}^+(A)$.

COROLLARY 3.6. *Suppose* $\delta_1, \delta_2, \in \mathbf{R} \setminus \mathfrak{D}_A$ *and* $\delta_1 \leqslant \delta_2$. *If* $U \in CD_{t,\delta_2}^p(E|\Gamma)$ *then* $M_{\delta_2}^+ U - M_{\delta_1}^+ R^+ M_{\delta_2}^+ U \in N_{\delta_1}^{p\#}(A)$.

PROOF. By Proposition 3.2, $M_{\delta_2}^+ U \in K_{\delta_2}^+(A) \subset K_{\delta_1}^+(A)$, so we may apply Proposition 3.3 with $\delta = \delta_1$ to $u = M_{\delta_2}^+ U$.

PROPOSITION 3.7. *Suppose* $\delta_1, \delta_2 \in \mathbf{R} \setminus \mathfrak{D}_A$ *and* $\delta_1 \leqslant \delta_2$. *If* $U \in H_{\delta_1}^+ \cap CD_{t,\delta_2}^p \cdot (E|\Gamma)$, *then* $U = R^+ u$ *where* $u \in K_{\delta_1}^+$ *is of the form* $u = u_1 + u_2$ *with*

$$u_1 \in N_{\delta_1}(A) \qquad \textit{and} \qquad u_2 \in K_{\delta_2}^+(A) .$$

PROOF. Both $M^+_{\delta_1}U$ and $M^+_{\delta_2}U$ are defined and $U = R^+ M^+_{\delta_1} U_{\delta_1}$ by Corollary 3.5. Let $U_2 = R^+ M^+_{\delta_2} U - U$, so by Proposition 3.1 we have $U_2 \in CD^p_{t,\delta_2} \cdot (E|\Gamma)$. We claim that

$$U_2 = R^+ M^+_{\delta_1} U_2 = -R^+ M^+_{\delta_1} R^- M^-_{\delta_2} U_2 . \tag{3.21}$$

If this is true, then letting $u_1 = M^+_{\delta_1} R^- M^-_{\delta_2} U_2 \in N_{\delta_1}(A)$ (by Proposition 3.4) and $u_2 = M^+_{\delta_2} U \in K^+_{\delta_2}(A)$ (by Proposition 3.2) we find that $u = u_1 + u_2$ satisfies

$$\begin{aligned} R^+ u &= R^+ M^+_{\delta_1} R^- M^-_{\delta_2} U_2 + R^+ M^+_{\delta_2} U \\ &= - U_2 + (U_2 + U) = U \end{aligned}$$

proving the proposition.

The first equality in (3.21) follows from Corollary 3.5 since $U_2 \in H^+_{\delta_1}$. Now for $v \in C^\infty_0(F)$ use (3.14) to obtain

$$\int_X \langle \mu_A(U_2), v \rangle_F \, dx = \int_{X^+} \langle M^+_{\delta_2} U_2 , A^* v \rangle \, dx + \int_{X^-} \langle M^-_{\delta_2} U_2 , A^* v \rangle \, dx .$$

But $M^+_{\delta_2} U_2 = M^+_{\delta_2} R^+ M^+_{\delta_2} U - M^+_{\delta_2} U \in N^{p\#}_{\delta_2}(A)$ by Proposition 3.3, and by (3.8)

$$\int_{X^-} \langle M^-_{\delta_2} U_2 , A^* v \rangle_E \, dx = -\int_X \langle \mu_A(R^- M^-_{\delta_2} U_2), v \rangle_F \, dx$$

since $A M^-_{\delta_2} U_2 = 0$ in $X^-$. Thus $\mu_A(U_2) = -\mu_A(R^- M^-_{\delta_2} U_2)$ and applying $Q_{\delta_1}$ to this equation establishes the second equality in (3.21) and completes the proof.

## 4. – Proof of Theorem 1.4.

Theorem 1.3 implies that for $\delta \in \mathbb{R} \setminus \mathfrak{D}_A$,

$$A : W^p_{t,\delta}(E^+) \to W^p_{s,\delta}(F^+) \tag{4.1}$$

has closed range of finite codimension and

$$B : K^+_\delta(A) \to H^p_{r-1/p}(G) \tag{4.2}$$

is Fredholm. We begin with the following

PROPOSITION 4.1. *If* $\delta \in \mathbb{R} \setminus \mathfrak{D}_A$ *then the codimension of* (4.1) *is equal to* $\dim N^{p'\#}_{-\delta}(A^*)$.

PROOF. Let $v_1, \ldots, v_M, \ldots, v_N$ be a basis for $N^{p'}_{-\delta}(A^*)$ with $v_1, \ldots, v_M$ linearly independent on $X^-$ and $v_{M+1}|X^- = \ldots = v_N|X^- = 0$; hence $\{v_{M+1}, \ldots, v_N\}$ forms a basis for $N^{p'\#}_{-\delta}(A^*)$. We have $M$ linearly independent functionals on $C_0^\infty(F|X^-)$

$$f \mapsto \int_{X^-} \langle f, v_j \rangle_F \, dx$$

so let us choose $f_1, \ldots, f_M \in C_0^\infty(F|X^-)$ such that

$$\int_{X^-} \langle f_i, v_j \rangle_F \, dx = \delta_{ij} .$$

Since $X^-$ is bounded we may find an extension operator $\mathcal{E}$ which is bounded from $W^p_{s,\delta}(F^+)$ to $W^p_{s,\delta}(F)$. Thus the operator Ext defined by

$$\operatorname{Ext}(f) = \mathcal{E}(f) - \sum_{j=1}^{M} \left( \int_X \langle \mathcal{E}(f), v_j \rangle_F \, dx \, f_j \right.$$

is bounded from $W^p_{s,\delta}(F^+)$ to $W^p_{s,\delta}(F)$ and satisfies

$$\int_X \langle \operatorname{Ext}(f), v_j \rangle_F \, dx = 0 \qquad (i \leqslant j \leqslant M) ,$$

$$\int_X \langle \operatorname{Ext}(f), v_j \rangle_F \, dx = \int_{X^+} \langle f, v_j \rangle_F \, dx \qquad (M + 1 \leqslant j \leqslant N) .$$

Thus $\int_{X^-} \langle f, v_j \rangle_F \, dx = 0$ for $M + 1 \leqslant j \leqslant N$ implies we can solve $Au = \operatorname{Ext}(f)$ for $u \in W^p_{t,\delta}(E)$, and hence $Au = f$ *in* $X^+$. Conversely, if $Au = f$ in $X^+$ then for $M + 1 \leqslant j \leqslant N$ $\int_{X^-} \langle f, v_j \rangle \, dx = \int_{X^+} \langle u, A^* v_j \rangle_F \, dx = 0$ since $Rv_j = 0$. Since $\dim N^{p'\#}_{-\delta}(A^*) = N - M$ this proves the proposition.

We now wish to investigate the index of (4.2) and how it changes with $\delta$. Let $\delta_1 < \delta_2$ so that $K^+_{\delta_2}(A) \subset K^+_{\delta_1}(A)$. The commutative diagram

$$\begin{array}{ccc} K^+_{\delta_2}(A) & \overset{B}{\searrow} & \\ \downarrow & & \mathrm{H}^p_{r-1/p}(G) \\ K^+_{\delta_1}(A) & \underset{B}{\nearrow} & \end{array}$$

implies that $K^+_{\delta_1}(A)/K^+_{\delta_2}(A)$ is finite-dimensional.

PROPOSITION 4.2. *If* $\delta_1, \delta_2 \in \mathbb{R}\setminus\mathfrak{D}_A$ *with* $\delta_1 \leqslant \delta_2$ *then*

$$\dim\left(K^+_{\delta_1}(A)/K^+_{\delta_2}(A)\right) = i_{\delta_1}(A) - i_{\delta_2}(A) - \dim N^{p'\#}_{-\delta_2}(A^*) + \dim N^{p'\#}_{-\delta_1}(A^*)\,. \tag{4.3}$$

PROOF. Let $K^+_{\delta_1}(A) = K^+_{\delta_2}(A) \oplus J$ and $\tilde{J} = R^+J$. Then

$$\dim J = \dim \tilde{J} + \dim N^{p\#}_{\delta_1}(A) - \dim N^{p\#}_{\delta_2}(A)\,. \tag{4.4}$$

By Proposition 3.1, $\tilde{J} \subset CD^p_{t,\delta_1}(E|\Gamma)$ so let $\tilde{J}_2 = \tilde{J} \cap CD^p_{t,\delta_2}(E|\Gamma)$, and write $\tilde{J} = \tilde{J}_1 \oplus \tilde{J}_2$.

Pick a basis $\{v_1, \ldots, v_L, \ldots, v_M, \ldots, v_N\}$ for $N^{p'}_{-\delta_2}(A^*)$ where

$$v_1, \ldots, v_L \in N^{p'}_{-\delta_1}(A^*)\,, \qquad v_{L+1}, \ldots, v_M \notin N^{p'}_{-\delta_1}(A^*)\,,$$

and $Rv_{M+1} = \ldots = Rv_N = 0$ so that $L = \dim N^{p'}_{-\delta_1}(A^*) - \dim N^{p'\#}_{-\delta_1}(A^*)$ and $M = \dim N^{p'}_{-\delta_2}(A^*) - \dim N^{p'\#}_{-\delta_2}(A^*)$. The functionals

$$\Phi_i(f) = \int_X \langle \mu_A(f), v_i\rangle_F\, dx \qquad (i = 1, \ldots, M)$$

are linearly independent on $CD^p_t(E|\Gamma)$. For $1 \leqslant i \leqslant L$ the $\Phi_i$ vanish on $\tilde{J}$ but are linearly independent for $L < i \leqslant M$, with nullspace equal to $\tilde{J}_2$. Hence

$$\dim \tilde{J}_1 = \dim N^{p'}_{-\delta_2}(A^*) - \dim N^{p'\#}_{-\delta_2}(A^*) - \dim N^{p'}_{-\delta_1}(A^*) + \dim N^{p'\#}_{-\delta_1}(A^*)\,. \tag{4.5}$$

If $u \in N^p_{\delta_1}(A)$ then $Ru \in H^+_{\delta_1} \cap H^- \subset H^+_{\delta_1} \cap CD^p_{t,\delta_2}(E|\Gamma)$ so

$$R^p_{\delta_1}(N(A))/R(N^p_{\delta_2}(A)) = R(N^p_{\delta_1}(A))/\left(R(N^p_{\delta_1}(A)) \cap H^+_{\delta_2}\right) \subset \left(H^+_{\delta_1} \cap CD^p_{t,\delta_2}(E|\Gamma)\right)/H^+_{\delta_2}\,.$$

On the other hand, by Proposition 3.7,

$$H^+_{\delta_1} \cap CD^p_{t,\delta_2}(E|\Gamma) \subset H^+_{\delta_2} + R(N^p_{\delta_1}(A))$$

so

$$\left(H^+_{\delta_1} \cap CD^p_{t,\delta_2}(E|\Gamma)\right)/H^+_{\delta_2} \subset R(N^p_{\delta_1}(A))/R(N^p_{\delta_2}(A))\,.$$

Since $\tilde{J}_2$ is isomorphic to $\left(H^+_{\delta_1} \cap CD^p_{t,\delta_2}(E|\Gamma)\right)/H^+_{\delta_2}$ we find

$$\begin{aligned}\dim \tilde{J}_2 &= \dim R(N^p_{\delta_1}(A)) - \dim R(N^p_{\delta_2}(A)) \\ &= \dim N^p_{\delta_1}(A) - \dim N^{p\#}_{\delta_1}(A) - \dim N^p_{\delta_2}(A) + \dim N^{p\#}_{\delta_2}(A)\,.\end{aligned} \tag{4.6}$$

Combining (4.4), (4.5), and (4.6) yields (4.3).

REMARK 4.3. Notice thet Proposition 4.2 only involves the boundary operator $B$ to establish:

$$(*) \qquad \dim\left(K^+_{\delta_1}(A)/K^+_{\delta_2}(A)\right) < \infty .$$

Thus formula (4.3) holds whenever (*) is known.

PROOF of THEOREM 1.4. Note that $i_\delta(A, B)$ is just the index of (4.2) minus the codimension of (4.1). Thus, invoking Propositions 4.1 and 4.2 we find $i_{\delta_1}(A, B) - i_{\delta_2}(A, B) = i_{\delta_1}(A) - i_{\delta_2}(A)$ as to be shown .

## 5. – Proof of Theorem 1.2.

Let $X^+ = \Omega \times \mathbb{R}^+$ and recall that solutions of (1.10) admit asymptotic expansions in terms of exponential solutions of the form (1.11). In particular, suppose $\delta_1, \delta_2 \in \mathbb{R} \setminus \mathfrak{D}_A$ with $\delta_1 \leqslant \delta_2$ and let $\lambda_1, \ldots, \lambda_N$ denote the points of $\mathcal{C}_A$ satisfying $\delta_1 < \operatorname{Im} \lambda_l < \delta_2$. For $u \in K^+_{\delta_1}(A)$ we can find $u_1, \ldots, u_N$ of the form (1.11) such that

$$u - \sum_{l=1}^{N} u_l \in K^+_{\delta_2}(A)$$

(cf. [2], [14], [18]). Thus $\dim\left(K^+_{\delta_1}(A)/K^+_{\delta_2}(A)\right) \leqslant N(\delta_1, \delta_2)$. Since the reverse inequality is clear we find

$$(5.1) \qquad \dim\left(K^+_{\delta_1}(A)/K^+_{\delta_2}(A)\right) = N(\delta_1, \delta_2) .$$

Now, since $X^+ = \Omega \times \mathbb{R}^+$, we have $N^{p\#}_{\delta}(A) = \{0\} = N^{p'\#}_{-\delta}(A^*)$ for any $\delta \in \mathbb{R}$. Indeed, if $u \in N^{p\#}_{\delta}(A)$ let $\delta_1 \in \mathbb{R} \setminus \mathfrak{D}_A$ with $\delta_1 \leqslant \delta$, so $u \in N^{p\#}_{\delta_1}(A)$. We may extend $u$ to $Y = \Omega \times \mathbb{R}$ by letting $u(\omega, z) = 0$ if $z < 0$. Since (2.1) is an isomorphism we must have $u = 0$. (The same proof works for $A^*$.) Thus we may use Remark 4.3 to conclude

$$(5.2) \qquad \dim\left(K^+_{\delta_1}(A)/K^+_{\delta_2}(A)\right) = i_{\delta_1}(A) - i_{\delta_2}(A) .$$

Combining this with (5.1) completes the proof.

## 6. – Perturbations of translation invariant operators.

In this section we consider an elliptic operator $A\colon C^\infty(E) \to C^\infty(F)$ which is a perturbation of an elliptic operator $A_\infty\colon C^\infty(E) \to C^\infty(F)$ which is translation invariant in $z > 0$ as treated in the preceding sections. If $A$ is elliptic of order $m \in \mathbb{N}$ and $d = \dim(E, F) = 1$ then in each coordinate patch $\mathring{X}_\nu$ $(\nu = 1, \ldots, N)$

$$\tag{6.1} \begin{cases} A|\mathring{X}_\nu = \sum_{q+|\beta| \leqslant m} a_{q\beta}(\omega, z) D_\omega^\beta D_z^q , \\ A_\infty|\mathring{X}_\nu = \sum_{q+|\beta| \leqslant m} \overset{\infty}{a}_{q\beta}(\omega) D_\omega^\beta D_z^q . \end{cases}$$

Suppose the coefficients $a_{q\beta}(\omega, z)$ satisfy

$$\tag{6.2} \sup\{|D_z^h D_\omega^\gamma a_{q\beta}(\omega, z)|:\ \omega \in \operatorname{supp} \varphi_\nu,\ z \in \mathbb{R}^+\} = C_{q\beta}^{h\gamma}(\nu) < \infty$$

for $h + |\gamma| \leqslant s \in \mathbb{N}$ and $q + |\beta| \leqslant m$, where $\varphi_\nu$ $(\nu = 1, \ldots, N)$ denotes the partition of unity introduced in Section 1. Then

$$\tag{6.3} \|\varphi_\mu A \varphi_\nu u\|_{W^p_{m,\delta}(F|X_\mu)} \leqslant C(\mu, \nu) \|\varphi_\nu u\|_{W^p_{s+m,\delta}(E|X_\nu)}$$

for $\mu, \nu = 1, \ldots, N$; so

$$\tag{6.4} A\colon\ W^p_{s+m,\delta}(E) \to W^p_{s,\delta}(F)$$

is bounded.

In fact we assume that the coefficients of $A$ are $C^\infty$ on $X$, and that for all $\nu = 1, \ldots, N$, $q + |\beta| \leqslant m$, and $h + |\gamma| \leqslant s$

$$\tag{6.5} \lim_{z \to \infty} |D_z^h D_\gamma^\omega (a_{q\beta}(\omega, z) - \overset{\infty}{a}_{q\beta}(\omega))| = 0$$

uniformly in $\omega \in \operatorname{supp} \varphi_\nu$. The purpose of this section is to compare the maps

$$\tag{6.6} A\colon\ W^p_{s+m,\delta}(E) \to W^p_{s,\delta}(F) ,$$

$$\tag*{$(6.6)_\infty$} A_\infty\colon\ W^p_{s+m,\delta}(E) \to W^p_{s,\delta}(F) .$$

As we shall see, these maps are Fredholm for exactly the same values of $\delta$, and moreover their Fredholm indices agree up to a constant independent of $\delta$.

More generally, suppose (1.7) holds, $A\colon C^\infty(E) \to C^\infty(F)$ and $A_\infty\colon C^\infty(E) \to C^\infty(F)$ are both elliptic with respect to the system of orders $(t, s)$, and $A_\infty$ is translation invariant in $z > 0$. In each $X_\nu$ the coefficients of $A_{ij}$ and $(A_\infty)_{ij}$ are matrices; if the corresponding matrix entries satisfy (6.5) for all $q + |\beta| \leqslant t_j - s_i$ and $|h| + |\gamma| \leqslant s_i$ then we say *$A$ is asymptotic to $A_\infty$* and write

$A \sim A_\infty$. Again we wish to compare

$$(6.7) \qquad A\colon \; W^p_{t,\delta}(E) \to W^p_{s,\delta}(F)\,,$$

$$(6.7)_\infty \qquad A_\infty\colon \; W^p_{t,\delta}(E) \to W^p_{s,\delta}(F)\,.$$

We shall prove the following.

THEOREM 6.1. *The maps* (6.7) *and* $(6.7)_\infty$ *are Fredholm for exactly the same values of* $\delta$ *and their Fredholm indices differ by a constant independent of* $\delta$.

PROOF. Suppose $(6.7)_\infty$ is Fredholm. Let $\varphi_R \in C^\infty(X)$ with $\varphi_R \equiv 1$ on $X_R = X_0 \cup (\Omega \times [0, R])$ and $\operatorname{supp} \varphi_R \subset X_{2R}$. Using (6.2) and the openness of the Fredholm group, we take $R \gg 0$ so that $A'_\infty = A_\infty + (1-\varphi_R)(A - A_\infty)$ is Fredholm for $\delta$, and $i_\delta(A'_\infty) = i_\delta(A_\infty)$. Notice that $A'_\infty = A$ for $z > 2R$. Let $P_1$ be a parametrix for $A$ in $X_{4R}$, $P_2$ be a Fredholm inverse for $A'_\infty$: $W^p_{t,\delta} \to W^p_{s,\delta}$ and

$$Tf = \psi_1 P_1(\varphi_{2R} f) + \psi_2 P_2((1-\varphi_{2R}) f)$$

where $\psi_1 = 1$ on $\operatorname{supp} \varphi_{2R}$, $\psi_1 \in C_0^\infty(X_{4R})$, $\psi_2 \equiv 1$ on $\operatorname{supp}(1-\varphi_{2R})$, and $\psi_2 = 0$ on $X_{2R}$. Then $T$ defines a Fredholm inverse for $A$, so (6.7) is Fredholm. Let $\mathfrak{D}_A = \mathfrak{D}_{A_\infty}$ as in Section 1.

Now suppose $\delta_1, \delta_2 \in \mathbb{R} \setminus \mathfrak{D}_A$ with $\delta_1 \leqslant \delta_2$ and choose $R_1$ so large that for all $R \geqslant R_1$ we have $A_\infty + \varphi_R(A - A_\infty)$ elliptic and

$$i_{\delta_j}(A) = i_{\delta_j}(A_\infty + \varphi_R(A - A_\infty)) \qquad \text{for } j = 1, 2\,.$$

Letting $X^+ = \Omega \times (2R, \infty)$ we can apply Proposition 4.2 and (5.2) to conclude

$$\begin{aligned} i_{\delta_1}(A) - i_{\delta_2}(A) &= i_{\delta_1}(A_\infty + \varphi_R(A - A_\infty)) - i_{\delta_2}(A_\infty + \varphi_R(A - A_\infty)) \\ &= \dim\,(K^+_{\delta_1}(A_\infty)/K^+_{\delta_2}(A_\infty)) \\ &= i_{\delta_1}(A_\infty) - i_{\delta_2}(A_\infty)\,. \end{aligned}$$

So the indices of $A$ and $A_\infty$ differ by a constant independent of $\delta$. In particular, since $i_\delta(A_\infty)$ changes as $\delta$ crosses a point of $\mathfrak{D}_A$, we can see that $A$ cannot be Fredholm for $\delta \in \mathfrak{D}_A$. This completes the proof.

As a corollary we see that Theorems 1.1 and 1.2 remain true for such perturbations.

THEOREM 6.2. *If $A$ is elliptic with respect to $(t, s)$ and $A \sim A_\infty$ where $A_\infty$ is elliptic with respect to $(t, s)$ and translation invariant in $z > 0$, then* (6.7) *is Fredholm if and only if $\delta \in \mathbb{R}\setminus\mathfrak{D}_A$ ($\mathfrak{D}_A = \mathfrak{D}_{A_\infty}$). Moreover, if $\delta_1, \delta_2 \in \mathbb{R}\setminus\mathfrak{D}_A$ with $\delta_1 \leqslant \delta_2$ then $i_{\delta_1}(A) - i_{\delta_2}(A) = N(\delta_1, \delta_2)$.*

Similarly we find that Theorems 1.3 and 1.4 and Corollary 1.5 remain true for perturbations.

THEOREM 6.3. *If $A$ satisfies the hypotheses of Theorem* 6.2 *and $(A, B)$ is elliptic on $X^+$ with respect to $(t, s, r)$, then*

$$(A, B)\colon\ W^p_{t,\delta}(E^+) \to W^p_{s,\delta}(F^+) \times H^p_{r-1/p}(G) \tag{6.8}$$

*is Fredholm if and only if $\delta \in \mathbb{R}\setminus\mathfrak{D}_A$. Moreover, if $\delta_1, \delta_2 \in \mathbb{R}\setminus\mathfrak{D}_A$ with $\delta_1 < \delta_2$ then $i_{\delta_1}(A, B) - i_{\delta_2}(A, B) = N(\delta_1, \delta_2)$.*

## 7. – Some index theory.

We collect here some easy results on the stability of nullspaces and the index, and show how to compute the index of an operator whose principal symbol is self-adjoint. Throughout this section $A$ denotes an operator satisfying the hypotheses of Theorem 6.2. Thus (6.7) is bounded for all $\delta \in \mathbb{R}$, and Fredholm for $\delta \in \mathbb{R}\setminus\mathfrak{D}_A$ with Fredholm index $i_\delta(A) = \dim N^p_\delta(A) - \dim N^{p'}_{-\delta}(A^*)$ where

$$A^*\colon\ W^{p'}_{-s,-\delta}(F) \to W^{p'}_{-t,-\delta}(E) \tag{7.1}$$

is defined with respect to the Hermitian structures of Section 3:

$$\int_X \langle u, A^*v\rangle_E\, dx = \int_X \langle Au, v\rangle_F\, dx$$

for $u \in W^p_{t,\delta}(E)$ and $v \in W^{p'}_{-s,-\delta}(F)$.

LEMMA 7.1. *If the closed interval $[\delta_1, \delta_2] \subset \mathbb{R}\setminus\mathfrak{D}_A$, then $i_{\delta_1}(A) = i_{\delta_2}(A)$, $N^p_{\delta_1}(A) = N^p_{\delta_2}(A)$, and $N^{p'}_{-\delta_1}(A^*) = N^{p'}_{-\delta_2}(A^*)$.*

PROOF. Theorems 1.2 and 6.2 show $i_{\delta_1}(A) = i_{\delta_2}(A)$. Now $N^p_{\delta_1}(A) \supset N^p_{\delta_2}(A)$ and $N^{p'}_{-\delta_1}(A^*) \subset N^{p'}_{-\delta_2}(A^*)$, so

$$\dim N^p_{\delta_1}(A) - \dim N^p_{\delta_2}(A) \geqslant 0 \quad\text{and}\quad \dim N^{p'}_{-\delta_2}(A^*) - \dim N^{p'}_{-\delta_1}(A^*) \geqslant 0\,.$$

Since

$$i_{\delta_1}(A) - i_{\delta_2}(A) = \dim N^p_{\delta_1}(A) - \dim N^p_{\delta_2}(A) + \dim N^{p'}_{-\delta_2}(A^*) - \dim N^{p'}_{-\delta_1}(A^*) = 0$$

we conclude that $N^p_{\delta_1}(A) = N^p_{\delta_2}(A)$ and $N^{p'}_{-\delta_1}(A^*) = N^{p'}_{-\delta_2}(A^*)$.

For the next lemma note that $t = (t_1, \ldots, t_J) \geqslant C$ means $t_j \geqslant C$ for all $j$.

LEMMA 7.2. *There is a continuous embedding* $W^p_{t,\delta_2}(E) \to W^p_{\hat{t},\delta_1}(E)$ *if:*

i) $t - \hat{t} \geqslant n/p - n/q$,

ii) $t - \hat{t} \geqslant 0$, *and either*

iii) $1 < p \leqslant q < \infty$ *with* $\delta_1 \leqslant \delta_2$, or

iii)′ $1 < q < p < \infty$ *with* $\delta_1 < \delta_2$.

PROOF. By the classical Sobolev embedding theorem it suffices to consider

$$W^p_{t_j,\delta_2}(E_j|\mathring{X}_\nu) \to W^p_{\hat{t}_j,\delta_1}(E_j|\mathring{X}_\nu) \tag{7.2}$$

for $\nu = 1, \ldots, N$ and $j = 1, \ldots, J$. In case iii) we use the classical embedding theorem (with measure $\exp[\delta_2 z]\, d\omega\, dz$) and the embedding

$$W^q_{t_j,\delta_2}(E_j|X_\nu) \to W^p_{\hat{t}_j,\delta_1}(E_j|X_\nu)$$

to conclude (7.2) is continuous.

In case iii)′ we use Hölder's inequality to show

$$\|u\|_{W^q_{0,\delta_1}} \leqslant \left( \int_{X_\nu} \exp\left[\frac{pq}{p-q}(\delta_1 - \delta_2) z\right] d\omega\, dz \right)^{(p-q)/pq} \|u\|_{W^p_{0,\delta_2}}. \tag{7.3}$$

Since $p > q$ and $\delta_1 < \delta_2$ this shows that (7.2) is continuous whenever $\hat{t}_j = 0$, and the general case follows by replacing $u$ in (7.3) by $D^\alpha u$.

Notice that $\mathfrak{D}_A$ does not depend on $p$, and the notation $i_\delta(A)$ suggests that the index does not depend on $p$. We now see that this is indeed the case, but for the moment let $i_\delta(A; p)$ denote the index of (6.7).

LEMMA 7.3. *If* $\delta \in \mathbb{R} \setminus \mathfrak{D}_A$ *and* $1 < p,\ q < \infty$ *then* $N^p_\delta(A) = N^q_\delta(A)$ *and* $i_\delta(A; p) = i_\delta(A; q)$.

PROOF. Assume $p > q$. Choose $\varepsilon > 0$ so that $[\delta, \delta + \varepsilon] \subset \mathbb{R} \setminus \mathfrak{D}_A$. By Lemma 7.1 we find $N^p_\delta(A) = N^p_{\delta+\varepsilon}(A)$ and by Lemma 7.2 (with $t = \hat{t}$ and iii)′) we find $N^p_{\delta+\varepsilon}(A) \subset N^q_\delta(A)$, so $N^p_\delta(A) \subset N^q_\delta(A)$. Similarly, since $q' > p'$

we have $N^{p'}_{-\delta}(A^*) \supset N^{q'}_{-\delta}(A^*)$. Hence, since

$$i_\delta(A; p) = \dim N^p_\delta(A) - \dim N^{p'}_{-\delta}(A^*) \text{ and } i_\delta(A; q) = \dim N^q_\delta(A) - \dim N^{q'}_{-\delta}(A^*)$$

we have that $i_\delta(A; p) \leqslant i_\delta(A; q)$ and we have equality if and only if $N^p_\delta(A) = N^q_\delta(A)$ and $N^{q'}_{-\delta}(A^*) = N^{p'}_{-\delta}(A^*)$. Thus we need only to show that $i_\delta(A; p) \geqslant i_\delta(A; q)$.

To do this we first approximate $A$ in the operator norm by $\hat{A}$ with $C^\infty$-coefficients such that $i_\delta(\hat{A}; p) = i_\delta(A; p)$, $i_\delta(\hat{A}; q) = i_\delta(\hat{A}; p)$ and in each $X_\nu$ $(\nu = 1, \ldots, N)$ the coefficients satisfy

$$\lim_{z\to\infty} |D^h_z D^\gamma_\omega(\hat{a}_{\alpha\beta}(\omega, z) - \overset{\infty}{a}_{\alpha\beta}(\omega))| = 0 \tag{7.4}$$

uniformly in $\omega \in \operatorname{supp} \varphi_\nu$, for *all* $h + |\gamma| < \infty$. By elliptic regularity, $N^q_\delta(\hat{A})$ is also the nullspace of

$$\hat{A}\colon W^q_{\hat{t},\delta}(E) \to W^q_{\hat{s},\delta}(F)$$

where $\hat{t} = t + l$ and $\hat{s} = s + l$ for $l \in \mathbb{N}$. In particular with $l \geqslant n/q - n/p$ we find $N^q_\delta(\hat{A}) \subset W^q_{\hat{t},\delta}(E) \subset W^p_{t,\delta}(E)$ so $N^q_\delta(\hat{A}) \subset N^p_\delta(\hat{A})$. Similarly $N^{p'}_{-\delta}(\hat{A}^*) \subset N^{q'}_{-\delta}(\hat{A}^*)$. Thus $i_\delta(\hat{A}; p) \geqslant i_\delta(\hat{A}; q)$. But this means $i_\delta(A; p) \geqslant i_\delta(A; q)$ which completes the proof.

Now suppose $A = A_0 + A_1$ where $A_0$ is the principal part of $A$ and $A_1$ has lower order (i.e., each $(A_1)_{ij}$ involves only derivatives of order strictly less than $t_j - s_i$). Consider the homotopy through elliptic operators satisfying the hypotheses of Theorem 6.2: $A(\tau) = A_0 + \tau A_1$ for $0 \leqslant \tau \leqslant 1$. The points in $\mathfrak{D}_{A(\tau)}$ vary continuously in $\tau$ and the index $i_\delta(A(\tau))$ is constant in $\tau \in [\tau_1, \tau_2]$ provided $\delta \notin \{\mathfrak{D}_{A(\tau)}\colon \tau_1 \leqslant \tau \leqslant \tau_2\}$. For each fixed $\tau$ we have the change of index formula, so theoretically we can compute $i_\delta(A)$ from $i_\delta(A_0)$.

If $A\colon C^\infty(E) \to C^\infty(E)$ and $A_0 = A_0^*$ then we can apply the following to determine $i_\delta(A_0)$.

THEOREM 7.4. *Let* $A\colon C^\infty(E) \to C^\infty(E)$ *satisfy the hypotheses of Theorem* 6.2 *and be self-adjoint*: $A = A^*$. *Let* $\varepsilon \geqslant 0$ *such that* (6.7) *is Fredholm for* $\varepsilon \geqslant |\delta| > 0$. *Then*

$$i_{-\varepsilon}(A) = \tfrac{1}{2} \sum \{d(\lambda)\colon \operatorname{Im} \lambda = 0\}\,. \tag{7.5}$$

*In particular, if* (6.7) *is Fredholm for* $\delta = \varepsilon = 0$ *then* $i_0(A) = 0$.

PROOF. Taking $p = 2$ the formula (7.5) follows from Theorem 6.2 and the calculation

$$i_{-\varepsilon}(A) - i_\varepsilon(A)$$
$$= (\dim N^2_{-\varepsilon}(A) - \dim N^2_\varepsilon(A)) - (\dim N^2_\varepsilon(A) - \dim N^2_{-\varepsilon}(A)) = 2i_{-\varepsilon}(A) .$$

## II. GENERALIZATIONS AND APPLICATIONS

### 8. - Manifolds with multiple ends.

In this section we suppose that $X$ has multiple cylindrical ends at infinity. More precisely, $X$ contains an open submanifold $\mathring{X}_0$ whose closure in $X$, $X_0$, is compact and satisfies: $X \setminus X_0$ is a disjoint union $\bigcup_{l=1}^{L} \Omega_l \times \mathring{\mathbb{R}}^+$ where each $\Omega_l$ is an $(n-1)$-dimensional compact $C^\infty$-manifold without boundary. We also suppose that we are given 2 vector bundles $E$ and $F$ over $X$ and a differential operator of order $(t, s)$, $A\colon C_0^\infty(E) \to C_0^\infty(F)$, which is translation invariant in each end $\Omega_l \times \mathbb{R}^+$. (As before, $t$ and $s$ may be vectors if $E$ and $F$ decompose into direct sums.)

We may define weighted Sobolev spaces with different weights on each end. Let $\varphi_0, \varphi_1, \ldots, \varphi_L$ denote a $C^\infty$-partition of unity with $\operatorname{supp} \varphi_0$ compact and $\operatorname{supp} \varphi_l \subset \Omega_l \times \mathring{\mathbb{R}}^+$ for $l = 1, \ldots, L$. For $\delta = (\delta(1), \ldots, \delta(L)) \in \mathbb{R}^L$ let $W^p_{s,\delta}(E)$ be the closure of $C_0^\infty(E)$ under the norm

$$\|u\|_{W^p_{s,\delta}} = \|\varphi_0 u\|_{H^p_s(E)} + \sum_{l=1}^{L} \|\varphi_l u\|_{W^p_{s,\delta(l)}(E|\Omega_l \times \mathring{\mathbb{R}}^+)} .$$

We find that

$$A\colon W^p_{t,\delta}(E) \to W^p_{s,\delta}(F) \tag{8.1}$$

is a bounded operator.

Now assume that $A$ is elliptic with respect to $(t, s)$. Let $A(l) = A|X_l$ and $\mathcal{C}_{A(l)} \subset \mathbf{C}$ denote the « generalized eigenvalues » as defined in Section 1. Let $\mathcal{D}_{A(l)} = \{\delta = \operatorname{Im} \lambda \in \mathbb{R}\colon \lambda \in \mathcal{C}_{A(l)}\}$, and for $\lambda(l) \in \mathcal{C}_{A(l)}$ denote the dimension of « exponential solutions » (1.11) in $\Omega_l \times \mathring{\mathbb{R}}^+$ with exponent $\lambda(l)$ by $d[\lambda(l)]$. Define $\mathcal{D}_A = \{\delta = (\delta(1), \ldots, \delta(L))\colon$ for at least one $l$, $\delta(l) = \operatorname{Im} \lambda(l)$

where $\lambda(l) \in \mathcal{C}_{A(l)}\}$, and for $\delta_1, \delta_2 \in \mathbb{R}^L \setminus \mathfrak{D}_A$ with $\delta_1 \leqslant \delta_2$ (i.e. $\delta_1(l) \leqslant \delta_2(l)$ for every $l$) let

$$N(\delta_1, \delta_2) = \sum \{d[\lambda(l)]:\ \lambda(l) \in \mathcal{C}_{A(l)} \text{ with } \delta_1(l) < \operatorname{Im} \lambda(l) < \delta_2(l)\} .$$

(If $L > 1$ then $\mathfrak{D}_A$ is no longer discrete but rather a union of $(L-1)$-dimensional hyperplanes in $\mathbb{R}^L$.)

With these definitions we can state the generalization of Theorems 1.1 and 1.2:

THEOREM 8.1. *If $A$ is elliptic with respect to $(t, s)$ and is translation invariant in each $\Omega_l \times \mathbb{R}^+$, then* (8.1) *is Fredholm if and only if $\delta \in \mathbb{R}^L \setminus \mathfrak{D}_A$. If $\delta_1, \delta_2 \in \mathbb{R}^L \setminus \mathfrak{D}_A$ with $\delta_1 \leqslant \delta_2$ then the change in the Fredholm index of* (8.1) *is given by*

$$i_{\delta_1}(A) - i_{\delta_2}(A) = N(\delta_1, \delta_2) .$$

Similarly we may generalize Theorems 1.3 and 1.4 by considering $\Gamma$, an $(n-1)$-dimensional compact $C^\infty$-submanifold without boundary, which is contained in $X_0$ and such that $\Gamma = \partial X^\pm$ where $X = X^+ \cup X^-$ and $X^-$ is compact. If $G$ is a vector bundle over $\Gamma$, and $B$: $C_0^\infty(E|X^+) \to C^\infty(G)$ is a boundary operator of order $(t, r)$, then we may consider the bounded operator

$$(8.2) \qquad (A, B):\ W^p_{t,\delta}(E^+) \to W^p_{s,\delta}(F^+) \times H^p_{r-1/p}(G) .$$

If we define $(A, B)$ to be elliptic with respect to $(t, s, r)$ exactly as before, then we obtain:

THEOREM 8.2. *If $A$ satisfies the hypotheses of Theorem* 8.1 *and $(A, B)$ is elliptic with respect to $(t, s, r)$ in $X^+$, then* (8.2) *is Fredholm if and only if $\delta \in \mathbb{R}^L \setminus \mathfrak{D}_A$: If $\delta_1, \delta_2 \in \mathbb{R}^L \setminus \mathfrak{D}_A$ with $\delta_1 < \delta_2$ then the change in the Fredholm index of* (8.2) *is given by*

$$i_{\delta_1}(A, B) - i_{\delta_2}(A, B) = N(\delta_1, \delta_2) .$$

These 2 theorems are proved exacly as in Part I (where $L = 1$): for example, to prove Theorem 8.1 let $X^+ = X \setminus X_0$ and use multiple layer potentials to show $i_{\delta_1}(A) - i_{\delta_2}(A) = \dim (K^+_{\delta_1}(A)/K^+_{\delta_2}(A))$, and then asymptotic expansions to equate this with $N(\delta_1, \delta_2)$.

## 9. – Elliptic systems in $\mathbb{R}^n$.

Suppose $A = (A_{ij})$ is an $N \times N$ system of operators in $\mathbb{R}^n$ which is uniformly elliptic (in the sense of Douglis-Nirenberg) with respect to the system of orders $t = (t_1, \ldots, t_N)$ and $s = (s_1, \ldots, s_N)$. Letting

$$A_{ij} = \sum_{|\alpha| \leqslant t_j - s_i} a_\alpha^{ij}(x) D_x^\alpha$$

we make the following assumptions on the coefficients $a_\alpha^{ij}$: for each $i, j, \alpha$ we have $a_\alpha^{ij} \in C^{s_i}(R^n)$ and there is a continuous function $h_\alpha^{ij}$ on the unit sphere $S^{n-1}$ such that in spherical coordinates

$$x = (\omega, r)\,, \qquad \omega \in S^{n-1}\,, \qquad 0 < r < \infty\,,$$

we have

$$\lim_{r=|x|\to\infty} \left| r^{|\gamma|} D_x^\gamma \left( r^{t_j - s_i - |\alpha|} a_\alpha^{ij}(x) - h_\alpha^{ji}(\omega)\right)\right| = 0 \tag{9.1}$$

for all $|\gamma| \leqslant s_i$. Let us define

$$W^p_{t,\delta-t}(\mathbb{R}^n, dx_e) = \prod_{j=1}^{N} W^p_{t_j,\delta-t_j}(\mathbb{R}^n, dx_e)$$

where the factors on the right are defined as the closure of $C_0^\infty(\mathbb{R}^n)$ under the norm

$$\left( \sum_{|\alpha| \leqslant t_j} \int_{\mathbb{R}^n} \left| (1 + |x|)^{\delta - t_j + |\alpha|} D_x^\alpha u(x) \right|^p dx \right)^{1/p}. \tag{9.2}$$

Then $A$ defines a bounded operator

$$A\colon\; W^p_{t,\delta-t}(\mathbb{R}^n, dx_e) \to W^p_{s,\delta-s}(\mathbb{R}^n, dx_e) \tag{9.3}$$

and we may ask: when is this map Fredholm, and how does its index depend on $\delta$?

In the case

$$h_\alpha^{ij} = \begin{cases} \text{constant} & \text{for } |\alpha| = t_j - s_i\,, \\ 0 & \text{for } |\alpha| < t_j - s_i\,, \end{cases} \tag{9.4}$$

it was shown in [17] that if $A$ is elliptic with respect to $(t, s)$ then (9.3) is

Fredholm whenever $\delta \in \mathbb{R} \setminus \mathfrak{D}_A$ where

(9.5)
$$\mathfrak{D}_A = \left\{ \delta \in \mathbb{R} : -\delta + t_j - \frac{n}{p} \in \mathbb{N} \text{ if } \delta - t_j \leqslant -\frac{n}{p} \right.$$
$$\left. \cdot \text{or } \delta - s_j - \frac{n}{p'} \in \mathbb{N} \text{ if } \delta - t_j > -\frac{n}{p} \right\}$$

that is exactly when the operator

(9.6)
$$A_\infty \colon W^p_{t,\delta-t}(\mathbb{R}^n, dx_e) \to W^p_{s,\delta-s}(\mathbb{R}^n, dx_e)$$

is Fredholm where

$$(A_\infty)_{ij} = \sum_{|\alpha| = t_j - s_i} h^{ij}_\alpha D^\alpha_x .$$

(In fact, it was claimed in [17] that the index of (9.3) agrees with that of (9.6) but this is not correct; cf. [17*a*].)

In order to prove a similar result for the more general coefficients (9.1), let $A_\infty$ be an elliptic operator with $C^\infty$-coefficients satisfying

$$(A_\infty)_{ij} = \sum_{|\alpha| \leqslant t_j - s_i} r^{|\alpha| + s_i - t_j} h^{ij}_\alpha(\omega) D^\alpha_x$$

for $|x| > 1$. Then

(9.7)
$$A_\infty \colon W^p_{t,\delta-t}(\mathbb{R}^n, dx_e) \to W^p_{s,\delta-s}(\mathbb{R}^n, dx_e)$$

is a bounded operator. Let $\varrho(x) \in C^\infty(\mathbb{R}^n)$ be a positive function satisfying $\varrho(x) = r = |x|$ if $|x| > 1$, and let

$$\varrho^t(x) = (\varrho^{t_1}(x), \ldots, \varrho^{t_N}(x)) .$$

Thus, multiplication $\varrho^t \colon W^p_{t,\delta}(\mathbb{R}^n, dx_e) \to W^p_{t,\delta-t}(\mathbb{R}^n, dx_e)$ defines an isomorphism and we may replace (9.3) and (9.7) by

(9.8)
$$\tilde{A} = \varrho^{-s} A \varrho^t ,$$

(9.9)
$$\tilde{A}_\infty = \varrho^{-s} A_\infty \varrho^t .$$

In $|x| > 1$ we may write $\tilde{A}_\infty = \varrho^{-s} A_\infty \varrho^t$ in spherical coordinates (locally for $\omega \in S^{n-1}$) as

(9.10)
$$(\tilde{A}_\infty)_{ij} = \sum_{q + |\beta| \leqslant t_j - s_i} \overset{\infty}{a}{}^{ij}_{q\beta}(\omega) (rD_r)^q D^\beta_\omega$$

where ellipticity implies

$$\det\Big(\sum_{\alpha+|\beta|=t_j-s_i} a^{ij}_{\alpha\beta}(\omega)\,\zeta^\alpha\xi^\beta\Big)\neq 0$$

for $\zeta\in\mathbb{R}$, $\xi\in\mathbb{R}^{n-1}$ with $|\zeta|+|\xi|\neq 0$. Let

$$(\tilde{A}_\infty(\lambda))_{ij}=\sum_{\alpha+|\beta|\leqslant t_j-s_i}\overset{\infty}{a}{}^{ij}_{\alpha\beta}(\omega)\,\lambda^\alpha D^\beta_\omega .$$

These operators may be put into the framework of Part I as follows. Let $X$ denote the manifold obtained by gluing together the half-sphere $X_0=S^n_-=\{(x_1,\dots,x_{n-1},z)\in S^n\colon z<0\}$ and the half-cylinder $S^{n-1}\times\mathbb{R}^+$. With a regularization along the seam we may assume $X$ is $C^\infty$. Let $\Phi\colon X\to\mathbb{R}^n$ be a diffeomorphism such that $z>0$ implies $\Phi(\omega,z)=(\omega,r)$ where $r=e^z$. Then $\tilde{A}$ and $\tilde{A}_\infty$ can be realized as operators on $E=X\times\mathbf{C}^N$, and (using $dx=r^{n-1}\,dr\,d\omega=e^{nz}\,dz\,d\omega$) we find that (9.3) is equivalent to

$$\tilde{A}\colon W^p_{t,\tilde\delta}(E)\to W^p_{s,\tilde\delta}(E) \tag{9.11}$$

where $\tilde\delta=\delta+n/p$. But the results of Part I determine that (9.11) is Fredholm if and only if $\tilde\delta\in\mathbb{R}\setminus\mathfrak{D}_{\tilde A}$ and the change of index is given by $\tilde N(\tilde\delta_1,\tilde\delta_2)$ for $\tilde\delta_1<\tilde\delta_2$ with $\tilde\delta_1,\tilde\delta_2\in\mathbb{R}\setminus\mathfrak{D}_{\tilde A}$. Letting

$$\mathfrak{D}_A=\left\{\delta\in\mathbb{R}\colon \delta+\frac{n}{p}\in\mathfrak{D}_{\tilde A}\right\}\qquad\text{and}\qquad N(\delta_1,\delta_2)=\tilde N\left(\delta_1+\frac{n}{p},\,\delta_2+\frac{n}{p}\right)$$

we find

THEOREM 9.1. *If $A=(A_{ij})$ is uniformly elliptic in $\mathbb{R}^n$ with respect to $(t,s)$ and satisfies* (9.1), *then* (9.3) *is Fredholm if and only if $\delta\in\mathbb{R}\setminus\mathfrak{D}_A$. Moreover, if $\delta_1,\delta_2\in\mathbb{R}\setminus\mathfrak{D}_A$ with $\delta_1<\delta_2$, then $i_{\delta_1}(A)-i_{\delta_2}(A)=N(\delta_1,\delta_2)$.*

COROLLARY 9.2. *Under the hypotheses of Theorem* 9.1, (9.3) *is Fredholm if and only if* (9.7) *is Fredholm, and their Fredholm indices differ by a constant independent of $\delta$.*

We can similarly apply the results of Part I to the study of boundary-value problems in exterior domains. Namely, suppose $U^-$ is a compact domain with $C^\infty$-boundary $\Gamma=\partial U^-$ and let $U^+$ be the closure of $\mathbb{R}^n\setminus U^-$. Let

$$W^p_{t,\delta-t}(U^+)=\prod_{j=1}^N W^p_{t_j,\delta-t_j}(U^+)$$

where $W^p_{t_j,\delta-t_j}(U^+)$ is the closure of $C_0^\infty(U^+)$ under the norm (9.2). Suppose $r=(r_1, \ldots, r_\mu)$ is a set of positive integers (where $2\mu$ is the degree of the characteristic $(t, s)$ determinant of $A$), and suppose $B=(B_{kj})$ is a $\mu\times N$ system of differential boundary operators, where $B_{kj}$ is of order $\leqslant t_j-r_k$ and has coefficients in $C^{r_k}(\Gamma)$. Let

$$H^p_{r-1/p}(\Gamma)=\prod_{k=1}^{\mu} H^p_{r_k-1/p}(\Gamma)$$

so

$$(9.12)\qquad (A, B)\colon\ W^p_{t,\delta-t}(U^+)\to W^p_{s,\delta-s}(U^+)\times H^p_{r-1/p}(\Gamma)$$

is bounded. Using the above techniques we transform this to

$$(9.13)\qquad (\hat{A}, \hat{B})\colon\ W^p_{t,\delta+n/p}(E^+)\to W^p_{s,\delta+n/p}(E^+)\times H^p_{r-n/p}(G)$$

where $E^+=X^+\times\mathbb{C}^N$, $X^+=\Phi^{-1}(U^+)$, and $G=Y\times\mathbb{C}^\mu$, $Y=\Phi^{-1}(\Gamma)$. Clearly (9.12) and (9.13) are equivalent, so we may use the results of Part I (e.g. Theorem 6.3) to determine when (9.12) is Fredholm and a formula for the change in the Fredholm index.

## 10. – On Hodge theory for Riemannian manifolds with conic singularities.

Suppose that $X$ is an orientable non-compact $n$-dimensional Riemans nian manifold which topologically is as in Section 8. As vector bundle- we take $\Lambda^k=\Lambda^k(T^*X)$ and let $d\colon C^\infty(\Lambda^k)\to C^\infty(\Lambda^{k+1})$ be the exterior derivative. The metric $g$ induces a pointwise inner product $\langle\ ,\ \rangle$ as well as a measure $d\mu$ on $X$. For $u\in C^\infty(\Lambda^k)$ let

$$|u|^2=\langle u, u\rangle \quad\text{and}\quad \|u\|^2_{L^2}=\int_X |u|^2\,d\mu$$

and denote the closure of $C_0^\infty(\Lambda^k)$ in this norm by $L^2(\Lambda^k)$. Define the $L^2$-harmonic forms by

$$\mathcal{H}^k_\Delta=\{u\in L^2(\Lambda^k)\colon\ \Delta u=0\}$$

where

$$\Delta=d^*d+dd^*\colon\ C^\infty(\Lambda^k)\to C^\infty(\Lambda^k)$$

is the Laplacian and

$$d_k^* = (-1)^{n(k+1)+1} * d *$$

is the coderivative. (Elliptic regularity implies $u \in \mathcal{H}_\Delta^k$ is $C^\infty$.) On the other hand, denote the $L^2$-forms which are closed and coclosed by

$$\mathcal{H}_d^k = \{u \in L^2(\Lambda^k):\ du = d^* u = 0\}\ .$$

If $X$ is *complete* then it is well-known (cf. [26]) that

$$\mathcal{H}_\Delta^k = \mathcal{H}_d^k \tag{10.1}$$

just as in the $C^\infty$-compact case. On the other hand, (10.1) may fail to hold if $X$ is *incomplete*. For example, if

$$X \setminus X_0 = \Omega \times (0, 1) \quad \text{and} \quad g|\Omega \times (0,1) = dr^2 + r^2 h \tag{10.2}$$

where $h$ is a metric on $\Omega$, then the metric completion is a compact space with a conic singularity at $r = 0$ (which is also the « end » of $X$). In [6] and [7], Cheeger has proved a Strong Hodge Theorem for such singularities which may hold even when (10.1) does not (cf. [7], p. 317). In this section we investigate when (10.1) holds for (10.2).

We must introduce weighted Sobolev spaces of forms for manifolds as in (10.2). If $u$ is a $k$-form on $X$ write $u|\Omega \times (0,1) = (\varphi r^{k-1})\, dr \wedge v + (\psi r^k) w$ where $\varphi$, $\psi$ are functions and $v$ and $w$ are respectively $k-1$ and $k$ forms on $\Omega$. Then in $\Omega \times (0,1)$ we find $|u|^2 = \varphi^2 |v|_h^2 + \psi^2 |w|_h^2$ where $|\cdot|_h$ denotes the pointwise norm on $\Omega$ induced by $h$. In terms of a local orthonormal basis of 1-forms $\tau_1, \ldots, \tau_{n-1}$ in a coordinate chart $\Omega_\nu$ of $\Omega$, let $\mathring{X}_\nu = \Omega_\nu \times (0,1)$ and form a basis for $\Lambda^k(\mathring{X}_\nu)$ by taking wedge products from $\{dr, r\tau_1, \ldots, r\tau_{n-1}\}$; if $u$ is a $k$-form on $\mathring{X}_\nu$ then

$$|u|^2 = \sum_{j=1}^{[k]} (u_j)^2$$

where $u_1, \ldots, u_{[k]}$ denote the coefficients in this basis and $[k] = \binom{n}{k}$. This provides the trivialization with which to define $W^2_{s,\delta}(\Lambda^k)$ as in Section 1:

$$\|u\|^2_{W^2_{s,\delta}(\Lambda^k(\mathring{X}_\nu))} = \sum_{j=1}^{[k]} \sum_{a+|\beta| \leq s} \int_{\mathring{X}_\nu} |r^{\delta+a} D_r^a D_\omega^\beta u_j(r, \omega)|^2\, d\mu\ .$$

A calculation shows that for all $\delta \in \mathbb{R}$ and $s, k \in \mathbb{N}$

$$d\colon W^2_{s,\delta}(\Lambda^k) \to W^2_{s-1,\delta+1}(\Lambda^{k+1}) ,$$
$$*\colon W^2_{s,\delta}(\Lambda^k) \to W^2_{s,\delta}(\Lambda^{n-k}) ,$$

are bounded operators. Hence

$$d^*\colon W^2_{s,\delta}(\Lambda^k) \to W^2_{s-1,\delta+1}(\Lambda^{k-1})$$

and

$$\Delta\colon W^2_{s,\delta}(\Lambda^k) \to W^2_{s-2,\delta+2}(\Lambda^k) \tag{10.3}$$

are also bounded for all $\delta \in \mathbb{R}$ and $s, k \in \mathbb{N}$. In the above trivializations of $\Lambda^k(\mathring{X}_\nu)$ and $\Lambda^{k+1}(\mathring{X}_\nu)$, $rd$ is a system of differential operators involving derivatives on $\Omega_\nu$ and $r(\partial/\partial r)$. Thus

$$r^2\Delta|\mathring{X}_\nu = \sum_{q=0}^{2} A^{m-q}(\omega, D_\omega)(rD_r)^q$$

and we may apply the theory in Sections 1 and 2 to obtain the a priori inequality

$$\|u\|_{W^2_{2,\delta}} \leqslant C(\|\Delta u\|_{W^2_{0,\delta+2}} + \|u\|_{W^2_{0,\delta}}) \tag{10.4}$$

and that (10.3) is Fredholm if and only if $\delta \in \mathbb{R}\setminus\mathfrak{D}^k_\Delta$ where $\mathfrak{D}^k_\Delta$ is a discrete set.

Recall the Stokes formula

$$\int\langle du, v\rangle\, d\mu = \int\langle u, d^*v\rangle\, d\mu \tag{10.5}$$

which holds for $u \in C^\infty_0(\Lambda^k)$, $v \in C^\infty_0(\Lambda^{k+1})$, and hence by closing whenever $u \in W^2_{1,\delta}(\Lambda^k)$, $v \in W^2_{1,-\delta-1}(\Lambda^{k+1})$. With this observation we easily prove the following.

THEOREM 10.1 *If* (10.3) *is Fredholm for all* $-1 \leqslant \delta < 0$ *then* (10.1) *holds.*

PROOF. If $u \in \mathcal{H}^k_\Delta$ then $u \in W^2_{2,0}(\Lambda^k)$ by (10.4). Now in general $W^2_{2,-1}(\Lambda^k) \subset W^2_{2,0}(\Lambda^k)$ (since the weights are controlling growth at 0), however, since (10.3) is Fredholm for $-1 \leqslant \delta < 0$ we can conclude that $u \in W^2_{2,-1}(\Lambda^k)$. But then we can apply (10.5) as in the $C^\infty$-compact case

$$\int \langle \Delta u, u\rangle \, d\mu = \int |du|^2 \, d\mu + \int |d^*u|^2 \, d\mu$$

to conclude that $du = d^*u = 0$.

Following the calculation in [6] it is easy to see that $\mathfrak{D}_\Delta^k \cap [-1, 0) \neq \emptyset$ can only occur when a $L^2$-harmonic form on $\Omega \times (0, 1)$ exists of the form

$$u_1^- = r^{a_k^-(\mu)} w$$

or

$$u_4^- = r^{a_{k-2}^-(\mu)+1} \, dr \wedge v$$

where $w$ and $v$ are respectively coclosed and closed eigenforms for the Laplacian $\Delta_h$ on $\Omega$ with eigenvalue $\mu$, and $a_k^-(\mu) = \alpha_k - \nu_k(\mu)$, $\alpha_k = 1 + k - n/2$, $\nu_k(\mu) = (\alpha_k^2 + \mu)^{\frac{1}{2}}$.

Since

$$|u_1^-|^2 = 0(r^{2-n-2\nu_k(\mu)}) \, ,$$

$$|u_4^-| = 0(r^{2-n-2\nu_{k-2}(\mu)}) \, ,$$

we see immediately that $\mathfrak{D}_\Delta^k \cap [-1, 0) = \emptyset$ if $\nu_k(\mu), \nu_{k-2}(\mu) \geqslant 1$; in particular if $|k - n/2 \pm 1| \geqslant 1$. Otherwise, the condition $\mathfrak{D}_\Delta^k \cap [-1, 0) = \emptyset$ may hold if $\mu$ is sufficiently large. Let $0 \leqslant \mu_0^k \leqslant \mu_1^k < \mu_2^k < \ldots$ denote the eigenvalues for $\Delta_h$ on $\Lambda^k(\Omega)$, where $\mu_0^k$ is the smallest eigenvalue and $\mu_1^k$ is the smallest *positive* eigenvalue (i.e., $\mu_0^k = 0$ or $\mu_0^k = \mu_1^k$). It is easy to verify the following.

THEOREM 10.2. *Suppose $n$ is an even integer. If $k - n/2 \neq \pm 1$ then $\mathfrak{D}_\Delta^k \cap [-1, 0) = \emptyset$ and (10.1) hold. If $k - n/2 = -1$ and $\mu_0^k \geqslant 1$, or if $k - n/2 = 1$ and $\mu_0^{k-1} \geqslant 1$, then again $\mathfrak{D}_\Delta^k \cap [-1, 0] = \emptyset$ and (10.1) hold.*

For $n$ odd the same calculation shows that if $|k - n/2 \pm 1| < 1$ and $\mu_0^k \geqslant \frac{3}{4}$ then $\mathfrak{D}_\Delta^k \cap [-1, 0) = \emptyset$. In fact, when $k - n/2 = -\frac{1}{2}$ (resp. $+\frac{1}{2}$), $\delta \in \mathfrak{D}_\Delta^k \cap [-1, 0)$ corresponds to $u_1^-$ (resp. $u_4^-$) with $0 \leqslant \mu < \frac{3}{4}$: But for $\mu = 0$, $u_1^- = w$ is harmonic so $du_1^- = d^*u_1^- = 0$ (similarly for $u_4^-$). Thus we obtain

THEOREM 10.3. *Suppose $n$ is an odd integer. If $k - n/2 \neq \pm \frac{1}{2}, \pm \frac{3}{2}$ then $\mathfrak{D}_\Delta^k \cap [-1, 0) = \emptyset$ and (10.1) hold. If $k - n/2 = -\frac{3}{2}$ and $\mu_0^k \geqslant \frac{3}{4}$, or if $k - n/2 = \frac{3}{2}$ and $\mu_0^{k-1} \geqslant \frac{3}{4}$, then again $\mathfrak{D}_\Delta^k \cap [-1, 0) = \emptyset$ and (10.1) hold. If $k - n/2 = \pm \frac{1}{2}$ and $\mu_1^{(n-1)/2} \geqslant \frac{3}{4}$ then (10.1) holds (although $\mathfrak{D}_\Delta^k \cap [-1, 0) = \emptyset$ may fail).*

These three theorems generalize immediately to manifolds with a finite number of conic singularities by appealing to Section 8: the conditions on $\delta$ and $\mu$ in each theorem must then be imposed on each end of $X$.

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