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Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 12, nº 1 (1985), p. 43-80

<http://www.numdam.org/item?id=ASNSP_1985_4_12_1_43_0>

Algebraic Varieties of Dimension Three Whose Hyperplane Sections Are Enriques Surfaces.

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Introduction.

In recent years there has been a renewed interest in the study of Fano threefolds; see for instance [24], [14], [15] and [13]. From the modern point of view Fano threefolds are projective varieties of dimension three with ample anti-canonical class. Of special interest are those which can be embedded in projective space by the anti-canonical class itself; such varieties have then as hyperplane sections K_3-surfaces. From this point of view it is then also natural to study threefolds whose hyperplane sections are Enriques surfaces. In fact this has been done by Fano himself in a paper [11] published in 1938. Like his papers on Fano threefolds this paper of Fano's is very interesting and full of geometry; by means of ingenious arguments and constructions Fano obtains striking results and gives a classification of varieties of the mentioned type. However, as in the case of his papers on Fano threefolds, also this paper contains serious gaps and from a modern point of view the arguments of Fano give only an indication for a possible proof.

It is our purpose to take up again this paper of Fano's and to provide proofs for the theorems he stated. In this paper we treat the general theory for such threefolds with Enriques surfaces as hyperplane sections.

The first two sections are preparations on surfaces. In section three we begin to state explicitely the conditions under which the main theorem (7.2) is proved. These conditions (see also sections 4 and 5) seem to have

(*) This work has been supported by a joint contract between the University of Torino and the University of Leiden, financed by the Italian C.N.R. (Consiglio Nazionale delle Ricerche); the first named auther has also been supported by a grant of the Dutch Z.W.O. (Nederlandse Stichting voor Zuiver-Wetenschappelijk Onderzoek).

Pervenuto alla Redazione il 6 Giugno 1983.

been made also by Fano himself, partially explicitely but also partially implicitely. Roughly speaking they may be summarized by saying that we do restrict our study to the « general case ».

As has been shown by Godeaux [12] and Fano, such threefolds carry also a linear system of surfaces which have the property that their linear curve sections are canonically embedded curves. It should be noted, however, that these surfaces are *singular*, a fact which does *not* seem to have bothered Godeaux and Fano. The key points of our paper, treated in section 4, 5 and 6, are on the one hand the study of the nature of these surfaces and on the other hand the study of the rational map associated to this linear system. The surfaces turn out to be indeed, as predicted by Godeaux and Fano, K_3-surfaces.

The present paper differs considerably from our preliminary paper [5]. A major difference is that *there* we *assumed* that the above mentioned map is birational, whereas *here* we *prove* this as a fact, provided the genus of the linear curve section is larger than five.

Section 7 contains the main theorem, due to Fano [11], saying that the hypothesis that the hyperplane sections of a threefold W are Enriques surfaces implies that W itself has precisely eight singular points which are quadruple and which have as tangent cone the cone over the Veronese surface. Furthermore the above mentioned birational transformation associated to the linear system of K_3-surfaces transforms W into a (singular) Fano threefold, in the classical sense, in such a way that the « images » of the eight singular points are eight planes.

Finally in section 8 we give the list of the known examples of such threefolds with Enriques surfaces as hyperplane sections. The list is again due to Fano and, moreover, according to him this is the complete list. In this paper we do not enter into this classification problem, intending to return to it on a future occasion.

Part of the work has been done during a stay of both authors at the Mittag-Leffler Institute in Djursholm; we like to thank this Institute for its support and for the splendid hospitality.

Some conventions and notations.

We work over an algebraically closed field k of characteristic zero.

If V is a variety and D a divisor on V then we often write shortly $H^i(\mathcal{O}_V(D))$ instead of $H^i(V, \mathcal{O}_V(D))$; for its dimension we write $h^i(D)$. For the rational map associated with the linear system $|D|$ we write λ_D or sometimes $\lambda_{|D|}$. For linear equivalence we use the symbol $\sim$ and for numerical equivalence the symbol $\equiv$. Finally, we denote the canonical class of V by K_V.

1. – Auxiliary results on special surfaces.

In this section we prove some results for a special type of singular surfaces which will be needed in the following. In our applications these surfaces are either the surfaces obtained by taking a sufficiently general hyperplane section through one of the singular points P_i (see 3.3) of the threefold W or a sufficiently general member of the system $|\varphi|$ (see 3.8).

LEMMA 1.1. *Let S be a (projective) surface with* one *singular point P. Let $\pi\colon \tilde{S} \to S$ be a resolution of singularities:*

$$\begin{array}{ccc} C & \hookrightarrow & \tilde{S} \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ P & \in & S \end{array}$$

Assume:

i) $\pi^{-1}(P) = C$ *is an* irreducible, smooth *curve,*

ii) *for the canonical class $K_{\tilde{S}}$ we have $K_{\tilde{S}} \equiv \varrho C$ (numerical equivalence) with $\varrho \in \frac{1}{2}\mathbb{Z}$.*

Then, at most, the following cases are possible:

a) $\varrho = -2$; *then $\tilde{S}$ is ruled and $g(C) > 1$.*

b) $\varrho = -1$; *then $\tilde{S}$ either is ruled or rational and* $g(C) = 1$.

c) $\varrho = -\frac{1}{2}$; *then $\tilde{S}$ is rational,* $g(C) = 0$ *and* $C^2 = -4$.

d) $\varrho = 0$; *then* $g(C) = 0$ *and* $C^2 = -2$.

e) $\varrho = 1$; *then* $g(C) = 0$ *and* $C^2 = -1$ [1].

PROOF. Consider the Stein factorization:

$$\begin{array}{ccc} \tilde{S} & \xrightarrow{\ \pi'\ } & S' = \operatorname{Spec} \pi^*(\mathcal{O}_{\tilde{S}}) \\ {\scriptstyle \pi}\downarrow & \swarrow{\scriptstyle \mu} & \\ S & & \end{array}$$

[1] In case $\varrho = 1$ the resolution is not the minimal one because now we can blow down the curve C. Also note that in case the surface S is normal then it is automatically smooth in case $\varrho = 1$ (by ZMT).

where S' is the normalization of S. Since the fibres of π are connected we have that μ is one to one and the fibres of π and π' are th esame. Hence by Mumford [18] we have that $C^2 = -s$ with $s > 0$.

Next consider the arithmetic genus of C. We have

$$g(C) = p_a(C) = \tfrac{1}{2}(C^2 + C\cdot K_{\tilde{S}}) + 1 = -\tfrac{1}{2}(1+\varrho)s + 1 \geqq 0 \tag{1}$$

Hence $(1+\varrho)s \leqq 2$ and the following cases are left

$$\begin{cases} \text{i)}\ \varrho = 1 \Rightarrow s = 1\,,\ g(C) = 0\,,\ C^2 = -1\,, \\ \text{ii)}\ \varrho = 0 \Rightarrow s = 2\,,\ g(C) = 0\,,\ C^2 = -2\,, \\ \text{iii)}\ \varrho = -\tfrac{1}{2} \Rightarrow s = 4\,,\ g(C) = 0\,,\ C^2 = -4\,, \\ \text{iv)}\ \varrho = -1 \Rightarrow \text{no information on } s\,,\ g(C) = 1\,, \\ \text{v)}\ \varrho < -1 \Rightarrow g(C) > 1\,. \end{cases} \tag{2}$$

It suffices now to consider the case $\varrho < 0$. From $K_{\tilde{S}} \equiv \varrho C$ we have that S must be *ruled* or *rational.*

Consider first *the case that* $\tilde{S}$ *is ruled.* So let $\tau\colon \tilde{S} \to B$, with a curve B with $g(B) > 0$ and for a general $b \in B$ the fibre $\tau^{-1}(b) = l$ is a line. By the adjunction formula

$$K_l = (K_{\tilde{S}} + l)\cdot l \equiv \varrho C\cdot l\,. \tag{3}$$

Let $\#(C\cdot l)$ denote the intersection number. We have $\varrho\cdot\#(C\cdot l) = -2$, so C is not on a fibre of the ruling and we have the following possibilities

$$\begin{aligned} \#(C\cdot l) &= 1 \Rightarrow \varrho = -2\,, \\ \#(C\cdot l) &= 2 \Rightarrow \varrho = -1\,, \\ \#(C\cdot l) &= 4 \Rightarrow \varrho = -\tfrac{1}{2}\,. \end{aligned}$$

It suffices now to see that $\varrho = -\frac{1}{2}$ is not possible (in case $\tilde{S}$ is ruled). Applying the Hurwitz formula for $C \to B$, we get

$$g(C) = 4g(B) - 3 + \tfrac{1}{2}\deg(R)$$

where R denotes the ramification. Hence $g(C) \geqq 1$, but we have already seen that $g(C) = 0$ (see (2)), hence $\varrho \neq -\frac{1}{2}$.

Next consider *the case that* $\tilde{S}$ *is rational.*

Suppose first that $\tilde{S}$ dominates *a minimal model* that is *ruled* and consider again a general fibre of the ruling l. Applying the same formula (3) and using again $l^2 = 0$ we have again only the possibilities $\varrho = -2$, $\varrho = -1$ and $\varrho = -\frac{1}{2}$. Now we must see that $\varrho = -2$ is not possible. We have $\tau\colon \tilde{S} \to B$ with $B = \mathbb{P}^1$; $\varrho = -2$ gives $\#(C\cdot l) = 1$ and hence C and B are birational, hence $g(C) = g(B) = 0$, but we have already seen in (2) that $g(C) > 1$, a contradiction.

Finally suppose that $\tilde{S}$ dominates only the minimal model $\mathbb{P}^2$. Now let l be a general line in $\mathbb{P}^2$ and denote its proper transform in $\tilde{S}$ by the same letter. We get

$$K_l = (K_{\tilde{S}} + l)\cdot l = \varrho C\cdot l + l^2 \,. \tag{3'}$$

Since now $\#(l^2) = 1$ we have $\varrho\cdot\#(C\cdot l) = -3$ and, always for $\varrho < 0$, we have the following possibilities

$$\#(C\cdot l) = 1 \Rightarrow \varrho = -3\,,$$

$$\#(C\cdot l) = 3 \Rightarrow \varrho = -1\,,$$

$$\#(C\cdot l) = 6 \Rightarrow \varrho = -\tfrac{1}{2}\,.$$

We have only to show that $\varrho = -3$ is *not* possible. For $\varrho = -3$, we have $\#(C\cdot l) = 1$, hence

$$C = l + a_1E_1 + a_2E_2 + \ldots + a_nE_n$$

with $a_i \in \mathbb{Z}$ and $E_1, E_2, \ldots, E_n$ curves obtained by blowing up in that order. Then

$$K_{\tilde{S}} = -3C = -3l - 3a_1E_1 - \ldots - 3a_nE_n\,.$$

However from the known expression for $K_{\tilde{S}}$ on such a $\tilde{S}$ we know that the coefficient of E_n in $K_{\tilde{S}}$ must be $+1$. Hence $a_n = 0$, next $a_{n-1} = 0$, etc. Hence $C = l$, but this is impossible because now $C^2 = 1$ and we must have $C^2 < 0$.

1.2. Next we consider the case of a finite number of singular points.

LEMMA 1.2. *Let S be a (projective) surface with isolated singularities $P_1, P_2, \ldots, P_n$ $(n > 1)$. Let $\pi\colon \tilde{S} \to S$ be a resolution. Assume:*

i) $H^1(S, \mathcal{O}_S) = 0$, $\dim H^2(S, \mathcal{O}_S) = 1$,

ii) $\pi^{-1}(P_i) = C_i$ *is an irreducible, smooth curve* $(i = 1, \ldots, n)$,

iii) $K_{\tilde{S}} \equiv \varrho\left(\sum C_i\right)$, *with* $\varrho \in \mathbb{Z}$,

iv) *the points* P_i *are « similar » (see remark below).*

Then we have:

1) S *is a normal surface,*

2) *at most the following cases are possible:*

a) $\varrho = -1$; *then* $n = 2$ *and* $\tilde{S}$ *is a ruled surface over a curve* B *and* $g(C_1) = g(C_2) = g(B) = 1$,

b) $\varrho = 0$; *then* $g(C_i) = 0$ *and* $C_i^2 = -2$ $(i = 1, \ldots, n)$,

c) $\varrho = 0$; *then* $g(C_i) = 0$ *and* $C_i^2 = -1$ $(i = 1, \ldots, n)$.

REMARK. « Similar » means that all points P_i behave in the same way, *i.e.* in particular $g(C_i) = g(C_j)$, $C_i^2 = C_j^2$, etc.

PROOF. As in the proof of 1.1 we consider the Stein factorization S' of the morphism π and we have the same diagram. As in 1.1 we see that μ is one to one, i.e. S and S' are the same (Zariski) topological space. Consider on $S = S'$ the exact sequence

$$0 \to \mathcal{O}_S \to \mathcal{O}_{S'} \to Q \to 0$$

where Q is the quotient sheaf. From the corresponding exact cohomology sequence and using the fact that $H^1(S, \mathcal{O}_S) = 0$ we get $H^0(S, Q) = 0$ and since Q is a skyscraper sheaf we get $Q = 0$, hence $S = S'$ as algebraic variety. Hence S is normal since S' is normal.

From the result of Mumford we have again that $C_i^2 = -s_i$ with $s_i > 0$; by similarity we have $s_i = s$ $(i = 1, \ldots, n)$. Next looking again to the arithmetic genus of C_i we get from the formula (1) the following possibilities:

$$(2') \quad \begin{cases} \text{i)}\ \varrho = 1; & \text{then } s = 1\,,\ g(C_i) = 0\,,\ C_i^2 = -1\,, \\ \text{ii)}\ \varrho = 0; & \text{then } s = 2\,,\ g(C_i) = 0\,,\ C_i^2 = -2\,, \\ \text{iii)}\ \varrho = -1; & \text{then } g(C_i) = 1\,, \\ \text{iv)}\ \varrho < -1; & \text{then } g(C_i) > 1\,. \end{cases}$$

For $\varrho = 1$, by blowing down the curves C_i we get a non-singular surface which, since S is normal, is isomorphic with S by ZMT, hence S was already non-singular.

It suffices now to consider the *case* $\varrho < 0$. From $K_{\tilde{S}} \equiv \varrho\left(\sum_i C_i\right)$ we see that $\tilde{S}$ is *ruled* or *rational.*

Case $\tilde{S}$ *ruled*: $\tau\colon \tilde{S} \to B$. It is impossible that a curve C_i is lying on a fibre of the ruling because then $g(C_i) = 0$ contradicting the fact that $g(C_i) > 0$ in the case $\varrho < 0$. Hence $\#(C_i \cdot l) > 0$. For a general fibre of the ruling l we have

$$K_l = (K_{\tilde{S}} + l)\cdot l \equiv \varrho \sum_i C_i \cdot l \,.$$

By looking to degrees left and right we see $\varrho \geqq -2$; next also $\varrho = -2$ is impossible because there are at least two points P_1, P_2 (for: $n > 1$). Hence, if $\varrho < 0$, then $\varrho = -1$ and there are two points P_1 and P_2 and $\#(C_i \cdot l) = 1$. Hence C_1 and C_2 are birational with B and $g(C_1) = g(C_2) = g(B) = 1$.

Case $\tilde{S}$ *rational.* We use the spectral sequence (cf. [1]) for the morphism π and obtain an exact sequence

$$0 \to H^1(S, \pi_* \mathcal{O}_{\tilde{S}}) \to H^1(\tilde{S}, \mathcal{O}_{\tilde{S}}) \to H^0(S, R^1\pi_* \mathcal{O}_{\tilde{S}}) \to H^2(S, \pi^* \mathcal{O}_{\tilde{S}}) \to H^2(\tilde{S}, \mathcal{O}_{\tilde{S}}) \to 0 \,.$$

Hence we get, since S is normal and hence $\pi_* \mathcal{O}_{\tilde{S}} = \mathcal{O}_S$,

$$\chi(S) - \chi(\tilde{S}) = \dim H_0(S, R^1\pi_* \mathcal{O}_{\tilde{S}}) \,. \tag{4}$$

The sheaf $R^1\pi_* \mathcal{O}_{\tilde{S}}$ is concentrated in the points P_i $(i = 1, \ldots, n)$ and hence the stalks are equal to their formal completion, so by the theorem of Grothendieck on formal functions we get ([13, Th. 11.1, p. 277)

$$(R^1\pi_* \mathcal{O}_{\tilde{S}})_{P_i} = \varprojlim_n H^1(C_i, \mathcal{O}_{C_{i,n}}) \,,$$

where $C_{i,n} = \tilde{S} \times_S \operatorname{Spec}(\mathcal{O}_{S,P_i}/\mathcal{M}^n_{S,P_i})$. On the other hand we have an exact sequence

$$0 \to \mathcal{K}_n \to \mathcal{O}_{C_{i,n}} \to \mathcal{O}_{C_{i,n-1}} \to 0$$

(for some sheaf $\mathcal{K}_n$ on $C_{i,n}$) and this gives via induction that

$$\dim H^1(C_i, \mathcal{O}_{C_{i,n}}) \geqq \dim H^1(C_i, \mathcal{O}_{C_i}) = p_a(C_i) = g(C_i) \,.$$

Hence (4) gives

$$\chi(S) - \chi(\tilde{S}) \geqq \sum_i g(C_i) .$$

In our case $\chi(S) = 2$, $\chi(\tilde{S}) = 1$. Therefore $1 \geqq \sum g(C_i)$, hence (by similarity) $g(C_i) = 0$ $(i = 1, ..., n)$ since $n > 1$. However, this is impossible since we have seen above in (2′) that for $\varrho < 0$ the $g(C_i) > 0$.

REMARK. In the meantime surfaces of the nature described in this section have been studied by Epema, a student of one of ours (see his forthcoming paper [9]). Rational surfaces of this type have alredy been studied in 1933 by Du Val ([8]).

2. – Auxiliary facts about Enriques surfaces.

2.1. For the convenience of the reader we collect some known facts about (smooth) Enriques surfaces which will be needed later on.

First recall that on an Enriques surface S we have $K_S \neq 0$, but $2K_S = 0$ in Pic (S).

LEMMA 2.1. *Let S be an Enriques surface and $\Gamma \subset S$ a smooth, irreducible curve of genus $p > 1$. Let $K_S \cdot \Gamma \neq 0$ in* Pic (Γ). *Then*:

1) $\Gamma^2 = 2p - 2$,
2) $h^0(\Gamma) = p$, $h^1(\Gamma) = 0$, $h^2(\Gamma) = 0$,
3) *there exists a $\Gamma' \in |K_S + \Gamma|$; such a Γ' is connected, $\Gamma'^2 = 2p - 2$, $h^0(\Gamma') = p$ and $p_a(\Gamma') = p$.*

PROOF. The assertion about Γ^2 follows immediately from the adjunction formula. Next look at the exact sequence of sheaves:

$$0 \to \mathcal{O}_S(-\Gamma) \to \mathcal{O}_S \to \mathcal{O}_\Gamma \to 0$$

and at the corresponding exact cohomology sequence

$$\begin{array}{c} 0 \to H^1(\mathcal{O}_S(-\Gamma)) \to H^1(\mathcal{O}_S) \to H^1(\Gamma, \mathcal{O}_\Gamma) \to H^2(\mathcal{O}_S(-\Gamma)) \to H^2(\mathcal{O}_S) \to 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad H^0(\mathcal{O}_S(\Gamma'))^\vee \end{array}$$

We see dim $|\Gamma + K_S| = p - 1$, hence there is $\Gamma' \in |K_S + \Gamma|$; moreover we have $h^0(\Gamma') = p$ and $\Gamma'^2 = 2p - 2$. Next look at the exact sequence

$$0 \to \mathcal{O}_S \to \mathcal{O}_S(\Gamma) \to \mathcal{O}_\Gamma(\Gamma^2) \to 0$$

and the corresponding exact sequence

$$\begin{aligned} 0 \to H^0(\mathcal{O}_S) \to H^0(\mathcal{O}_S(\Gamma)) \to H^0(\Gamma, \mathcal{O}_\Gamma(\Gamma^2)) \to \\ \to H^1(\mathcal{O}_S) \to H^1(\mathcal{O}_S(\Gamma)) \to H^1(\Gamma, \mathcal{O}_\Gamma(\Gamma^2)) \to 0 \\ \| \\ H^0(\Gamma, \mathcal{O}_\Gamma(K_\Gamma - \Gamma^2))^\vee . \end{aligned}$$

Since $K_S \cdot \Gamma \neq 0$ we have $\Gamma^2 \neq K_\Gamma$ and we obtain from this sequence immediately $h^0(\Gamma) = p$, $h^1(\Gamma) = 0$. Moreover, $h^2(\Gamma) = 0$ is trivial. Now look at

$$0 \to \mathcal{O}_S(-\Gamma') \to \mathcal{O}_S \to \mathcal{O}_{\Gamma'} \to 0 .$$

From $H^1(\mathcal{O}_S(-\Gamma')) = H^1(\mathcal{O}_S(\Gamma))$ we get, using $h^1(\Gamma) = 0$, that

$$\dim H^0(\Gamma', \mathcal{O}_{\Gamma'}) = 1 ,$$

hence Γ' is connected and

$$p_a(\Gamma') = \dim H^1(\Gamma', \mathcal{O}_{\Gamma'}) = \dim H^2(\mathcal{O}_S(-\Gamma')) = \dim H^0(\mathcal{O}_S(\Gamma)) = p .$$

REMARK 2.2. Let S be an Enriques surface and put $\Gamma = S \cdot H$ for a sufficiently general hyperplane section. Then Γ satisfies the condition of 2.1.

PROOF. Since $K_S \neq 0$ we have $\Gamma \cdot K_S \neq 0$ by the Weil equivalence criterium which can be applied in the case of an Enriques surface ([27, p. 120). Also since $p = h^0(\Gamma) = h^0(\mathcal{O}(1))$ by the same argument as before, we have $p > 1$.

COROLLARY 2.3. *Let S be an Enriques surface embedded in projective space such that the hyperplanes cut out a complete system. Let $\Gamma = S \cdot H$ (H hyperplane). Then:*

1) *$|\Gamma + K_S|$ has no fixed components and no base points,*

2) *there exists a $\Gamma' \in |\Gamma + K_S|$ which is smooth and irreducible,*

4) *such a Γ' is a canonically embedded curve,*

4) *$\lambda_{\Gamma'}$ is a finite morphism.*

PROOF. Take H sufficiently general such that Γ is smooth and irreducible. Look at the same exact sequence of sheaves as in the proof of 2.1, tensor with $\mathcal{O}_S(\Gamma')$. We get an exact sequence of cohomology groups

$$H^0(S, \mathcal{O}_S(\Gamma')) \to H^0(\Gamma, \mathcal{O}_\Gamma(\Gamma')) \to H^1(S, \mathcal{O}_S(-\Gamma + \Gamma'))$$

and

$$H^1(S, \mathcal{O}_S(-\Gamma + \Gamma')) = H^1(S, \mathcal{O}_S(K_S)) = H^1(S, \mathcal{O}_S)^\vee = 0 \,.$$

Hence $\mathrm{Tr}_\Gamma |\Gamma'|$ is complete, but since $|\Gamma'|$ is the adjoint system of $|\Gamma|$, this is the canonical system on Γ. Hence it has no base points, but then $|\Gamma'|$ itself has no fixed components.

Next suppose $P_0 \in S$ is a base point of $|\Gamma'|$. Take a hyperplane H_0 through P_0 sufficiently general such that $\Gamma_0 = H_0 \cdot S$ is a smooth *irreducible* curve. This is possible because by Bertini and the transversality of H_0 with the tangent plane to S at P_0 it follows that Γ_0 is smooth and, since Γ_0 is an ample divisor, that it is connected. We repeat the above argument with Γ_0 instead of Γ and we obtain that $\mathrm{Tr}_{\Gamma_0} |\Gamma'|$ is the canonical system; since this is free of base points the system $|\Gamma'|$ is itself free of base points.

Next, in order to prove 2, it suffices, by Bertini, to see that $|\Gamma'|$ is not composed with a pencil. However, then the image of S by $|\Gamma'|$ is a curve and then by taking hyperplane sections we have that the Γ' are disconnected, contrary to 2.1.

As to 3, writing again p for the genus of Γ it follows from $h^0(\Gamma) = p$ that S spans $\mathbb{P}^{p-1}$. Since Γ' cannot be lying in a hyperplane the Γ' spans also $\mathbb{P}^{p-1}$, moreover by 2.1 we have also $g(\Gamma') = p$ and the system $|\mathcal{O}(1)| = |\Gamma|$ cuts out canonical divisors; hence Γ' is canonically embedded in $\mathbb{P}^{p-1}$.

Finally, 4 follows from the fact that $|\Gamma'|$ is base-point free (by 1) and ample by Nakai's criterion since $\Gamma' \equiv \Gamma$ (numerical equivalence).

REMARKS 2.4.

i) For further reference we recall that $S \subset \mathbb{P}^{p-1}$ if $g(\Gamma) = p$ for $\Gamma = S \cdot H$.

ii) Since we have seen above that $|\Gamma'|$ is not composed with a pencil we have that the corresponding rational map

$$\lambda_{\Gamma'}\colon S \to \mathbb{P}^{p-1}$$

has a 2-dimensional image.

iii) $\nexists \Gamma'_0 \in |\Gamma'|$ such that we have a splitting $\Gamma'_0 = \Gamma'_{01} + \Gamma'_{02}$ in effective curves with $\lambda_{\Gamma'}$ *contracting* Γ'_{02} into a point.

(PROOF. Follows immediately from the fact that $\lambda_{\Gamma'}$ is finite.)

3. – Basic assumptions and preliminary results.

3.1. BASIC ASSUMPTIONS. In the following $W \subset \mathbb{P}^N$ denotes a three dimensional algebraic variety (shortly: threefold) satisfying the following assumptions:

i) W is *projectively normal*,

ii) if H is a (sufficiently general) hyperplane then $W \cdot H = F$ is a *smooth Enriques* surface,

iii) the genus of a «general curve section» is denoted by p, i.e. $g(W \cdot H \cdot H') = p$,

iv) W is not a cone.

LEMMA 3.2. *W has isolated singularities.*

PROOF. First of all W has at most isolated singularities, for $W \cdot H = F$ is smooth. Next: W is singular. For, suppose W was smooth. By the adjunction formula $(K_W + F) \cdot F = K_F$. Write $T = K_W + F$ then, since $2K_F = 0$ we have $2T \cdot F = 0$. Hence by Weil's equivalence criterium ([26], Theorem 2) we have $2T = 0$. Therefore $T \equiv 0$ (numerical equivalence), and $-K_W \equiv F$. Hence $-K_W$ is ample (Nakai-criterium). However, then W is a Fano threefold. For a Fano threefold Pic (W) has no torsion ([14, I. 1.1.15); this yields $T = 0$, then $K_F = 0$, which is a contradiction.

3.3. NOTATIONS. In the following $P_1, P_2, \ldots, P_n$ denote the singular points of W.

Furthermore $\Gamma = W \cdot H \cdot H' = F \cdot H'$ denotes a (sufficiently) general curve section, so by the above assumption $g(\Gamma) = p$.

LEMMA 3.4. *$H^i(W, \mathcal{O}_W(n)) = 0$ for $i = 1, 2, 3$ and $n \geqq 0$.*

PROOF. Consider the exact sequence

$$0 \to \mathcal{O}_W(n) \to \mathcal{O}_W(n+1) \to \mathcal{O}_F(n+1) \to 0$$

and the corresponding exact sequence for cohomology

$$\begin{aligned} H^0(W, \mathcal{O}_W(n+1)) &\to H^0(F, \mathcal{O}_F(n+1)) \to H^1(W, \mathcal{O}_W(n)) \to H^1(W, \mathcal{O}_W(n+1)) \to \\ &\to H^1(F, \mathcal{O}_F(n+1)) \to H^2(W, \mathcal{O}_W(n)) \to H^2(W, \mathcal{O}_W(n+1)) \to \\ &\to H^2(F, \mathcal{O}_F(n+1)) \to H^3(W, \mathcal{O}_W(n)) \to H^3(W, \mathcal{O}_W(n+1)) \to 0\,. \end{aligned}$$

Now apply decreasing induction starting with $n \gg 0$, and use the fact that $H^i(F, \mathcal{O}_F(n+1)) \cong H^{2-i}(F, \mathcal{O}_F(-n-1+K_F)) = 0$ for $i = 1, 2$ and $n \geqq 0$ by Kodaira vanishing theorem; we get the result for $i = 2$ and 3. On the other hand we have $H^1(W, \mathcal{O}_W(-1)) = 0$ since W is normal, by a result of Mumford [19]. Therefore applying usual induction, starting with $n = -1$ and using $H^1(F, \mathcal{O}_F) = 0$, we get the result also for $i = 1$. Moreover, this argument gives also:

COROLLARY 3.5. *$H^0(W, \mathcal{O}_W(n)) \to H^0(F, \mathcal{O}_F(n))$ is surjective for $n > 0$.*

COROLLARY 3.6. *If $p = g(\Gamma)$ with $\Gamma = W \cdot H \cdot H' = F \cdot H'$ then $N = p$ and the degree of W is $(2p-2)$, i.e. we have $W_3^{2p-p} \subset \mathbb{P}^p$.*

PROOF. Note that the hyperplanes cut out a complete linear system on W by assumption and now also on F by Corollary 3.5. Apply now Lemma 2.1, then we see that $W \subset \mathbb{P}^N$ with $N = p$.

LEMMA 3.7 (Godeaux [12]). *On W there exists a linear system $|\varphi|$ of Weil-divisors φ such that*

i) $\dim |\varphi| = p - 1$,

ii) *for a (sufficiently) general φ the hyperplane section $\varphi \cdot H = \Gamma'$ is a canonically embedded curve* (2),

iii) *for a (sufficiently) general F we have*

$$\mathrm{Tr}_F |\varphi| = |\Gamma'|$$

where $|\Gamma'|$ is the system of Lemma 2.1.

iv) *$|\varphi|$ has no base points except possibly in the singular points $P_1, \ldots, P_n$,*

v) $H^1(\varphi, \mathcal{O}_\varphi(n)) = 0, \; n \geqq 0;$

$H^2(\varphi, \mathcal{O}_\varphi(n)) = 0, \; n > 0;$

$\dim H^2(\varphi, \mathcal{O}_\varphi) = 1$.

PROOF. We follow the reasoning given by Godeaux and Fano. Consider a general *pencil* $F_\lambda = W \cdot H_\lambda$ of Enriques surfaces on W and let $\Gamma_0 = F_{\lambda_1} \cdot F_{\lambda_2}$ be the axis of this pencil, then $p = g(\Gamma_0)$. Choose on Γ_0 $(p-1)$ points

(2) Both Godeaux and Fano seem to draw from this fact immediately the conclusion that φ is a K_3-surface (see for instance [11], p. 42, line 7). However, this is not justified since φ may have singular points (namely the fixed points $P_1, \ldots, P_n$ of $|\varphi|$) and then there are also rational surfaces [8] and ruled surfaces [9] with canonical curve sections.

$A_1, \ldots, A_{p-1}$, independent and in general position (or, if you prefer, independent generic points in the sense of Weil). Note that this $(p-1)$-tuple determines a unique $(2p-2)$-tuple $A_1, A_2, \ldots, A_{p-1}, A_p, \ldots, A_{2p-2}$ which makes up a canonical divisor on Γ_0. Now take a generic member F_λ of the pencil, then—since the points $A_1, \ldots, A_{p-1}$ are independent—there exists a *unique* curve Γ'_λ from the system $|K_{F_\lambda} + \Gamma_0|$ going through $A_1, \ldots, A_{p-1}$ (and « a fortiori » also through $A_p, \ldots, A_{2p-2}$). Varying the surface F_λ within the pencil the locus of the curve Γ'_λ is a surface φ on W.

By construction $\varphi \cdot F_\lambda = \Gamma'_\lambda$. Moreover, by choosing the points $A_1, \ldots, A_{p-1}$ generically independent on Γ_0 we get a generic member Γ'_λ of the system. By Corollary 2.3 this proves ii). Another choice $A^*_1, \ldots, A^*_{p-1}$ gives a surface φ^* such that $\varphi^* \cdot F_\lambda \sim \varphi \cdot F_\lambda$ (linear equivalence) hence, by [26], Theorem 2, we have $\varphi^* \sim \varphi$.

Now consider the linear system $|\varphi|$. Since we can get, in the above way, a generic member $\Gamma'_\lambda \in |K_{F_\lambda} + \Gamma_0|$ we get by specializing that

$$\mathrm{Tr}_{F_0} |\varphi| = |\Gamma'_\lambda|$$

which proves iii). Moreover, this shows that $\dim |\varphi| \geqq p - 1$. However, looking at the exact sequence

$$0 \to \mathcal{O}_W(\varphi - F_\lambda) \to \mathcal{O}_W(\varphi) \to \mathcal{O}_{F_\lambda}(\varphi) \to 0$$

and at the corresponding cohomology sequence we see that $\dim |\varphi| \geqq p$ should imply that there exists a divisor $\varphi > F_\lambda$. Now $\deg \varphi = \deg (\Gamma \cdot \Gamma')$ $= 2p - 2 = \deg (F_\lambda)$, hence this should imply that $|\varphi| = |F|$ which is a contradiction since in that case on F_λ we would have $|\Gamma'| = |\Gamma|$. This proves i).

For iv), let $P_0 \neq P_i$ $(i = 1, \ldots, n)$ be a base point. Take a hyperplane section H_0 generic through P_0; this gives a (non-singular) Enriques surface F_0. By specialization we have on F_0

$$\mathrm{Tr}_{F_0} |\varphi| = |\Gamma'| ,$$

and hence a base point of $|\varphi|$ different from the singular points gives a base point of $|\Gamma'|$, which contradicts 2.3.

Finally v) is a consequence of the following lemma.

LEMMA 3.8. *Let S be a surface with at most isolated singularities, spanning $\mathbf{P}^p$. Assume that $S \cdot H = \Gamma'$ is a canonical curve. Then*

i) $H^1(S, \mathcal{O}_S(n)) = 0,\ n \geqq 0;$

ii) $H^2(S, \mathcal{O}_S(n)) = 0,\ n > 0$ *and* $\dim H^2(S, \mathcal{O}_S) = 1.$

PROOF. Consider the exact sequence

$$0 \to \mathcal{O}_S(n) \to \mathcal{O}_S(n+1) \to \mathcal{O}_{\Gamma'}(n+1) \to 0 .$$

Looking at the corresponding cohomology sequence and using the fact that Γ' is a canonical curve, so that, by Noether's theorem, it is projectively normal, the assertions follow easily by applying decreasing induction on n.

3.9. NOTATIONS. Let $\tilde{W}$ be the variety obtained by blowing up W in the singular points $P_1, \ldots, P_n$; let $\pi\colon \tilde{W} \to W$ be the corresponding morphism and $E_i = \pi^{-1}(P_i)$ $(i = 1, \ldots, n)$.

If $A \subset W$ is a subvariety then we denote by $\tilde{A}$ the *proper* transform of A on $\tilde{W}$; we denote the restriction of π to $\tilde{A}$—in order to simplify notations—by the same letter π.

We continue to denote by $|F|$ the linear system of hyperplane sections of W and we denote by $|F_i| \subset |F|$ the linear subspace consisting of the hyperplane sections *going through* P_i: Note that according to the above introduced convention of notations we have now $|\tilde{F}_i| \not\subset |\tilde{F}|$ since

$$\tilde{F} \sim \tilde{F}_i + (\text{multiple of } E_i) .$$

3.10. FURTHER ASSUMPTIONS.

1) $\tilde{W}$ *is smooth.*

2) E_i *is smooth.*

(REMARK. Actually 2 implies 1.)

LEMMA 3.11.

i) *A general member* $\tilde{F}_i$ *of* $|\tilde{F}_i|$ *is smooth,*

ii) $|\tilde{F}| = |\tilde{F}_i + E_i|$,

iii) *for general* F_i *we have* $\tilde{F}_i \cdot E_i = \Delta_i$ *is smooth and irreducible and* $|\Delta_i|$ *is very ample on* E_i.

PROOF. We have $W \subset \mathbb{P}^N$. Let $\tilde{\mathbb{P}}$ be the blow up of $\mathbb{P}^N$ in P_i, write again $\pi\colon \tilde{\mathbb{P}} \to \mathbb{P}^N$, and put $B_i = \pi^{-1}(P_i)$. Now $B_i \simeq \mathbb{P}^{N-1}$ and if $|H_i|$ is the system of hyperplanes through P_i then $\mathrm{Tr}_{B_i} |\tilde{H}_i|$ is the complete system of hyperplanes on B_i and therefore

$$\mathrm{Tr}_{E_i} |\tilde{F}_i| = \mathrm{Tr}_{E_i} (\mathrm{Tr}_{B_i} |\tilde{H}_i|)$$

has no fixed points, hence $|\tilde{F}_i|$ has no fixed points, which—since $\tilde{W}$ is smooth—proves i) by Bertini.

For ii) it suffices to see that a general member of $|\tilde{H}_i|$ is not everywhere tangent to E_i, and hence it suffices to see that a general member of $\mathrm{Tr}_{B_i}|\tilde{H}_i|$ does not have this property; but this is clear for the hyperplane system on B_i.

As to iii), again from the fact that $\mathrm{Tr}_{B_i}|\tilde{H}_i|$ is the complete hyperplane system it follows that for general F_i we have $\tilde{F}_i \cdot E_i = \Delta_i$ is smooth, irreducible and $|\Delta_i|$ is very ample.

LEMMA 3.12.

i) *In* Pic (W) *we have* $2F = 2\varphi$ (*the system from* 3.7) (*here* Pic (W) *denotes the Picard group in the sense of Weil*).

In Pic $(\tilde{W})$ *we have*:

ii) $2\tilde{F} = 2\tilde{\varphi} + \sum_i t_i E_i \quad (t_i \in \mathbb{Z})$.

iii) $K_{\tilde{W}} = -\tilde{\varphi} + \sum_i r_i E_i \quad (r_i \in \mathbb{Z})$.

PROOF. On W we consider a generic F^* (which is by our basic assumption an Enriques surface) and we have (with the notations of 1.2) that $F \cdot F^* = \Gamma$ and $\varphi \cdot F^* = \Gamma'$, hence $(F - \varphi)\cdot F^* = K_{F^*}$ and hence by the equivalence criterium ([26], Theorem 2) $2F = 2\varphi$ which proves i). From this ii) follows since the kernel of π_*: Pic $(\tilde{W}) \to$ Pic (W) is generated by the divisors E_i $(i = 1, \dots, n)$.

For iii) we start again with $(F - \varphi)\cdot F^* = K_{F^*}$; since $\tilde{F}^* \simeq F^*$ we have also $(\tilde{F} - \tilde{\varphi})\cdot \tilde{F}^* = K_{\tilde{F}^*}$, hence by the adjunction formula $(K_{\tilde{W}} + \tilde{\varphi})\cdot \tilde{F}^* = 0$. Projecting down we get by the equivalence criterium ([26], Theorem 2) $\pi_*(K_{\tilde{W}}) = -\varphi$. Again by lifting this equality in Pic (W) to $\tilde{W}$ we get iii).

REMARK 3.13. $\pi_*(K_{\tilde{W}}) = K_W$ is the canonical class (as Weil divisor) on W; cf. also with [28], p. 31-32.

REMARK 3.14. Putting $\tau_i = \frac{1}{2} t_i$ $(i = 1, \dots, n)$ we can rewrite 3.12 ii) in Pic $(\tilde{W}) \otimes_{\mathbb{Z}} \mathbb{Q}$ as:

$$\tilde{F} \equiv \tilde{\varphi} + \sum_i \tau_i E_i \qquad (\equiv: \text{ numerical equivalence}).$$

COROLLARY 3.15. *Put* $C_i = \tilde{\varphi} \cdot E_i$ $(i = 1, \dots, n)$. *Then on* E_i:

i) $C_i \equiv \tau_i \Delta_i$ (*see* 3.11),

ii) $K_{E_i} \equiv -(\tau_i + r_i + 1)\cdot \Delta_i$.

PROOF.

i) $C_i = \tilde{\varphi}\cdot E_i \equiv (\tilde{F} - \sum \tau_j E_j)\cdot E_i = -\tau_i E_i E_i$

$$= -\tau_i(\tilde{F} - \tilde{F}_i) E_i = +\tau_i \Delta_i .$$

ii) From the adjunction formula

$$K_{E_i} = (K_{\tilde{W}} + E_i)\cdot E_i = \{-\tilde{\varphi} + \sum r_j E_j + E_i\}\cdot E_i \equiv -(\tau_i + r_i + 1)\Delta_i .$$

COROLLARY 3.16. *On $\tilde{F}_i$ we have $K_{\tilde{F}_i} \equiv \varrho_i \Delta_i$ with*

$$\varrho_i = \tau_i + r_i - 1 \qquad (i = 1, \ldots, n) .$$

PROOF. By the adjunction formula and 3.12 we have

$$K_{\tilde{F}_i} = (K_{\tilde{W}} + \tilde{F}_i)\cdot \tilde{F}_i = \{-\tilde{\varphi} + \sum r_j\cdot E_j + \tilde{F} - E_i\}\cdot \tilde{F}_i \equiv$$

$$\equiv (\tau_i + r_i - 1) E_i\cdot \tilde{F}_i = (\tau_i + r_i - 1)\Delta_i .$$

LEMMA 3.17. *The Picard variety* $\mathrm{Pic}^0(\tilde{W}) = 0$ *(and hence $H^1(\tilde{W}, \mathcal{O}_{\tilde{W}}) = 0$).*

PROOF. Apply the result of Matsusaka [17], (Theorem on page 167), to $\tilde{W}$ and to the system $|\tilde{F}|$ on $\tilde{W}$; note that the characteristic system of $|\tilde{F}|$ is the system $|\Gamma|$ on $\tilde{F} \gtrsim F$ and hence the characteristic system contains an irreducible curve. By this theorem we have that $\mathrm{Pic}^0(\tilde{W})$ is isogeneous to $\mathrm{Pic}^0(F)$, hence 0.

4. – Study of the system $|\tilde{\varphi}|$.

4.1. ASSUMPTION 3. The singular points P_i $(i = 1, \ldots, n)$ are « similar ». This assumption means that all points P_i « behave » in the same way and this implies in particular that for $i \neq j$ we have $\tau_i = \tau_j$, $r_i = r_j$, $p_a(C_i) = p_a(C_j)$, etc. Therefore we write from now on $\tau_i = \tau$, $r_i = r$, etc.

LEMMA 4.2.

i) *The P_i are base points of $|\varphi|$.*

ii) $\tau = \tau_i > 0$.

PROOF. ii) follows from i) since $C_i = \tau_i \Delta_i$ by 3.15. For i) we are going to reason by contradiction. First note that by Assumption 3 either each P_i is a base point or none of the P_i are base points. Also note that if the

P_i are not base points then $|\varphi|$ has no base points at all (see 3.7 iv)) and hence a general φ is smooth.

CLAIM *a*). If the P_i are not base points then a general φ is a K_3-surface.

PROOF. $\varphi\cdot F = \varphi\cdot H = \Gamma'$ is a canonical curve (3.7 and 2.3), hence

$$\Gamma'\cdot H = K_{\Gamma'} = (K_\varphi + \Gamma')\cdot\Gamma' = (K_\varphi+\Gamma')\cdot H\,, \qquad \text{i.e. } K_\varphi\cdot H = 0\,.$$

Since $p_g(\varphi) = 1$ (3.7 *v*)), we have $|K_\varphi| \neq \emptyset$ and hence $K_\varphi = 0$. Since moreover also $q(\varphi) = 0$ (3.7 v)), the φ is a K_3-surface.

CLAIM *b*). If the P_i are not base points, then for a general φ the $\tilde\varphi$ is a K_3-surface.

PROOF. Immediate because now $\varphi = \tilde\varphi$.

CLAIM *c*). If the P_i are not base points then for two general members φ_1 and φ_2 of $|\varphi|$ we have $p_a(\varphi_1\cdot\varphi_2) = p-2$.

PROOF. We have $2\varphi = 2F$ in Pic (W) (3.12). Hence $p_a(2\varphi) = p_a(2F)$ $= 2p_a(F) + p_a(F_1\cdot F_2) = 0 + p_a(\Gamma) = p$. On the other hand $p_a(2\varphi_1) = 2p_a(\varphi_1)$ $+ p_a(\varphi_1\cdot\varphi_2)$, hence $p_a(\varphi_1\cdot\varphi_2) = p-2$.

PROOF OF i). Suppose the P_i are not base points of $|\varphi|$, then $t_i = 2\tau_i = 0$ (see 3.15), hence we have $2\tilde F = 2\tilde\varphi$ in Pic $(\tilde W)$. Intersecting with a general $\tilde\varphi^*$ we get $\tilde F\cdot\tilde\varphi^* \equiv \tilde\varphi\cdot\tilde\varphi^*$ (numerical equivalence). Now $\tilde F\cdot\tilde\varphi^*$ is a curve of type Γ'. Hence $p = p_a(\Gamma') = p_a(\tilde F\cdot\tilde\varphi^*) = p_a(\tilde\varphi\cdot\tilde\varphi^*) = p-2$ by Claim *c*); a contradiction.

4.3. ASSUMPTION 4.

i) $|\tilde\varphi|$ has no base points,

ii) for a general φ the curve $\tilde\varphi\cdot E_i = C_i$ is irreducible ($i = 1, \ldots, n$).

REMARKS.

a) The assumption i) means again that we are in the « general » situation, namely we have seen that $|\varphi|$ has base points but now we assume that there are no infinitesimal base points.

b) The assumptions imply (by Bertini) that:

α) for a general φ the $\tilde\varphi$ is smooth,

β) $\mathrm{Tr}_{E_i}|\tilde\varphi|$ has no fixed components and no base points ($i = 1, \ldots, n$),

γ) For a general φ the C_i is a smooth, irreducible curve ($i = 1, \ldots, n$).

Note also that β) is in fact equivalent to i) since $|\varphi|$ has no base points outside the points P_i $(i = 1, ..., n)$.

LEMMA 4.4. *For a general* φ *we have* $K_{\tilde{\varphi}} = r \sum_i C_i$.

PROOF. The $\tilde{\varphi}$ is smooth and the lemma follows immediately from the adjunction formula and 3.12.

4.5. REMARK. *From now on we can and shall restrict ourselves to the case* $p > 5$. We can do this for the following reason. According to our basic assumption 3.1 ii) we have that for a general hyperplane section H the $W \cdot H = F$ is a *smooth* Enriques surface and $F \subset \mathbb{P}^{p-1}$ by 3.6. Clearly there is no smooth Enriques surface in $\mathbb{P}_3$, hence $p > 4$. Next: from [13], page 434, we see that if F is a smooth Enriques surface in $\mathbb{P}^4$ and if F has degree d then d must satisfy the equation $d^2 - 10d + 12 = 0$; since this equation has no solution in integers this is impossible and hence $p > 5$ (3).

4.6. Consider now the rational map

$$\tilde{\lambda} = \lambda_{|\tilde{\varphi}|} : \tilde{W} \to M \subset \mathbb{P}^{p-1} \tag{6}$$

determined by the linear system $|\tilde{\varphi}|$ and where we have written M for its image; by Assumption 4 i) this map is a *morphism*.

PROPOSITION 4.7. *Under the previous Assumptions* 1 *up to* 4, *we have that* $\dim M = 3$. *Moreover, under these assumptions* $\tilde{\lambda}|\tilde{F}$ *is birational.*

4.8. PROOF. Take a general hyperplane section $F = W \cdot H$. By 3.7 iii) we have $\mathrm{Tr}_F |\varphi| = |\Gamma'|$; furthermore $|\Gamma'|$ has no fixed parts and no base points (2.3) and $\Gamma'^2 = \Gamma^2 = 2p - 2 > 8$ since $p > 5$. Hence $\lambda_\varphi|F$ is birational by [7], Thm. 5.1 and Lemma 5.2.6. In particular we have:

a) $\dim M \geqq 2$,

b) $\lambda_\varphi|\Gamma'$ is birational.

Suppose now that $\dim M = 2$; then $\lambda_\varphi(W) = \lambda_\varphi(F) = S$ (say) and for $s \in S$ we have

$$\lambda_\varphi^{-1}(s) = \bigcup_i B_{s,i} \cup \Big(\bigcup_i P_i\Big)$$

(3) Note that in case we should allow the hyperplane section to be a *singular* Enriques surface then the case $p = 4$ is possible [11] and leads to the so-called *Enriques threefold*, an (in)famous threefold studied extensively by several authors (see for instance [20]). On the contrary we do not know whether $p = 5$ is possible with a *singular* Enriques surface as hyperplane section.

with *curves* $B_{s,i}$. Since $\lambda_\varphi|F$ is birational and $F = W \cdot H$, with H a *general hyperplane*, we have that $\lambda_\varphi^{-1}(s) \cap H$ consists of one point, hence there is only one $B_{s,i}$, write B_s, and B_s is a *line* and W is *ruled* over F.

Now restrict λ_φ to a general $\varphi_0 \in |\varphi|$, then we have $\lambda_\varphi(\varphi_0) = \lambda_\varphi(\Gamma_0')$ with $\Gamma_0' = \varphi_0 \cdot F$ and φ_0 is a *ruled surface* over Γ_0'.

Claim. φ_0 is *singular*. For, if not, then we have by 3.7 v) that $p_g(\varphi_0) = 1$ which is impossible for a ruled surface. By Bertini we have that the only singular points of φ_0 can be the points P_i (which are fixed points) ($i = 1, \ldots, n$). Furthermore the ruling on φ_0 is given by the lines B_s ($s \in \lambda_\varphi(\Gamma_0')$). Now look at the corresponding surface $\tilde{\varphi}_0$ on $\tilde{W}$. Take a *general* point $s \in \lambda_\varphi(\Gamma_0')$, take the line B_s and its proper transform $\tilde{B}_s$.

Claim. B_s passes through a singular point P_i. For, if not, then $\tilde{B}_s \cap E_i = \emptyset$ for $i = 1, \ldots, n$ and hence we have $K_{\tilde{B}_s} = (K_{\tilde{\varphi}} + \tilde{B}_s) \cdot \tilde{B}_s = 0$ by Lemma 4.4 which is a contradiction. Hence a general rule B_s of φ_0 passes through some singular point P_i, hence—by the similarity Assumption 3—through all P_i. Clearly then $n = 1$, i.e. there is only one singular point P and W is a cone with vertex P. However this is contrary to our assumptions; hence $\dim M = 3$.

Corollary 4.9. *Under the Assumptions* 1, ..., 4 *we have for a general* φ *that* $\mathrm{Pic}^0(\tilde{\varphi}) = 0$.

Proof. Apply the theorem of Matsusaka ([17], Theorem 2, p. 167) to the variety $\tilde{W}$ and to the linear system $|\tilde{\varphi}|$ (or, strictly speaking, to a general 2-dimensional subsystem of it). The general $\tilde{\varphi}$ is smooth, moreover, by 4.7, the characteristic linear system of $|\varphi|$ is not composed with a pencil and contains therefore an irreducible curve $\Gamma'' = \tilde{\varphi}_1 \cdot \tilde{\varphi}_2$ which is moreover smooth. So we can apply the theorem and hence $\mathrm{Pic}^0(\tilde{\varphi})$ is isogenous to $\mathrm{Pic}^0(\tilde{W})$, hence $\mathrm{Pic}^0(\tilde{\varphi}) = 0$ by 3.17.

Corollary 4.10. *With the same assumptions as in* 4.9 *for a general* φ *the* $\tilde{\varphi}$ *either is a* K_3*-surface or a rational surface. Moreover the case of a rational surface occurs only if* $r = -1$ (*the* r *from* 3.12) *and then there is only* one *point* $P_i = P$ (i.e. $n = 1$).

(Remark. For a more complete result see 6.5.)

Proof. By 3.8 and 4.4 (and the previous Assumptions, in particular 4) the φ and $\tilde{\varphi}$ fulfill the conditions of Lemma 1.1 and 1.2. Applying these lemmas we have to consider the two cases $r < 0$ and $r \geqq 0$.

$r < 0$: then $\tilde{\varphi}$ is a rational surface because by 4.9 a ruled surface is impossible. Moreover we have the fact $n = 1$ by 1.2 and $r = -1$ by 1.1.

$r \geqq 0$: according to lemmas 1.1 and 1.2 we get either $r = 0$ or $r = 1$. In both cases we have $\dim H^1(\varphi, \mathcal{O}_\varphi) = \dim H^1(\tilde{\varphi}, \mathcal{O}_{\tilde{\varphi}}) = 0$ by 3.8. If $r = 0$ then $K_{\tilde{\varphi}} = 0$ by 4.4, hence—since $q(\tilde{\varphi}) = 0$—we have that $\tilde{\varphi}$ is a K_3-surface. If $r = 1$ then we can blow down the curves C_i and we get $\pi' : \varphi \to \varphi'$ with φ' a K_3-surface, hence φ also a K_3-surface.

LEMMA 4.11. *Making the same assumptions as in* 4.9, *we get* $\deg M \geqq \geqq p - 3$.

PROOF. This is an immediate consequence of

a) $\dim |\tilde{\varphi}| = \dim |\varphi| = p - 1$ (3.7), hence M spans $\mathbb{P}^{p-1}$;

b) the well-known inequality $\deg X \geqq \operatorname{codim} X + 1$.

So in our case $\deg M \geqq p - 4 + 1 = p - 3$.

5. – Study of the system $|\tilde{\varphi}|$ (*continued*).

5.1. In this section *we always make the Assumptions* 1, ..., 4. Our goal is to prove:

THEOREM 5.1. *Under the above assumptions we have that* $\tilde{\lambda} = \lambda_{\tilde{\varphi}}$ (*and hence also* $\lambda = \lambda_\varphi$) *is* birational.

5.2. First of all we remark that we can apply Lemma 1.1 and 1.2 to our surfaces φ and $\tilde{\varphi}$. Since by 4.10 the surface $\tilde{\varphi}$ can *not* be ruled we have by 1.1 and 1.2 for the *integer* r (introduced in 3.12) at most the following possibilities:

$$r = -1, 0, 1 .$$

We proceed case by case.

LEMMA 5.3. *With the assumptions as above,* $\tilde{\lambda}$ *is birational for* $r = -1$.

5.4. The proof shall be given only after some « sublemmas ».

First of all the relations of 3.12 read for $r = -1$ (since now $n = 1$ by 1.2):

$$2\tilde{F} = 2\tilde{\varphi} + tE$$

or with $\tau = \frac{1}{2}t$

$$\tilde{F} \equiv \tilde{\varphi} + \tau E . \qquad \text{(num. eq.)}$$

Next apply Lemma 1.1 to F_1 and $\tilde{F}_1$. We have now $\varrho = \tau + r - 1 = = \tau - 2$ (3.16). Writing Δ_E^2 for the self-intersection number of Δ *on* E we get from 1.1 the following table of possibilities

ϱ	-2	-1	$-\frac{1}{2}$	0	1
τ	0	1	$\frac{3}{2}$	2	3
Δ_E^2			4	2	1

Case $\varrho = -2$. Impossible for $\tau > 0$, by 4.2.

Case $\varrho = -1$. Again our aim is to prove that this is *impossible* but this will require some preparations.

SUBLEMMA 5.5. *If* $\varrho = -1$ (*and always in the case* $r = -1$) *then* E *is a rational surface.*

PROOF. By 3.12 we have $K_{\tilde{W}} = -\tilde{\varphi} + rE = -\tilde{\varphi} - E$. Hence

$$K_E = (K_{\tilde{W}} + E)\cdot E \doteq -\tilde{\varphi}\cdot E = -C = -\Delta$$

(since now $\tau = 1$) and Δ is ample on E by 3.11. Hence E is a Del Pezzo surface, hence rational.

5.6. For the following we have to use a general fact:

LEMMA A. *Let* V *be a* smooth *projective variety and* $i\colon D_1 \to V$ *a* smooth *subvariety of codimension* one. *Let* D_2 *be a* positive *divisor on* V *such that* $i^*(D_2)$ *is defined. Then we have*

$$p_a(D_1 + D_2) = p_a(D_1) + p_a(D_2) + p_a(i^*(D_2)) \,. \tag{7}$$

PROOF. The divisors may, under the present assumptions, be considered as schemes. Moreover, the notion of arithmetic genus of such a divisor coincides with the notion of arithmetic genus of the corresponding scheme ([29], p. 581). Now recall that for a scheme D defined over a field and of dimension r we have ([13], p. 230)

$$p_a(D) = (-1)^r\{\chi(\mathcal{O}_D) - 1\} \,.$$

Formula (7) then reads

$$\chi(\mathcal{O}_{D_1+D_2}) = \chi(\mathcal{O}_{D_1}) + \chi(\mathcal{O}_{D_2}) - \chi(\mathcal{O}_{i^*(D_2)})$$

and follows immediately from the exact sequence

$$0 \to \mathcal{O}_V(-D_1-D_2) \to \mathcal{O}_V(-D_2) \to \mathcal{O}_{D_1}(-i^*(D_2)) \to 0 .$$

SUBLEMMA 5.7. *If* $r=-1$ *and* $\varrho=-1$ *then we have for general* φ *and general* F_1 *that* $p_a(\tilde{\varphi})=p_a(\tilde{F}_1)$.

PROOF. First of all from 3.12 we have now $2\tilde{\varphi}+2E=2\tilde{F}=2\tilde{F}_1+2E$, hence $2\tilde{\varphi}=2\tilde{F}_1$.

Next take two *independent general* members φ and φ_0 of $|\varphi|$ and similarly F_1 and F_1^0 of $|F_1|$.

CLAIM 1. $p_a(\tilde{F}_1\cdot\tilde{F}_1^0)=p_a(\tilde{\varphi}\cdot\tilde{\varphi}_0)$. (Here the left hand side is the arithmetic genus of a *divisor on* $\tilde{F}_1^0$; the right hand side is that of a *divisor on* $\tilde{\varphi}_0$.)

PROOF OF CLAIM 1. First since $2\tilde{\varphi}=2\tilde{F}_1^0$, we have on the smooth surface $\tilde{\varphi}_0$ that $\tilde{F}_1^0\cdot\tilde{\varphi}_0\equiv\tilde{\varphi}\cdot\tilde{\varphi}_0$ (numerical equivalence), and hence $p_a(\tilde{F}_1^0\cdot\tilde{\varphi}_0)=$ $=p_a(\tilde{\varphi}\cdot\tilde{\varphi}_0)$ as *divisors* on $\tilde{\varphi}_0$. Next we remark that the divisor $\tilde{F}_1^0\cdot\tilde{\varphi}_0$ on $\tilde{\varphi}_0$ has only components of multiplicity one. For this look first at $F_1^0\cdot\varphi_0$: since F_1^0 is obtained via a *general hyperplane section through* P this divisor has only components of multiplicity one ([26], Lemma 3, p. 100). Hence the same is true for $\tilde{F}_1^0\cdot\tilde{\varphi}_0$ since there are no extra components on E, where the $\tilde{F}_1^0$ is cut out by a *general* hyperplane of B (with the notations of the proof of 3.11). Therefore the $p_a(\tilde{F}_1^0\cdot\tilde{\varphi}_0)$ can be considered *at choice* as the arithmetic genus of a *divisor on* $\tilde{\varphi}_0$ or as the arithmetic genus of the (possibly reducible) variety $\tilde{F}_1^0\cap\tilde{\varphi}_0$ (see [29], Section 11), or—for the same reason—as the arithmetic genus of $\tilde{F}_1^0\cdot\tilde{\varphi}_0$ as *divisor on* $\tilde{F}_1^0$. (Remark for this that as divisor on $\tilde{\varphi}_0$ or as divisor on $\tilde{F}_1^0$ we get for each component the same coefficient, namely the intersection multiplicity of the component in $\tilde{F}_1^0\cap\tilde{\varphi}_0$ on $\tilde{W}$). Finally, from $2\tilde{\varphi}_0=2\tilde{F}_1$, we get on the smooth surface $\tilde{F}_1^0$ that $p_a(\tilde{F}_1\cdot\tilde{F}_1^0)=p_a(\tilde{\varphi}_0\cdot\tilde{F}_1^0)$. Combining this with our previous equality $p_a(\tilde{F}_1^0\cdot\tilde{\varphi}_0)=p_a(\tilde{\varphi}\cdot\tilde{\varphi}_0)$ on the surface $\tilde{\varphi}_0$, and using the above remarks about the interpretation of the integer $p_a(\tilde{F}_1^0\cdot\tilde{\varphi}_0)$, we conclude the proof of Claim 1.

Next we apply Lemma A the case $V=\tilde{W}$, $D_1=\tilde{F}_1^0$ and $D_2=\tilde{F}_1$ (always independent general members of $|\tilde{F}_1|$). Formula (7) gives

$$p_a(2\tilde{F}_1^0)=2p_a(\tilde{F}_1)+p_a(\tilde{F}_1\cdot\tilde{F}_1^0) .$$

Similarly applying Lemma A to $V=\tilde{W}$, $D_1=\tilde{\varphi}_0$ and $D_2=\tilde{\varphi}$ we get

$$p_a(2\tilde{\varphi}_0)=2p_a(\tilde{\varphi})+p_a(\tilde{\varphi}\cdot\tilde{\varphi}_0) .$$

Using $2\tilde{F}_1^0 = 2\tilde{\varphi}_0$, the above expressions and Claim 1 we conclude the proof of Sublemma 5.7.

SUBLEMMA 5.8. *If $r = -1$ and $\varrho = -1$ then, for a general F_1, the $\tilde{F}_1$ is a rational surface.*

PROOF. Since $\varrho = -1$, by 1.1 the $\tilde{F}_1$ is ruled or rational. Hence it suffices to see that $p_a(\tilde{F}_1) = 0$. However since $r = -1$ we have, by 1.1 and 4.10, that $\tilde{\varphi}$ is rational for a general φ and hence by 5.7 we have $p_a(\tilde{F}_1) = p_a(\tilde{\varphi}) = 0$.

5.9. *Proof that, for $r = -1$, the case $\varrho = -1$ is impossible.*

$\varrho = -1$ implies, by 1.1, that $g(\Delta) = 1$. We have $\tilde{F} = \tilde{F}_1 + E$. Since $\tilde{F}$ is an Enriques surface we have

$$0 = p_a(\tilde{F}) = p_a(\tilde{F}_1) + p_a(E) + p_a(\tilde{F}_1 \cdot E) = 0 + 0 + p_a(\Delta) = 1 ,$$

which contradicts 5.8 and 5.5.

5.10. *The remaining cases $\varrho = -\frac{1}{2}, 0, 1$ (always $r = -1$).*

For those we shall use the following formulae. We always have (for $r = -1$) $\tilde{F} \equiv \tilde{\varphi} + \tau E$ by 3.12. Hence, since $\tilde{F} \cdot \tilde{E} = 0$, using 3.6 we get:

$$\tilde{\varphi}^3 = \tilde{F}^3 - \tau^3 E^3 = 2p - 2 - \tau^3 E^3 .$$

Furthermore

$$E^3 = E^2 \cdot (\tilde{F} - \tilde{F}_1) = -E^2 \cdot \tilde{F}_1 = -E \cdot_{\tilde{W}} \Delta = -(\tilde{F} - \tilde{F}_1) \cdot_{\tilde{W}} \Delta = \tilde{F}_1 \cdot_{\tilde{W}} \Delta$$
$$= \Delta \cdot_E \Delta = \Delta_E^2 .$$

Hence

$$\tilde{\varphi}^3 = 2p - 2 - \tau^3 \Delta_E^2 . \tag{8}$$

From this we see immediately:

Case $\varrho = -\frac{1}{2}$ is impossible. We get $\varphi^3 = 2p - 2 - 27/2$; however clearly φ^3 is an integer.

5.11. CASE $\varrho = 0$. Claim: $\lambda_{\tilde{\varphi}}$ is birational.

PROOF. Write $\tilde{\lambda} = \lambda_{\tilde{\varphi}}$. We *always have* for a morphism the relation

$$\deg \tilde{\lambda} = \frac{\tilde{\varphi}^3}{\deg M} . \tag{9}$$

By 4.11 we have $\deg M \geqq p-3$. From (8) we get in the present case $\tilde{\varphi}^3 = 2p-2-8\cdot 2 = 2p-18$. Hence

$$\deg \tilde{\lambda} \leqq \frac{2p-18}{p-3} < 2\,.$$

Thus $\deg \tilde{\lambda} = 1$.

5.12. CASE $\varrho = 1$. Claim: $\lambda_{\tilde{\varphi}}$ is birational.

PROOF. From (8) we get here $\tilde{\varphi}^3 = 2p-2-27 = 2p-29$; as before $\deg M \geqq p-3$ and as before we get $\deg \tilde{\lambda} = 1$.

5.13. This *completes* now the *proof of* 5.3 that $\tilde{\lambda}$ is birational if $r = -1$. Moreover for further use we also state: *if* $r = -1$ *then from the table of* 5.4 *at most the cases* $\varrho = 0$ *and* $\varrho = 1$ *are possible.*

5.14. *Next we turn to the case* $r \geqq 0$ (*i.e.* $r = 0$ or $r = 1$). By 4.10 we have now that $\tilde{\varphi}$ *is a* K_3*-surface* if $\varphi \in |\varphi|$ is sufficiently general. We write again $\tilde{\lambda} = \lambda_{\tilde{\varphi}}$.

LEMMA 5.14. *If* $r \geqq 0$ *then we have*

i) $\tilde{\varphi}^3 = 2p-6$;

ii) $\deg \tilde{\lambda} = 1$ *or* 2;

iii) *correspondingly* $\deg M = 2p-6$ *or* $p-3$.

PROOF. Consider the curve $\tilde{\Gamma}''' = \tilde{\varphi}_1 \cdot \tilde{\varphi}_0$ on a sufficiently general $\tilde{\varphi}_0$ (with $\tilde{\varphi}_1$ another independent member of $|\tilde{\varphi}|$). By Bertini $\tilde{\Gamma}'''$ is a *smooth* curve; put $g = g(\tilde{\Gamma}''')$. By well-known theorems on K_3-surfaces ([2], p. 129) we have $g+1 = h^0(\tilde{\varphi}_0, \tilde{\Gamma}''')$. On the other hand by 3.17 it follows that $\mathrm{Tr}_{\tilde{\varphi}_0} |\tilde{\varphi}| = |\tilde{\Gamma}'''|$ and hence $h^0(\tilde{\varphi}_0, \tilde{\Gamma}''') = p-1$. Hence $g = p-2$. Since $p > 4$ we have $g \geqq 3$ and hence ([2], p. 129-130) we have $\tilde{\varphi}^3 = \tilde{\Gamma}'''^2 = 2g-2 = 2p-6$ and $\deg \tilde{\lambda} = 1$ or 2; finally using (9) we get $\deg M = 2p-6$ if $\deg \tilde{\lambda} = 1$, resp. deg. $M = p-3$ if $\deg \tilde{\lambda} = 2$.

COROLLARY 5.15 (Case I). *If* $\deg \tilde{\lambda} = 1$ *then for a sufficiently general* φ *the image* $S = \tilde{\lambda}(\tilde{\varphi})$ *is a* K_3*-surface which is* normal (*and has in fact at worst rational double points*).

PROOF. [22], p. 615 and Theorem 6.1 on p. 623.

COROLLARY 5.16 (Case II). *If* $\deg \tilde{\lambda} = 2$ *then for a sufficiently general* φ

the image $S = \tilde{\lambda}(\tilde{\varphi})$ *is of the following type*:

i) *a Veronese surface,*

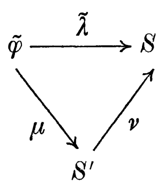
ii) *a rational normal scroll,*

iii) *a cone over a rational normal twisted cubic* (*then our* $p = 6$),

iv) *a cone over a rational normal twisted quartic* (*then our* $p = 7$).

PROOF. We have $\tilde{\Gamma}''^2 = 2p - 6 \geqq 6$. Now apply [22], Theorem 5.2, 5.6 and 5.7.

REMARK 5.17. For the following we have to use (again) the *theory of arithmetic genera of divisors* $p_a(D)$. This can be done clearly on a *smooth* variety but *also on a normal variety* (provided we work with Weil divisors) as is shown in [29]. Now note that the surfaces S in 5.15 and in 5.16 i), ii) and iii) either are smooth or normal. In 5.16 iv) the surface is not normal (since a twisted quartic curve is not projectively normal), however in case of 5.16 iii) and iv) we can again reduce to the case of a *smooth* surface by using the following factorization ([21], Lemma 2.1 on p. 429):



Here $S' = F_3$, resp. F_4, μ is a *morphism* and ν is the birational transformation contracting the curve with negative self intersection on F_n ($n = 3, 4$) into the vertex of the cone S.

5.18. We need the following general lemma.

LEMMA B. *Let* $\dim V = 2$ *with* V *smooth. Let* D *and* Z *be divisors on* V *and* $m > 0$. *Then we have*

$$p_a(Z + mD) = p_a(Z) + mp_a(D) + m(Z \cdot D) + \binom{m}{2}(D \cdot D) - m\,.$$

Special case:

$$p_a(mD) = mp_a(D) + \binom{m}{2}(D \cdot D) - (m-1)\,.$$

(REMARK. $p_a(0) = +1$, see [29], p. 581).

PROOF. This follows immediately by using the fact that for a divisor on a smooth surface $p_a(D) = 1 + \frac{1}{2}D\cdot(D+K)$, ([13], p. 366).

5.19. Now apply Lemma 1.1 to the surfaces F_i and $\tilde{F}_i$ ($i = 1, \ldots, n$). Since $\varrho = \tau + r - 1$ (see 3.16) we have the following possibilities:

Case $r = 0$:

ϱ	-2	-1	$-\frac{1}{2}$	0	1
τ	-1	0	$\frac{1}{2}$	1	2
Δ_E^2			4	2	1

Case $r = 1$:

ϱ	-2	-1	$-\frac{1}{2}$	0	1
τ	-2	-1	$-\frac{1}{2}$	0	1
Δ_E^2			4	2	1

Since, by 4.2, $\tau > 0$ we have for $r = 0$ that $\varrho \neq -2, -1$ and for $r = 1$ that $\varrho \neq -2, -1, -\frac{1}{2}$ and 0.

Furthermore from 3.12 we have $\tilde{F} = \tilde{\varphi} + \tau \sum_{i=1}^{n} E_i$ and by 5.14 this gives

$$(10) \qquad 2p - 6 = \tilde{\varphi}^3 = (\tilde{F} - \tau \sum E_i)^3 = 2p - 2 - \tau^3 \sum_{i=1}^{n} \Delta_E^2 .$$

5.20. *Case* $r = 0$ *and* $\varrho = -\frac{1}{2}$.

LEMMA 5.20. *In case* $r = 0$ *and* $\varrho = -\frac{1}{2}$ *we have*:

i) $\deg \tilde{\lambda} = 1$,

ii) $n = 8$,

iii) $\tilde{\lambda}(E_i)$ *are planes*,

iv) *the tangent cone to* W *in the* P_i ($i = 1, \ldots, 8$) *is the cone over the Veronese surface.*

PROOF. From formulae (10) and the table in 5.19 we get at once: $2p - 3 = 2p - 2 - \frac{1}{8}\cdot 4n$, hence $n = 8$. On $\tilde{W}$ we have

$$2\tilde{F} = 2\tilde{\varphi} + \sum_{1}^{8} E_i .$$

After applying $\tilde{\lambda}$ we get in Pic (M) the relation

$$2\tilde{\lambda}_*(\tilde{F}) = 2\tilde{\lambda}_*(\tilde{\varphi}) + \sum_1^8 \tilde{\lambda}_*(E_i)\,.$$

Now note that M has *no singularities in codim.* 1 since $\tilde{\varphi}_0$ being a K_3-surface (for general $\varphi_0 \in |\varphi|$) we have that $S_0 = \tilde{\lambda}(\varphi_0)$ has no singularities in codim. 1 by [22], see 5.15, 5.16 and 5.17; the above relation is in the *Weil*-Pic (M).

Now first consider *the case* deg $(\tilde{\lambda}) = 2$. Then we have (since $\tilde{\lambda}|F$ is birational, see 4.7) that $\tilde{\lambda}_*(\tilde{F}) = \tilde{\lambda}(\tilde{F})$. On the other hand now $\tilde{\lambda}_*(\tilde{\varphi}) = 2\tilde{\lambda}(\tilde{\varphi})$ so we get in Pic (M):

$$2\tilde{\lambda}(\tilde{F}) = 4\tilde{\lambda}(\tilde{\varphi}) + \sum_1^8 \tilde{\lambda}_*(E_i)\,.$$

Taking degrees left and right, using that $\deg \tilde{\lambda}(\tilde{F}) = \Gamma'^2 = 2p - 2$ and 5.14 we get $2(2p-2) = 4(p-3) + 8 \deg \tilde{\lambda}_*(E_i)$; hence $\tilde{\lambda}|E_i$ is birational and the $\tilde{\lambda}(E_i)$ are planes; in particular the $\tilde{\lambda}(C_i)$ are lines and $\tilde{\lambda}_*(C_i) = \tilde{\lambda}(C_i)$ because $C_i^2 = \tau^2 \Delta_i^2 = 1$.

Now take a general $\varphi_0 \in |\varphi|$ and intersect with $\tilde{\varphi}_0$: We have (write E instead of E_i by similarity):

$$2\tilde{F}\cdot\tilde{\varphi}_0 = 2\tilde{\varphi}\cdot\tilde{\varphi}_0 + \sum_1^8 E\cdot\tilde{\varphi}_0\,. \tag{11}$$

Applying $\tilde{\lambda}$ we get on the surface $S_0 = \tilde{\lambda}(\tilde{\varphi}_0)$ with $\tilde{\Gamma}'' = \tilde{\varphi}\cdot\tilde{\varphi}_0$:

$$2\tilde{\lambda}(\tilde{\Gamma}') = 4\tilde{\lambda}(\tilde{\Gamma}'') + \sum_1^8 \tilde{\lambda}(C_i)\,. \tag{11'}$$

Assume now first that we are in cases i) or ii) of 5.16 so that S_0 is a *smooth* surface. We are going to compute the $p_a(\text{—})$ of the L.H.S. and of the R.H.S. of (11′). We get

$$p_a(2\tilde{\lambda}(\tilde{\Gamma}')) = 2p_a(\tilde{\lambda}(\tilde{\Gamma}')) + \tilde{\lambda}(\tilde{\Gamma}')^2 - 1 = 2p + 2p - 2 - 1 = 4p - 3\,.$$

For the R.H.S. we use Lemma B with $Z = \sum_1^8 \tilde{\lambda}(C_i)$. First of all we have $p_a(Z) \geqq -7$ and then, since now the $\tilde{\lambda}(\tilde{\Gamma}'')$ are rational curves (and hyperplane sections on S_0):

$$p_a(Z + 4\tilde{\lambda}(\tilde{\Gamma}'')) \geqq p_a(Z) + 4p_a(\tilde{\lambda}(\tilde{\Gamma}'')) +$$
$$+ 4\cdot\tilde{\lambda}(\tilde{\Gamma}'')\cdot Z + \binom{4}{2}\tilde{\lambda}(\tilde{\Gamma}'')^2 - 4 \geqq -7 + 0 + 4\sum_1^8 1 + 6(p-3) - 4 \geqq 6p + 3\,.$$

Hence $4p - 3 \geqq 6p + 3$; impossible.

Next we turn to the cases iii) and iv) of 5.16. Now we replace the surface S by S' as in Remark 5.17 in order to be on a *smooth* surface (for the use of intersection theory). Applying now μ (instead of $\tilde{\lambda}$) on the relation (11) on $\tilde{\varphi}_0$ we get on S':

$$(11'') \qquad 2\mu(\tilde{\Gamma}') = 4\mu(\tilde{\Gamma}'') + \sum_1^8 \mu(C_i)\,.$$

Again we compute the $p_a(-)$ of L.H.S. and R.H.S. Note that the intersection $\tilde{\Gamma}'_1\cdot\tilde{\Gamma}'_2$ of the general members of $|\tilde{\Gamma}'|$ is outside the counter-image $\tilde{\lambda}^{-1}(s_0)$ of the vertex $s_0 \in S_0$ because looking on φ_0 we take general hyperplane sections F_1 and F_2 on W. Therefore we get exactly *the same numerical* values for $p_a(LHS)$ and $p_a(RHS)$ in (11$''$) as before by (11$'$), which is again impossible. Therefore $\deg(\tilde{\lambda}) = 1$ proving i) of the Lemma.

Finally, turning to the *case* $\deg(\tilde{\lambda}) = 1$ we have now in Pic (M) the relation:

$$2\tilde{\lambda}(\tilde{F}) = 2\tilde{\lambda}(\tilde{\varphi}) + \sum_1^8 \tilde{\lambda}_*(E_i)\,.$$

Taking degrees left and right we get $2(2p-2) = 2(2p-6) + 8\deg\tilde{\lambda}_*(E)$, so again $\tilde{\lambda}_*(E_i) = \tilde{\lambda}(E_i)$ and $\tilde{\lambda}(E_i)$ is a plane. Furthermore $C_E^2 = (\frac{1}{2}\Delta_E)^2 = 1$ and $K_{E_i} = -\frac{3}{2}\Delta_i$ (by 3.15). Hence $K_{E_i}^2 = 9$, $E_i \cong \mathbb{P}^2$ and the Δ_i are conics, so we get that the tangent cone to W at P_i is the cone over the Veronese surface.

LEMMA 5.21. *The case $r = 0$ and $\varrho = 0$ is impossible.*

PROOF. From the table in 5.19 we have $\tau = 1$ and $\Delta_E^2 = 2$; formula (10) gives now

$$2p - 6 = \tilde{\varphi}^3 = \Big(\tilde{F} - \sum_1^u E_i\Big)^3 = 2p - 2 - \sum_1^n 2,$$

hence $n = 2$. Hence 3.12 reads

$$2\tilde{F} = 2\tilde{\varphi} + 2E_1 + 2E_2\,.$$

Since $\tilde{F} = \tilde{F}_i + E_i$ $(i = 1, 2)$ this can be rewritten as

$$\tilde{F}_1 + \tilde{F}_2 = 2\tilde{\varphi} + E_1 + E_2 . \tag{12}$$

Now the E_i are del Pezzo surfaces because (being by 3.12, $K_{\tilde{W}} = -\tilde{\varphi}$) we have $K_E = (-\tilde{\varphi} + E)\cdot E = -C + E\cdot(\tilde{F} - \tilde{F}_1) = -C - \Delta = -2\Delta$; moreover it is ample by 3.11. By Lemma A we get now $p_a(E_1 + E_2) = 1$. Next we compute the arithmetic genera of the LHS and RHS of (12). For the LHS first remark that $2\tilde{F} = \tilde{F}_1 + \tilde{F}_2 + E_1 + E_2$ and that by Lemma A we have $p_a(2\tilde{F}) = 2p_a(\tilde{F}) + p_a(\tilde{\Gamma}) = p$. Next, applying Lemma A with $D_1 = E_1$ we get firstly

$$p_a(\tilde{F}_1 + \tilde{F}_2 + E_1) = p_a(E_1) + p_a(\tilde{F}_1 + \tilde{F}_2) + p_a(\Delta_1) = p_a(\tilde{F}_1 + \tilde{F}_2)$$

and secondly, applying again Lemma A with $D_1 = E_2$ we get

$$p_a(\tilde{F}_1 + \tilde{F}_2 + E_1 + E_2) = p_a(E_2) + p_a(\tilde{F}_1 + \tilde{F}_2 + E_1) + p_a(\Delta_2) = p_a(\tilde{F}_1 + \tilde{F}_2) .$$

Hence, for the LHS we get $p_a(\tilde{F}_1 + \tilde{F}_2) = p$. Turning to the RHS of (12), write $2\tilde{\varphi} = \tilde{\varphi}_* + \tilde{\varphi}_{**}$ with two independent general members φ_* and φ_{**} of $|\varphi|$. We have

$$p_a(2\tilde{\varphi}) = 2p_a(\tilde{\varphi}_*) + p_a(\tilde{\varphi}_*\cdot\tilde{\varphi}_{**}) = 2 + p_a(\tilde{\Gamma}'') = 2 + p - 2 = p$$

(see the proof of 5.14). Next, applying Lemma A with $D_1 = E_1$ we get

$$\begin{aligned} p_a(\tilde{\varphi}_* + \tilde{\varphi}_{**} + E_1) &= p_a(E_1) + p_a(\tilde{\varphi}_* + \tilde{\varphi}_{**}) + p_*(C_* + C_{**}) = \\ &= 0 + p + 2p_a(C_*) + C_*\cdot C_{**} - 1 = p + 0 + 2 - 1 = p + 1 \end{aligned}$$

(note $\tau = 1$ so $C \equiv \Delta$ and $\Delta_E^2 = 2$). Repeating the argument gives

$$p_a(\tilde{\varphi}_* + \tilde{\varphi}_{**} + E_1 + E_2) = p + 2 .$$

Hence $p_a(RHS) = p + 2$ contradicting our previous result $p_a(LHS) = p$. Hence $\varrho \neq 0$ if $r = 0$.

LEMMA 5.22. $r = 0$, $\varrho = 1$ *is impossible.*

PROOF. Now $\tau = 2$, $\Delta_E^2 = 1$. From (10) we get

$$2p - 6 = \tilde{\varphi}^3 = \Big(\tilde{F} - 2\sum_1^n E_i\Big)^3 = 2p - 2 - 8\cdot\Big(\sum_1^n 1\Big) = 2p - 2 - 8n ;$$

impossible.

LEMMA 5.23. *$r = 1$, $\varrho = 1$ is impossible.*

PROOF. Now $\tau = 1$, $\Delta_E^2 = 1$. Formula (10) gives

$$2p - 6 = \tilde{\varphi}^3 = \Bigl(\tilde{F} - \sum_1^n E_i\Bigr)^2 = 2p - 2 - n\,,$$

hence $n = 4$. Then 3.12 gives

$$2\tilde{F} = 2\tilde{\varphi} + 2(E_1 + E_2 + E_3 + E_4)\,.$$

Intersecting with a general $\tilde{\varphi}$ gives

$$\tilde{\Gamma}' \equiv \tilde{\Gamma}'' + (C_1 + C_2 + C_3 + C_3) \qquad \text{(num. eq.)}$$

Taking arithmetic genera we get $p_a(\tilde{\Gamma}') = p$, and on the other side

$$p_a(\tilde{\Gamma}'' + \sum C_i) = p_a(\tilde{\Gamma}'') + p_a(\sum C_i) + (\tilde{\Gamma}'' \cdot \sum C_i) - 1 =$$
$$= p - 2 - 3 + 4 - 1 = p - 2$$

(because $\tilde{\Gamma}'' \cdot C = C_2^E = \Delta_E^2$); contradiction.

5.24. *Summary of the results of Section* 5.

Under the *Assumptions* 1, 2, 3, 4 we have:

a) $\tilde{\lambda} = \lambda_{|\tilde{\varphi}|}$ is birational.

b) For the numbers n (3.3), $r = r_i$ (3.12) and $\varrho = \varrho_i$ (3.16) there are, at most, the following possibilities:

i) $r = -1$; then $n = 1$, i.e., there is one singular point $P = P_i$. For the surface F_1 we have then $\varrho = 0$ or $\varrho = 1$. For a general φ the $\tilde{\varphi}$ is a rational surface.

ii) $r = 0$. Then for the surfaces F_i ($i = 1, \ldots, n$) we have $\varrho = -\frac{1}{2}$.

Moreover:

α) For a general φ the $\tilde{\varphi}$ is a K_3-surface.

β) $n = 8$; *i.e.* there are eight singular points $P_1, \ldots, P_8$.

γ) $\tilde{\lambda}(E_i)$ are planes ($i = 1, \ldots, 8$).

δ) The tangent cones to W at P_i are cones over the Veronese surface ($i = 1, \ldots, 8$).

6. – Study of the system $|\tilde{\varphi}|$. (Elimination of the case $r = -1$.)

6.1. We make the same assumptions as in Section 5. We have the following commutative diagram of mappings (see 3.9 and 4.6):

$$\begin{array}{ccc} & \tilde{W} & \\ {}^{\pi}\swarrow & & \searrow^{\tilde{\lambda} = \lambda_{|\tilde{\varphi}|}} \\ \mathbf{P}^p \supset W & \underset{\lambda = \lambda_{|\varphi|}}{\dashrightarrow} & M \subset \mathbf{P}^{p-1} \end{array}$$

In particular we have now that $\tilde{\lambda}$ (and hence also λ) is a birational transformation (5.24).

LEMMA 6.2 (Weil's equivalence criterion for the system $|\varphi|$). *Let X be a (Weil-) divisor on W such that $X\cdot\varphi_t = 0$ in* Pic (φ_t) *for a generic φ_t in $|\varphi|$. Then $X = 0$ in* Pic (W).

PROOF (cf. [26], p. 111-112). Take a generic pencil Λ in $|\varphi|$, i.e. Λ is spanned by φ_{t_1} and φ_{t_2} with t_1 and t_2 independent generic points in the dual space $\check{\mathbf{P}}^{p-1}$ of $\mathbf{P}^{p-1}$; let $C = \varphi_{t_1} \cap \varphi_{t_2}$ be the axis of the pencil. There is a function f on W with div $(f) = \varphi_{t_1} - \varphi_{t_2}$ and this gives a rational map $f\colon W \dashrightarrow \mathbf{P}^1$ which is a morphism on $W_0 = W - C$. Let W^* be the graph Γ_f of f; let $p_1\colon W^* \to W$ and $p_2\colon W^* \to \mathbf{P}^1$ be the projections. We lift each component of the divisor X, with its multiplicity, to W^* and obtain in this way a divisor X^* on W^* with $p_1(X^*) = X$.

For each $s \in \mathbf{P}^1$ we consider on W^* the *divisor* $W^*\cdot(W\times s) = \varphi_s^*$ (the intersection is on $W\times\mathbf{P}^1$) and we denote the (possibly reducible) underlying *variety* by $\{\varphi_s^*\}$. We have $p_1(\varphi_s^*) = \varphi_s$ and, if we denote by $\{\varphi_s\}$ also the underlying variety of φ_s, then we have an isomorphism $p_1\colon \{\varphi_s^*\} \xrightarrow{\sim} \{\varphi_s\}$. Now let t be generic on $\mathbf{P}^1$ (over the field $k(t_1, t_2) = k_1$). Let $Y_t^* = X^*\cdot\varphi_t^*$, then Y_t^* can be considered as a divisor on the irreducible variety $\{\varphi_t^*\}$ and Y_t^* and $Y_t = X\cdot\varphi_t$ correspond to each other by the isomorphism $p_1\colon \{\varphi_t^*\} \xrightarrow{\sim} \{\varphi_t\}$; also note that these varieties $\{\varphi_t^*\}$ and $\{\varphi_t\}$ have (at most) points as singularities. Since $Y_t = 0$ in Pic $(\{\varphi_t\})$ we have on $\{\varphi_t^*\}$ that $Y_t^* = \operatorname{div}(\psi_t^*)$, with a function ψ_t^* on $\{\varphi_t^*\}$ defined over the field $k_1(t)$ ([26], Cor. 2, p. 265). Therefore, if $(x, t) = x^*$, with $t = f(x)$, is a generic point of $\{\varphi_t^*\}$ over $k_1(t)$, then $\psi_t^*(x^*) \in k_1(x^*) = k_1(x)$ and hence, since also x^* is a generic point of W^* over k_1, the ψ_t^* determines a function Ψ^* on W^* and a function Ψ on W (corresponding to each other by the isomorphism $k_1(W) \xrightarrow{\sim} k_1(W^*)$). Consider now on W^* the divisor $Z^* = X^* - \operatorname{div}(\Psi^*)$, then $Z^* \cdot (W \times t) = 0$. Hence every component of Z^* is contained in a variety of type $\{\varphi^*_{s_i}\}$ with $s_i \in \mathbb{P}^1$. Now we state:

CLAIM A. $\{\varphi^*_{s_i}\}$ is an irreducible variety and occurs with multiplicity one in the divisor $\varphi^*_{s_i}$ (for each $s_i \in \mathbb{P}^1$).

CLAIM A $\Rightarrow$ LEMMA 6.2. We have now $Z^* = X^* - \operatorname{div}(\Psi^*) = \sum_i m_i \varphi^*_{s_i}$. Projecting down to W we get in $\operatorname{Pic}(W)$ that $X = \sum_i m_i \varphi_{s_i}$. Moreover, we have (always in $\operatorname{Pic}(W)$) that $\varphi_{s_i} = \varphi_{s_0}$ for some fixed $s_0 \in \mathbb{P}^1$, hence $X = m\varphi_{s_0}$, for some integer m. Intersecting again with a generic φ_t we get, in $\operatorname{Pic}(\varphi_t)$, that $0 = Y_t = X \cdot \varphi_t = m\varphi_{s_0} \cdot \varphi_t$. However, $\varphi_{s_0} \cdot \varphi_t$ is an effective curve on φ_t, hence $m = 0$, i.e. $X = 0$ in $\operatorname{Pic}(W)$.

PROOF OF CLAIM A.

CLAIM B. There does *not* exist a $\varphi_{u_0} \in |\varphi|$ such that $\varphi_{u_0} = V_1 + V_2$ with positive divisors V_1 and V_2 such that for the underlying variety of one of them, say for V_2, we have that $\lambda(V_2)$ is a *point* (where, as always, $\lambda = \lambda_{|\varphi|}$).

CLAIM B $\Rightarrow$ CLAIM A. Look to the diagram in 6.1. Put $\mathfrak{D}_i = \{y \in M: \dim \lambda^{-1}(y) \geqq i\}$ for $i = 1, 2, \ldots$. Let $\mathfrak{D}_i = \bigcup_j D_j^{(i)}$ be the decomposition in irreducible components. We have $\mathfrak{D}_2 = \emptyset$ by Assumption B, so we have only to consider $\mathfrak{D}_1$.

A *generic* pencil Λ' of hyperplanes $\{H_u\}_{u \in \Lambda'}$ in the space $\mathbb{P}^{p-1}$ of M has the following properties:

For *all* $u \in \Lambda'$ we have:

i) $H_u \cap M$ is an *irreducible* variety ([26], p. 102, Lemma 4),

ii) this variety occurs with multiplicity *one* in $H_u \cdot M$ (ibid., same lemma),

iii) $H_u \not\supset$ those $D_j^{(1)}$ which have *dimension at least one* (because such varieties span in $\mathbb{P}^{p-1}$ a linear space of dimension at least one, *defined over k*, and the conditions that this span is contained in a member H_u of this *generic* (over k) pencil are *two* independent linear conditions).

Now look to the *birational* transformation $\lambda\colon W \dashrightarrow M$; by property ii) above we have that the irreducible variety $M \cap H_u$ is non-singular on M, hence λ is biregular at the generic point of $M \cap H_u$. From this and from property iii) it follows that the underlying variety $\{\varphi_u^*\}$ of the divisor φ_u^* is irreducible for all $u \in \mathbb{P}^1$. Finally, again using ii) and the biregularity of λ (at the genric point), we get that the multiplicity of $\{\varphi_u^*\}$ in φ_u^* is *one* for all $u \in \mathbb{P}^1$.

PROOF OF CLAIM B. Suppose such $\varphi_{u_0} \in |\varphi|$ exists. Intersect with a generic $F = W \cdot H$ (H generic in $\mathbb{P}^p$, the space of W). We have $\mathrm{Tr}_F\,|\varphi| = |\Gamma'|$ (3.7) and we get $\varphi_{u_0} \cdot F = (V_1 + V_2) \cdot F = \Gamma_1' + \Gamma_2'$ with $\lambda(\Gamma_2')$ contracting to a point, contradicting Remark 2.4 iii).

LEMMA 6.3. *For the F_i we have that* $\varrho_i = \varrho = \tau + r - 1 = -\frac{1}{2}$ (cf. 3.16) $(i = 1, \ldots, n)$.

PROOF. By 3.16 we can apply Lemma 1.1 to a general F_i going through a point P_i. It follows that if $\varrho \neq -\frac{1}{2}$ then $\varrho \in \mathbb{Z}$ and hence $\tau \in \mathbb{Z}$. Then $\left\{\tilde{\varphi} + \tau\left(\sum_i E_i\right)\right\} \in \mathrm{Pic}\,(\tilde{W})$ (and not merely in $\mathrm{Pic}\,(\tilde{W}) \otimes_{\mathbb{Z}} \mathbb{Q}$). Take now a general $\tilde{\varphi}_0$, then we have by 3.14 in $\mathrm{Pic}\,(\tilde{\varphi}_0)$:

$$\left\{\tilde{F} - \left(\tilde{\varphi} + \tau\left(\sum_i E_i\right)\right)\right\} \cdot \tilde{\varphi} \equiv 0 \qquad \text{(numerical eq.)}\,.$$

By 4.10 $\tilde{\varphi}_0$ is either rational or a K_3-surface, hence numerical equivalence coincides with linear equivalence. Hence

$$\left\{\tilde{F} - \left(\tilde{\varphi} + \tau\left(\sum_i E_i\right)\right)\right\} \cdot \tilde{\varphi}_0 = 0\,.$$

Projecting down to $\varphi_0 = \pi(\tilde{\varphi}_0)$ we get in $\mathrm{Pic}\,(\varphi_0)$:

$$(F - \varphi) \cdot \varphi_0 = 0\,,$$

hence by using Weil's equivalence criterium (in extended form 6.2)

$$F = \varphi$$

in $\mathrm{Pic}\,(W)$. However, now take a general F_0 and intersect, then we get in $\mathrm{Pic}\,(F_0)$

$$\Gamma = \Gamma'\,,$$

a contradiction. Hence $\varrho = -\frac{1}{2}$.

COROLLARY 6.4. *On a general F_i we have $\Delta_{i]}^2 = -4$, $g(\Delta_i) = 0$ and the P_i are rational, quadruple points* ($i = 1, \dots, n$).

PROOF. Since $\varrho_i = \varrho = -\frac{1}{2}$ this follows now at once from 1.1.

COROLLARY 6.5. $r_i = r = 0$ (*hence* $n = 8$) *and for a general* φ *the* $\tilde{\varphi}$ *is a* K_3*-surface.*

PROOF. We have seen (in 5.24) that the only other possibility $r = -1$ implies $\varrho = 0$ or 1, contrary to Lemma 6.3. The other assertions follow now from 5.24.

7. – Main theorem and some remarks.

7.1. Let W be a threefold satisfying the basic assumptions (see 3.1) and the Assumptions 1, ..., 4.

For the morphisms (and maps) we refer to the diagram in 6.1. With these assumptions and notations we get:

THEOREM 7.2. *W is a variety of degree $2p - 2$, spans $\mathbb{P}^p$ and has 8 singular points $P_1, \dots, P_8$. Each point P_i is a quadruple point and its tangent cone is a cone over the Veronese surface.*

W carries a linear system $|\varphi|$ of Weil divisors, the general members of which are K_3-surfaces. This system is of dimension $(p-1)$ and the base points of it are the points P_i ($i = 1, \dots, 8$); the associated rational map $\lambda = \lambda_{|\varphi|}$ is a birational morphism. The points P_i are rational double points on a general φ.

Let $M = \lambda(W)$ be the image. Then M spans $\mathbb{P}^{p-1}$, has degree $(2p-6)$ and has K_3-surfaces as (general) hyperplane sections (i.e., M is a Fano-variety in the classical sense).

Furthermore M contains 8 planes $\pi_1, \dots, \pi_8$ which are the « images » of the points $P_1, \dots, P_8$.

PROOF. Almost all the assertions have already been proved in the previous sections, see notably 6.5 and 5.24. As to the assertion about the degree see 5.14. Finally: since the $\tilde{\varphi}$ are K_3-surfaces the hyperplane sections $M \cdot H$ are K_3-surfaces having at most rational double points ([22], p. 615) for a general H (i.e., $\tilde{\lambda}(\tilde{\varphi})$ for a general φ), hence M is a « Fano-variety » in the classical sense.

7.3. Remarks.

i) From the relation $2\tilde{F} = 2\tilde{\varphi} + \sum_{i=1}^{8} E_i$ we see that the system 2φ is cut out on W by the *quadrics* going simply *therough* the 8 points $P_1, \ldots, P_8$.

ii) We have seen (Claim B) in the proof of 6.2 that by $\lambda = \lambda_\varphi$, and hence also by $\tilde{\lambda} = \lambda_{\tilde{\varphi}}$, at worst curves can be contracted into points. If A is such a curve then the « intersection number » of A and 2φ must be zero and hence *the only curves which are contracted are the lines P_iP_j provided these lines are contained in W.* Since a general φ does not contain such a line (because the only base points of the system $|\varphi|$ are the points P_i) we see that for a general H the $M \cdot H$ is a *smooth* K_3-surface.

iii) From the above remark we see that M has at most *isolated singularities.*

iv) Two points P_i and P_j are called *associated* if the line P_iP_j is contained in W. It follows from the above that if P_i and P_j are associated then the planes π_i and π_j *have a point in common,* namely the point corresponding to the contraction of P_iP_j. Also note that by our similarity assumption every point is associated to the same number of other points.

8. – The known examples of algebraic threefolds whose hyperplane sections are Enriques surfaces.

8.1. In this paper [11] Fano gives examples of threefolds with Enriques hyperplane sections and gives also a scheme of classification of them claiming that they do exist only for $p = 4, 6, 7, 9$ and 13.

The case $p = 4$, as we have already remarked, is a kind of exception and leads to the classical Enriques threefold.

8.2. The case $p = 6$ is indeed possible. It is treated in all details in Part II of our Mittag Leffler Report [5]. Here $W_3^{10} \subset \mathbb{P}^6$ can be obtained as a 3-dimensional variety parametrising the dual conics of $\mathbb{P}^3$ degenerating into a couple of points. Its hyperplane sections are Enriques surfaces of the type called « Reye congruences ». It was Fano himself who realised the first, in [10], that Reye congruences are Enriques surfaces. A modern treatment of the theory of Reye congruences is contained in Cossec's thesis [7]. The corresponding variety $M_3^6 \subset \mathbb{P}^5$ turns out to be the intersection of a quadric and a cubic hypersurface, i.e. identifying the quadric with the Grassmanian $G(1, 3)$, a cubic complex of lines. Since it contains 8 planes, it is the complex of lines lying on some quadric of a net in $\mathbb{P}^3$ (the 8 planes correspond to the lines through the 8 base points of the net).

8.3. The case $p = 7$ is again possible. Here $W_3^{12} \subset \mathbb{P}^7$ is the image of $\mathbb{P}^3$ by the linear system of all sextic surfaces containing one fixed plane cubic and going doubly through the edges of a tetrahedron, whilst $M_3^8 \subset \mathbb{P}^6$ is the intersection of 3 quadric with 8 planes in common.

8.4. In the case $p = 9$ the $W_3^{16} \subset \mathbb{P}^9$ has as hyperplane sections Enriques surfaces which can be obtained as quotients modulo an involution from the intersection of 3 quadrics of $\mathbb{P}^5$. The Fano variety $M_3^{12} \subset \mathbb{P}^8$ is the intersection of the Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^2$ with a quadric containing 4 planes of each of its two families. A complete analysis of the case can be found in [6].

8.5. Finally, in the case $p = 13$, $W_3^{24} \subset \mathbb{P}^{13}$ is the image of $\mathbb{P}^3$ by the linear system of all sextic surfaces going doubly through the edges of a tetrahedron, whilst $M_3^{20} \subset \mathbb{P}^{12}$ is the image of $\mathbb{P}^3$ by the linear system of all quartic surfaces going simply through the edges of the same tetrahedron.

8.6. Fano claims in his paper that the above list is *complete*. His method is based on an analysis of the configuration of the 8 planes $\pi_1, \ldots, \pi_8$ lying on M and on his own classification of Fano threefolds. As has been mentioned in the introduction in this paper we have restricted ourselves to the *general theory* of threefolds whose hyperplanes sections are Enriques surfaces, leaving the *classification* problem for a future occasion.

ADDED IN FEBRUARY 1984. The referee has pointed out that Fano's list is not complete and that in fact an example with $p = 10$ can be constructed.

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