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Bombieri's Theorem in Short Intervals.

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Introduction.

The well known theorem of Bombieri - A. I. Vinogradov (see Bombieri [1], p. 57) states, in the stronger form due to Bombieri, that

$$\sum_{q\leqslant Q}\max_{(a,q)=1}\max_{z\leqslant x}\left|\psi(z,q;a)-\frac{z}{\varphi(q)}\right|\ll\frac{x}{(\log x)^A},\tag{1}$$

where A is an arbitrary positive constant, $Q=x^{\frac12}/(\log x)^B$ with $B=B(A)>0$ and $\psi(x,q;a)=\sum_{\substack{n\leqslant x\\ n\equiv a\,(\mathrm{mod}\,q)}}\Lambda(n)$; Λ is von Mangoldt's function.

The original proof of (1) was based on zero-density estimates for Dirichlet L-functions, which in turn follow from large sieve inequalities. Later Gallagher [3] and Vaughan [13] obtained new proofs of (1) using only the large sieve and some identities for $(L'/L)(s,\chi)$. Recently Vaughan [14] gave an essentially elementary proof of Bombieri's theorem by means of his identity for $(L'/L)(s,\chi)$, combined with the large sieve and with inequalities for bilinear forms.

The problem of finding a result analogous to (1) for short intervals was first investigated by Jutila [7]. Using the zero-density method he obtained an estimate of the form

$$\sum_{q\leqslant Q}\max_{(a,q)=1}\max_{h\leqslant y}\max_{\frac12 x<z\leqslant x}\left|\psi(z+h,q;a)-\psi(z,q;a)-\frac{h}{\varphi(q)}\right|\ll\frac{y}{(\log x)^A},\tag{2}$$

where, putting

$$y=x^{\theta}\quad\text{and}\quad Q=\frac{x^{\varphi}}{(\log x)^B},$$

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Q is subject to the restriction $\psi < \dfrac{4c\theta + 2\theta - 1 - 4c}{6 + 4c}$. Here

$$c = \inf\{\xi;\, \zeta(\tfrac{1}{2} + it) \ll t^{\xi}\}\,.$$

Soon after, Motohashi [10] obtained some estimates of the form (2) by Gallagher's method. The results of Jutila and Motohashi were improved by Huxley-Iwaniec [6] and Ricci [12]. Huxley-Iwaniec also used the zero-density method and obtained the following bounds:

$$\psi \leqslant \theta - \frac{1}{2} \quad \text{if} \quad \frac{3}{4} \leqslant \theta \leqslant 1\,,$$

$$\psi \leqslant \left(\frac{1}{5} + \left\{\frac{3}{5}\left(\theta - \frac{3}{5}\right)\right\}^{\frac{1}{2}}\right)^{2} - \varepsilon \quad \text{if} \quad \frac{29}{48} < \theta \leqslant \frac{3}{4}$$

$$\psi \leqslant 3\theta - \frac{7}{4} - \varepsilon \quad \text{if} \quad \frac{7}{12} < \theta \leqslant \frac{29}{48}\,.$$

Following the Gallagher-Vaughan method Ricci improved the estimates of Huxley-Iwaniec in the range $\frac{3}{5} + \alpha \leqslant \theta \leqslant \frac{3}{4}$, with a certain $\alpha > 0$; precisely he obtained

$$\psi \leqslant \min\left(\theta - \frac{1}{2}, \frac{1}{2}\,(5\theta - 3)\right) \qquad \text{if } \theta > \frac{3}{5}\,.$$

It is interesting to note that, while the Huxley-Iwaniec' method works till $\theta = \frac{7}{12}$, Ricci's method stops at $\theta = \frac{3}{5}$.

The Generalized Density Hypothesis would imply (see [6], (2.23)-(2.24)) that (2) is true if

$$\psi \leqslant \theta - \frac{1}{2}\,, \tag{3}$$

for $\theta > \frac{1}{2}$. Huxley-Iwaniec and Ricci obtained (3) for $\theta \geqslant \frac{3}{4}$ and $\theta \geqslant \frac{2}{3}$, respectively.

Using Heath-Brown's extension of Vaughan's identity (see [4] and [5]) we improve the range of validity of (3). Precisely we obtain the following

THEOREM. *The estimate* (2) *holds if* $\psi \leqslant \theta - \frac{1}{2}$ *and* $\theta > \frac{3}{5}$.

REMARK. Indeed, as it is clear from the proof, we obtain (2) if (3) holds and $y \geqslant x^{3/5}(\log x)^{C}$, $C = C(A)$.

PROOF OF THE THEOREM. We begin with the usual reduction to primitive characters. In the sequel c will denote a suitable absolute positive constant, whose value is not necessarily the same at each occurrence; we will also use the notation $L = \log x$.

We start from the well known relation

$$\psi(z+h, q; a) - \psi(z, q; a) - \frac{h}{\varphi(q)} = \frac{1}{\varphi(q)} \sum_{\chi (\operatorname{mod} q)} \bar{\chi}(a) E(z, h, \chi)$$

where

$$E(z, h, \chi) = \begin{cases} \sum_{z<n\leqslant z+h} \Lambda(n)\chi(n) & \text{if } \chi \text{ is non-principal} \\ \sum_{z<n\leqslant z+h} \Lambda(n)\chi(n) - h & \text{if } \chi \text{ is principal}. \end{cases}$$

If χ_1, primitive $(\operatorname{mod} q_1)$, induces χ $(\operatorname{mod} q)$, then

$$E(z, h, \chi_1) - E(z, h, \chi) \ll \log q \log z \ll L^2 .$$

Moreover

$$\sum_{\substack{q\equiv 0 (\operatorname{mod} q_1) \\ q\leqslant Q}} \frac{1}{\varphi(q)} \ll \frac{\log Q}{\varphi(q_1)} .$$

Hence we have

$$\text{(4)} \quad \sum_{q\leqslant Q} \max_{(a,q)=1} \max_{h\leqslant y} \max_{\frac{1}{2}x<z\leqslant x} \left| \psi(z+h, q; a) - \psi(z, q; a) - \frac{h}{\varphi(q)} \right|$$
$$\ll \sum_{q\leqslant Q} \frac{\log Q}{\varphi(q)} \sum_{\chi (\operatorname{mod} q)}^{*} \max_{h\leqslant y} \max_{\frac{1}{2}x<z\leqslant x} |E(z, h, \chi)| + QL^3$$
$$\ll \max_{Q'\leqslant Q} \frac{L^c}{Q'} \sum_{\frac{1}{2}Q'<q\leqslant Q}^{*} \sum_{\chi (\operatorname{mod} q)}^{*} \max_{h\leqslant y} \max_{\frac{1}{2}x<z\leqslant x} |E(z, h, \chi)| + QL^3 .$$

Moreover, it follows from the results of Huxley-Iwaniec [6] that we may assume

$$\text{(5)} \qquad Q \gg x^{\delta}$$

with a sufficiently small fixed positive constant $\delta = \delta(\theta)$.

Hence, by (4), our task reduces to proving that

$$\text{(6)} \qquad \sum_{\frac{1}{2}Q'<q\leqslant Q'} \sum_{\chi (\operatorname{mod} q)}^{*} \max_{h\leqslant y} \max_{\frac{1}{2}x<z\leqslant x} |E(z, h, \chi)| \ll yQ' L^{-A-c}.$$

In the sequel we will always suppose A sufficiently large.

Let us recall Heath-Brown's identity (see [4] and [5]). Let k be any integer $\geqslant 1$, and

$$M(s) = M_X(s) = \sum_{n\leqslant X} \mu(n) n^{-s}$$

where $\mu(n)$ denotes the Möbius function; then

$$(7)\qquad \frac{\zeta'}{\zeta}(s) = \sum_{j=1}^{k}(-1)^{j-1}\binom{k}{j}\zeta(s)^{j-1}\zeta'(s)M(s)^j + \frac{\zeta'}{\zeta}(s)\big(1-\zeta(s)M(s)\big)^k .$$

We choose $k = 4$ and $X^4 = 2x$; since

$$\zeta(s)M(s) = 1 + \sum_{n>X} c_n n^{-s}, \qquad \text{with } c_n = \sum_{\substack{d/n\\ d\leqslant X}} \mu(d)\,,$$

it is clear that the last term of (7) does not contribute to the coefficient of n^{-s}, when $n \leqslant x$. We apply (7) to the sum $S = \sum_{z<n\leqslant z+h} \Lambda(n)\chi(n)$, picking out the relevant coefficients of n^{-s}. On splitting up each range of summation into intervals of the form $N < n \leqslant 2N$, we find that S is a linear combination of $O(L^8)$ sums of the form

$$(8)\qquad S^* = \sum_{\substack{n_i\in I_i\\ z<n_1\ldots n_8\leqslant z+h}} \chi(n_1)\log n_1\chi(n_2)\ldots\chi(n_4)\chi(n_5)\mu(n_5)\ldots\chi(n_8)\mu(n_8)$$

in which

$$(9)\qquad I_i = (N_i, 2N_i]\,, \qquad x \ll \prod_{i=1}^{8} N_i \ll x \qquad \text{and} \qquad N_i \leqslant (2x)^{\frac14} \quad \text{if } i = 5, \ldots, 8\,.$$

Hence it is sufficient to prove, for any S^*,

$$(10)\qquad \sum_{\frac12 Q'<q\leqslant Q'} \sum_{\chi(\mathrm{mod}\, q)}^{*} \max_{h\leqslant y} \max_{\frac12 x<z\leqslant x} |S^*| \ll yQ'L^{-A-c}\,.$$

Suppose now that there exists an i such that

$$N_i \geqslant yL^{-3A}\,.$$

By (9) we have $1 \leqslant i \leqslant 4$, since $y \gg x^{3/5}$. We may suppose $i = 1$, the other cases being similar.

By partial summation and using the Pólya-Vinogradov inequality (see [2]) we get

$$S^* = \sum_{n_i\in I_i} \chi(n_2)\ldots\chi(n_4)\chi(n_5)\mu(n_5)\ldots\chi(n_8)\mu(n_8) \sum_{\substack{n_i\in I_i\\ z/(n_2\ldots n_8)<n_1\leqslant (z+h)/(n_2\ldots n_8)}} \chi(n_1)\log n_1$$

$$\ll \prod_{i=2}^{8} N_i (Q')^{\frac12}\log Q \log x \ll \frac{x}{y} L^{3A+c}(Q')^{\frac12}\,,$$

so that

$$\sum_{\frac{1}{2}Q'<q\leqslant Q'} \sideset{}{^*}\sum_{\chi(\mathrm{mod}\, q)} \max_{h\leqslant v} \max_{\frac{1}{2}x<z\leqslant x} |S^*| \ll \frac{x}{y}(Q')^{\frac{5}{2}} L^{3A+c} \ll Q' x^{1-\theta+\frac{3}{2}(\frac{1}{2}-\theta)} L^{-\frac{3}{2}B+3A+c}. \tag{11}$$

(11) is certainly $\ll Q'yL^{-A-c}$ if $\theta \geqslant \frac{1}{2}$ and $B \geqslant cA$, and the Theorem follows in this case.

Hence we may suppose that

$$N_i < yL^{-3A} \quad \text{for every } i, \tag{12}$$

and in this case we evaluate S^* by Perron's formula (see Prachar [11], p. 376).

Let $a_1(n) = \log n$, $a_i(n) = 1$ for $2 \leqslant i \leqslant 4$ and $a_i(n) = \mu(n)$ for $5 \leqslant i \leqslant 8$. Then, putting $\delta = L^{-1}$, we get

$$S^* = \frac{1}{2\pi i} \int_{1+\delta-ix^2}^{1+\delta+ix^2} \prod_{i=1}^{8} \left(\sum_{n_i \in I_i} \frac{\chi(n_i)a_i(n_i)}{n_i^s} \right) \frac{(z+h)^s - z^s}{s}\, ds + O\left(\frac{L^c}{x}\right).$$

Shifting the line of integration to $\sigma = \frac{1}{2}$ the contribution R^* of the horizontal sides is

$$R^* \ll L \prod_{i=1}^{8} \left(\sum_{n_i \in I_i} n_i^{-\frac{1}{2}} \right) \frac{x}{x^2} \ll Lx^{-\frac{1}{2}}. \tag{13}$$

From (13) we have

$$\sum_{\frac{1}{2}Q'<q\leqslant Q'} \sideset{}{^*}\sum_{\chi(\mathrm{mod}\, q)} \max_{h\leqslant v} \max_{\frac{1}{2}x<z\leqslant x} |R^*| \ll L(Q')^2 x^{-\frac{1}{2}} \ll Q'yL^{-A-c},$$

hence

$$\begin{aligned} \sum_{\frac{1}{2}Q'<q\leqslant Q'} \sideset{}{^*}\sum_{\chi(\mathrm{mod}\, q)} \max_{h\leqslant v} \max_{\frac{1}{2}x<z\leqslant x} |S^*| \\ \ll \sum_{\frac{1}{2}Q'<q\leqslant Q'} \sideset{}{^*}\sum_{\chi(\mathrm{mod}\, q)} \max_{h\leqslant v} \max_{\frac{1}{2}x<z\leqslant x} |S_1^*| + Q'yL^{-A-c}, \end{aligned} \tag{14}$$

where

$$S_1^* = \frac{1}{2\pi i} \int_{\frac{1}{2}-ix^2}^{\frac{1}{2}+ix^2} f(s) w(z, h, s)\, ds \tag{15}$$

$$f(s) = \prod_{i=1}^{8} \left(\sum_{n_i \in I_i} \frac{\chi(n_i)a_i(n_i)}{n_i^s} \right) = \prod_{i=1}^{8} f_i(s)$$

$$w(z, h, s) = \frac{(z+h)^s - z^s}{s}.$$

For $\sigma = \frac{1}{2}$ we have

$$\max_{h\leqslant v}\max_{\frac{1}{2}x<z\leqslant x}|w(z,h,s)| \ll \min\left(\frac{y}{x^{\frac{1}{2}}}, \frac{x^{\frac{1}{2}}}{|t|}\right). \tag{16}$$

We subdivide the interval $[-x^2, x^2]$ in (15) into $O(L)$ not necessarily disjoint subintervals of the form $[T_1/2, T_1]$, $|T_1| > 10$, and the interval $[-10, 10]$. We deal first with the last interval.

Let us group the largest factor of $f(s)$ (i.e. the factor with the largest N_i) in one block, say $N(s)$, and all the other factors in a second block, $M(s)$. We denote by N and M the lengths of the Dirichlet polynomials $N(s)$ and $M(s)$ respectively.

Using the large sieve mean-value theorem (see Montgomery [9], theorem 7.1) we get, recalling (5),

$$\sum_{\frac{1}{2}Q'<q\leqslant Q'}\ \sideset{}{^*}\sum_{\chi(\mathrm{mod}\,q)}\max_{h\leqslant v}\max_{\frac{1}{2}x<z\leqslant x}\int_{\frac{1}{2}-10i}^{\frac{1}{2}+10i}|f(s)w(z,h,s)||ds|$$

$$\ll \frac{y}{x^{\frac{1}{2}}}\left(\sum_{\frac{1}{2}Q'<q\leqslant Q'}\ \sideset{}{^*}\sum_{\chi(\mathrm{mod}\,q)}\int_{\frac{1}{2}-10i}^{\frac{1}{2}+10i}|M(s)|^2|ds|\right)^{\frac{1}{2}}\left(\sum_{\frac{1}{2}Q'<q\leqslant Q'}\ \sum_{\chi(\mathrm{mod}\,q)}\int_{\frac{1}{2}-10i}^{\frac{1}{2}+10i}|N(s)|^2|ds|\right)^{\frac{1}{2}}$$

$$\ll \frac{y}{x^{\frac{1}{2}}}\big((Q')^2+M\big)^{\frac{1}{2}}\big((Q')^2+N\big)^{\frac{1}{2}}L^c \ll yQ'\left\{\frac{Q'}{x^{\frac{1}{2}}}+\left(\frac{\max(M,N)}{x}\right)^{\frac{1}{2}}+(Q')^{-1}\right\}L^c$$

$$\ll Q'yL^{-A-c}$$

since $N \ll yL^{-3A} \ll xL^{-3A}$ by (12) and $M \ll x^{7/8}$ by the construction of $M(s)$.

Now we consider the integrals

$$\sum_{\frac{1}{2}Q'<q\leqslant Q'}\ \sideset{}{^*}\sum_{\chi(\mathrm{mod}\,q)}\max_{h\leqslant v}\max_{\frac{1}{2}x<z\leqslant x}\int_{\frac{1}{2}+iT_1/2}^{\frac{1}{2}+iT_1}|f(s)w(z,h,s)||ds|,$$

with $10\leqslant T_1\leqslant x^2$, the treatment of the integrals with $-x^2\leqslant T_1\leqslant -10$ being clearly similar. Let

$$T = \frac{x}{y} \quad \text{and} \quad Y = \max\left(\frac{T_1}{T}, 1\right). \tag{17}$$

We will need the following

LEMMA. *If the Dirichlet polynomials*

$$M(s) = \sum_{M\ll m\ll M}\frac{\chi(m)a(m)}{m^s}, \quad N(s) = \sum_{N\ll n\ll N}\frac{\chi(n)b(n)}{n^s}$$

satisfy

i) $MN \leqslant x$ and $\sum_{M \ll m \ll M} \frac{|a(m)|^2}{m}, \quad \sum_{N \ll n \ll N} \frac{|b(n)|^2}{n} \ll L^c,$

ii) $\max(M, N) \ll \frac{x}{T_1} Y^2 L^{-3A},$

then

$$I(T_1) = \sum_{\frac{1}{2}Q' < q \leqslant Q'} \sum_{\chi (\mathrm{mod}\, q)}^{*} \max_{h \leqslant y} \max_{\frac{1}{2}x < z \leqslant x} \int_{\frac{1}{2}+i(T_1/2)}^{\frac{1}{2}+iT_1} |M(s)N(s)w(z, h, s)||ds| \ll yQ'L^{-A-c} \tag{18}$$

if $\psi \leqslant \theta - \frac{1}{2}$ *and* $B \geqslant cA$.

PROOF. Using the mean-value theorem in the same way as before, and recalling (5) and (16), we get

$$\begin{aligned} I(T_1) &\ll \min\left(\frac{y}{x^{\frac{1}{2}}}, \frac{x^{\frac{1}{2}}}{T_1}\right)\left((Q')^2 T_1 + M\right)^{\frac{1}{2}}\left((Q')^2 T_1 + N\right)^{\frac{1}{2}} L^c \\ &\ll \min\left(\frac{y}{x^{\frac{1}{2}}}, \frac{x^{\frac{1}{2}}}{T_1}\right)\left((Q')^2 T_1 + Q'(\max(M, N) T_1)^{\frac{1}{2}} + x^{\frac{1}{2}}\right) L^c \\ &\ll \min\left(\frac{y}{x^{\frac{1}{2}}}, \frac{x^{\frac{1}{2}}}{T_1}\right)\left\{Q' x^{\theta - \frac{1}{2}} L^{-B} Y x^{1-\theta} + Q' Y (x L^{-3A})^{\frac{1}{2}} + x^{\frac{1}{2}}\right\} L^c \\ &\ll yQL^{-A-c} \end{aligned}$$

if $\psi \leqslant \theta - \frac{1}{2}$ and $B \geqslant cA$.

The problem is then to show that it is always possible to group the factors of $f(s)$ into two blocks satisfying the conditions of the Lemma, since the Theorem clearly follows from (18). Moreover, it is easy to see that condition i) is certainly satisfied in our case at every possible grouping of terms $f_i(s)$, $i = 1, 2, \ldots, 8$.

We have to consider various cases, according to the length of the factors of $f(s)$. We recall that we may assume that $N_i < yL^{-3A}$ for every i.

Case I. There exists a factor $f_j(s)$ with $N_j > (T_1/Y^2)L^{3A}$. In this case let

$$N(s) = f_j(s) \quad \text{and} \quad M(s) = \prod_{i \neq j} f_i(s).$$

Then

$$\text{length of } N(s) = N_j < yL^{-3A} = \frac{x}{T} L^{-3A} \leqslant \frac{x}{T_1} YL^{-3A} \leqslant \frac{x}{T_1} Y^2 L^{-3A},$$

$$\text{length of } M(s) \ll \frac{x}{N_j} \leqslant \frac{x}{T_1} Y^2 L^{-3A},$$

and the Lemma is applicable in this case.

Case II.

$$N_i < \frac{T_1}{Y^2} \qquad \text{for every } i. \tag{19}$$

We have to split Case II into various subcases.

*Case II/*1. There exists some i with

$$N_i > x^{0.3} Y^{\frac{1}{2}} L^{-2A} \qquad (> (QT)^{\frac{1}{2}}). \tag{20}$$

This means that $1 \leqslant i \leqslant 4$. Let us choose one such factor, say

$$f_j(s) = \sum_{n_j \in I_j} \frac{\chi(n_j) \log^g n_j}{n_j^s},$$

$g = 1$ or $g = 0$ according to $j = 1$ or $2 \leqslant j \leqslant 4$.

We use the approximate functional equation for the L-functions given by Lavrik [8], theorem 1 and corollary 1. We consider explicitly only the case $g = 1$, the case $g = 0$ being very similar and even simpler.

Differentiating both sides of the approximate functional equation (5) of [8] and then using it twice for $x = N_j$ and $x = 2N_j$, and subtracting both relations we get

$$\left| \sum_{N_j < n \leqslant 2N_j} \frac{\chi(n_j) \log n_j}{n_j^{\frac{1}{2}+it}} \right| \ll L \left| \sum_{qt/4\pi N_j < n \leqslant qt/2\pi N_j} \frac{\bar{\chi}(n)}{n^{\frac{1}{2}-it}} \right| + \left| \sum_{qt/4\pi N_j < n \leqslant qt/2\pi N_j} \frac{\bar{\chi}(n) \log n}{n^{\frac{1}{2}-it}} \right| + L^{2A}. \tag{21}$$

(21) is obtained using Cauchy's integral expression of the derivative of an analytic function, choosing the path of integration as a circle of radius L^{-1}, together with the bounds (19) and (20). (We remark that it would be also possible to use the original form and partial summation).

Let $a_k^*(n) = \log^k n$, $k = 0$ or 1. Using Perron's formula again we have

$$\begin{aligned}
&\left| \sum_{qt/4\pi N_j < n \leqslant qt/2\pi N_j} \frac{\bar{\chi}(n) a_k^*(n)}{n^{\frac{1}{2}-it}} \right| \\
&\ll \left| \int_{-ix^{10}}^{ix^{10}} \left(\sum_{Q'T_1/20\pi N_j < n \leqslant Q'T_1/\pi N_j} \frac{\bar{\chi}(n) a_k^*(n)}{n^{\frac{1}{2}-it+iv}} \right) \frac{(qt/4\pi N_j)^{iv} - (qt/2\pi N_j)^{iv}}{iv} \, div \right| + x^{-8} \\
&\ll \int_{-x^{10}}^{x^{10}} \left| \sum_{Q'T_1/20\pi N_j < n \leqslant Q'T_1/\pi N_j} \frac{\bar{\chi}(n) a_k^*(n)}{n^{\frac{1}{2}-it+iv}} \right| \left| \frac{2^{iv} - 1}{v} \right| dv + x^{-8} \\
&\ll L \left| \sum_{Q'T_1/20\pi N_j < n \leqslant Q'T_1/\pi N_j} \frac{\bar{\chi}(n) a_k^*(n) n^{-iv_k}}{n^{\frac{1}{2}-it}} \right| + x^{-8}
\end{aligned}$$

for some $v_k \in [-x^{10}, x^{10}]$. Let $a_k'(n) = a_k^*(n)n^{-iv_k}$; from (21) and (22) we get

$$(23)\quad \left|\sum_{N_j<n\leqslant 2N_j} \frac{\chi(n)\log n}{n^{\frac{1}{2}+it}}\right| \ll L^2 \sum_{k=0,1}\left|\sum_{Q'T_1/20\pi N_j<n\leqslant Q'T_1/\pi N_j} \frac{\bar{\chi}(n)a_k'(n)}{n^{\frac{1}{2}-it}}\right| + L^{2A}.$$

The length of the new Dirichlet polynomial in (23) is then

$$(24)\qquad \ll \frac{Q'T_1}{N_j} \ll \frac{Q'T_1L^{2A}}{x^{0.3}Y^{\frac{1}{2}}} \ll x^{0.2}Y^{\frac{1}{2}}L^{2A-B}.$$

If we define the longest of the remaining polynomials as $M(s)$ and we group the other six polynomials into $N(s)$ then we have clearly for their length M and N respectively

$$M < \frac{T_1}{Y^2}L^{3A} < \frac{x}{T_1}Y^2L^{-7A}$$

and

$$N \ll \left(\frac{x}{x^{0.3}Y^{\frac{1}{2}}L^{-2A}}\right)^{6/7} \ll yL^{-7A} \ll \frac{x}{T_1}Y^2L^{-7A}$$

if $y \geqslant x^{0.6}L^B$, $B \geqslant cA$ as we supposed.

Hence we have, in the same way as the Lemma,

$$\sum_{\frac{1}{2}Q'<q\leqslant Q'}\ \sideset{}{^*}\sum_{\chi(\mathrm{mod}\, q)}\ \max_{h\leqslant y}\ \max_{\frac{1}{2}x<z\leqslant x} \int_{\frac{1}{2}+i(T_1/2)}^{\frac{1}{2}+iT_1} \left| L^{2A}\prod_{i=2}^{8} f_i(s)w(z,h,s)\right| |ds| \ll yQ'L^{-A-c}.$$

We are now ready to finish the proof.

We observe that if we apply the above reflection method to $j=1$, then, as one sees from (23), we obtain the sum of two shortened polynomials. However, we remark that the treatment of the two polynomials is completely analogous. Hence, for the sake of semplicity, we will always suppose to have a single polynomial.

Case II/1/a. There are at least two factors with $N_i > x^{0.3}Y^{\frac{1}{2}}L^{-2A}$.

Let $f_{j,1}$ and $f_{j,2}$ be two such factors. In this case we use the approximate functional equation for $f_{j,1}$ and afterwards we group the shortened polynomial in one block with $f_{j,2}$. Call this block $N(s)$, and let $M(s)$ be the block of all the other factors. Then, since $\theta \geqslant 3/5$, by (19) and (24)

$$\text{length of } N(s) \ll x^{0.2}Y^{\frac{1}{2}}L^{2A-B}x^{0.4}L^{3A} \ll yYL^{-3A} \ll \frac{x}{T_1}Y^2L^{-3A}$$

$$\text{length of } M(s) \ll \frac{x}{x^{0.6}YL^{-4A}} \ll \frac{x}{T_1}Y^2L^{-3A},$$

and the Lemma is applicable in this case.

Case II/1/b. There is just one factor with $N_i > x^{0.3} Y^{\frac{1}{2}} L^{-2A}$.

Again using the approximate functional equation we reflect this factor into a Dirichlet polynomial of length $\ll x^{0.2} Y^{\frac{1}{2}} L^{2A-B}$, and we are reduced to the next case, namely

Case II/2. $N_i < x^{0.3} Y^{\frac{1}{2}} L^{-2A}$ for every i.

We have to split up Case II/2 into two subcases.

Case II/2/a. There are at least two factors with $N_i > x^{0.2} L^{2A}$.

Then we group two of such factors in one block, say $N(s)$, and the remaining factors in another block, $M(s)$. Then

$$\text{length of } N(s) \ll x^{0.6} Y L^{-4A} \ll y Y L^{-3A},$$

$$\text{length of } M(s) \ll \frac{x}{x^{0.4} L^{4A}} = x^{0.6} L^{-4A} \ll y Y L^{-3A}$$

and the Lemma is applicable.

Case II/2/b. There is at most one factor with $N_i > x^{0.2} L^{2A}$.

Let us group this factor and afterwards fill the block, say $N(s)$, with other factors until we just reach $x^{0.4} L^{-A}$; the remaining factors form the block $M(s)$. Then

$$\text{length of } N(s) \ll x^{0.6} L^{A} \ll y Y L^{-3A},$$

$$\text{length of } M(s) \ll \frac{x}{x^{0.4} L^{-A}} = x^{0.6} L^{A} \ll y Y L^{-3A}$$

if $y \geqslant x^{0.6} L^{4A}$, and the Lemma is applicable.

The proof is now complete.

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