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Boundary Estimates for Solutions of Monge-Ampère Equations in the Plane.

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Dedicated to Professor E. HEINZ *on his sixtieth birthday*

Introduction and statement of the theorem.

Let Ω be a bounded open subset of the x, y-plane of class $C^{2,\alpha}$ $(0<\alpha<1)$. We shall consider the Dirichlet problem for elliptic Monge-Ampère equations

$$(1) \qquad Ar + 2Bs + Ct + (rt - s^2) = E\,, \qquad z|_{\partial\Omega} = \varphi\,,$$

for solutions $z(x, y) \in C^{2,\alpha}(\bar{\Omega})$ with boundary values in $C^{2,\alpha}(\partial\Omega)$. The coefficients A, B, C, E are assumed to be of class C^{α} with respect to the five variables x, y, z, p, q. Adopting Monge's notation, p, q; r, s, t represent the first and second derivatives of $z(x, y)$.

We shall impose the following quantitative *assumptions*:

The functions A, B, C, E are bounded in absolute value by a constant a and their Hölder semi-norms are bounded by b.

Ellipticity of (1) means that

$$\Delta\big(x, y, z(x, y), p(x, y), q(x, y)\big) := AC - B^2 + E \geqslant \frac{1}{c} > 0$$

for $(x, y) \in \Omega$.

Furthermore

$$\|z\|_{C^2(\Omega)} \leqslant K\,, \qquad \|\varphi\|_{C^{2,\alpha}(\partial\Omega)} \leqslant k\,.$$

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Then we can state the boundary estimates:

THEOREM. *The second derivatives of $z(x, y)$ satisfy the Hölder conditions $|r(x', y') - r(x'', y'')|, \ldots, |t(x', y') - t(x'', y'')| \leq H\big((x'-x'')^2 + (y'-y'')^2\big)^{\alpha/2}$ for $(x', y'), (x'', y'') \in \Omega$, where the constant H depends only on α, a, b, c, k, K and Ω.*

The proof consists of a refinement of the techniques developed in [9], [10], [11], where interior estimates were derived for applications to geometrical problems. However, the present paper is independent of the ones cited above, and we would like to note that the interior estimates can now be derived a little more simply by using the differential equation (5).

In addition to the works quoted in [11], we should mention Pogorelov [8], chapters X-XIII, who treated the Dirichlet problem for strongly elliptic Monge-Ampère equations. Aubin [2] and Delanoë [4] also treated the two-dimensional case. Of current interest are the boundary estimates in n variables, which have recently been derived by Caffarelli, Nirenberg and Spruck [3], and Krylov [6] (see also Delanoë [5]).

The purpose of the present paper is to cover also the case of merely Hölder continuous coefficients and $C^{2,\alpha}$-boundary in the plane. The case of differentiable data is contained in Nirenberg's work [7].

We shall use the *notation*

$$[z]^{\Omega}_{k,\alpha} := \sup \sum_{m+n=k} \frac{|(\partial^k z/\partial x^m \partial y^n)(x', y') - (\partial^k z/\partial x^m \partial y^n)(x'', y'')|}{\big((x'-x'')^2 + (y'-y'')^2\big)^{\alpha/2}}$$

for the Hölder semi-norms of $z(x, y)$ ($k = 0, 1, 2, \ldots$; $0 < \alpha < 1$). The letter C denotes various constants, which may change from line to line. Unless otherwise stated, constants are assumed to be ≥ 1.

1. – Proof of the theorem.

Let $D_R = D_R(x_0, y_0)$ be the circular disc of radius $R > 0$ and centre $(x_0, y_0) \in \bar{\Omega}$. The assumption, $\Omega \in C^{2,\alpha}$ $(0 < \alpha < 1)$, means that for some R_0, $0 < R_0 \leq 1$:

$$\Omega_{R_0} := \Omega \cap D_{R_0} = \{(x, y) \in D_{R_0} | G(x, y) < 0\},$$

where $G(x, y) \in C^{2,\alpha}(\bar{D}_{R_0})$, $G_x^2 + G_y^2 > 0$. We shall assume that

$$\|G\|_{C^{2,\alpha}(D_{R_0})} \leq \varkappa, \qquad G_x^2 \geq \frac{1}{\varkappa} \qquad \big((x, y) \in D_{R_0}\big).$$

Then the transformation ψ:

$$\begin{cases} \xi = G(x, y) \\ \eta = y - y_0 \end{cases} \quad \big((x, y) \in D_{R_0}\big),$$

straightens $\partial\Omega \cap D_{R_0}$:

LEMMA 1. (i) *ψ is a $C^{2,\alpha}$-diffeomorphism of D_{R_0} onto the image $\psi(D_{R_0})$, such that*

$$\psi(\Omega \cap D_{R_0}) \subset \{(\xi, \eta) | \xi < 0\},$$
$$\psi(\partial\Omega \cap D_{R_0}) \subset \{(\xi, \eta) | \xi = 0\}.$$

(ii) *Let (x', y'), $(x'', y'') \in D_{R_0}$, $\xi' = \xi(x', y'), \ldots, \eta'' = \eta(x'', y'')$.*

Then we have the dilatation estimates

$$(\xi' - \xi'')^2 + (\eta' - \eta'')^2 \leqslant \varkappa_1^2\big((x' - x'')^2 + (y' - y'')^2\big),$$
$$(x' - x'')^2 + (y' - y'')^2 \leqslant \varkappa_2^2\big((\xi' - \xi'')^2 + (\eta' - \eta'')^2\big),$$

with constants $\varkappa_1, \varkappa_2 \geqslant 1$, depending only on $\varkappa$.

(iii) *Hence the inclusions*

$$\psi(\Omega_{R/\varkappa_1}) \subset D_R^-(\xi_0, \eta_0), \qquad D_{R/\varkappa_2}^-(\xi_0, \eta_0) \subset \psi(\Omega_R)$$

hold for all R, $0 < R \leqslant R_0$. Here

$$D_R^-(\xi_0, \eta_0) := \{(\xi, \eta) \in D_R(\xi_0, \eta_0) | \xi < 0\}.$$

(iv) *The function*

$$\hat{z}(\xi, \eta) := z(x, y) - p(x_0, y_0)(x - x_0)$$

is of class $C^{2,\alpha}(\overline{D}^-_{R_0/\varkappa_2}(\xi_0, \eta_0))$, solving the Monge-Ampère equation

$$(\hat{z}_{\xi\xi} + \hat{C})(\hat{z}_{\eta\eta} + \hat{A}) - (\hat{z}_{\xi\eta} - \hat{B})^2 = \hat{\Delta}, \tag{2}$$

where

$$\begin{pmatrix} \hat{C}(\xi, \eta) & -\hat{B}(\xi, \eta) \\ -\hat{B}(\xi, \eta) & \hat{A}(\xi, \eta) \end{pmatrix} := \frac{1}{G_x^2} \begin{pmatrix} 1 & 0 \\ -G_y & G_x \end{pmatrix} \begin{pmatrix} C & -B \\ -B & A \end{pmatrix} \begin{pmatrix} 1 & -G_y \\ 0 & G_x \end{pmatrix},$$

and

$$\hat{\Delta}(\xi,\eta) := \frac{1}{G_x^2}\left(\Delta - \hat{z}_\xi\big(G_{yy}(r+C) - 2G_{xy}(s-B) + G_{xx}(t+A)\big) + \hat{z}_\xi^2(G_{xx}G_{yy} - G_{xy}^2)\right).$$

PROOF. The mean value theorem yields

$$\begin{pmatrix}\xi'-\xi''\\ \eta'-\eta''\end{pmatrix} = \begin{pmatrix}G_x(\tilde{x},\tilde{y}) & G_y(\tilde{x},\tilde{y})\\ 0 & 1\end{pmatrix}\begin{pmatrix}x'-x''\\ y'-y''\end{pmatrix},$$

where $(\tilde{x},\tilde{y})$ is a point on the segment joining (x',y') and (x'',y''). Parts (i)-(iii) are immediate consequences. Part (iv) follows easily by calculating

$$\begin{pmatrix}\hat{p}\\ \hat{q}\end{pmatrix} = \begin{pmatrix}G_x & 0\\ G_y & 1\end{pmatrix}\begin{pmatrix}\hat{z}_\xi\\ \hat{z}_\eta\end{pmatrix},$$

$$\begin{pmatrix}r & s\\ s & t\end{pmatrix} = \begin{pmatrix}G_x & 0\\ G_y & 1\end{pmatrix}\begin{pmatrix}\hat{z}_{\xi\xi} & \hat{z}_{\xi\eta}\\ \hat{z}_{\xi\eta} & \hat{z}_{\eta\eta}\end{pmatrix}\begin{pmatrix}G_x & G_y\\ 0 & 1\end{pmatrix} + \hat{z}_\xi\begin{pmatrix}G_{xx} & G_{xy}\\ G_{xy} & G_{yy}\end{pmatrix}. \quad \square$$

Note that we consider the function $\hat{z}(\xi,\eta)$, in order to ensure ellipticity of the equation (2) in a neighbourhood of (ξ_0,η_0). Therefore we have to deal with the boundary function

$$\hat{\varphi}(\xi,\eta) := \varphi(x,y) - p(x_0,y_0)(x-x_0),$$

defined for $(\xi,\eta)\in D_{R_0/\varkappa_2}(\xi_0,\eta_0)$. Bounds for the absolute values of $\hat{A}$, $\hat{B}$, $\hat{C}$ and for their Hölder semi-norms will be denoted by $\hat{a}$, $\hat{b}$ respectively. $\hat{K}$ is a bound for $\|\hat{z}\|_{C^2(\bar{D}_{R_0/\varkappa_2})}$, and $\hat{k}$ is a bound for $\|\hat{\varphi}\|_{C^{2,\alpha}(D_{R_0/\varkappa_2})}$.

We proceed to freeze the coefficients $\hat{A}$, $\hat{B}$, $\hat{C}$ by putting

$$\hat{A}_0 := \hat{A}(\xi_0,\eta_0),\ \dots,\ \hat{E}_0 := \hat{E}(\xi_0,\eta_0), \qquad \hat{\Delta}_0 := \hat{\Delta}(\xi_0,\eta_0),$$

where

$$\hat{E}(\xi,\eta) := \frac{1}{G_x^2}\left(E - \hat{z}_\xi\big(G_{yy}(r+C) - 2G_{xy}(s-B) + G_{xx}(t+A)\big) + \hat{z}_x^2(G_{xx}G_{yy} - G_{xy}^2)\right).$$

LEMMA 2. (i) *The function*

$$\tilde{z}(\xi,\eta) := \hat{z}(\xi,\eta) + \tfrac{1}{2}(\hat{C}_0\xi^2 - 2B_0\xi\eta + \hat{A}_0\eta^2) + 2(\hat{a}+\hat{K})\eta$$

solves the Monge-Ampère equation

$$(3)\qquad \tilde{z}_{\xi\xi}\tilde{z}_{\eta\eta} - \tilde{z}_{\xi\eta}^2 = \tilde{f}(\xi,\eta) \qquad ((\xi,\eta)\in \bar{D}_{R_0/\varkappa_2}),$$

where

$$\hat{f}(\xi,\eta) := \hat{\Delta}_0 + \big((\hat{A}_0-\hat{A})\hat{z}_{\xi\xi} + 2(\hat{B}_0-\hat{B})\hat{z}_{\xi\eta} + (\hat{C}_0-\hat{C})\hat{z}_{\eta\eta} - (\hat{E}_0-\hat{E})\big)\,.$$

(ii) *The equation* (3) *is elliptic, i.e., the inequality*

$$\tilde{f}(\xi,\eta) \geqslant \frac{1}{2\varkappa^2 c} =: \frac{1}{\tilde{c}}$$

holds for $(\xi,\eta)\in D^-_{\tilde{R}}(\xi_0,\eta_0)$, $\tilde{R} = \tilde{R}(\alpha, a, b, c, R_0, \varkappa, K)$.

(iii) *Furthermore we have*

$$\|\tilde{z}\|_{C^2(D^-_{\tilde{R}})} \leqslant \tilde{K}\,, \qquad \tilde{z}_\eta \geqslant 1 \qquad \big((\xi,\eta)\in D^-_{\tilde{R}}\big)\,,$$

where $\tilde{K}$ *depends only on known quantities.*

PROOF. Part (i) is a simple calculation. In order to show ellipticity, we make use of the inequality

$$|\hat{z}_\xi| \leqslant 2\hat{K}\big((\xi-\xi_0)^2 + (\eta-\eta_0)^2\big)^{\frac{1}{2}}\,.$$

We estimate

$$(4)\qquad \tilde{f}(\xi,\eta) \geqslant \frac{1}{\varkappa^2 c} - \Big(4\hat{b}\hat{K} + \varkappa\big(\varkappa_2 b + 2\hat{K}(4\varkappa(K+a) + 2\varkappa^2\hat{K})\big)\Big) \cdot\big((\xi-\xi_0)^2 + (\eta-\eta_0)^2\big)^{\alpha/2} \geqslant \frac{1}{\varkappa^2 c} - \frac{1}{2\varkappa^2 c} = \frac{1}{\tilde{c}}\,,$$

if $(\xi,\eta)\in D^-_{\tilde{R}}(\xi_0,\eta_0)$,

$$\tilde{R} := \min\left\{\frac{R_0}{\varkappa_2}\,,\ \frac{1}{(2C\varkappa^2 c)^{1/\alpha}}\right\}\,,$$

where C is the constant appearing in (4). This proves the lemma.

Now we can apply the transformation T:

$$\begin{cases} u = \xi \\ v = \tilde{z}_\eta(\xi,\eta) \end{cases}$$

to the function $\tilde{z}(\xi,\eta)$ in $D^-_{\tilde{R}}(\xi_0,\eta_0)$. The following lemma lists some properties. Compare also the transformation lemma of [11].

LEMMA 3. (i) *T maps $D^-_{\tilde{R}}(\xi_0,\eta_0)$ diffeomorphically onto the image $T(D^-_{\tilde{R}})$,* *such that*

$$T(D^-_{\tilde{R}}) \subset \{(u, v) | u < 0\} ,$$

$$T(D^-_{\tilde{R}} \cap \{\xi = 0\}) \subset \{(u, v) | u = 0\} .$$

(ii) *For* (ξ', η'), $(\xi'', \eta'') \in D^-_{\tilde{R}}$, *we have the dilatation estimates*

$$(u' - u'')^2 + (v' - v'')^2 \leqslant \gamma_1^2 \big((\xi' - \xi'')^2 + (\eta' - \eta'')^2\big) ,$$

$$(\xi' - \xi'')^2 + (\eta' - \eta'')^2 \leqslant \gamma_2^2 \big((u' - u'')^2 + (v' - v'')^2\big) ,$$

with constants $\gamma_1, \gamma_2 \geqslant 1$, *depending only on* $\tilde{c}$, $\tilde{K}$.

(iii) *Hence the inclusions*

$$T\big(D^-_{R/\gamma_1}(\xi_0, \eta_0)\big) \subset D^-_R(u_0, v_0) ,$$

$$D^-_{R/\gamma_2}(u_0, v_0) \subset T\big(D^-_R(\xi_0, \eta_0)\big)$$

hold for all R, $0 < R \leqslant \tilde{R}$.

(iv) *The function* $\eta(u, v) \in C^{1,\alpha}\big(\bar{D}^-_{\tilde{R}/\gamma_2}(u_0, v_0)\big)$ *is a weak solution of the equation*

$$\eta_{uu} + (\hat{f}\eta_v)_v = 0 . \tag{5}$$

PROOF. For later purposes let us only note that

$$\begin{pmatrix} \xi_u & \xi_v \\ \eta_u & \eta_v \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\tilde{z}_{\xi\eta}/\tilde{z}_{\eta\eta} & 1/\tilde{z}_{\eta\eta} \end{pmatrix} . \tag{6}$$

The equation (5) follows easily. □

We proceed to calculate the boundary values $\phi(v)$ of $\eta(u, v)$. Assuming that we can take $\gamma_2/2$ instead of γ_2, we have

$$\phi(v) = \eta = \tilde{z}_\eta^{-1}(0, v) = \tilde{\varphi}_\eta^{-1}(0, v) \qquad \big((0, v) \in D_{2\tilde{R}/\gamma_2}(u_0, v_0)\big) ,$$

where

$$\tilde{\varphi}(\xi, \eta) = \hat{\varphi}(\xi, \eta) + \tfrac{1}{2}\hat{A}_0\eta^2 + 2(\hat{a} + \hat{K})\eta \qquad \big((\xi, \eta) \in D_{\tilde{R}}(\xi_0, \eta_0)\big) .$$

It is convenient to extend $\phi(v)$ by setting

$$\phi(u, v) := \phi(v) \qquad \big((u, v) \in D_{\tilde{R}/\gamma_2}(u_0, v_0)\big) .$$

Furthermore we calculate

$$\phi_v(u, v) = \frac{1}{\tilde{z}_{\eta\eta}(0, \eta)} = \frac{1}{\tilde{\varphi}_{\eta\eta}(0, \eta)}.$$

We introduce zero boundary data by

$$\mathring{\eta}(u, v) := \eta(u, v) - \phi(u, v)$$

and rewrite the equation (5):

LEMMA 4. (i) $\mathring{\eta}(u, v)$ *solves the equation*

$$\mathring{\eta}_{uu} + (\hat{\Delta}_0 \mathring{\eta}_v)_v = g_v, \tag{7}$$

where

$$g(u, v) := \big((\hat{A} - \hat{A}_0)\hat{z}_{\xi\xi} + 2(\hat{B} - \hat{B}_0)\hat{z}_{\xi\eta} + (\hat{C} - \hat{C}_0)\hat{z}_{\eta\eta} - (\hat{E} - \hat{E}_0)\big)\frac{1}{\tilde{z}_{\eta\eta}} - \frac{\hat{\Delta}_0}{\tilde{\varphi}_{\eta\eta}(0, \eta)}.$$

(ii) *The equation* (7) *is elliptic, i.e., the inequalities*

$$\frac{1}{(\varkappa c)^2} \leqslant \xi_1^2 + \hat{\Delta}_0 \xi_2^2 \leqslant (2\varkappa a)^2$$

hold for $\xi_1^2 + \xi_2^2 = 1$.

Now we can apply the Schauder estimates of Agmon-Douglis-Nirenberg [1], chapter III, for equations of divergence structure. By employing a version of [9], auxiliary theorem 4 of the appendix, we obtain

LEMMA 5. *The inequalities*

$$[\mathring{\eta}]_{1,\alpha}^{D_{R/4\varkappa^2 ac}^{-}(u_0, v_0)} \leqslant C(\varkappa, a, c)\left(\frac{[\mathring{\eta}]_0^{D^-}}{R^{1+\alpha}} + \frac{[g]_0^{D^-}}{R^\alpha} + [g]_\alpha^{D^-}\right)$$

hold for $0 < R \leqslant \tilde{R}/\gamma_2$, *where* $D^- = D_R^-(u_0, v_0)$.

By virtue of

$$|\tilde{z}_{\eta\eta}| \geqslant \frac{1}{\tilde{c}\tilde{K}},$$

we can estimate the quantities $[\mathring{\eta}]_0^{D^-}$, $[g]_0^{D^-}$ and $\|\phi\|_{C^{1,\alpha}(D^-)}$. Then we reintroduce the variables ξ, η in order to obtain the inequalities

$$[\eta_u]_\alpha^{D_R^-/\tilde\gamma(\xi_0,\eta_0)} + [\eta_v]_\alpha^{D_R^-/\tilde\gamma(\xi_0,\eta_0)} \leqslant C\left(\frac{1}{R^{1+\alpha}} + [g]_\alpha^{D_R^-}\right)$$

for $0 < R \leqslant \tilde R$. Here $\tilde\gamma = 4\varkappa^2 ac\gamma_1\gamma_2$, and $C = C(\alpha, a, b, c, \varkappa, k, K)$.

We proceed to estimate $[g]_\alpha^{D_R^-}$ in terms of $[\hat z]_{2,\alpha}^{D_R^-}$. For (ξ', η'), $(\xi'', \eta'') \in D_R^-(\xi_0, \eta_0)$, we set

$$\hat A' := \hat A(\xi', \eta'), \ldots, \hat z'_{\xi\xi} := \hat z_{\xi\xi}(\xi', \eta'), \ldots, \varphi''_{\eta\eta} := \varphi_{\eta\eta}(0, \eta'') .$$

Then we have

$$\begin{aligned}
|g(\xi', \eta') - g(\xi'', \eta'')| &\leqslant |(\hat A' - \hat A'')\hat z'_{\xi\xi} + \ldots - (\hat E' - \hat E'')| \left|\frac{1}{\tilde z'_{\eta\eta}}\right| \\
&\quad + \left|(\hat A'' - \hat A_0)\left(\frac{\hat z'_{\xi\xi}}{\tilde z'_{\eta\eta}} - \frac{\hat z''_{\xi\xi}}{\tilde z''_{\eta\eta}}\right) + \ldots - (\hat E'' - \hat E_0)\left(\frac{1}{\tilde z'_{\eta\eta}} - \frac{1}{\tilde z''_{\eta\eta}}\right)\right| + \hat\Delta_0 \left|\frac{1}{\tilde\varphi'_{\eta\eta}} - \frac{1}{\tilde\varphi''_{\eta\eta}}\right| \\
&\leqslant C\left(1 + R^\alpha [\hat z]_{2,\alpha}^{D_R^-}\right)\left((\xi' - \xi'')^2 + (\eta' - \eta'')^2\right)^{\alpha/2}.
\end{aligned}$$

Because of (6), we can also estimate

$$[\tilde z_{\eta\eta}]_\alpha^{D_R^-/\tilde\gamma(\xi_0,\eta_0)} \leqslant C[\eta_v]_\alpha^{D_R^-/\tilde\gamma(\xi_0,\eta_0)} .$$

Whence, using (6) again, and by taking the differential equation (3) into account, that

$$[\tilde z]_{2,\alpha}^{D_R^-/\tilde\gamma(\xi_0,\eta_0)} \leqslant C\left(\frac{1}{R^{1+\alpha}} + R^\alpha [\hat z]_{2,\alpha}^{D_R^-}\right) \quad (0 < R \leqslant \tilde R) .$$

Re-introducing the variables x, y we arrive at

LEMMA 6. *The inequalities*

(8) $$[z]_{2,\alpha}^{\Omega_{R/\gamma}(x_0,y_0)} \leqslant C\left(\frac{1}{R^{1+\alpha}} + R^\alpha [z]_{2,\alpha}^{\Omega_R(x_0,y_0)}\right)$$

hold for $0 < R \leqslant \varkappa_2 \tilde R$, *where* $\gamma = 4\varkappa^2 ac\gamma_1\gamma_2\varkappa_1\varkappa_2$, *and* C *depends only on the data.*

PROOF OF THE THEOREM. A standart covering argument shows, that we can choose R_0 independent of (x_0, y_0). Let

$$R := \min\left\{\varkappa_2 \tilde{R},\ \frac{1}{(2C)^{1/\alpha}}\right\},$$

where C is the constant appearing in (8). There exist two points (x', y'), $(x'', y'') \in \bar{\Omega}$ such that

$$[z]^{\Omega}_{2,\alpha} = \frac{|r(x', y') - r(x'', y'')| + 2\,|s(x', y') - s(x'', y'')| + |t(x', y') - t(x'', y'')|}{\left((x' - x'')^2 + (y' - y'')^2\right)^{\alpha/2}}.$$

In the case

$$(x' - x'')^2 + (y' - y'')^2 < \left(\frac{R}{\gamma}\right)^2,$$

we may conclude from (8) the asserted estimate

$$[z]^{\Omega}_{2,\alpha} \leqslant C(\alpha, a, b, c, k, K, \Omega).$$

The theorem is thus proved by taking also the case

$$(x' - x'')^2 + (y' - y'')^2 \geqslant \left(\frac{R}{\gamma}\right)^2$$

into account.

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